AST4320 COSMOLOGY AND EXTRAGALACTIC ASTRONOMY ASSIGNMENT 1

Deadline: Friday, September 13

1. Exercises to Support Lecture 1-5.

Exercise 1. Let the unperturbed quantities ρ_0 (density), ϕ_0 (gravitational potential), $\mathbf{v}_0$ (velocity) and p_0 (pressure) obey the continuity, Euler & Poission equations:

$$\frac{d\rho_0}{dt} + \rho_0 \nabla \cdot \mathbf{v}_0 = 0 \tag{1}$$
$$\frac{d\mathbf{v}_0}{dt} = -\frac{1}{\rho_0}\nabla p_0 - \nabla\phi_0$$
$$\nabla^2\phi_0 = 4\pi G\rho_0,$$

where $\frac{d}{dt}$ denotes the 'total' derivative, which is defined as $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$.

(1) Using the continuity equation, show that in a Universe that is undergoing Hubble expansion (i.e. $\mathbf{v} = H\mathbf{r}$), that $\bar{\rho}(t) = \bar{\rho}(t = t_0)a^{-3}$, where t_0 denotes the age of the Universe today.

(2) In the lecture I introduced perturbed $\rho \equiv \rho_0 + \delta\rho$ (density), $\phi \equiv \phi_0 + d\phi$ (gravitational potential), $\mathbf{v} \equiv v_0 + \delta v$ (velocity) and $p_0 + \delta p$ (pressure), and showed how the perturbed overdensity $\delta \equiv \delta\rho/\bar{\rho}$ became:

$$\frac{d\delta}{dt} = -\nabla \cdot \delta\mathrm{v},$$

where $\delta \equiv \delta\rho/\rho_0$.

Derive the expressions for the Perturbed Poisson & Euler equations (see Eq 29 in the lecture notes 1 and 2), by substituting perturbed quantities into them, and simplifying as much as possible.

Exercise 2. In the lecture I sketched how we can get a second order differential equation that describes $\delta(t)$

$$\frac{d^2\delta}{dt^2} + 2\frac{\dot{a}(t)}{a(t)}\frac{d\delta}{dt} = \delta(4\pi G\rho_0 - k^2c_{\mathrm{s}}^2),$$

where $k \equiv 2\pi/\lambda$ denotes the wavenumber of the perturbation.

(1) Use the lecture notes to obtain an expression for the term $\frac{\dot{a}}{a}$ from the Friedmann equations for a cosmology with $(\Omega_{\rm m}, \Omega_\Lambda) = (1.0, 0.0)$, $(\Omega_{\rm m}, \Omega_\Lambda) = (0.3, 0.7)$, and with $(\Omega_{\rm m}, \Omega_\Lambda) = (0.8, 0.2)$.

(2) Insert this expression into the differential equation. Solve and plot the time evolution of $\log \delta(a)$ vs $\log a$ in both cosmologies, by numerically integrating forward in time the diferential equation. For the boundary conditions, you can assuming that $\log \delta = -3.0$ at $\log a = -3.0$, and that the perturbation of interest is much larger than the Jeans length (i.e. ignore the pressure term $k^2 c_{\rm s}^2$). For the boundary condition for $\dot{\delta}$ at $\log a = -3.0$, assume for simplicity that $\delta \propto a$ at early times, and derive the resulting boundary condition for $\dot{\delta}$ from this.

(3) The growth factor f is defined as $f \equiv \frac{d \ln \delta}{d \ln a}$. Plot f as a function of redshift z.

Exercise 3. In the lecture we derived expressions for the Jeans length & mass. After decoupling (which occurred at $z = 1090$) the temperatures of the cosmic radiation background and the gas evolved differently.

(1) Derive & plot the time-evolution of the gas and radiation temperatures within the range $\log a = -4.0 - 0.0$, assuming that both fluids evolve adiabatically during the expansion of the Universe.

(2) Obtain expressions for the $z-$dependence of the cosmological Jeans mass & length. Compare these to their evolution (and amplitude) before decoupling/recombination.

Exercise 4. We followed the non-linear time-evolution of a spherical overdensity inside an Einstein-de-Sitter $[\Omega_{\rm m} = 1.0$ and $\Omega_\Lambda = 0.0]$. We assumed that this overdensity was confined to a sphere of radius $R(t) \equiv b(t)R_0 \equiv b(t)$ (i.e. we define $R_0 = 1$). Birkhoff's law states that this overdensity effectively behaves as a closed-universe with $\Omega_{\rm m} > 1.0$. Note that we introduced the 'local' scale factor $b(t)$ for clarity. We showed in the lecture that initially $b(t) \sim a(t)$, where $a(t)$ denotes the scale factor of the background Universe. The acceleration of the radius of the sphere is given by

$$\ddot{R} = -\frac{GM}{R^2},$$

where M is the total enclosed mass, and is therefore $M = \frac{4\pi}{3}\rho_{\rm m,0} R_0^3 = \frac{4\pi}{3}\rho_{\rm m,0}$. Show that the following parameterised solutions satisfy the above equation.

$$\begin{aligned} R &= A(1 - \cos\theta) \\ t &= B(\theta - \sin\theta) \\ A^3 &= GMB^2. \end{aligned}$$

Exercise 5. Using the parametrised solution for the collapse of a spherical top-hat, compute the infall velocity v when material first reaches the virial radius. Phrase your answer in terms of G, M and R.

Exercise 6. Show that the gravitational binding energy of uniform sphere of radius R and mass M equals $U = -\frac{3GM^2}{5R}$.