FYS3120 Classical Mechanics and Electrodynamics

Problem set 8

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March 15, 2019

Problem 1 A monocromatic light source is at rest in the laboratory and sends photons with frequency ν_0 towards a mirror which has its reflective surface perpendicular to the beam direction. The mirror moves away from the light source with velocity v. Use the transformation formula for four-momentum $p^\mu = (E/c, \vec{p})$ and the Planck relation $E = h\nu$ to:

a) Find the frequency of the emitted and reflected light in the rest frame of the mirror.

Solution **1.a**

Assuming the mirror and the light moving in the x-direction for simplicity, $\hat{\mathbf{v}} = \hat{\mathbf{x}}$. In the lab frame, RF S, the light has four-momentum p^μ, which transforms to $p^{'\mu}$ in the mirror's frame, RF $S^{'}$, as follows

$$p^\mu = \left(\frac{E}{c}, \vec{\mathbf{p}}\right) \quad \rightarrow \quad p^{'\rho} = L^\rho{}_\mu p^\mu$$

where

$$p^0 = \frac{E}{c} = \frac{h\nu_0}{c}, \quad m = 0 \Rightarrow E = pc \Rightarrow \vec{\mathbf{p}} = \left(\frac{h\nu_0}{c}, 0, 0\right), \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}.$$

Notice that $p^0 = p^1$. Transforming each component I find

$$\begin{aligned} p^{'0} &= \gamma\big(p^0 - \beta p^1\big) = \gamma\left(\frac{h\nu_0}{c} - \beta\frac{h\nu_0}{c}\right) = \frac{h\nu_0}{c}\gamma(1-\beta) \\ \text{similarly} \quad p^{'1} &= \gamma\big(p^1 - \beta p^0\big) = \frac{h\nu_0}{c}\gamma(1-\beta) \\ p^{'2} &= p^2 \quad \text{and} \quad p^{'3} = p^3 \end{aligned}$$

The relationship between the frequencies in the two frames can be found from the relation between the p^0s

$$\begin{aligned} p^{'0} &= \frac{h\nu_0^{'}}{c} = \frac{h\nu_0}{c}\gamma(1-\beta) \\ \text{so:} \quad \nu_0^{'} &= \nu_0\gamma(1-\beta) \end{aligned}$$

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b) Find the frequency of reflected light in the lab system.

Solution **1.b**

I find the frequency in the lab system by transforming back into the lab RF

$$p^\nu = L^\nu{}_\rho p^{'\rho}.$$

Here one needs to mind the sign in the velocity, as the lab frame moves in the opposite direction of the mirror, $\beta = -v/c$. I only need the zero'th component to find the frequency

$$p^0 = \gamma\Big(p'^0 - \beta p'^1\Big) = \gamma\bigg(\frac{h\nu_0}{c}(1-\beta) - \beta\frac{h\nu_0}{c}(1-\beta)\bigg)$$
$$= \frac{h\nu_0}{c}\gamma^2(1-\beta)^2 \quad \text{using} \quad \gamma^2 = \frac{1}{(1-\beta)(1+\beta)}$$
$$= \frac{h\nu_0}{c}\frac{1-\beta}{1+\beta}.$$

Let the expression for p^0 be $\frac{h\nu_R}{c}$ for the received frequency ν_R, then

$$\nu_R = \nu_0\frac{1-\beta}{1+\beta} \quad \text{where} \quad \beta = -\frac{v}{c}$$

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Problem 2 Figure 1 shows a particle with mass m and (relativistic) kinetic energy K in the laboratory frame S. The particle is moving towards another particle, with the same mass m, which is at rest in S.

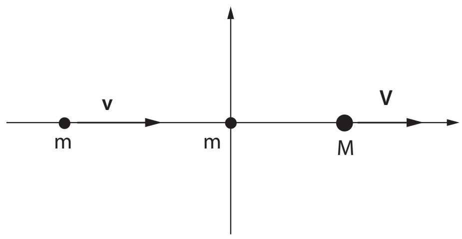

Figure 1: Collision between two particles of mass m resulting in a particle with mass M.

a) Find the velocity v of the first particle expressed in terms of the dimensionless quantity $\alpha = K/mc^2$ (and the speed of light).

Solution **2.a**

The relativistic kinetic energy is the relativistic energy minus the rest mass energy. Let m_1 be the mass of particle 1.

$$K = E - m_1c^2 = \gamma m_1c^2 - m_1c^2 = m_1c^2(\gamma - 1)$$
$$\frac{K}{m_1c^2} = \alpha = \gamma - 1 \quad \Rightarrow \frac{1}{\sqrt{1-\beta^2}} = \alpha + 1$$
$$1 - \beta^2 = \frac{1}{(\alpha+1)^2}$$
$$\beta = \sqrt{1 - \frac{1}{(\alpha+1)^2}} \quad \Rightarrow v = c\sqrt{1 - \frac{1}{(\alpha+1)^2}}$$

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First we will assume that the particles collide in such a way that they form one particle after the collision (a totally inelastic collision).

b) Determine the compound particle's energy E, momentum P, velocity V and mass M in terms of the velocity v (or γ). Find the change in the total kinetic energy of the system due to the collision.

Solution **2.b**

To determine the properties of the compound particle I use conservation of relativistic energy and momentum through the four-momentum, which conserves the two quantities separately

$$\sum_i p_i^\mu = \sum_f p_f^\mu$$

where index i and f describes the initial and final quantities before and after the collision. In these calculations I constantly use that $\vec{\mathbf{v}}_2 = 0$ and $m_1 = m_2$.
First I find the final energy, $E_f = E = \gamma_f M c^2$, through the zero'th component of p^μ:

$$\begin{aligned} E &= \gamma_f M c^2 = \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2 \quad \text{where} \quad \gamma_2 = 1 \\ \Rightarrow E &= (\gamma_1 + 1) m_1 c^2 \end{aligned} \tag{1}$$

The momentum is easily found from the vector component of p^μ:

$$\begin{aligned} \vec{\mathbf{P}} &= \gamma_f M \vec{\mathbf{V}} = \gamma_1 m_1 \vec{\mathbf{v}}_1 + \gamma_2 m_2 \vec{\mathbf{v}}_2 = \gamma_1 m_1 \vec{\mathbf{v}}_1 \\ \Rightarrow \vec{\mathbf{P}} &= \gamma_1 m_1 \vec{\mathbf{v}}_1 \end{aligned} \tag{2}$$

Using the two last results, eqs. (1) and (2), I find the velocity:

$$\begin{aligned} \vec{\mathbf{P}} &= \vec{\mathbf{p}}_i \\ \gamma_f M \vec{\mathbf{V}} &= \gamma_1 m_1 \vec{\mathbf{v}} \\ \vec{\mathbf{V}} &= \frac{\gamma_1 m_1 \vec{\mathbf{v}}_1}{\gamma_f M} \quad \text{from eq. (1)} \quad \gamma_f M = (\gamma_1 + 1) m_1 \\ \Rightarrow \vec{\mathbf{V}} &= \frac{\gamma_1}{\gamma_1 + 1} \vec{\mathbf{v}}_1 \end{aligned} \tag{3}$$

Rewriting the relativistic energies as $E = \sqrt{m^2c^4 + p^2c^2}$, using the result from eq. (2) to express the compound energy and $E_2 = m_2c^2$ as the rest energy of body 2 I find

$$\begin{aligned} E_f = E_1 + E_2 = \sqrt{M^2c^4 + \vec{\mathbf{P}}^2c^2} &= \sqrt{m_1^2c^4 + \vec{\mathbf{p}}_1^2 m_1^2 c^2} + m_2c^2 \\ \sqrt{M^2c^4 + (\gamma_1 m_1 \vec{\mathbf{v}}_1)^2 c^2} &= \sqrt{m_1^2c^4 + (\gamma_1 m_1 \vec{\mathbf{v}}_1)^2 c^2} + m_2c^2 \\ M^2c^4 + \cancel{(\gamma_1 m_1 \vec{\mathbf{v}})^2 c^2} &= m_1^2c^4 + \cancel{\gamma_1^2 m_1^2 \vec{\mathbf{v}}_1 c^2} + 2m_2c^2\sqrt{m_1^2c^4 + (\gamma_1 m_1 \vec{\mathbf{v}}_1)^2 c^2} + m_2^2c^4 \\ m_1 = m_2, \quad M^2c^4 &= 2m_1^2c^4 + 2m_1c^2\sqrt{m_1^2c^4 + (\gamma_1 m_1 \vec{\mathbf{v}}_1)^2c^2} \,\Big|\, \cdot \frac{1}{2mc^2} \\ \frac{M^2c^2}{2m_1} &= m_1c^2 + \sqrt{m_1^2c^4 + (\gamma_1 m_1 \vec{\mathbf{v}}_1)^2 c^2} \\ &= E_2 + E_1 = (\gamma_1 + 1) m_1 c^2 \\ \Rightarrow M &= m\sqrt{2(\gamma_1 + 1)} \end{aligned} \tag{4}$$

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In the rest of the exercise we will assume that the situation before the collision is as described earlier, but that the particles now collide elastically, *i.e.* after the collision the two particles are the same as before the collision, with no change in their masses. The collision happens in such a way that the particles after the collision make the same angle, θ, with the x-axis in the lab frame S, see Fig. 2.

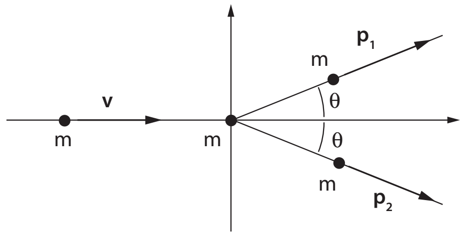

Figure 2: Collision between two particles of mass m resulting in scattering at angle θ.

c) Show that after the collision the particles have the same magnitude of momentum ($|\vec{p}_1| = |\vec{p}_2|$) and energy ($E_1 = E_2$).

Solution 2.c

To avoid confusion with the indices I will write the quantities before the collision with a bar.
I show $|\vec{\mathbf{p}}_1| = |\vec{\mathbf{p}}_2| = p$ by conservation of momentum

$$\bar{\vec{\mathbf{p}}}_1 = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2$$

writing out the equation on component form, as the collision is elastic these can be solved separately;

$$\bar{p}_1(1,0,0) = p_1(\cos(\theta), \sin(\theta), 0) + p_2(\cos(\theta), -\sin(\theta), 0) \tag{5}$$

reveals the relation in the second component

$$\begin{aligned}\bar{p}_1 \cdot 0 = 0 &= p_1 \sin(\theta) - p_2 \sin(\theta) = \sin(\theta)(p_1 - p_2) \\ &\Rightarrow p_1 = p_2\end{aligned}$$

where $\theta = 0$ is not considered.
The energies after the collision can be written as

$$\begin{aligned}E_1^2 &= p_1^2 c^2 + m_1^2 c^4 \\ E_2^2 &= p_2^2 c^2 + m_2^2 c^4.\end{aligned} \tag{6}$$

As $m_1 = m_2$ and I've shown $|\vec{\mathbf{p}}_1| = |\vec{\mathbf{p}}_2| = p_1 = p_2$ these energies are identical.
This problem can also be solved (maybe in a nicer way) by looking at the system from the center-of-mass RF (CoM).

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d) Determine $E \equiv E_1 = E_2$ and $p \equiv |\vec{p}_1| = |\vec{p}_2|$.

Solution **2.d**

To determine the magnitude of momentum p I use the first component of eq. (5) and that the magnitude of the momenta after the collision are equal. Having the expression for p I can use eq. (6) to express the energy by the same argument as in **2.c**:

$$\bar{p} = 2p\cos(\theta)$$
$$\Rightarrow p = \frac{\bar{p}}{2\cos(\theta)}, \quad E = \sqrt{\frac{\bar{p}^2c^2}{4\cos^2(\theta)} + m^2c^4}$$

where I have stopped using subscript 1 and 2 as all the quantities discussed here are equal for both particles.

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e) Determine the angle θ. Find θ in the limiting cases when $\alpha = K/mc^2$ goes to zero and to infinity. Show that $\theta < \pi/4$.

Solution **2.e**

To determine the angle I will look at the conserved energy. In this calculation there are a lot of algebra and substitutions. Other approaches probably exists where the answer is easier to interpret, but this is the one I've found:

$$\begin{aligned} 2E &= \bar{E}_1 + \bar{E}_2 \\ 2\sqrt{\frac{\bar{p}^2c^2}{4\cos^2(\theta)} + m^2c^4} &= \sqrt{m^2c^4 + \bar{p}^2c^2} + mc^2 \quad \text{square both} \\ 4\left(\frac{\bar{p}^2c^2}{4\cos^2(\theta)} + m^2c^4\right) &= m^2c^4 + \bar{p}^2c^2 + 2mc^2\sqrt{m^2c^4 + \bar{p}^2c^2} + m^2c^4 \\ \frac{\bar{p}^2c^2}{\cos^2(\theta)} - \bar{p}^2c^2 &= 2mc^2\left(\sqrt{m^2c^4 + \bar{p}^2c^2} - mc^2\right) \qquad (*) \end{aligned}$$

where in the last step I gathered all the terms with mc^2 on the right hand side, which contains the kinetic energy:

$$\begin{aligned} 2mc^2\left(\sqrt{m^2c^4 + \bar{p}^2c^2} - mc^2\right) &= 2mc^2\bar{K} \\ &\text{where} \quad \bar{K} = mc^2(\bar{\gamma}_1 - 1) \quad \text{from } \mathbf{2.a} \\ &= 2m^2c^4(\bar{\gamma}_1 - 1). \end{aligned}$$

Rewriting eq. (*) using the kinetic energy and $\bar{p}^2 = \bar{\gamma}_1^2 m^2 \bar{v}_1^2$ I find

$$\begin{aligned} \bar{\gamma}_1^2 m^2 \bar{v}_1^2 c^2\left(\frac{1}{\cos^2(\theta)} - 1\right) &= 2m^2c^4(\bar{\gamma}_1 - 1) \\ \bar{\gamma}_1^2\bar{\beta}^2\left(\frac{1}{\cos^2(\theta)} - 1\right) &= 2(\bar{\gamma}_1 - 1) \\ \Rightarrow \cos^2(\theta) &= \frac{1}{\frac{2(\bar{\gamma}_1 - 1)}{\bar{\gamma}_1^2\bar{\beta}^2} + 1} = \frac{\bar{\gamma}_1^2\bar{\beta}^2}{\bar{\gamma}_1^2\bar{\beta}^2 + 2(\bar{\gamma}_1 - 1)}. \end{aligned}$$

where $\bar{\beta} = \frac{\bar{v}_1}{c}$. I have now expressed the angle using only the velocity of body 1 before the collision, so to reduce clutter I will stop using the bar notation. With $\gamma_1 = \frac{1}{\sqrt{1-\beta^2}}$ I find an expression for the angle θ:

$$\begin{aligned}\cos^2(\theta) &= \frac{\beta^2}{(1-\beta^2)\left[\frac{\beta^2}{1-\beta^2} + \frac{2}{\sqrt{1-\beta^2}} - 2\right]} \\ &= \frac{\beta^2}{\beta^2 + 2\sqrt{1-\beta^2} - 2(1-\beta^2)}.\end{aligned}$$

For $\alpha \to \infty$ the kinetic energy is infinite and so $\beta \to 1$ as the velocity approaches c. Then

$$\begin{aligned}\lim_{\beta\to 1} \cos^2(\theta) = \lim_{\beta\to 1} \frac{\beta^2}{\beta^2 + 2\sqrt{1-\beta^2} - 2(1-\beta^2)} = 1 \\ \theta = \arccos\left(\sqrt{1}\right) = 0.\end{aligned}$$

In the limit $\alpha \to 0$ the kinetic energy is zero, that means $\beta = \frac{v}{c} \to 0$. To find θ in the limits I use L'Hôpital's rule as I have a $0/0$ expression:

$$\begin{aligned}\lim_{\beta\to 0} \cos^2(\theta) &\overset{\text{(H)}}{=} \lim_{\beta\to 0} \frac{2\beta}{2\beta - 2\beta(1-\beta^2)^{-1/2} + 4\beta} \\ &= \lim_{\beta\to 0} \frac{2\beta}{6\beta - 2\beta(1-\beta^2)^{-1/2}} = \lim_{\beta\to 0} \frac{2}{6 - \frac{2}{\sqrt{1-\beta^2}}} = \frac{1}{2} \\ \theta &= \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.\end{aligned}$$

As $0 < \alpha < \infty$ I have that $\theta < \frac{\pi}{4}$ as I wanted to show.

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