Bathtub Galaxy Model: Next Steps

From Toy Units to Physical Cosmology

Overview

Your current model works well in dimensionless units. The next step is to place your galaxy inside a **growing dark matter halo** with physical units (Solar masses, kpc, Gyr). This will let you reproduce results similar to Pandya et al. (2022) and connect your model to real observables.

The work is organized into four phases. Complete each phase before moving to the next.

Contents

1 Phase 1: The Cosmological Backbone

The goal here is to build a module that tracks how dark matter halos grow over cosmic time.

The Big Picture

At every timestep, your code will calculate the "virial properties" of the halo. The logic flows one direction:

$$\text{Time } (t) \rightarrow \text{Redshift } (z) \rightarrow \text{Halo Mass } (M_h) \rightarrow R_{\text{vir}} \rightarrow V_{\text{vir}} \rightarrow T_{\text{vir}}$$

These physical values will later replace your ad-hoc parameters.

1.1 Step 1.1: Define Your Unit System

Before writing any physics, establish a consistent unit system. I recommend:

Quantity	Unit
Mass	$M_\odot$ (solar masses)
Length	kpc
Time	Gyr
Velocity	km/s

Your Task

Create a constants file or class that defines:

- G in units compatible with [kpc, $M_\odot$, km/s]
- k_B (Boltzmann constant) in cgs
- m_p (proton mass) in grams
- $\mu = 0.6$ (mean molecular weight for ionized gas)

You'll also need the conversion factor: 1 km/s/Mpc $\approx$ 0.0010227 Gyr^{-1}.

1.2 Step 1.2: Cosmology—Time vs. Redshift

Your ODE integrator works in time (t), but cosmological formulas use redshift (z). You need functions to convert between them.

Recommended approach: Use `astropy.cosmology` to handle this cleanly.

```
from astropy.cosmology import Planck15 as cosmo
from astropy.cosmology import z_at_value
from astropy import units as u

def redshift_from_time(t_gyr):
    """Convert cosmic time (Gyr) to redshift."""
    return z_at_value(cosmo.age, t_gyr * u.Gyr)

def critical_density(z):
    """Return rho_crit in M_sun / kpc^3."""
    return cosmo.critical_density(z).to(u.M_sun / u.kpc**3).value
```

Note: Calling `z_at_value` repeatedly can be slow. Consider building an interpolation table if performance becomes an issue.

1.3 Step 1.3: The Mass Accretion History (MAH)

Use the Correa et al. (2015) formula for halo growth:

$$M_{\rm halo}(z) = M_0\,(1+z)^{\alpha}\,e^{\beta z} \tag{1}$$

where M_0 is the halo mass *today* (at $z=0$), and α, β control the growth history.

Starting values for a Milky Way–like halo:

$$M_0 = 10^{12}\,M_\odot, \quad \alpha = 1.6, \quad \beta = -1.2$$

Your Task

Implement a function `mass_at_z(z)` that returns $M_{\rm halo}(z)$ using Eq. (1).

1.4 Step 1.4: The Accretion Rate

Your ODE solver needs $\dot{M}_{\rm halo}$ (mass per time). Use the chain rule:

$$\frac{dM}{dt} = \frac{dM}{dz} \times \frac{dz}{dt} \tag{2}$$

Key relations:

- From Eq. (1): $\dfrac{dM}{dz} = M(z)\left[\dfrac{\alpha}{1+z} + \beta\right]$
- From cosmology: $\dfrac{dz}{dt} = -H(z)(1+z)$

Your Task

1. Derive dM/dz by differentiating Eq. (1). Verify you get the expression above.
2. Implement a function `accretion_rate(z)` that returns dM/dt in $M_\odot$/Gyr.
3. **Watch your units!** You must convert $H(z)$ from km/s/Mpc to Gyr^{-1}.

Sanity Check

For a $10^{12}\,M_\odot$ halo at $z=0$, you should get $\dot{M} \approx 20$–$40\ M_\odot$/yr (or 2–4 $\times\ 10^{10}\ M_\odot$/Gyr). If you get 10^{15}, you forgot the unit conversion. If negative, check your signs.

1.5 Step 1.5: Virial Quantities

Once you have $M_{\rm halo}(z)$, calculate the virial properties. A halo is defined as a region with mean density $\Delta_{\rm vir} \times \rho_{\rm crit}(z)$, where typically $\Delta_{\rm vir} = 200$.

Virial Radius:

$$R_{\rm vir} = \left(\frac{3\,M_{\rm halo}}{4\pi \cdot 200\,\rho_{\rm crit}}\right)^{1/3} \tag{3}$$

Virial Velocity:

$$V_{\rm vir} = \sqrt{\frac{G\,M_{\rm halo}}{R_{\rm vir}}} \tag{4}$$

Virial Temperature:

$$T_{\rm vir} = \frac{\mu\,m_p\,V_{\rm vir}^2}{2\,k_B} \tag{5}$$

Your Task

Implement a function `get_virial_properties(z)` that returns $(M, R_{\rm vir}, V_{\rm vir}, T_{\rm vir})$.
Be careful with units when computing $T_{\rm vir}$—you may need to convert $V_{\rm vir}$ to cm/s.

1.6 Step 1.6: Validation

Before proceeding to Phase 2, verify that Phase 1 works correctly.

Deliverable: Three-Panel Plot

Create a plot with three panels, spanning $t = 1$–13.7 Gyr for a halo with $M_0 = 10^{12}\,M_\odot$:

1. **Panel 1:** $M_{\rm halo}$ vs. time (should rise to 10^{12})
2. **Panel 2:** $\dot{M}_{\rm halo}$ vs. time (should peak early and decline)
3. **Panel 3:** $T_{\rm vir}$ vs. time (should rise steadily toward $\sim 10^6$ K)

2 Phase 2: Reformulating the Baryonic Equations

Now rewrite your ODEs to use physical quantities from Phase 1.

2.1 Step 2.1: Baryon Inflow

Replace your current `mdot_IGM` with the cosmological baryon accretion rate:

$$\dot{M}_{\rm IGM} = f_b \cdot \dot{M}_{\rm halo}(t) \tag{6}$$

where $f_b \approx 0.16$ is the universal baryon fraction.

2.2 Step 2.2: Dynamic Timescales and Star Formation

Replace fixed `t_dep` with a physically motivated star formation law that depends on the *state of the galaxy itself*, not just halo properties.

Estimating ISM surface density: You can estimate the gas surface density from your state variables by assuming the ISM lives in a disk of size $R_{\rm disk} \approx \lambda R_{\rm vir}$ (where $\lambda \sim 0.02$–0.05 is related to the halo spin):

$$\Sigma_{\rm ISM} \sim \frac{M_{\rm ISM}}{\pi(\lambda R_{\rm vir})^2}$$

Physical considerations: The star formation rate (or depletion time) should reflect local ISM conditions. Options to explore:

- **Surface density** $\Sigma_{\rm ISM}$: Schmidt–Kennicutt gives $\dot{\Sigma}_* \propto \Sigma_{\rm gas}^{1.4}$
- **ISM pressure**: $P_{\rm ISM} \sim \Sigma_{\rm ISM}^2$ in a self-gravitating disk
- **CGM pressure**: Does the weight of the CGM compress the disk and enhance SF?

Bursty Star Formation

Real galaxies—especially low-mass ones—don't form stars smoothly. They exhibit "bursty" behavior: episodes of intense star formation followed by quiescent periods.
Think about what functional forms might produce oscillations or threshold behavior in your model. For example:

- A **threshold**: SF only occurs above some critical $\Sigma_{\rm ISM}$ or $P_{\rm ISM}$
- A **strongly nonlinear** dependence: $\dot{M}_* \propto \Sigma_{\rm ISM}^n$ with $n > 1$
- **Coupling to feedback**: If feedback depletes ISM quickly, SF shuts off until gas re-accumulates

Your Task

1. Implement $\Sigma_{\rm ISM}$ as a diagnostic in your code.
2. Choose a star formation law that depends on $\Sigma_{\rm ISM}$ or $P_{\rm ISM}$.
3. Experiment: can you find a functional form that produces bursty behavior? What controls the burst amplitude and duty cycle?

2.3 Step 2.3: Feedback Scaling

Replace constant η_M and η_E parameters with functions of *ISM and CGM properties*.

Physical considerations: Feedback efficiency depends on local conditions where SNe explode and where winds propagate. Consider:

- **ISM density or pressure**: Does a denser ISM make it harder to launch outflows (lower η_M)? Or does higher pressure lead to more efficient momentum coupling?
- **CGM temperature** $T_{\rm CGM}$: Hot CGM may allow energy to escape more easily (higher η_E?), while cool CGM radiates away injected energy.
- **CGM density**: A denser CGM might entrain more mass but also suffer more radiative losses.
- **Pressure contrast**: The ratio $P_{\rm ISM}/P_{\rm CGM}$ determines how easily gas can be pushed out of the disk.

Your Task

1. Implement η_M and/or η_E as functions of quantities your model tracks: $M_{\rm ISM}$, $M_{\rm CGM}$, $T_{\rm CGM}$, $\Sigma_{\rm ISM}$, etc.
2. Start simple—one dependence—and add complexity only if needed.
3. For each scaling, write down the physical argument. Does higher ISM density increase or decrease mass loading? Why?

Why Not $V_{\rm vir}$?

Classic semi-analytic models scale feedback with $V_{\rm vir}$ because it correlates with observed trends. But correlation isn't causation. The *actual* physics involves ISM and CGM conditions—$V_{\rm vir}$ just happens to correlate with these. Your model can test whether local conditions are sufficient to explain the trends, or whether halo depth matters independently.

3 Phase 3: The Energy Equation

The key insight from Pandya et al. is tracking CGM *energy*, not just mass or temperature. This phase is more advanced—take your time.

3.1 Step 3.1: Physical Cooling

Your current cooling term is likely a placeholder. You need a physical cooling function $\Lambda(T, Z)$.

Simple approach: Use a power-law approximation for the relevant temperature range, or interpolate from tabulated cooling curves (e.g., Sutherland & Dopita 1993).

The cooling luminosity is:

$$L_{\text{cool}} = n_H \, n_e \, \Lambda(T) \cdot V_{\text{CGM}} \tag{7}$$

Your Task

Look up or implement a simple cooling function. How does $\Lambda(T)$ behave for $T \sim 10^5$–10^7 K?

3.2 Step 3.2: The Energy ODE

Reformulate your temperature equation as an energy equation:

$$\dot{E}_{\text{CGM}} = \dot{E}_{\text{inflow}} + \dot{E}_{\text{feedback}} - L_{\text{cool}} - P \, dV \text{ work} \tag{8}$$

Temperature then comes from:

$$E_{\text{CGM}} = \frac{3}{2} N k_B T \tag{9}$$

Advanced (optional): Pandya et al. split CGM energy into thermal and turbulent components. Start with just thermal; add turbulence later if time permits.

4 Phase 4: Running and Analyzing

4.1 Step 4.1: Update the Solver

Your functions are now explicitly time-dependent: $f(t, \mathbf{y})$. Update your `solve_ivp` call accordingly.

Make sure your time array spans cosmic time: from $z \approx 10$ ($t \approx 0.5$ Gyr) to $z = 0$ ($t \approx 13.7$ Gyr).

4.2 Step 4.2: Diagnostic Plots

Final Deliverables

1. **Stellar mass–halo mass relation:** Plot $M_*/M_{\rm halo}$ vs. $M_{\rm halo}$ at $z = 0$. Compare to observed relations.
2. **Cooling ratio:** Plot $t_{\rm cool}/t_{\rm ff}$ vs. time. This shows when the CGM transitions from cold accretion to hot hydrostatic equilibrium.
3. **Halo comparison:** Run a $10^{11}\,M_\odot$ halo and a $10^{13}\,M_\odot$ halo. Does the CGM temperature behave differently? This bifurcation is a key prediction of modern galaxy formation theory.

Summary Checklist

- ☐ **HaloHistory class**: Returns $M(t)$, $R_{\rm vir}(t)$, $V_{\rm vir}(t)$, $T_{\rm vir}(t)$
- ☐ **Physics functions**: $\Lambda(T)$, $\eta_M(V_{\rm vir})$, $\dot{M}_{\rm IGM}(t)$
- ☐ **Validation plot**: Three-panel check of Phase 1
- ☐ **Full integration**: $[\dot{M}_{\rm ISM}, \dot{M}_{\rm CGM}, \dot{E}_{\rm CGM}, \dot{M}_*, \dot{Z}]$ over 13 Gyr
- ☐ **Comparison**: 10^{11} vs. $10^{13}\,M_\odot$ halos

Tips for Success

- **Units are treacherous.** When in doubt, write out the units explicitly and verify they cancel correctly.
- **Test each step.** Don't write all of Phase 1 before running anything. Test `mass_at_z` before implementing the accretion rate.
- **Print intermediate values.** If something looks wrong, print M, $R_{\rm vir}$, $V_{\rm vir}$, and $T_{\rm vir}$ at a few redshifts and check they're reasonable.
- **Compare to literature.** A Milky Way halo should have $V_{\rm vir} \sim 150$ km/s and $T_{\rm vir} \sim 10^6$ K at $z = 0$.