Exel

$$W(x) = \begin{cases} 1 & |x| < R \\ 0 & \text{else} \end{cases}$$

The ~~Fourier~~ Fourier tranformed of this window function from definition:

$$\tilde{W}(k) = \int_{-\infty}^{\infty} W(x) e^{-ikx} dx \quad ,$$

piecewise function, split in three regions

$$= \int_{-\infty}^{-R} 0 \cdot e^{-ikx} dx + \int_{-R}^{R} 1 \cdot e^{-ikx} dx + \int_{R}^{\infty} 0 \cdot e^{-ikx} dx$$

$$= \int_{-R}^{R} e^{-ikx} dx = \frac{i e^{-ikx}}{k} \Big|_{-R}^{R} = \frac{i}{k}\left[e^{-ikR} - e^{ikR} \right]$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

or:

$$= -\frac{1}{ki}\left[e^{ikR} - e^{-ikR} \right]$$

$$\tilde{W}(k) = \frac{2\sin(kR)}{k}$$

Problem when $k \to 0$?
↳ Test with L'Hôpitals:

$$\lim_{k\to 0} \tilde{W}(k) = \lim_{k\to 0} \frac{\frac{d}{dk} 2\sin kR}{\frac{d}{dk} k} = \lim_{k \to 0} \frac{2R\cos(kR)}{1}$$

$$= 2R$$

↳ Well behaved on hole real line.

Finding the FWHM numerically is quite easy.
As the maximum is at $k=0$, we find the index of k where $\hat{w}(k) = \frac{2R}{2} = R$.

Then the full width here is twice the value for k at $\hat{w}(k) = R$.

$(\hat{w}(0) = \hat{w}_{max} = 2R)$

Numerically we find $FWHM = \underline{3,79}$

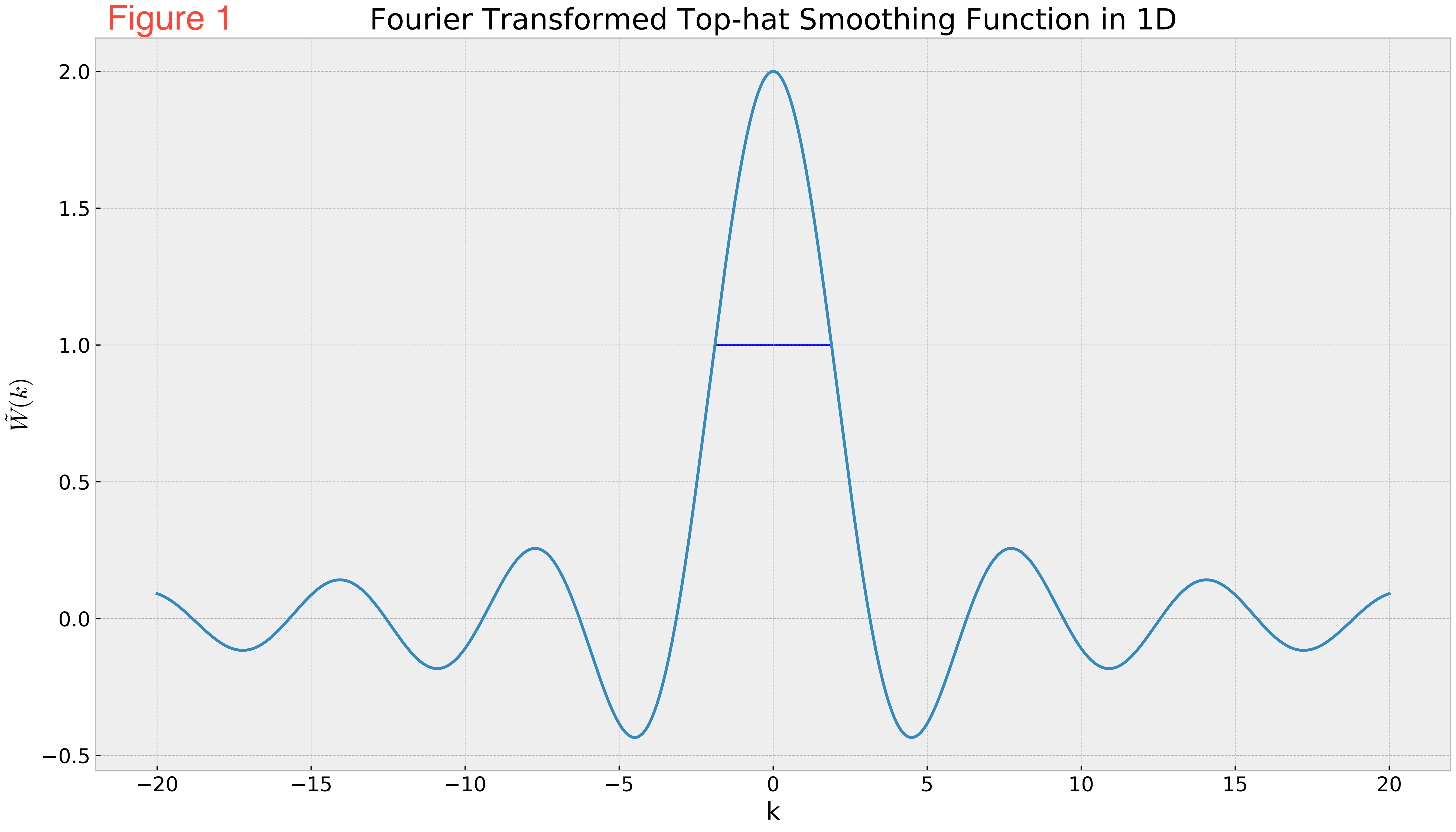
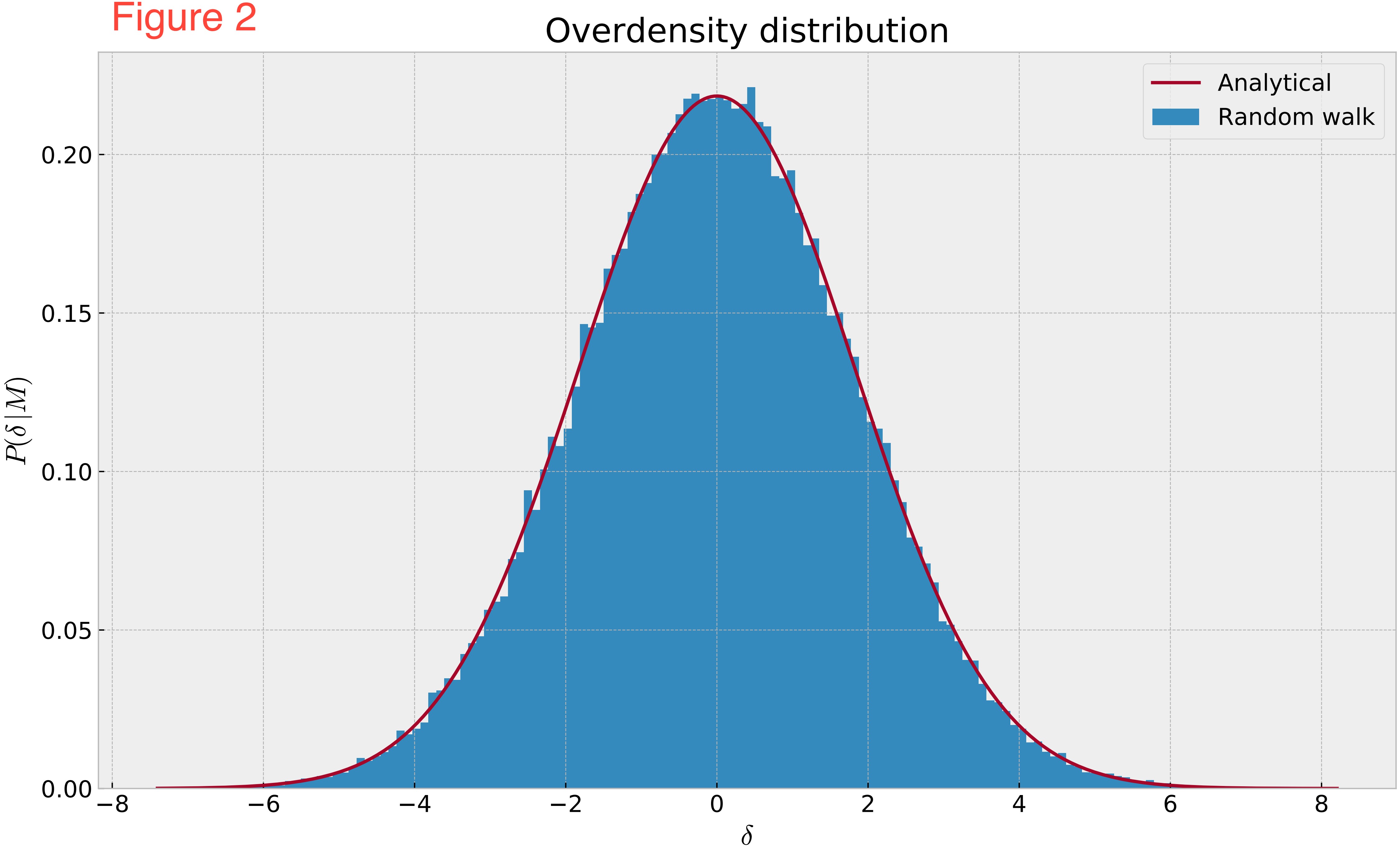


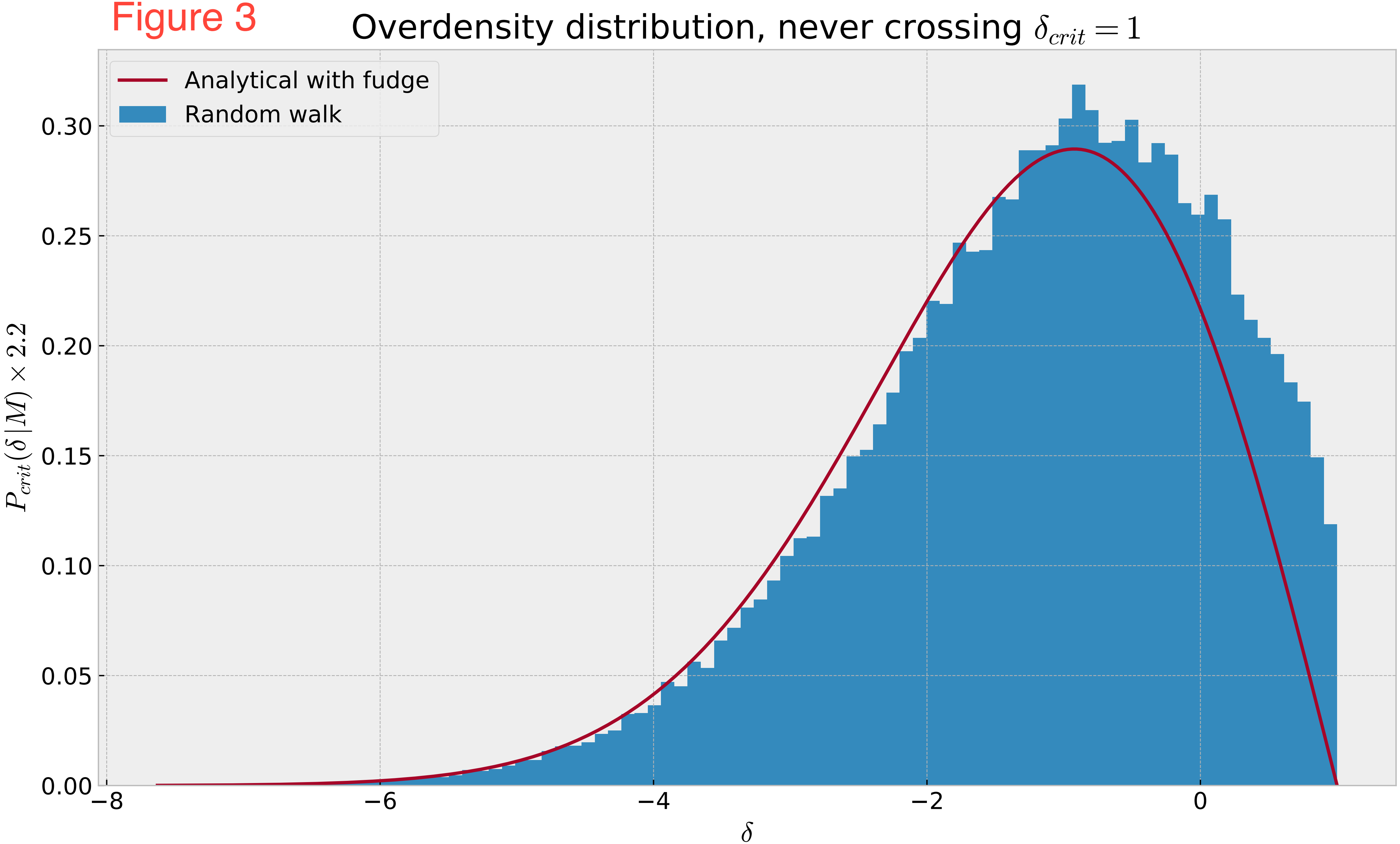
Figure 1
Fourier Transformed Top-hat Smoothing Function in 1D
$\tilde{W}(k)$
k
2.0
1.5
1.0
0.5
0.0
−0.5
−20
−15
−10
−5
0
5
10
15
20

Figure 2



Overdensity distribution
Analytical
Random walk
0.20
0.15
0.10
0.05
0.00
−8
−6
−4
−2
0
2
4
6
8
$P(\delta\,|\,M)$
δ

Figure 3



Overdensity distribution, never crossing $\delta_{crit} = 1$
Analytical with fudge
Random walk
$P_{crit}(\delta|M) \times 2.2$
δ
0.00
0.05
0.10
0.15
0.20
0.25
0.30
−8
−6
−4
−2
0

Exe2

$$\sigma^2 = \frac{\pi}{S_c^4} < 10^{-4}, \quad S_c = \left(\frac{\pi}{\sigma^2}\right)^{1/4} = \frac{2\pi}{k}$$

$$\hookrightarrow k = 2\pi \left(\frac{\pi}{\sigma^2}\right)^{-1/4}$$

I start by choosing $\sigma^2_{init} = 0{,}95 \cdot 10^{-4}$

$$\varepsilon = \begin{cases} 0{,}1 & \text{for crossing} \\ 0{,}05 & \text{for non-crossing} \end{cases}$$

Using the provided steps I get, using script exe2.py, the random walk results included in fig 2 and 3.

In fig. 2, the distribution for δ clearly match the analytical expression for the GRF with mean zero and $\sigma^2 = \frac{\pi}{S_c^4}$, where the last obtained value for the smoothing scale was used.

When only keeping walks with $\delta_{nc} < \delta_{crit} = 1$, we find the distribution seen in fig 3. Here the analytical expression doesn't fit so well, and I use a fudge factor of 2.2 to force it to fit better. We see that the calculated distribution is a bit skewed to the right, with a bit too sharp cut of at $\delta_{nc} = 1$. This cut of is expected from the filtered restrictions we implement in the code, but should ideally be a bit smoother. I've reduced the ε-value for this run, which helps but also slows down the code significantly. With even lower ε the results should look better.

Exe 3

a) From exercise 2 we had distribution $P_{nc}(\delta|M)$, which gives the probability of finding/having $\delta < \delta_{crit}$. These δx are not over the critical density, which means the mass at position $\vec{x}$ with $\delta < \delta_{crit}$ should probably not collapse, with probability

$$\int_{-\infty}^{\delta_{crit}} P_{nc}(\delta|M)\,d\delta$$

The mass at $\vec{x}$ does either collapse, or doesn't collapse, so the probability for collapse must be 1 minus the probability of not collapsing. This gives us the probability

$$P(>M) = 1 - \int_{-\infty}^{\delta_{crit}} P_{nc}(\delta|M)\,d\delta$$

Scanned with CamScanner

b)

Using the expression from a) we have

$$P(>M) = 1 - \int_{-\infty}^{\delta_c} P_{nc}(\delta | M)\, d\delta$$

$$= 1 - \int_{-\infty}^{\delta_c} \frac{d\delta}{\sqrt{2\pi}\,\sigma}\left(\exp\left(-\frac{\delta^2}{2\sigma^2}\right) - \exp\left(-\frac{(2\delta_c - \delta)^2}{2\sigma^2}\right)\right)$$

$$= \underbrace{1 - \int_{-\infty}^{\delta_c} \frac{\exp\left(-\frac{\delta^2}{2\sigma^2}\right)}{\sqrt{2\pi}\,\sigma}\, d\delta}_{I_1} + \underbrace{\int_{-\infty}^{\delta_c} \frac{\exp\left(-\frac{(2\delta_c - \delta)^2}{2\sigma^2}\right)}{\sqrt{2\pi}\,\sigma}\, d\delta}_{I_2}$$

Now the integral is separated into two similar pieces, wich we solve separately:

I_1: Perform substitution
$$\begin{cases} u = \dfrac{\delta}{\sqrt{2}\,\sigma} \\ \sqrt{2}\,\sigma\, du = d\delta \\ u(\delta_c) = \dfrac{\delta_c}{\sqrt{2}\,\sigma} = \dfrac{\nu}{\sqrt{2}} \end{cases}$$

$$\Rightarrow I_1 = \int_{-\infty}^{\frac{\nu}{\sqrt{2}}} \frac{\exp(-u^2)\,\cancel{\sqrt{2}\,\sigma}\, du}{\cancel{\sqrt{2}\,\sigma}\sqrt{\pi}} = \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\nu/\sqrt{2}} \exp(-u^2)\, du$$

$$= \frac{1}{2}\left[\underbrace{\frac{2}{\sqrt{\pi}}\int_{-\infty}^{0} \exp(-u^2)\, du}_{\operatorname{erf}(x)\big|_{-\infty}^{0} = 0-(-1) = 1} + \underbrace{\frac{2}{\sqrt{\pi}}\int_{0}^{\nu/\sqrt{2}} \exp(-u^2)\, du}_{\operatorname{erf}\left(\frac{\nu}{\sqrt{2}}\right)}\right]$$

per def

$$\Rightarrow I_1 = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right]$$

I_2: Similar integrand, similar substitution

$$u = \frac{2\delta_c - \delta}{\sqrt{2}\,\sigma} \Rightarrow \frac{du}{d\delta} = \frac{-1}{\sqrt{2}\,\sigma}$$

$$-\sqrt{2}\,\sigma\, du = d\delta$$

upper: $u(\delta_c) = \frac{1\delta_c}{\sqrt{2}\,\sigma} = \frac{\nu}{\sqrt{2}}$

lower: $u(-\infty) = \lim_{\delta \to -\infty} \frac{2\delta_c - \delta}{\sqrt{2}\,\sigma} \Rightarrow \infty$! change lower limit!

$$I_2 = \int_{+\infty}^{\nu/\sqrt{2}} \frac{\exp(-u^2)(-\sqrt{2}\,\sigma)\,du}{\sqrt{2\pi}\,\sigma} = -\int_{+\infty}^{\nu/\sqrt{2}} \frac{\exp(-u^2)\,du}{\sqrt{\pi}}$$

Turn around the limits using the sign

$$= \int_{\nu/\sqrt{2}}^{\infty} \frac{\exp(-u^2)}{\sqrt{\pi}}\,du = \frac{1}{2}\,\underbrace{\frac{2}{\sqrt{\pi}}\int_{\nu/\sqrt{2}}^{\infty} \exp(-u^2)\,du}_{\text{complementary error function } \mathrm{erfc}(\nu/\sqrt{2})}$$

complementary error function $\mathrm{erfc}(\nu/\sqrt{2})$

defined $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$

$$= \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right]$$

Combining the results:

$$P(>M) = 1 - I_1 + I_2 = 1 - \frac{1}{2}\left[\cancel{1} + \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right] + \frac{1}{2}\left[\cancel{1} - \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)\right]$$

$$= 1 - \mathrm{erf}\left(\frac{\nu}{\sqrt{2}}\right)$$