Exe 1

(1) Continuity eq: $\frac{d\bar{\rho}}{dt} + \bar{\rho}\,\vec{\nabla}\cdot\vec{v} = 0$

Huble expansion: $\vec{v} = H\vec{r} = \frac{\dot{a}}{a}\vec{r}$

$$\Rightarrow \frac{d\bar{\rho}}{dt} + \bar{\rho}\,\vec{\nabla}\cdot(H\vec{r}) = 0 \quad \Big|\cdot\frac{1}{\bar{\rho}}$$

$$\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dt} + \frac{\dot{a}}{a}\vec{\nabla}\cdot\vec{r} = 0 \quad , \quad \vec{\nabla}\cdot\vec{r} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}x_i = 3$$

$$\hookrightarrow \frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dt} + \frac{3}{a}\frac{da}{dt} = 0$$

$$\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{da}\cancel{\frac{da}{dt}} = -3\,a^{-1}\cancel{\frac{da}{dt}} \Rightarrow \int_{\bar{\rho}}^{\bar{\rho}_0}\frac{1}{\bar{\rho}}\,d\bar{\rho} = -3\int_{a}^{a_0}\frac{1}{a}\,da$$

$$\Rightarrow \ln\left(\frac{\bar{\rho}_0}{\bar{\rho}}\right) = -3\ln\left(\frac{a_0}{a}\right) = 3\ln a \quad , \quad (a_0 = 1)$$

$$\hookrightarrow \frac{\bar{\rho}_0}{\bar{\rho}} = \left(e^{\ln a}\right)^3 = a^3 \Rightarrow \underline{\underline{\bar{\rho}(t) = \bar{\rho}(t=t_0)\,a^{-3}}}$$

(2) Perturbed quantities, $Q = \bar{Q} + \delta Q$

Poisson eq:

$$\nabla^2 \Phi = 4\pi G \rho \quad \Rightarrow \quad \underbrace{\nabla^2 (\bar{\Phi} + \delta\Phi)}_{\text{Linear operator}} = 4\pi G(\bar{\rho} + \delta\rho)$$

$$\cancel{\nabla^2 \bar{\Phi}} + \nabla^2 \delta\Phi = 4\pi G(\cancel{\bar{\rho}} + \delta\rho)$$

| Unperturbed eq

$$\Rightarrow \quad \underline{\nabla^2 \delta\phi = 4\pi G \delta\rho}$$

Euler eq:

Let me write the velocity as $\vec{v} = \bar{v} + \delta v$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{1}{\rho} \nabla P - \nabla \phi$$

Inserting perturbations and clean up:

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial \delta v}{\partial t} + (\bar{v} + \delta v) \cdot \vec{\nabla} (\bar{v} + \delta v) = -\frac{1}{\bar{\rho} + \delta\rho} \nabla (P + \delta P) - \nabla(\bar{\phi} + \delta\phi)$$

$$= \frac{\partial \bar{v}}{\partial t} + \frac{\partial \delta v}{\partial t} + \bar{v} \cdot \vec{\nabla}(\bar{v} + \delta v) + \delta v \cdot \vec{\nabla}(\bar{v} + \delta v)$$

$$= \underbrace{\frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \vec{\nabla} \bar{v}}_{\frac{d\bar{v}}{dt}} + \underbrace{\bar{v} \cdot \vec{\nabla} \delta v + \frac{\partial \delta v}{\partial t} + \delta v \cdot \vec{\nabla} \delta v}_{\frac{d\delta v}{dt}} + \delta v \cdot \vec{\nabla} \bar{v}$$

Unperturbed total derivative

Perturbed total derivative

Also: $\frac{1}{\bar{\rho}+\delta\rho} = \frac{1}{\bar{\rho}(1+\frac{\delta\rho}{\bar{\rho}})} \simeq \frac{1}{\bar{\rho}}\left(1-\frac{\delta\rho}{\bar{\rho}}\right) \simeq \frac{1}{\bar{\rho}}$

as $\bar{\rho} \gg \delta\rho$

↳ I then can simplify the eq:

$$\cancel{\frac{d\bar{v}}{dt}} + \frac{d\delta v}{dt} + \delta v \cdot \vec{\nabla}\bar{v} = -\frac{1}{\bar{\rho}}\nabla(\cancel{\bar{P}}+\delta P) - \nabla(\cancel{\bar{\phi}}+\delta\phi)$$

Unperturbed eq. leaves:

$$\frac{d\delta v}{dt} + \delta v \cdot \vec{\nabla}\bar{v} = -\frac{1}{\bar{\rho}}\nabla\delta P - \nabla\delta\phi$$

Exe 2

(1) Friedmann 1: $\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_{i0}\left(\frac{a}{a_0}\right)^{-3(1+\omega_i)}$

$i = [m, \Lambda]$, $a_0 = 1$, $\omega_m = 0$, $\omega_\Lambda = -1$, $H = \frac{\dot{a}}{a}$

↳ $\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\Omega_m a^{-3} + \Omega_\Lambda\right)$

So for the different cosmologies we have

$(\Omega_m; \Omega_\Lambda) = (1, 0) \Rightarrow \frac{\dot{a}}{a} = H_0 a^{-3/2}$

$(0{,}3, 0{,}7) \Rightarrow \frac{\dot{a}}{a} = H_0\left(0{,}3 \cdot a^{-3} + 0{,}7\right)^{1/2}$

$(0{,}8, 0{,}2) \Rightarrow \frac{\dot{a}}{a} = H_0\left(0{,}8 \cdot a^{-3} + 0{,}2\right)^{1/2}$

(2)

I start by a change of variable from time to scale factor, $\delta(t) \to \delta(a)$, and rewrite the second order eq into two coupled first order eqs.

$$\frac{d^2\delta}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta}{dt} = \delta\, 4\pi G \bar{\rho}, \text{ where:}$$

I ignored pressure term.
Assuming flat universe, background density is then the critical density
$\bar{\rho} = \rho_{crit} = \frac{3H^2}{8\pi G}$.
$\dot{\delta} = \frac{d\delta}{dt}$

$$\frac{d}{dt}\dot{\delta} + 2H\dot{\delta} = \delta\, 4\pi G \frac{3H^2}{8\pi G} = \delta \frac{3H^2}{2}$$

$$\frac{d\dot{\delta}}{da}\cancel{\frac{da}{dt}} + \frac{2}{a}\cancel{\frac{da}{dt}}\dot{\delta} = \delta \frac{3}{2}\frac{1}{a^2}\left(\frac{da}{dt}\right)^{\cancel{2}}$$

$$\underline{\frac{d\dot{\delta}}{da} = -\frac{2}{a}\dot{\delta} + \frac{3\delta}{2a}\frac{\dot{a}}{a}} \quad \text{- eq for } \dot{\delta}$$

$$\frac{d\delta}{dt} = \dot{\delta} = \frac{d\delta}{da}\frac{da}{dt}, \text{ where } \frac{da}{dt} = aH = a\frac{\dot{a}}{a}$$

$$\hookrightarrow \underline{\frac{d\delta}{da} = a\frac{\dot{\delta}}{\dot{a}}} \quad \text{- eq for } \delta$$

Boundary:

$\underline{\delta(a=10^{-3}) = 10^{-3}}$

Assuming $\delta \propto a \Rightarrow \dot{\delta} \propto \dot{a}$. Let the relation be as simple as possible, so I set $\dot{\delta}(a=10^{-3}) = \dot{a} = Ha$.

Now, with the boundary conditions and the coupled eqs as a function of a and $H = \frac{\dot{a}}{a}$ I can solve the system with an Euler-Chromer scheme.

from last exercise

Euler - Chromer

$$\dot{\delta}_{i+1} = \dot{\delta}_i + \frac{d\dot{\delta}}{da} \cdot \Delta a$$

$$\delta_{i+1} = \delta_i + \frac{d\delta}{da} \cdot \Delta a$$

for $a \in [10^{-3}, 1]$

Δa ~ time steps

Note that the boundary conditions for $\dot{\delta}$ is dependent on the cosmology.

Plot of δ and growth factor (next exe) on next page.

(3)

The growth factor may be rewritten into:

$$f = \frac{d \ln \delta}{d \ln a} = \frac{a}{\delta} \frac{d\delta}{da} = \frac{a}{\delta} \frac{d\delta}{dt} \frac{dt}{da} = \frac{a}{\delta \dot{a}} \dot{\delta}$$

For each cosmology I calculated each of these quantities in the range $a \in [10^{-3}, 1]$. I find the redshift as $z(a) = \frac{1}{a} - 1$, and can plot f together with the overdensities.

Exe 3

(1) Adiabatic cooling $\Rightarrow$ $T_{gas} \propto a^{-2} = (1+z)^2$

$T_\gamma \propto a^{-1} = (1+z)$

Let $T_\gamma = C_\gamma a^{-1}$, $T_{gas} = C_G a^{-2}$ | today the CMB has temperature approximately $T_\gamma \simeq 2{,}725 K$ with $a = 1$

$\Rightarrow \quad T_\gamma(a_0) = \underline{C_\gamma = 2{,}725} \ K$

At decoupling, the gas and radiation was at the same temperature.

$$T_{gas}(z=1090) = T_\gamma(z=1090)$$

$$C_G a^{-2} = C_\gamma \cancel{a^{-1}}$$

$\hookrightarrow \quad C_G = C_\gamma \cdot a = 2{,}725 \frac{1}{1+1090} = \frac{2{,}725 K}{1091}$

Extra exe 3.2

- $\lambda_J \equiv c_s\sqrt{\frac{\pi}{G\bar\rho}}$, $M_J = \frac{\pi^{5/2}}{6}\frac{c_s^3}{G^{3/2}\bar\rho^{1/2}}$

Assuming matter dominati[on]:

$$\bar\rho = \bar\rho_0 a^{-3} = \bar\rho_0 (1+z)^3$$

$$a = \frac{1}{1+z}$$

Before decoupling:

$c_s = \frac{c}{\sqrt{3}}$, Inserting into λ_J gives

- ↳ $\lambda_J = c\sqrt{\frac{\pi}{3G\bar\rho_0}}\,(1+z)^{-3/2} \Rightarrow \lambda_J \propto (1+z)^{-3/2}$

$$M_J = \frac{\pi^{5/2}}{6}\frac{c^3}{3^{3/2}G^{3/2}\bar\rho_0^{1/2}(1+z)^{3/2}}$$

$$= c^3\sqrt{\frac{\pi^5}{162\, G^3\bar\rho_0 (1+z)^3}} \Rightarrow M_J \propto (1+z)^{-3/2}$$

After decoupling:

Sound speed now obtained through EOS ideal gas

$c_s^2 = \frac{\partial p}{\partial \rho} = \frac{k_B T}{\mu m_p}$. This is after the decoupling, ↳ we temperature for gas from last exercise: $T_{gas} = \underbrace{\frac{2.725K}{1091}}_{\equiv C_G} \cdot a^{-2}$

$$= \frac{k_B}{\mu m_p} C_G (1+z)^2$$

- The z-dependence of the density stays the same. Inserting this new sound speed gives:

./.

$$\lambda_J' = \lambda_{J,after} = \sqrt{\frac{k_B}{\mu m_p} C_G (1+z)} \sqrt{\frac{\pi}{G\bar{\rho}_0}} (1+z)^{-3/2}$$

$$= \sqrt{\frac{k_B}{\mu m_p} \frac{\pi C_G}{G\bar{\rho}_0 (1+z)}} \Rightarrow \lambda_J' \propto (1+z)^{-1/2}$$

Jeans Mass also given as

$$M_J' = \frac{4\pi}{3}\left(\frac{\lambda_J'}{2}\right)^3 \bar{\rho} = \frac{4\pi}{3}\bar{\rho}_0 (1+z)^3 \frac{1}{8}\left[\frac{k_B}{\mu m_p}\frac{\pi C_G}{G\bar{\rho}_0 (1+z)}\right]^{3/2}$$

$$= \frac{\pi^{5/2}}{6\sqrt{\bar{\rho}_0}}\left(\frac{k_B}{\mu m_p}\frac{C_G}{G}\right)^{3/2} (1+z)^{3/2} \Rightarrow M_J' \propto (1+z)^{3/2}$$

$\Rightarrow$ Assuming matter domination both before and after decoupling I find

Before

$$\lambda_J \propto (1+z)^{-3/2}$$

$$M_J \propto (1+z)^{-3/2}$$

After

$$\lambda_J' \propto (1+z)^{-1/2}$$

$$M_J' \propto (1+z)^{3/2}$$

Difference in amplitude (I don't mind the terms present both before and after)

$$\lambda_J \sim \frac{c}{\sqrt{3}} \sim 1{,}73 \cdot 10^8 \quad \gg \quad \lambda_J' \sim \sqrt{\frac{k_B C_G}{\mu m_p}} \simeq 5{,}8$$

$\mu = 0{,}62$

$$M_J \sim \frac{c^3}{9 \cdot \sqrt{2}} \sim 2{,}1 \cdot 10^{24} \quad \gg \quad M_J' \sim \frac{1}{6}\left(\frac{k_B C_G}{\mu m_p}\right)^{3/2} \simeq 32$$

Exe 4

$$R = A(1-\cos\theta)$$
$$t = B(\theta - \sin\theta)$$
$$A^3 = GMB^2$$

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \dot\theta \frac{d}{d\theta}$$

$$\frac{1}{\dot\theta} = \frac{dt}{d\theta} = B(1-\cos\theta)$$

$$\Rightarrow \quad \dot R = \dot\theta \frac{d}{d\theta} R = \dot\theta A \sin\theta = \frac{A\sin\theta}{B(1-\cos\theta)}$$

$$\ddot R = \dot\theta \frac{d}{d\theta}\dot R = \dot\theta \left[\frac{A\cos\theta B(1-\cos\theta) - A\sin^2\theta B}{B^2(1-\cos\theta)^2}\right]$$

$$= \dot\theta AB\left[\frac{\cos\theta - \cos^2\theta - \sin^2\theta}{B^2(1-\cos\theta)^2}\right]$$

$$-\cos^2\theta - \sin^2\theta = -1$$
$$(1-\cos\theta)^2 = (\cos\theta - 1)^2$$

$$= \dot\theta AB\left[\frac{\cancel{(\cos\theta - 1)}}{B^2(\cos\theta-1)^{\cancel{2}}}\right] = \frac{A}{B^2(\cos\theta-1)(1-\cos\theta)}$$

$$= -\frac{A}{B^2(1-\cos\theta)^2}$$

$$-\frac{GM}{R^2} = -\frac{A^{\cancel{3}}}{B^2}\frac{1}{\cancel{A^2}(1-\cos\theta)^2}$$

inserting for GM and R^2

$$\ddot R = -\frac{GM}{R^2}$$

The parameterization holds.

Exe 5

Virial ~~radius~~ reached at $\Theta = \frac{3\pi}{2}$

At this point, $R(\Theta = \frac{3\pi}{2}) = R_{vir} = A(1-0) = A$

The infall velocity is found from $v = \frac{dR}{dt} = \dot{R}$,

where $\dot{R} = \frac{A \sin\Theta}{B(1-\cos\Theta)}$ is found in exercise 4.

$$\Rightarrow v\left(\frac{3\pi}{2}\right) = -\frac{A}{B} = v_{vir}, \text{ where } \frac{A^3}{B^2} = GM$$

$$\Rightarrow v_{vir}^2 = \left(\frac{A}{B}\right)^2 \Big| \cdot \frac{R_{vir}}{R_{vir}}$$

$$= \frac{A^2 \overset{=A}{R_{vir}}}{B^2 R_{vir}} = \frac{A^3}{B^2 R_{vir}} = \frac{GM}{R_{vir}}$$

$$\Rightarrow v\left(\frac{3\pi}{2}\right) = \pm\sqrt{\frac{GM}{R(\frac{3\pi}{2})}}$$

Usual choice of sign gives

$$v = -\sqrt{\frac{GM}{R}}, \quad v_{vir} = -\sqrt{\frac{GM}{R_{vir}}}$$

Exe 6

- Gravitational potential energy $U = -G\frac{Mm}{r}$

Potential energy of shell with mass dm at a distance r is then

$$dU = -G\frac{M(r)}{r}dm$$

where $M(r)$ is the mass within radius r,

- $$M(r) = \frac{4\pi r^3}{3}\rho_0$$

where ρ_0 is the uniform density of the sphere.

The mass dm is the mass of a shell with thickness dr and area $4\pi r^2$

- $$dm = 4\pi r^2 \rho_0 \, dr$$

$$\Rightarrow dU = -G\left(\frac{4\pi r^3}{3}\right)^2 \rho_0^2 \, dr = -G\left(\frac{4\pi\rho_0}{3}\right)^2 r^4 dr \;\Big|\cdot\frac{3R^6}{3R^6}$$

$$= -\frac{3G}{R^6}\underbrace{\frac{4\pi R^3\rho_0}{3}}_{M}\underbrace{\frac{4\pi R^3\rho_0}{3}}_{M} r^4 dr$$

$M \neq M(r)$
↳ Mass of hole sphere
$M = M(R)$

- $$\int_0^U dU = -\int_0^R \frac{3GM^2}{R^6} r^4 dr \Rightarrow U = -\frac{3GM^2}{R^6}\left[\frac{r^5}{5}\right]_0^R$$

$$= \underline{-\frac{3GM^2}{5R}}$$