AST5220 Cosmology II

Milestone 2 - The Recombination History

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1 Brief Introduction

The goal of this second milestone[1] is to compute what happens during and after the short period of the early universe called recombination. In this period the primordial plasma of free electrons and protons combined into neutral hydrogen. We also want to compute how rapidly this transition happened. To achieve this we will solve the Boltzmann equation for the number density of electrons, using the Saha approximation and Peebles equation.

The main results from these calculations are the optical depth and its first and second derivatives, as well as the visibility function and its derivatives, as functions of the logarithmic scale factor x. Using these expressions we will be able to find the time when the cosmic microwave background (CMB) was released from the opaque primordial plasma, defining the so called last scattering surface (LSS) where the Universe became transparent.

1.1 Note on previous milestones

As mentioned this is part two of four in this overarching project. We will therefore build upon the theory and results from milestone 1[Borg, 2020], describing the background evolution of the Universe, without further elaboration. One point we would like to mention that is adjusted in the code used from milestone 1 is the initial condition for the conformal time. We now use the more accurate approach

$$\eta(x_{start}) = \frac{c}{\mathcal{H}(x_{start})}.$$

2 Theoretical Background

2.1 The Model In Use

As mentioned, the main results we are interested in are the optical depth and the visibility function discussed in section 2.5. In our cosmological model the main source absorbing and disrupting photons are Thompson scattering with free electrons. We will model the optical depth simply according to this kind of scattering alone. Therefore, to calculate these quantities, we need the electron number density.

We are modeling the early Universe as a hot, relativistic primordial plasma with photons, and free electrons and free protons as its only ordinary matter components. That is we exclude helium and any other heavier elements, as well as neutrinos[2]. After the recombination process we assume we are left in a cool Universe with only hydrogen

[1] Second of four milestones.
[2] To be clear, neutrinos are not considered a matter component, but is also excluded from our model.

and photons, and a tiny amount of free electrons left. The interactions between particle species governing this transition, and hence their distributions, are described with the Boltzmann equation, which will be discussed next in section 2.2.

To clarify, the model still includes the dark matter and dark energy components, which is important for the background evolution computed in milestone 1, but is not important for the interactions modeled during recombination. The dark energy component, modeled as the cosmological constant, is only affecting the expansion of the Universe and is therefore not directly included in the interactions between particle species. The dark matter component could theoretically be included in some type of interaction with it self or other particles, but is not modeled that way here and the component is left unchanged.

2.2 The Boltzmann Equation

The Boltzmann equation can be written on the following form taken from [Dodelson, 2003, p. 61] describing the number density of particle species 1, in a process going both ways where species 1 and 2 interact and produce species 3 and 4

$$1 + 2 \rightleftharpoons 3 + 4$$

$$\frac{1}{a^3}\frac{\mathrm{d}a^3 n_1}{\mathrm{d}t} = -\langle\sigma v\rangle\left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{EQ}} n_3 n_4\right]. \tag{1}$$

Here n_i and $(n_i)_{\mathrm{EQ}}$ are the mentioned number densities and the known[3] equilibrium (EQ) solutions of the different number densities respectively. While $\langle\sigma v\rangle$ is the thermally averaged cross sections of the interactions, taken from quantum field theory. This equation is permutable in the indices so that it can be written for species 2, 3 and 4 as well. We will apply this equation to our model

$$e^- + p^+ \rightleftharpoons H + \gamma, \tag{2}$$

where the processes taking place are as follows. On the left side we have free electrons and protons combining into neutral hydrogen and a photon. On the right side a high energy photon interacts through Compton scattering with a bound electron in a hydrogen atom, ionizing the atom and producing a free electron and proton.

The Boltzmann equation can be rewritten, again following Dodelson [2003], into the Saha equation and Peebles equation described in the following sections 2.3 and 2.4, which we will use to compute how the number density of free electrons evolve. First we set up some of the more straight forward assumptions. The photons of course stay relativistic always, and so we can approximate their number density by the equilibrium solution $n_\gamma \approx n_\gamma^{\mathrm{EQ}}$. We also assume that the Universe is neutral, so that $n_e = n_p$.

2.2.1 Natural Units

We have to mention that all the equations listed in Dodelson [2003] are given in so called natural units, where the familiar constants $\hbar = c = k_b \equiv 1$. These constants are therefore neglected in the equations in the book. In all our calculations we use SI-units, and we must therefore rewrite the equations into proper form with the constants included. This is done through dimensional analyses. This is straight forward, but a bit tedious, and we therefore does not included the analyses for all the equations to follow. Instead we present one example of how this is done in appendix B.

2.3 The Saha Approximation and Equation

In the early radiation dominated Universe, the plasma was dense and hot with high interaction rates so all the species i are considered to be in EQ

$$\frac{\mathrm{d}a^3 n_i}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \frac{n_e n_p}{n_H n_\gamma} = \left(\frac{n_e n_p}{n_H n_\gamma}\right)_{\mathrm{EQ}}.$$

[3] Also found in [Dodelson, 2003, p. 61].

As the Universe expands and the temperature and interaction rates drops, the species will gradually fall out of EQ. Thus the number densities for the different species will start to evolve differently. Initially this can be modeled with the Saha approximation

$$\frac{n_e n_p}{n_H n_\gamma} \approx \left(\frac{n_e n_p}{n_H n_\gamma}\right)_{\mathrm{EQ}} \tag{3}$$

when the species are very close to following their equilibrium distributions. Defining the free electron fraction, X_e, as the electron number density over the baryon number density, we can define the Saha equation

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e k_b T_b}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{\epsilon_0}{k_b T_b}\right) \tag{4}$$

$$\text{where} \quad X_e = \frac{n_e}{n_b} = \frac{n_e}{n_p + n_H}, \quad X_e \in [0,\, 1]\,, \tag{5}$$

$$n_b = \frac{\Omega_{B,0}\rho_{c,0}}{m_H}\exp(-3x) \quad \text{is the baryon number density}^4, \tag{6}$$

$$\text{and} \quad \epsilon_0 = (m_e + m_p - m_h)\, c^2 = 13.6\,\mathrm{eV} \quad \text{is the binding energy of hydrogen.}$$

Equation (4) is valid for as long as the Saha approximation holds, quantitatively for as long as $X_e \approx 1$[5]. The temperature involved is specifically the temperature of the baryons. Here we employ another approximation, which is reasonable to a large degree considering all our other approximations, that the baryon temperature follow the radiation temperature

$$T_b(x) = T_r(x) = T_{\mathrm{CMB}}(x=0)\exp(-x) \tag{7}$$

where the temperature of the CMB today is $T_{\mathrm{CMB}}(x=0) = 2.725\,\mathrm{K}$. We will therefore just note the temperature as T from now on, referring to the baryon and radiation temperature as a function of the logarithmic scale factor.

2.4 The Peebles Equation

Going beyond the equilibrium solutions we can no longer assume the species to follow the Saha approximation, eq. (3). The Saha equation, eq. (4), predicts the outset of recombination pretty accurately, but fails for the evolution of the electron density as the system goes out of equilibrium. We will here employ the Peebles equation, describing the free electron fraction according to the Boltzmann equation with the relevant interactions included in our model. These are combinations of free electrons and protons into hydrogen in its first excited state[6]. Recombination into the ground state is not included as this will also produce a photon with high enough energy to ionize another hydrogen atom in its ground state, resulting in a zero net change in the number densities. The Peebles equation goes as follows

$$\frac{\mathrm{d}X_e}{\mathrm{d}x} = \frac{C_r(T)}{H(x)}\left[\beta(T)(1-X_e) - n_H\alpha^{(2)}(T)X_e^2\right], \tag{8}$$

where

$$C_r(T) = \frac{\Lambda_{2s\to1s} + \Lambda_\alpha}{\Lambda_{2s\to1s} + \Lambda_\alpha + \beta^{(2)}(T)}, \tag{9}$$

$$\Lambda_{2s\to1s} = 8.227\mathrm{s}^{-1}, \tag{10}$$

$$\Lambda_\alpha = \frac{H(x)}{(8\pi)^2 n_{1s}}\left(\frac{3\epsilon_0}{\hbar c}\right)^3, \tag{11}$$

$$n_{1s} = (1-X_e)n_H, \tag{12}$$

$$\beta^{(2)}(T) = \beta(T)\exp\left(\frac{3\epsilon_0}{4k_bT}\right), \tag{13}$$

$$\beta(T) = \alpha^{(2)}(T)\left(\frac{m_e k_b T}{2\pi\hbar^2}\right)^{3/2}\exp\left(-\frac{\epsilon_0}{k_bT}\right), \tag{14}$$

4 Known from milestone 1.

5 Note that the free electron fraction of course starts at $X_e = 1$ in the primordial plasma.

6 It is a bit unclear if the higher order excited states are included too, as Dodelson [2003] includes all excited states in his description on p. 71. Nevertheless the first excited state will contribute the most, and is what we have assumed in the lectures.

$$\alpha^{(2)}(T) = \frac{64\pi}{\sqrt{27\pi}}\sigma_T c\sqrt{\frac{\epsilon_0}{k_b T}}\phi_2(T), \tag{15}$$

$$\sigma_T = \frac{8\pi}{3}\frac{\alpha^2\hbar^2}{m_e^2 c^2}, \tag{16}$$

$$\phi_2(T) = 0.448\ln\left(\frac{\epsilon_0}{k_b T}\right). \tag{17}$$

and $H(x)$ is the Hubble parameter, discussed in detail in Borg [2020].

2.5 The Optical Depth and Visibility Function

The optical depth describes how much the intensity of incident light that moves a distance x through a medium will be reduced[7], quantitatively as $I(x) = I_0 \exp(-\tau)$ where I_0 is the intensity emitted at the source. As mentioned, in our model we look at absorption through Thompson scattering alone. The optical depth is then defined as

$$\tau(x) = \int_x^{x_{\mathrm{Today}=0}} \frac{\sigma_T n_e c}{H(x)}\,\mathrm{d}x \tag{18}$$

where σ_T is the Thompson cross section defined in eq. (16). The electron number density can be obtained using the free electron fraction from the Saha and Peebles equations multiplied by the baryon number density following eq. (5). There are two interesting regimes governed by the optical depth. If $\tau \gg 1$ we say the medium is optically thick, and thus the observed intensity $I(x)$ will be approximately zero. The other way around, if $\tau \ll 1$ the medium is optically thin, and $I(x) \approx I_0$. When $\tau = 1$ defines the transition between these two regimes, and can be thought of as when the medium goes from opaque to transparent.

With the optical depth we can also define the visibility function

$$\tilde{g}(x) = -\frac{\mathrm{d}\tau}{\mathrm{d}x}\exp(-\tau) = -\tau'\exp(-\tau) \tag{19}$$

where we have used the τ' notation for the derivative with respect to the logarithmic scale factor. We will continue to use this through the report where it is convenient. Equation (19) is a probability distribution, in the sense that $\int_{-\infty}^{0} \tilde{g}\mathrm{d}x = 1$. We interpret this function as the probability that a photon last scattered at time x, going backwards in time from today.

2.5.1 The Last Scattering Surface

We can now define the last scattering surface (LSS). This is describing a spherical shell at distance $x_\star$ centered around us where the recombination period happened. The "thickness" of the shell will correspond to the rapidity of the period. At this point in time the Universe went from being opaque to transparent, and the CMB photons scattered one last time before they where released from the chain of scattering. We can now use our definition of the optical depth transition from opaque to transparent, or alternatively the peak in the visibility function, to determine when this happened, defined as $x_\star$

$$\tau(x_\star) \equiv 1$$
$$\tilde{g}(x_\star) \equiv \max\{\tilde{g}(x)\}.$$

3 Method

As mentioned, this milestone builds upon the work done in milestone 1[Borg, 2020], calculating everything we need for the background cosmology. These calculations are performed by the class *BackgroundCosmology*. The

[7] By absorption, scattering or other means.

work in this milestone was carried out by the class *RecombinationHistory*, which takes as input an instance of the mentioned *BackgroundCosmology* class with all the required calculations already done[8].

With the equations we need defined in section 2 we can now describe our approach step for step to solve for the optical depth and the visibility functions and their derivatives.

3.1 Solving the Saha and Peebles equations

First we implemented a method to get the baryon number density at time x, using eq. (6). This is used in both eqs. (4) and (8), as well as solving for the electron number density once we have the free electron fraction.

Next we solve for the free electron fraction using eqs. (4) and (8) in an x-interval from -12 to 2 with 1×10^5 points. Here we have to consider multiple numerical stability issues. First we must define a transition between the two regimes for when the Saha equation is no longer valid, due to the Saha approximation being validated, and define this as when $X_e < 0.99$. Looking at eq. (4), this is a simple quadratic equation in X_e, which we solve as follows

$$\text{let} \quad F \equiv \frac{1}{n_b}\left(\frac{m_e k_b T_b}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{\epsilon_0}{k_b T_b}\right)$$

$$\text{then} \quad X_e = \frac{-F \pm \sqrt{F^2 + 4F}}{2} = \frac{-F + F\sqrt{1+\frac{4}{F}}}{2}$$

where we have used the positive solution as X_e can not be negative. In the early universe the temperature was extremely high and thus F is a huge number. This is a numerical issue, essentially we subtract a huge number from a huge number, which is unstable. We solve this by Taylor expanding the square root to the first order so the solution is well defined

$$\sqrt{1+\frac{4}{F}} \approx 1 + \frac{2}{F}$$

$$\Rightarrow \quad \frac{-F + F\sqrt{1+\frac{4}{F}}}{2} \approx \frac{-F + F + 2}{2} = 1.$$

We also want to compare the solution from using the more accurate approximation with the Peebles equation with the solution obtained through the Saha equation alone. Thus we have a problem later on, when the temperature drops and F goes to zero so $4/F$ diverges. This is all solved in the code by implementing it as follows

$$X_e = \begin{cases} 1 & \text{if} \quad F > 1 \times 10^9 \\ 1 \times 10^{-20} & \text{if} \quad F < 1 \times 10^{-20} \\ \frac{-F+F\sqrt{1+\frac{4}{F}}}{2} & \text{else} \end{cases}$$

where the exact thresholds for each regime is found through experimentation. Note that we don't set the solution to zero, but a very small number, when F is small. This is because we want to create splines of the logarithmic results in the end, as the logarithmic X_e varies smoother in x.

We solve the Saha equation in the full time interval, to have this solution to compare our more accurate solution with later, but we keep track of the exact index x_i of our time array where our $X_e < 0.99$ condition is met, and what value we have for X_e at this point, $X_{e,i}$. This is used in the following for solving the Peebles equation, eq. (8), which is an ordinary differential equation (ODE).

To solve the Peebles equation, we use the class *ODESolver* wrapping the GSL library providing multiple ODE algorithms. We chose specifically the well known explicit embedded Runge-Kutta-Fehlberg (4, 5) method[gsl, 2020]. To solve the system we use the mentioned recorded $X_{e,i}$ as initial condition, and create a new time array spanning from the time corresponding to the index x_i, $x_{\text{Peebles}} \in [x(x_i), 2]$. When implementing the eqs. (8) to (17) we have to take care of the exponential in eq. (13). When the temperature drops to a low value this will

[8] The class also takes as input the helium fraction, but this is set to zero and does not contribute anything.

diverge out of control. Therefore we implement eq. (13) writing out the full expression for eq. (14) so we can combine the two exponential terms

$$\beta^{(2)} = \alpha^{(2)}(T)\left(\frac{m_e k_b T}{2\pi\hbar^2}\right)^{3/2}\exp\left(-\frac{\epsilon_0}{4k_b T}\right).$$

As briefly mentioned, the logarithmic results from both the combined Peebles and Saha solution and the Saha solution alone are splined, as well as the logarithmic electron number density using eq. (5). These splines are used in the following when we need the electron number density, and so we implement a get method to easily convert back and fourth from the logarithmic and normal value.

3.2 Solving for the Optical Depth and Visibility Function

We solve for the optical depth and visibility function in the same time interval as in section 3.1 with the same number of points. Here we prepare to solve for both quantities, and their first derivatives. First we rewrite eq. (18) into an ODE

$$\frac{\mathrm{d}\tau}{\mathrm{d}x} = -\frac{\sigma_t n_e c}{H(x)} \tag{20}$$

and solve it with the same algorithm as we did for the Peebles equation. We don't know the initial condition for this equation, but we do know the numerical value of it today[9] at $x = 0$. We therefore chose one arbitrary and large initial value, $\tau_{\text{init}} = 1 \times 10^3$, and then normalized the result in the end by subtracting the computed value of $\tau(x = 0)$ from every grid point. To do this we find the index where $x = 0$ dynamically, call it x_{zero}, using the *lower bound* functionality from the standard library.

The first derivative of τ is obtained directly through the ODESolver, and before we normalize the optical depth we create the spline for this derivative. Using this spline we can compute the second derivative too using the *deriv_x* method of the spline. This way we can use the spline to compute the first derivative of the visibility function, all in the same loop as we are normalizing the optical depth and computing the visibility function it self. To do this we use the chain rule on eq. (19) to express its derivative analytically, and compute the quantities according to this scheme

$$\begin{aligned}
&\text{for } i \in [0, n-1]: \\
&\quad \bar{\tau}_i = \tau_i - \tau_{x_{\text{zero}}} \\
&\quad \tilde{g}_i = -\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)_i \exp(-\bar{\tau}_i) \\
&\quad \tilde{g}'_i = \left[\left(\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)_i\right)^2 - \left(\frac{\mathrm{d}^2\tau}{\mathrm{d}x^2}\right)_i\right]\exp(-\bar{\tau}_i)
\end{aligned}$$

where $\tilde{g}'_i$ is the first derivative of the visibility function with respect to x, subscripts indicate the grid point the quantities are evaluated at, and n is the number of points. $\frac{\mathrm{d}^2\tau}{\mathrm{d}x^2}$ is obtained through the spline of $\frac{\mathrm{d}\tau}{\mathrm{d}x}$ which we prepared before the loop.

In the end, we create splines of τ, $\tilde{g}$ and $\tilde{g}'$. As mentioned, the second derivative of τ is obtained through the spline of τ'. Likewise for the second derivative of the visibility function, it is obtained through the spline of $\tilde{g}'$. All the splines are then implemented into convenient get methods in the class.

3.3 Some Specific Times of Interest

Lastly we implement some convenient methods to fetch, print and write to file some specific time points. First is the time of the LSS from section 2.5.1, which we find from $\tau(x_\star) = 1$. We also want the time when recombination is half-way done, that is $X_e(x_{\text{rec}}) = 0.5$. We also find this for the solution using only the Saha equation, as

[9] Or in other words at our location, distance $x = 0$ away.

mentioned in section 3.1, so we can compare the two solutions. All of the mentioned times are found using the implemented splines and a *binary search for value* algorithm, and also converted to redshift according to

$$z = \frac{1}{\exp(x)} - 1.$$

We also find the specific time for the transition between the two regimes solved using the Saha and Peebles equation. This is done easily using the approach we have for solving both the Saha equation alone and the combination of Saha and Peebles equation.

4 Results

4.1 Common Results in all Figures

The results from all calculations are combined in figs. 1 to 3, with a few things in common between all the figures. First, as we did in milestone1[Borg, 2020], the background of the figures are coloured according the to dominating energy contributor to the given time period. Blue marks the radiation dominated regime, red marks the matter[10] dominated regime and the green marks the regime dominated by dark energy in the form of the cosmological constant.

The results from section 3.3 are shown in each plot as vertical lines, except the times obtained for the Saha equation only solution. Here we found

Half-way recombination, only Saha:	$x_{\text{rec,Saha}} = -7.230\,22$	$z_{\text{rec,Saha}} = 1379.53$
Half-way recombination:	$x_{\text{rec}} = -7.1649$	$z_{\text{rec}} = 1292.23$
Transition from Saha to Peebles regime:	$x_{\text{Peebles}} = -7.370\,29$	
Last scattering surface:	$x_\star = -6.986\,08$	$z_\star = 1080.48.$

All figures are plotted using "Python".

4.2 The Electron Density

The evolution of the free electron fraction and the electron number density are shown in fig. 1. The left y-axis show the value of X_e, and the right show n_e overplotted. Both solutions are shown, the one using only the Saha equation in dash-dotted pink and subscript Saha, as well as the full solution using the Peebles equation in blue.

Here we see how the free electron fraction stay constant at 1 in the early Universe, before recombination happens, while the electron number density drops in a linear way[11] proportional to the expansion a^{-3}. Then at $x_{\text{Peebles}} \approx -7.37$ we enter the Peebles regime, and shortly after we can see the free electron fraction start to evolve quite rapidly, along with a dramatic drop in the electron number density of several orders of magnitude. Here the temperature of the Universe has dropped significantly so that the free electrons and protons can start to form neutral hydrogen as discussed. Already at $x_{\text{rec}} = -7.16$ the recombination is half-way done, using the full solution, while the Saha only solution predicts it happening already at $x_{\text{rec,Saha}} = -7.230\,22$. This is as we know too early, and it makes sense as the Saha equation assumes the solution is close to EQ at all times which is not the case during recombination. Equation (4) has a exponential drop of which we can recognize in the figure, the $X_{e,\text{Saha}}$ solution drops off to zero with decreasing temperature. The full solution[12] does not fall off exponentially, and instead flattens out to a stable value of $X_e(x = 0) \approx 1.910\,83 \times 10^{-4}$. This flattening can be described by two phases. First the free electrons decouple from the rest of the Universe as their interaction rate with the free protons drops below the expansion rate of the Universe. Second the free electrons freeze out, where the free electron fraction stops evolving and becomes more or less constant.

[10] Baryons and dark matter.
[11] This is of course in logarithmic axes.
[12] Just called «the solution» from now on, the Saha only solution is neglected.

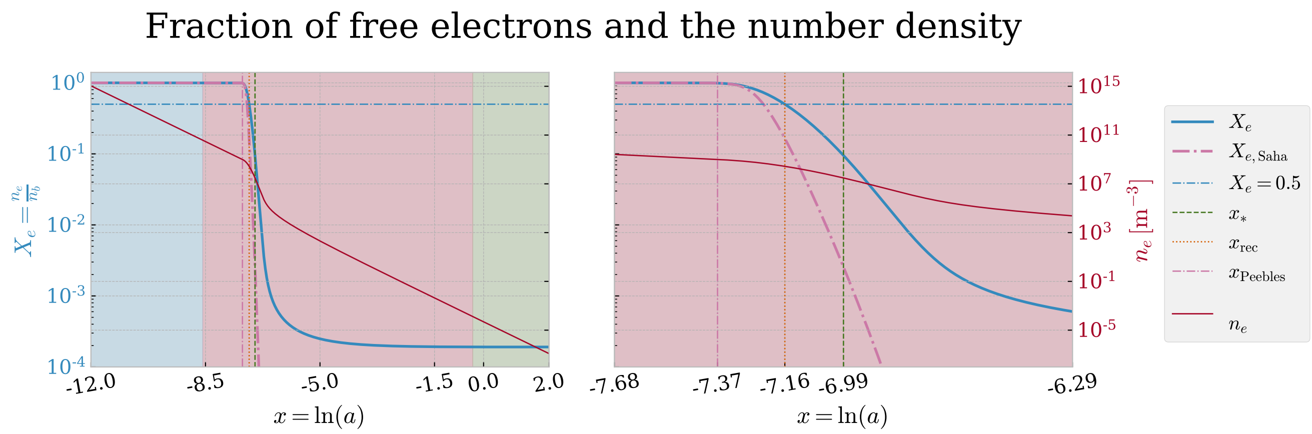

Figure 1: Plot showing the free electron fraction X_e, left y-axis, and electron number density n_e, right y-axis. The solution obtained using only Saha equation is included in the pink dash-dotted line, $X_{e,\mathrm{Saha}}$, which drops off exponentially to zero once the solution deviates from equilibrium. The transition between the Saha and Peebles regime is marked at $x_{\mathrm{Peebles}} \approx -7.37$. The horizontal blue dashed line at $X_e = 0.5$ indicate where recombination is half-way done, at $x_{\mathrm{rec}} \approx -7.16$. The LSS is indicated at $x_\star \approx -6.99$. The left panel zoom in centered on $x_\star$, showing a $10\,\%$ buffer in x on both sides.

After the recombination period we see that the electron number density goes back to evolving with the expansion as it did before recombination. If we neglect the recombination period and the time for freeze out to happen, we see that the free electron fraction stays constant as the electron number density only evolves with the expansion. Note also that at LSS the free electron fraction has dropped to $\sim 10^{-1}$.

The full computation of solving for X_e, computing both solutions, was done in $\sim 0.071\,742\,6\,\mathrm{s}$.

4.3 The Optical Depth

In fig. 2 the optical depth and its derivatives are plotted on the left y-axis, with the visibility function overplotted on the right y-axis in green to easily see how they are related. The horizontal dash-dotted blue line indicate the $\tau = 1$ transition.

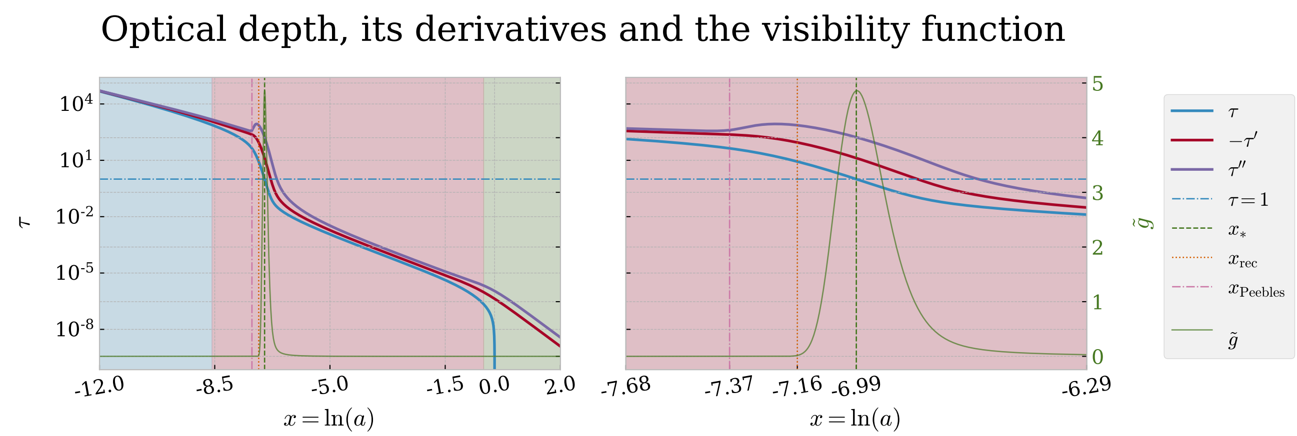

Figure 2: Plot showing the optical depth and its two first derivatives, left y-axis, and the visibility function, right y-axis. $x_{\mathrm{rec}} \approx -7.16$ indicate where recombination is half-way done. The LSS is indicated at $x_\star \approx -6.99$, located at the peak in $\tilde{g}$ and where $\tau = 1$. The transition between the Saha and Peebles regime is marked at $x_{\mathrm{Peebles}} \approx -7.37$. The left panel zoom in centered on $x_\star$, showing a $10\,\%$ buffer in x on both sides.

In the dense primordial Universe the mean free path of photons was really low due to rapid Thompson scattering on the abundant free electrons. This is reflected in the high optical depth, and high free electron fraction.

Here $\tau \gg 1$ and the Universe was highly opaque. From eq. (20) we see that the change in optical depth is proportional to the ratio of the Thompson scattering interaction rate, $\sigma_T n_e$, and the expansion rate, $H(x)$. From Borg [2020] we know that the expansion rate evolves in a linear way[13] following $H^2(x) \propto \exp(-4x)$ in the radiation dominated period, so we can focus our attention for now to n_e. As we saw in section 4.2 this just evolves with the expansion before recombination. This reflects the linear evolution in the optical depth and its derivatives in this period. Here the optical «thickness» of the Universe is only decreasing along with the expansion.

Then when recombination starts and n_e and X_e drops dramatically, it follows with a huge drop in the optical depth and its derivatives. This is consistent with less free electrons, less Thompson scattering and longer mean free path. Now the optical «thickness» of the Universe is changed dramatically due to the change in the constituents of the Universe, not only the expansion. At x_{rec} the optical depth has decreased to ~ 10, and at $x_\star$, $\tau = 1$, we have the LSS. As mentioned in section 2.5 this is where the Universe transitions from opaque to transparent, and corresponds to the peak in the visibility function. Note also the point where $\tau' = 1$. We will come back to this and the visibility function in section 4.4.

After recombination, and the free electron fraction freezes out, the optical depth goes back to decreasing with the expansion. Now we are in the matter dominated regime, and the expansion is proportional to $H^2(x) \propto \exp(-3x)$. Hence we see the optical depth goes back to decreasing linearly in the logarithmic plot, now slightly faster than before recombination, before we approach the todays value, where $\tau = 0$ by definition.

4.4 The Visibility Function

Figure 3 shows the visibility function and its scaled derivatives. The derivatives are scaled to fit into the same plot, with the first and second derivative having much larger values than the visibility function it self. $\tilde{g}$ is as mentioned a probability function, while the derivatives have negative values. Our choice of scaling is making the absolute value of the scaled first and second derivatives probability distributions as well, following

$$\overline{\tilde{g}'} = \frac{\tilde{g}'}{\int_{-\infty}^{\infty} |\tilde{g}'|\,\mathrm{d}x}$$
$$\overline{\tilde{g}''} = \frac{\tilde{g}''}{\int_{-\infty}^{\infty} |\tilde{g}''|\,\mathrm{d}x}.$$

As discussed in section 2.5, the visibility function describes the probability that a observed photon was last

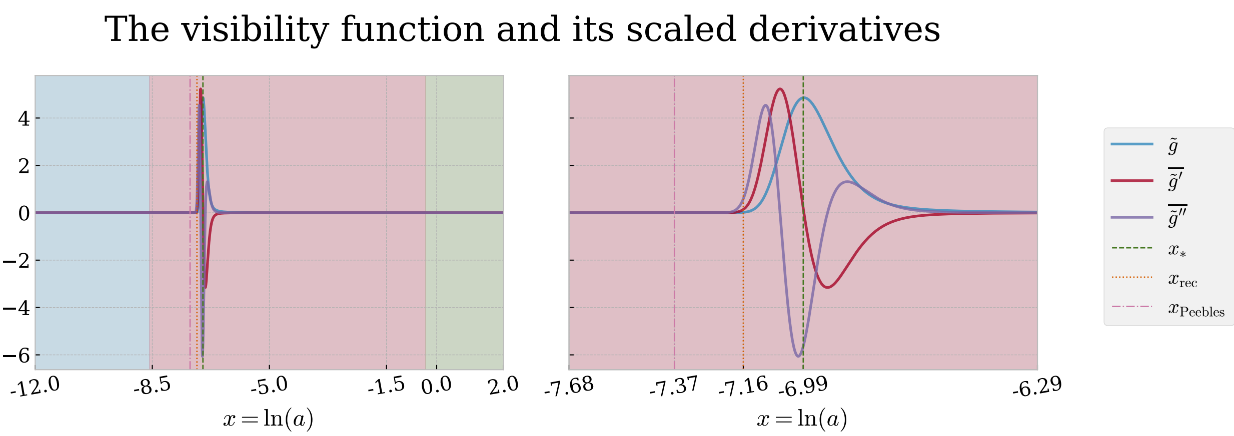

Figure 3: Plot showing the visibility function and its scaled two first derivatives. The transition between the Saha and Peebles regime is marked at $x_{\text{Peebles}} \approx -7.37$, the half-way recombination point is marked at $x_{\text{rex}} \approx -7.16$. The LSS surface is at $x_\star \approx -6.99$, which corresponds to the peak in the visibility function and where $\tau = 1$. The left panel zoom in centered on $x_\star$, showing a 10 % buffer in x on both sides.

scattered at time x, going backwards in time from today. From the left panel in fig. 3, also maybe better visible in fig. 2, this probability is more or less zero both before and after recombination. We can understand this as

[13] Again in logarithmic plots.

follows. Before recombination the Universe was dense and opaque, and photons scattered all the time, and thus the probability of a photon observed today scattered *last time* before recombination is minuscule. After recombination the Universe is transparent, huge and «empty». Here the Thompson interaction rate is lower than the expansion rate, as $\tau' < 1$, and so the chance for scattering is really low.

But as we see, at $x_\star$ the visibility function has its maximum, in accordance with $\tau = 1$ at this same point as we saw in fig. 2. As discussed this corresponds to the LSS, and from the sharp peak in the visibility function we understand that this surface, or shell around us, is really thin. We also see from the first derivative that the probability for last scattering starts to increase approximately at our defined x_{rec}, where the recombination is half-way done. The visibility function it self has a more sharply defined curve in the transition into recombination, and a less sharp curve in the transition out of recombination. We can understand this by looking at the sharp drop in X_e when recombination starts, and the more soft freeze out after recombination. This is also evident in τ', which drops below 1 slightly after recombination. As discussed this corresponds to the time where the Thompson scattering interaction rate drops below the expansion rate. As this happens slightly after recombination the probability of scattering *one more time* after $x_\star$ is slightly higher and the curve goes to zero in a more soft fashion. Note that as $\tau' < 1$ the visibility function is approximately back down to zero. These effects are also evident mathematically in the derivatives, while the physical interpretation of the derivatives are more unclear.

The total time for solving for both optical depth and the visibility function is done in $\sim 0.086\,735\,4\,\mathrm{s}$. The full run of everything in this milestone was done in $\sim 0.233\,127\,\mathrm{s}$.

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Appendices

A Testing our Results

In this milestone we do not have the same amount of good test of our implementation as we did in milestone 1. The best we can do is summarized here.

First we check that the results obtained for the visibility function actually is a probability distribution, by numerically integrating the results in "Python" using the simple trapezoidal method from *numpy*.

Second we test our results against the given results in the project description[pro, 2020], with a «toy-cosmology». To run the full calculations using this toy cosmology give the command line argument *Toy-Milestone2* when running the code, or the *toy2cmb* rule when running the makefile.

As a last line of checks, we compare our results to those of Callin [2006].

B Dimensional Analysis of the Saha Equation

Here follows an example of how to perform the dimensional analysis to rewrite the Saha equation as listed in [Dodelson, 2003, p. 70] in natural units into the form of eq. (4). First we do one simplifying step. As we neglect the helium atoms and assume a neutral universe we can write the sum of electron and hydrogen number densities as the baryon density, $n_e + n_H = n_p + n_H = n_b$.

$$\text{Original form} \qquad \underbrace{\frac{X_e^2}{1-X_e}}_{[\text{Unitless}]} = \underbrace{\frac{1}{n_b}}_{[\text{m}^3]} \underbrace{\left(\frac{m_e T_b}{2\pi}\right)^{3/2}}_{[\text{kg}^{3/2}\,\text{K}^{3/2}]} \underbrace{\exp\left(-\frac{m_e + m_p - m_H}{T_b}\right)}_{\exp\left([\text{kg}\,\text{K}^{-1}]\right)}$$

We must now balance the equation to get the right units on the left hand side of the equation. That is we must make the equation unitless. First, we have to remove the units inside the exponential function. Here we recognize the nominator as the binding energy of hydrogen, if we multiply with a factor c^2, thus the nominator is in units joule. To make the denominator have the same units we multiply with the Boltzmann constant.

$$\frac{(m_e + m_p - m_H)\,c^2}{k_b T_b} = \frac{\epsilon_0}{k_b T_b} \quad \Rightarrow \quad \frac{[\text{J}]}{[\text{J}\,\text{K}^{-1}\,\text{K}]} = [\text{Unitless}]$$

Next we have to make the product in front of the exponential unitless as well. The simplest way to do this is to pull the baryon number density inside the parenthesis, and then avoid the exponent $3/2$ all together.

$$\frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} \quad \Rightarrow \quad \underbrace{\left(\frac{m_e T_b}{n_b^{2/3}}\right)}_{[\text{kg}\,\text{K}\,\text{m}^2]}$$

To get rid of the temperature we need the Boltzmann constant, but we then introduce units of energy. To remove the unit of energy and the left unit of mass we need the square of the Planck constant.

$$\left(\frac{m_e k_b T_b}{n_b^{2/3}\hbar^2}\right) \quad \Rightarrow \quad \frac{[\text{kg}\,\text{K}\,\text{m}^2\,\text{J}]}{[\text{K}\,\text{J}^2\,\text{s}^2]} = \frac{[\text{kg}\,\text{m}^2]}{[\text{s}^2]}\frac{1}{[\text{J}]} = [\text{Unitless}]$$

Now we have the equation on its proper SI-unit form, as in eq. (4)

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e k_b T_b}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{\epsilon_0}{k_b T_b}\right).$$