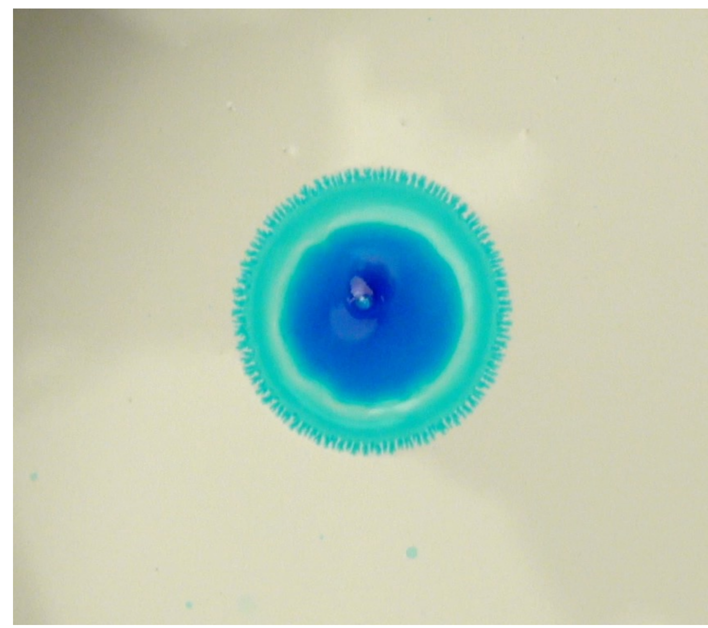
1. Fractal Fingers

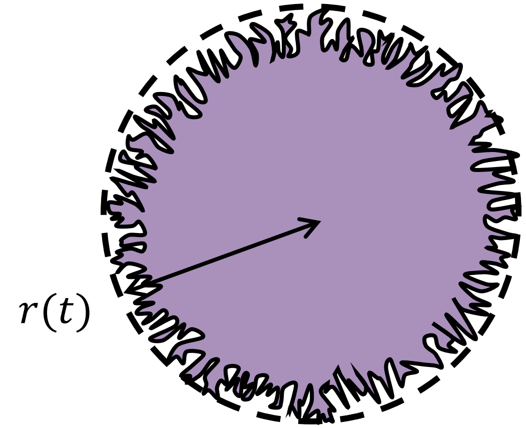
*"The effect of **fractal fingering** can be observed if a droplet of an **ink-alcohol mixture** is deposited onto diluted acrylic **paint**. How are the **geometry** and **dynamics** of the fingers influenced by **relevant parameters**?"*



Tasks

*"The effect of **fractal fingering** can be observed if a droplet of an **ink-alcohol mixture** is deposited onto diluted acrylic **paint**. How are the **geometry** and **dynamics** of the fingers influenced by **relevant parameters**?"*

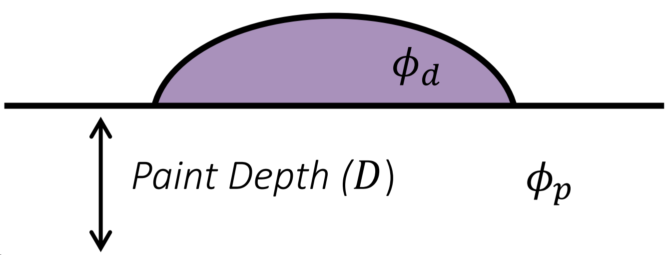
1. *Explore the **geometry** and **dynamics** of the fractal*
2. *Quantify **conditions** for branching*
3. *Create a **reproducible** experiment*



Parameters

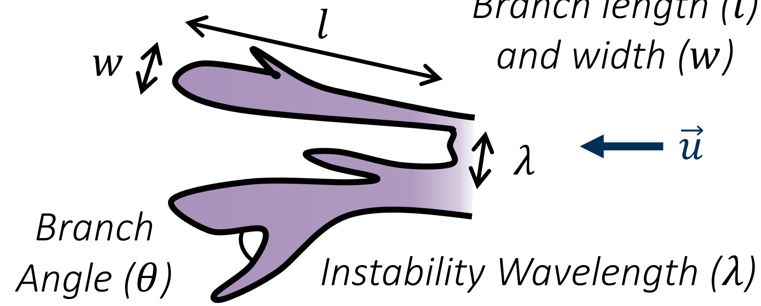
Viscosity Ratio (concentration of ink and paint) (ϕ_d: ϕ_p)

Initial Drop Volume (V)

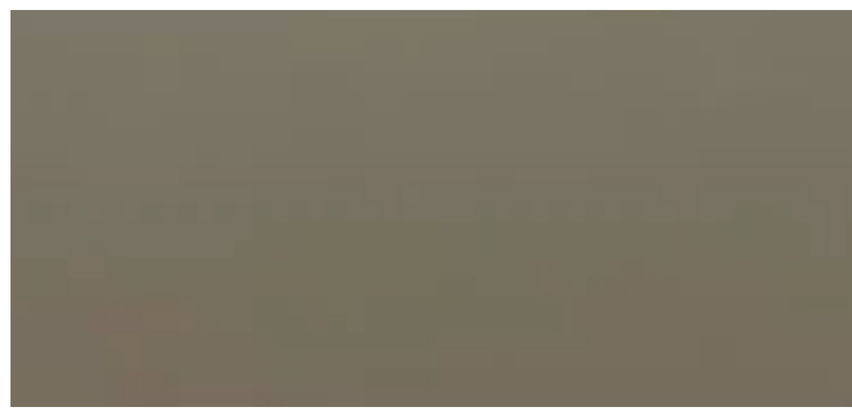


Dependent Variables

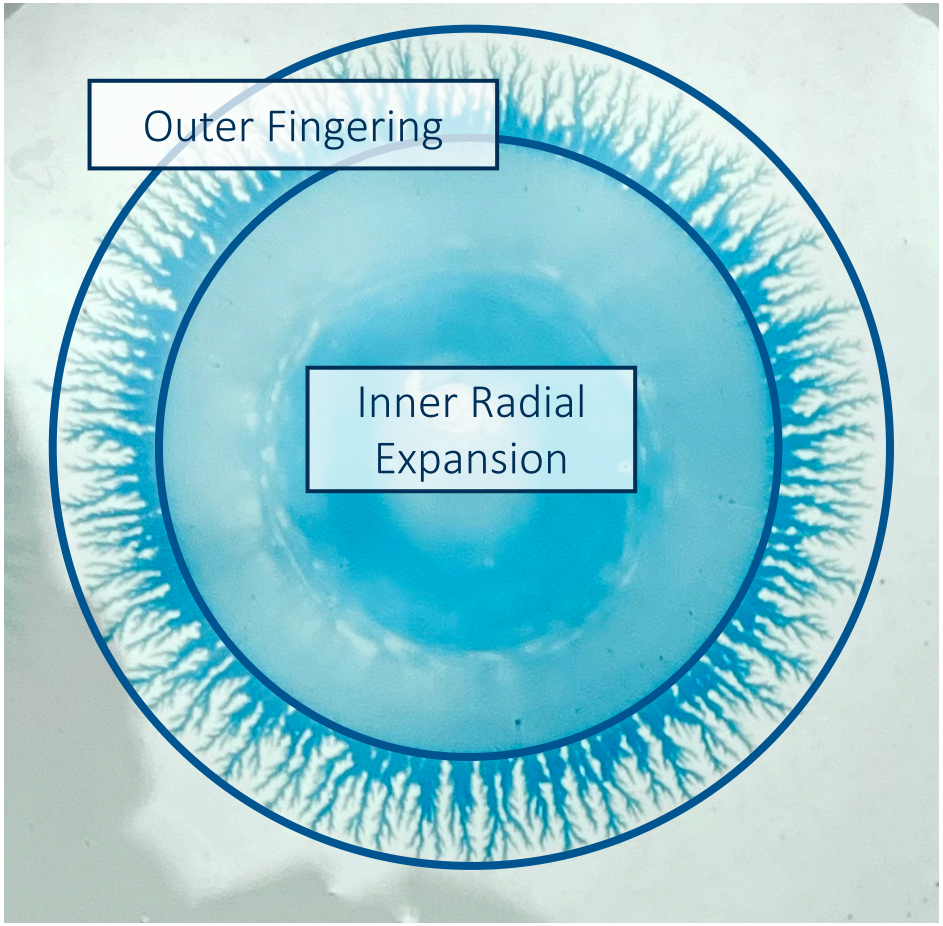
Radius $r(t)$ and flow field $u(r,t)$



Phenomenon



Phenomenon

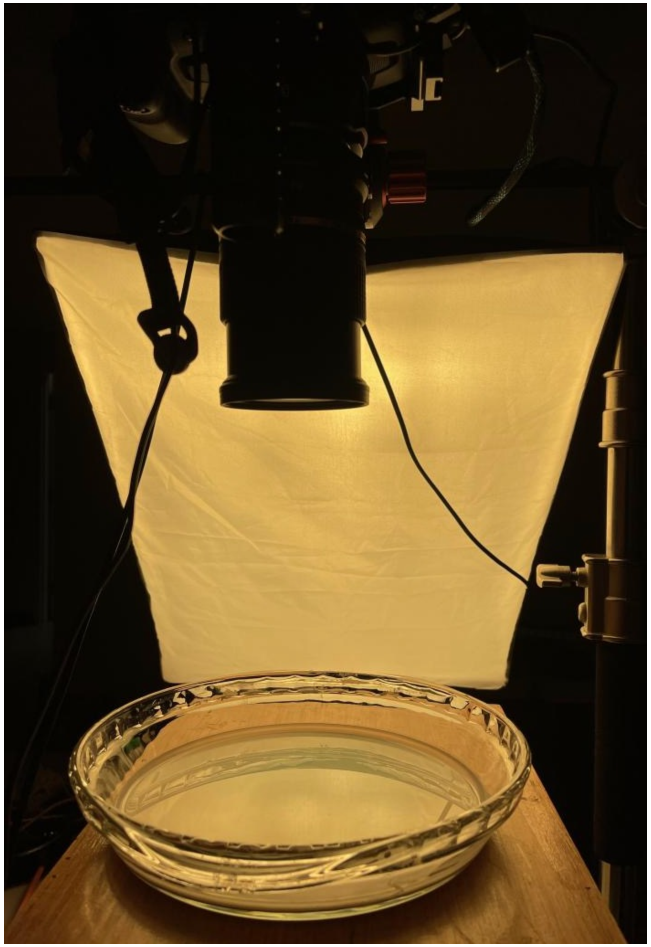


Overview

Experimental Setup

Experimental Setup

Camera



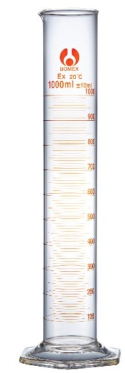
Lighting

Paint

Experimental Measurement

Digital Gram Scale
(±0.005g)

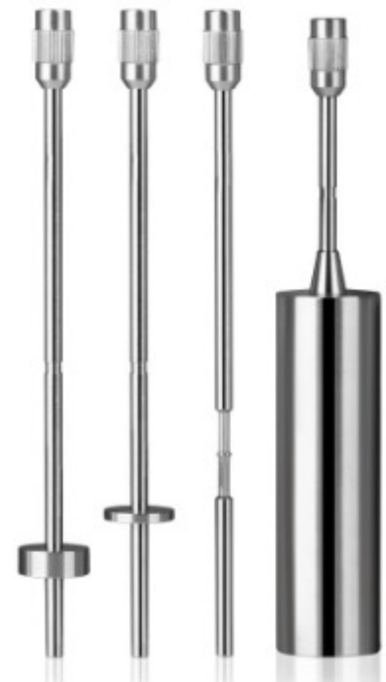
Micropipette
$(\pm 0.6\%)$



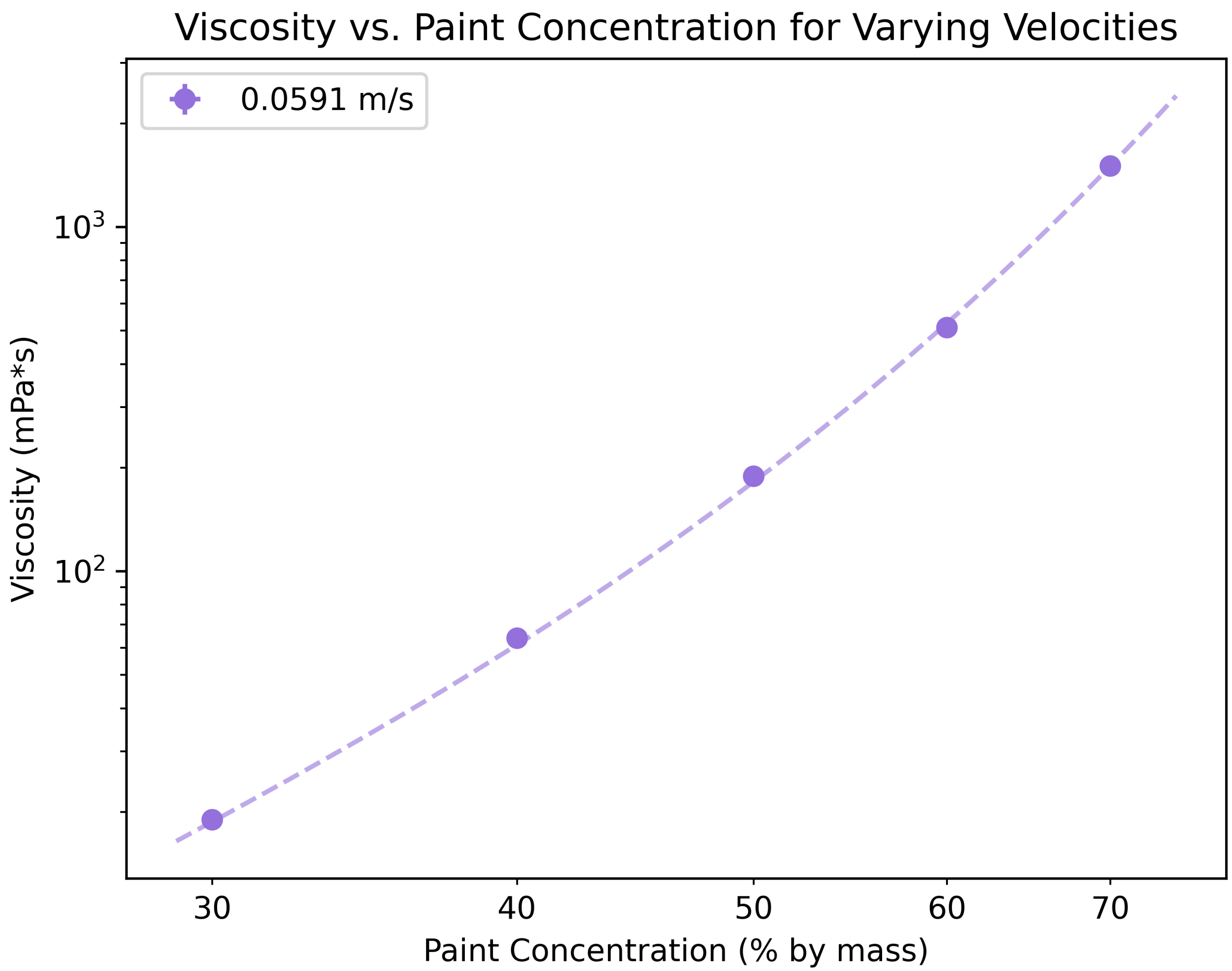
200 mL graduated cylinder
$(\pm 2\text{mL})$

Viscometry Setup

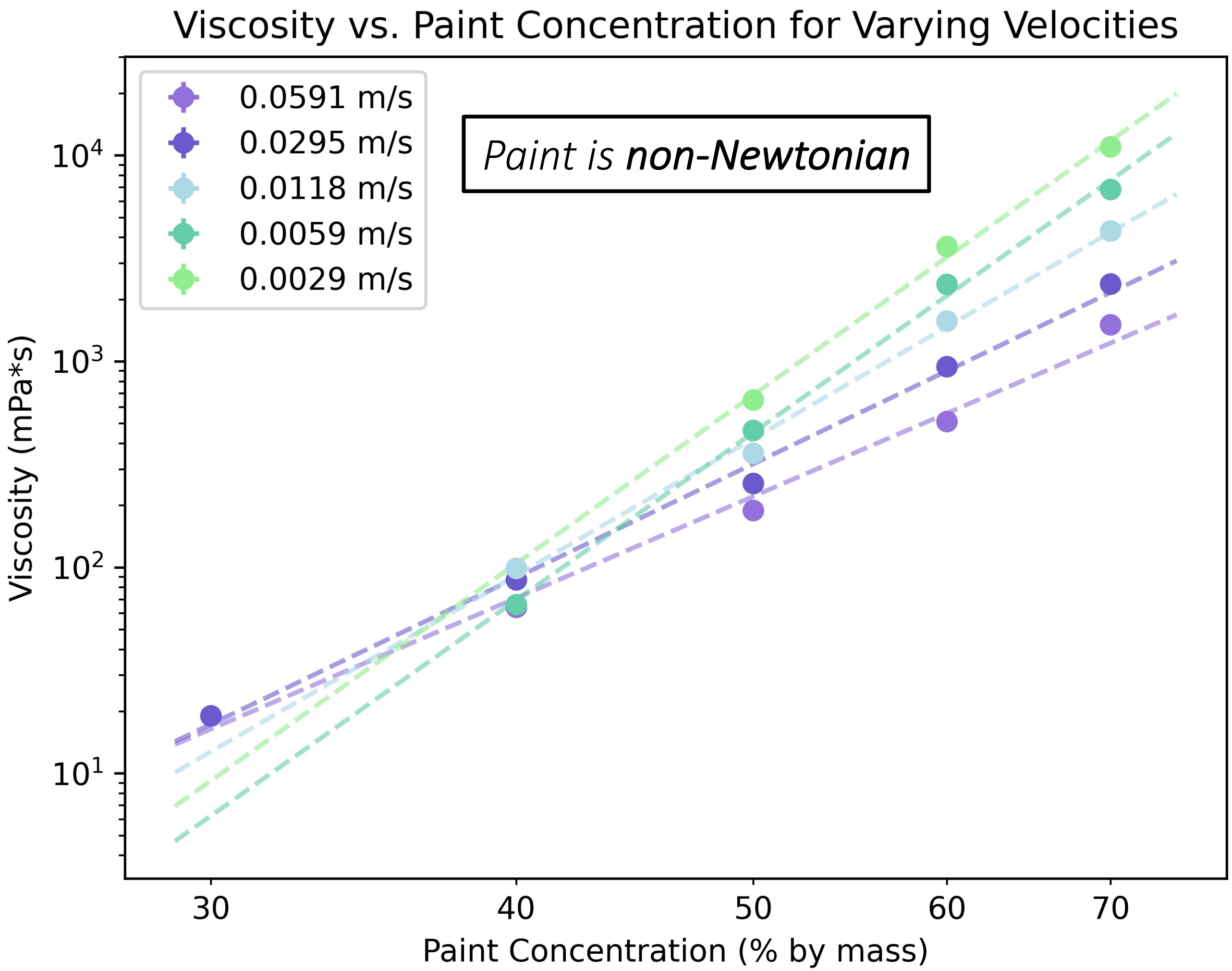
Varying Rotor Sizes



Viscosity of Paint

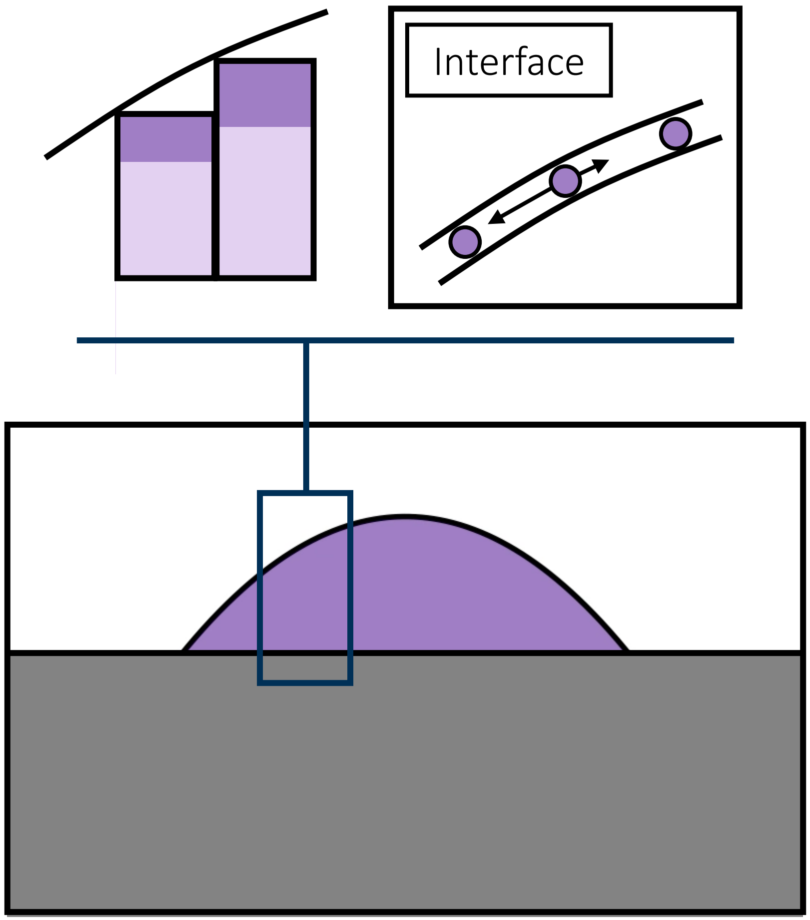


Viscosity of Paint

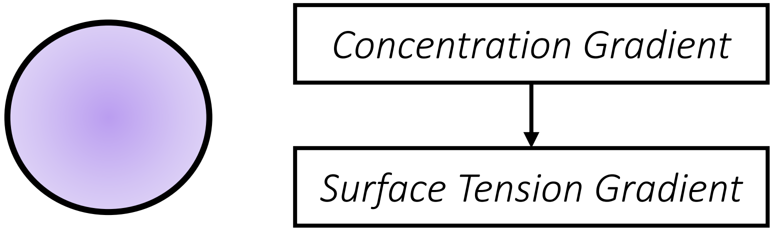


Droplet Dynamics

Qualitative Explanation



1. *Uneven evaporation along droplet profile*
2. *Radial gradients*



3. *Marangoni flow and radial expansion*

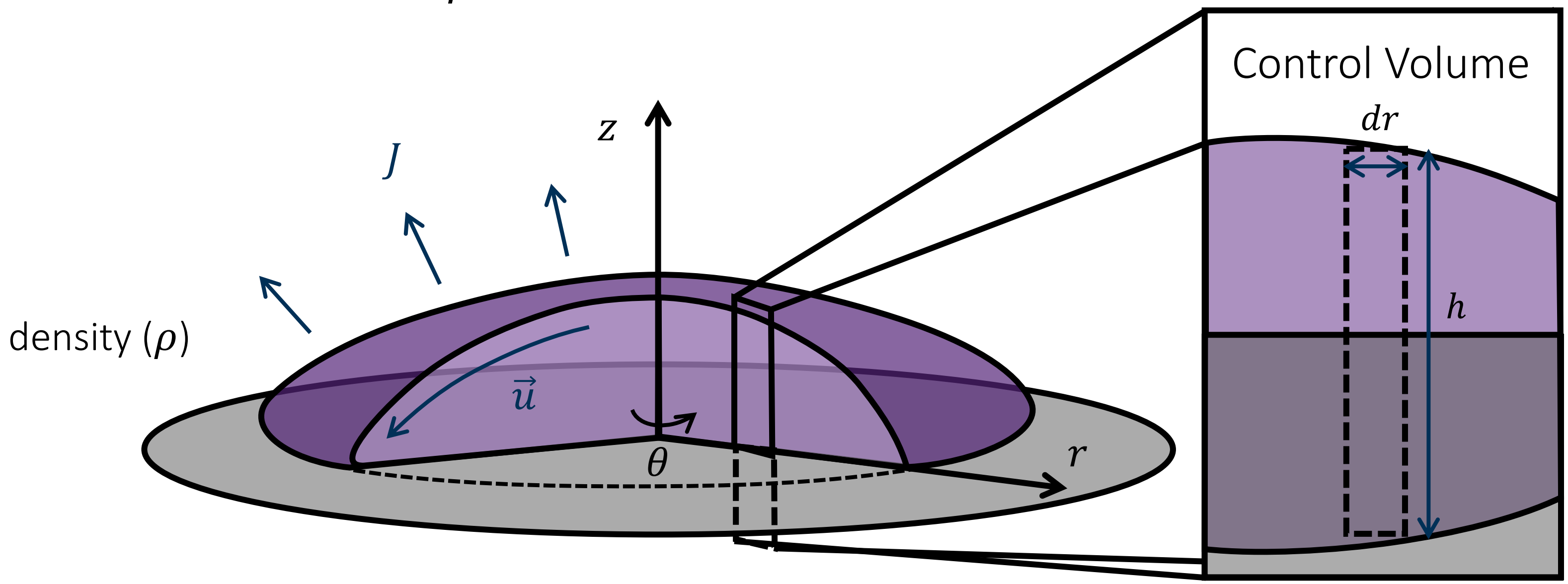
Assumptions

- *Evaporation of water is negligible*
- *Ambient room temperature and humidity remains constant*
- *The evaporation during the dropping process is negligible*

Continuity

For axisymmetric flow, by mass conservation,

$$\frac{\partial h}{\partial t} = -\frac{1}{\rho}J + \frac{1}{r}\frac{\partial Q}{\partial r} \quad \textit{where} \quad \vec{Q} = h\vec{u}$$

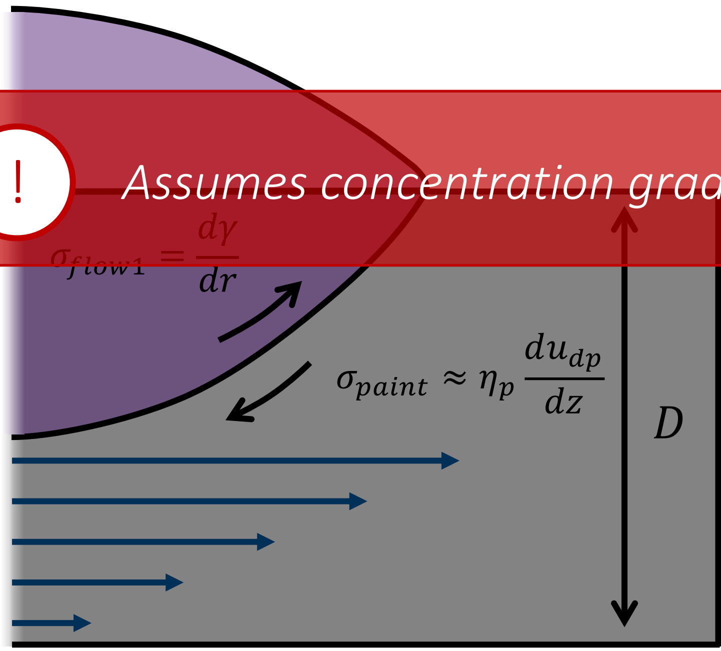


Volumetric Flow

! *Air is shear-free*

Equating shearing at drop-paint interface:

$$\frac{\partial \gamma}{\partial r} = \eta_p \frac{\partial u_{dp}}{\partial z}$$



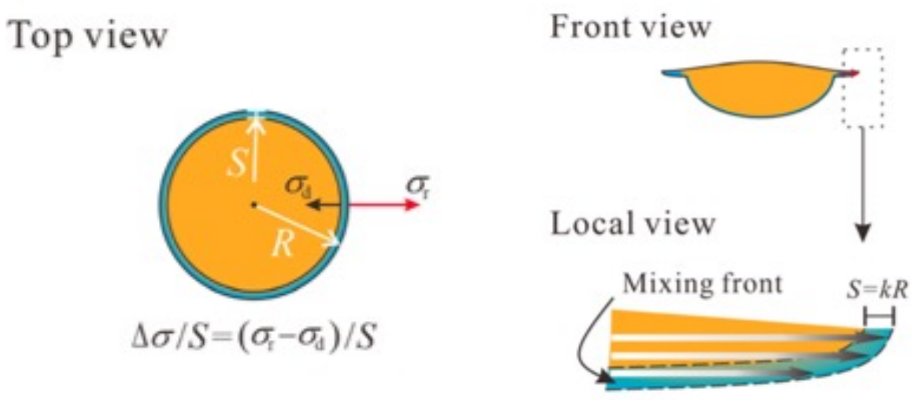
! *Assumes concentration gradient is linear over time and fluids are immiscible*

No slip condition: $u(z) = 0 \Big|_{z=-D}$ *so,* $\frac{\partial u}{\partial z} = \frac{u}{D}$

$$\boxed{u_{dp} = \frac{D}{\eta_p}\frac{\partial \gamma}{\partial r}}$$

Stage I: Marangoni Stress

- $\boldsymbol{\Delta\sigma}$ = *surface tension difference*
- σ_a= *surface tension of diluted acrylic paint*
- σ_d = *surface tension of droplet*
- S = *mixing length*
- R= *radius of droplet film*
- k= *constant of proportionality*
- μ_a = dynamic viscosity of diluted acrylic paint
- ρ_a = density of diluted acrylic paint
- t = time



When the droplet is first deposited onto the diluted acrylic paint bath,

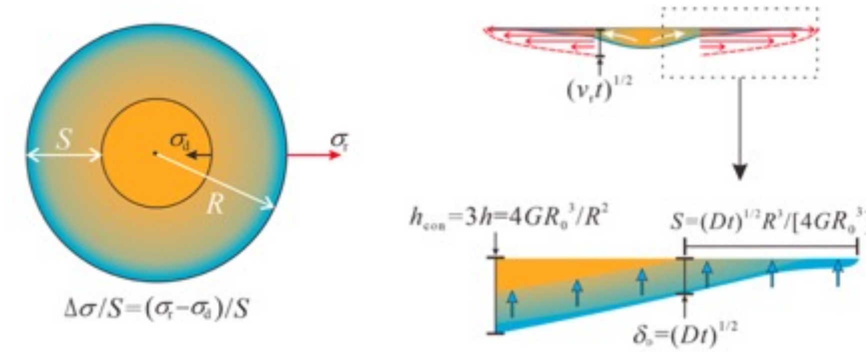
$$\frac{\boldsymbol{\Delta}\sigma}{S} = \frac{\sigma_a - \sigma_d}{kR} \sim \frac{\mu_a{}^{1/2}\rho_a{}^{1/2}R}{t^{3/2}}$$

The first stage power-law relation can be derived as,

$$R \sim \mathrm{k}^{1/2}(\sigma_a - \sigma_d)^{1/2}\mu_{\mathrm{a}}{}^{-1/4}\rho_{\mathrm{a}}{}^{-1/4}\mathrm{t}^{3/4}$$

Stage II: Normal Diffusion

- ***Δσ = surface tension difference***
- *σ_a= surface tension of diluted acrylic paint*
- *σ_d = surface tension of droplet*
- S = mixing length
- R= radius of droplet film
- k= constant of proportionality
- h_{cone}= height of cone
- δ_D= mass diffusion layer thickness
- D=binary–liquid mutual diffusion coefficient
- G= volume fraction of droplet



Once the normal diffusion dominates,

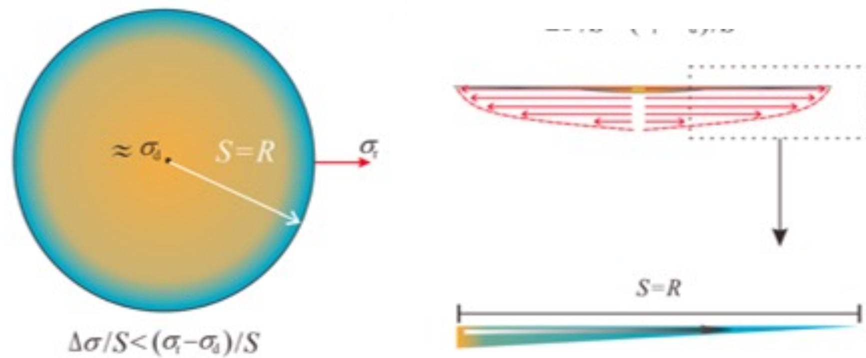
$$\frac{R}{S} = \frac{h_{cone}}{(Dt)^{1/2}} = \frac{4GR_0{}^3}{R^2(Dt)^{1/2}}$$

So,

$$R \sim \frac{(4(\sigma_a - \sigma_d)GR_0{}^3)^{\frac{1}{4}}}{(\mu_a\rho_a\mathrm{D})^{1/8}}\, t^{1/4}$$

Stage III: Tangential Stretch

- $\boldsymbol{\Delta\sigma}$ = *surface tension difference*
- σ_a= *surface tension of diluted acrylic paint*
- σ_d = *surface tension of droplet*
- S = mixing length
- R= radius of droplet film
- k= constant of proportionality
- h_{cone}= height of cone
- δ_{D}= mass diffusion layer thickness
- D=binary−liquid mutual diffusion coefficient
- G= volume fraction of droplet



Once the tangential stretch dominates,

$$\frac{\boldsymbol{\Delta\sigma}}{S} = \frac{(\sigma_a - \sigma_d)}{R} = \frac{4GR_0{}^3}{R^2(Dt)^{1/2}}$$

So,

$$R \sim \mu_a{}^{1/2}\rho_a{}^{1/2}t^{3/4}$$

Three-Stage Flow

Stage I: Marangoni Flow

$$R^* \sim k^{-1/2} {t^*}_1^{3/4}$$

$$t^*_1 = \frac{t}{{\mu_a}^3 (\sigma_a - \sigma_d)^{-2} {\rho_a}^{-1}}$$

Stage II: Normal Diffusion

$$R^* \sim {t^*}_2^{1/4}$$

$$t^*_2 = \frac{t}{\left({\mu_a}^{\frac{17}{2}} D^{\frac{1}{2}} (\sigma_a - \sigma_d)^{-5} {\rho_a}^{-\frac{7}{2}} {R_0}^{-3}\right)}$$

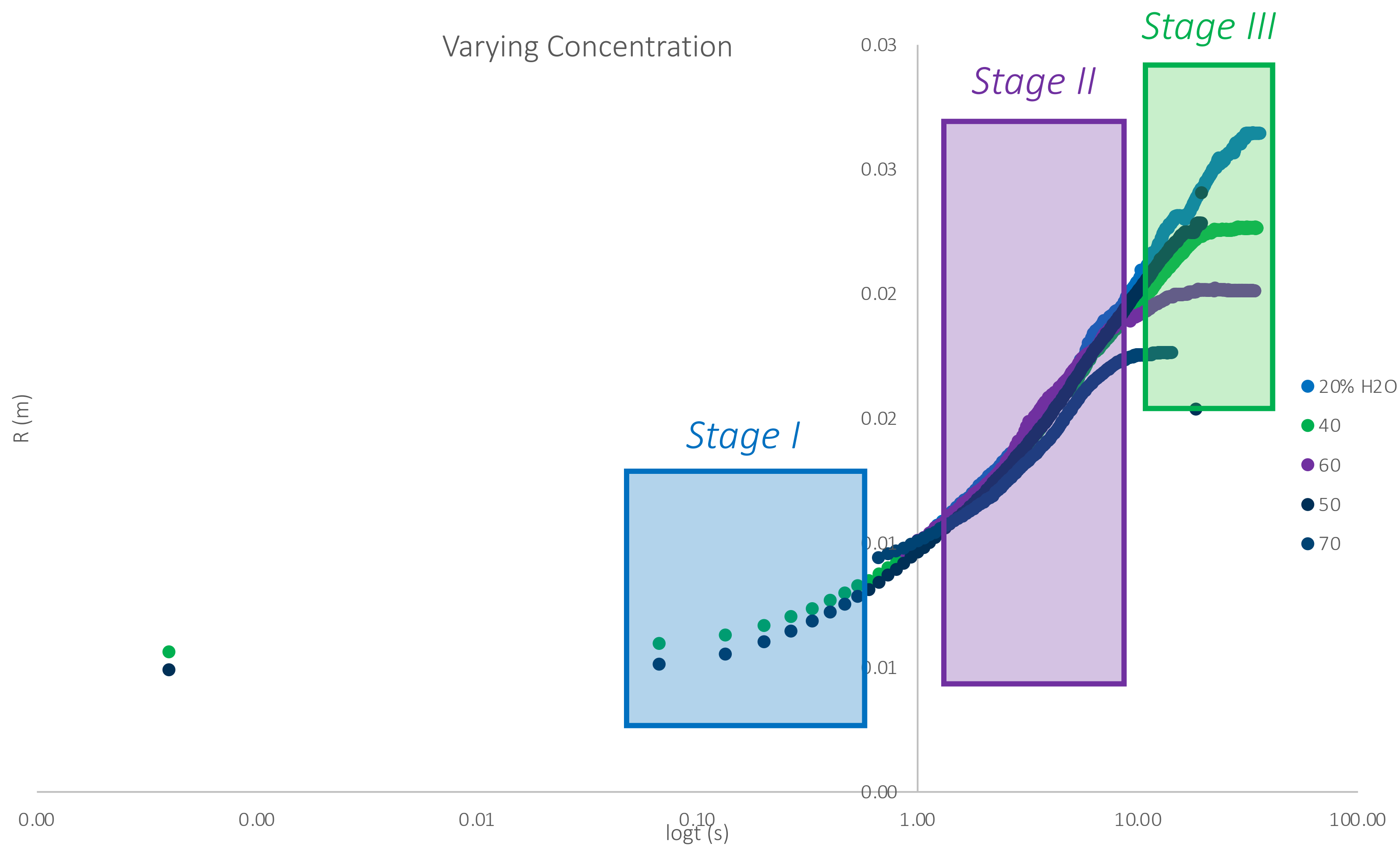
Stage III: Tangential Stretch

$$R^* \sim {t^*}_3^{3/4}$$

$$t^*_3 = \frac{t}{{\mu_a}^3 (\sigma_a - \sigma_d)^{-2} {\rho_a}^{-1}}$$

(Jia et al., 2022)

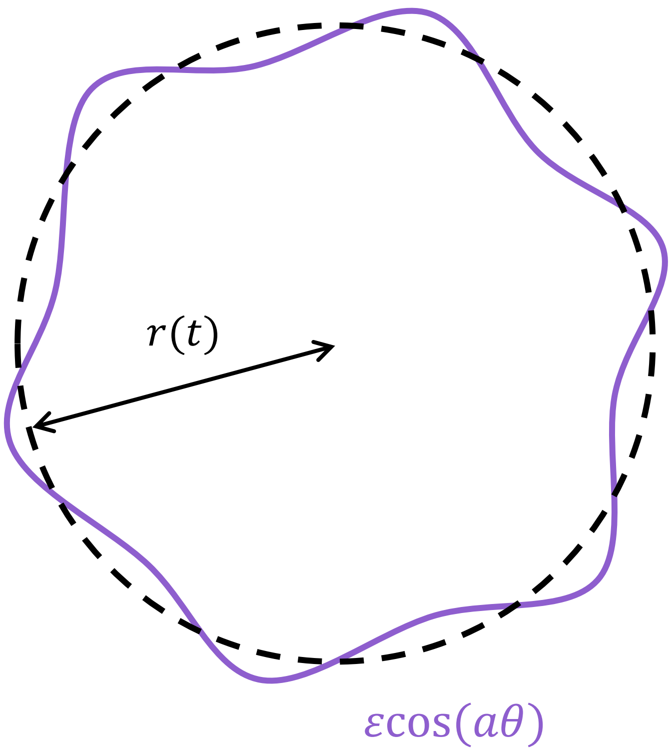
Experimental Verification



Branching Geometry

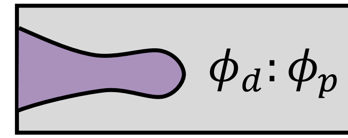
Saffman-Taylor Instability

Wavelike perturbation of magnitude ε



Viscosity Difference

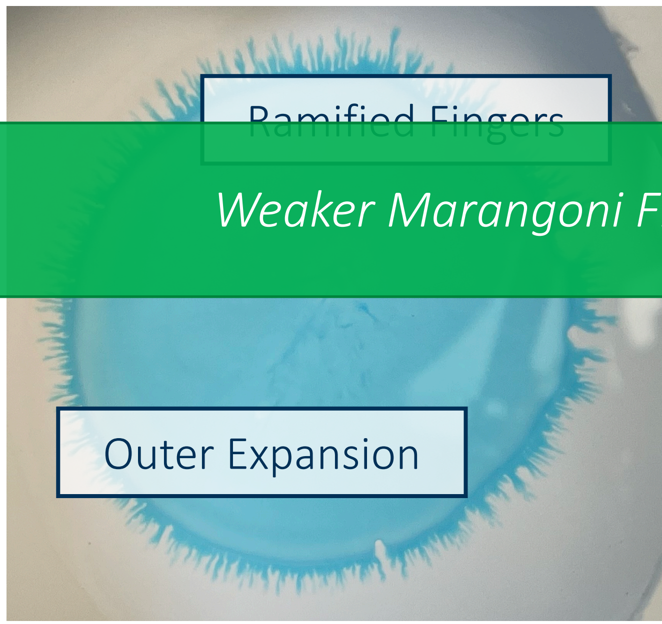
Drives linear instability



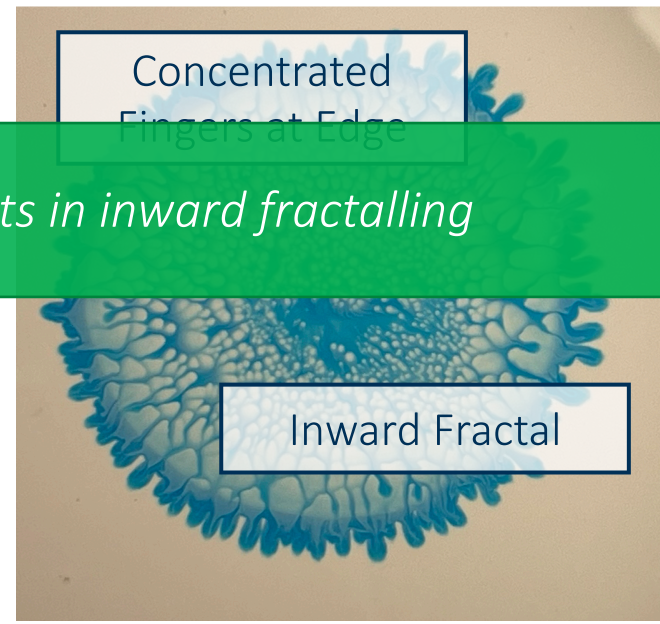
1. *Small **sinusoidal perturbation** in flow*
2. *Perturbations of **different wavelengths** grow/shrink depending on parameters*
3. ***Self similarity** of perturbations that grow **fastest** result in fractals*

Experimental Observations

80% Isopropanol

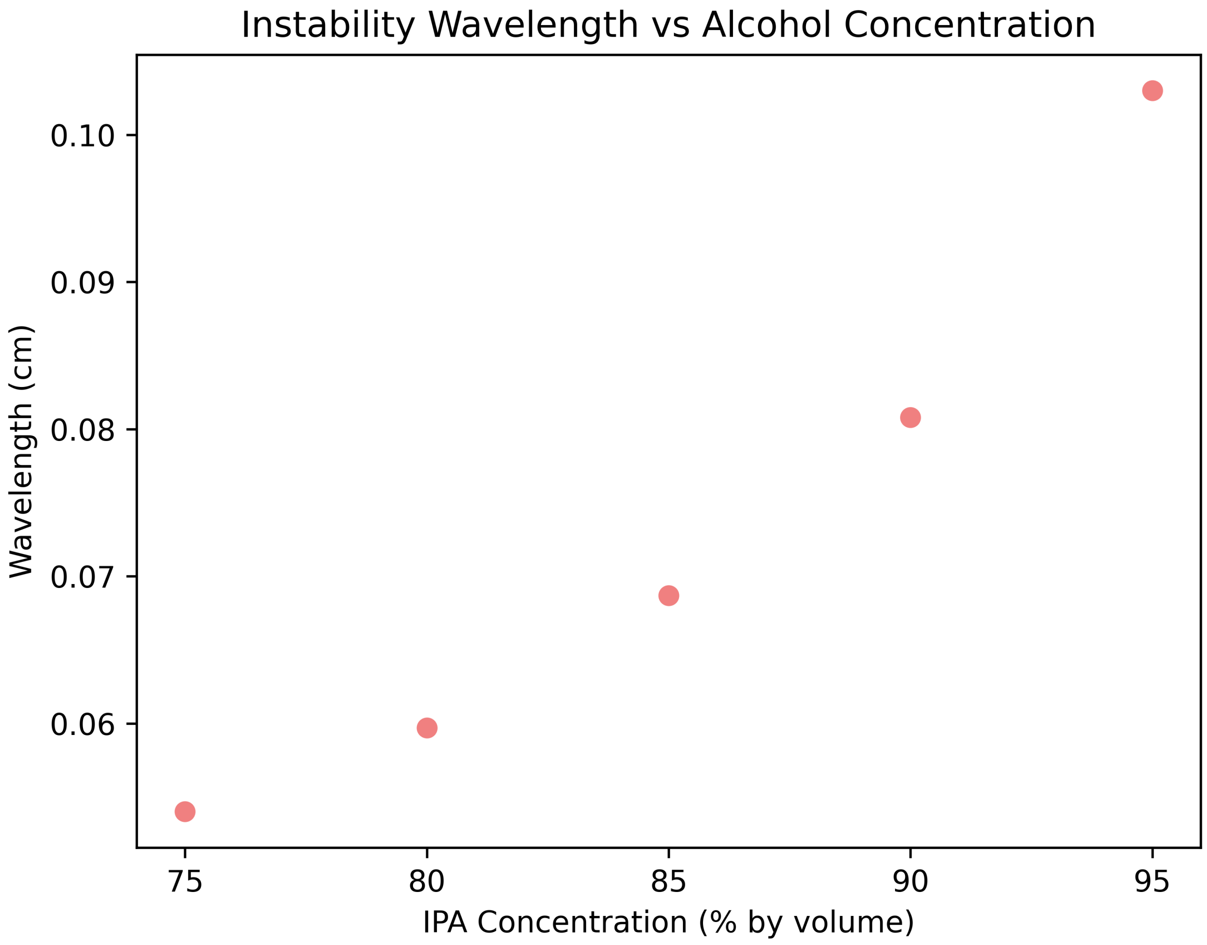


30% Isopropanol

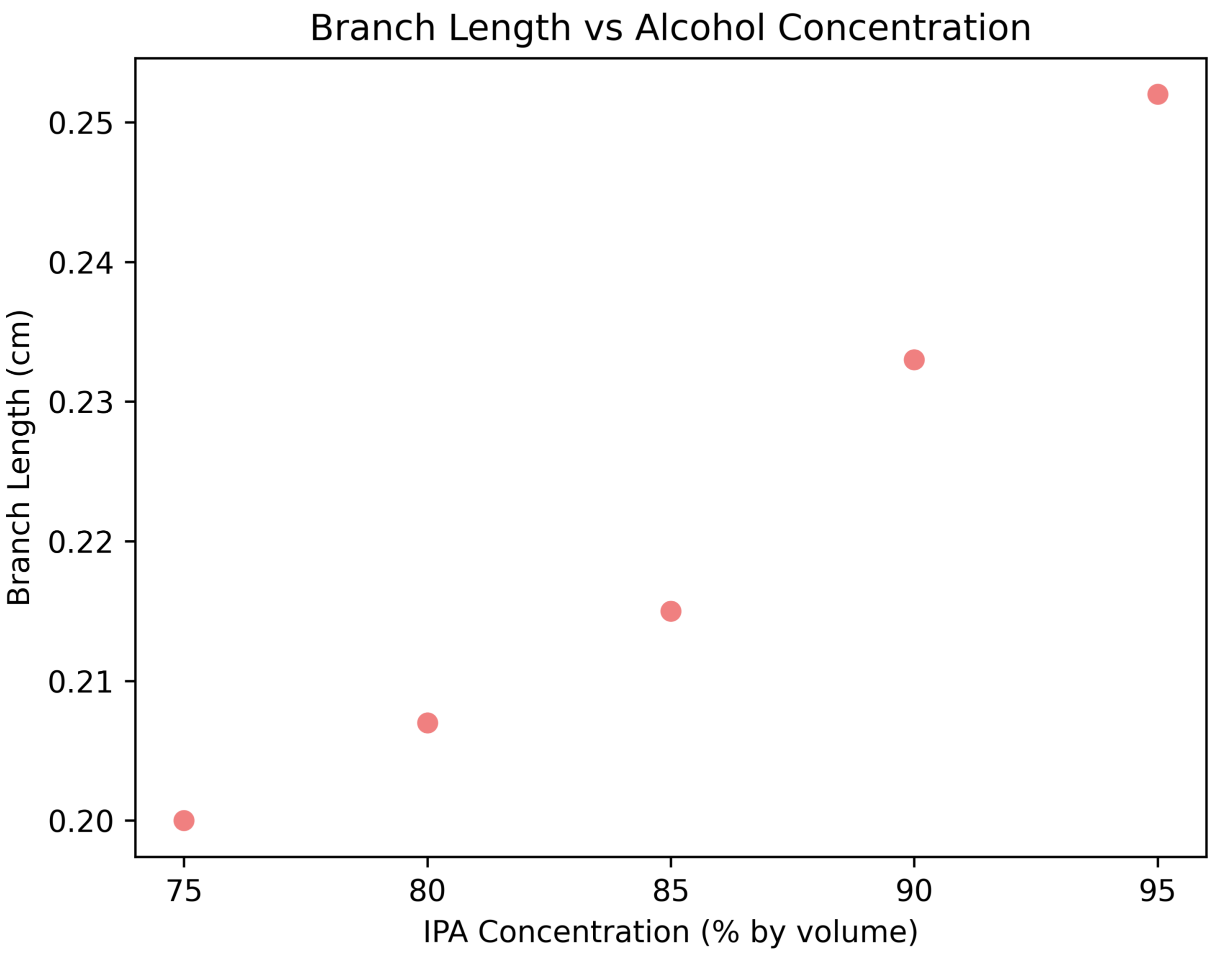


Weaker Marangoni Flow results in inward fractalling

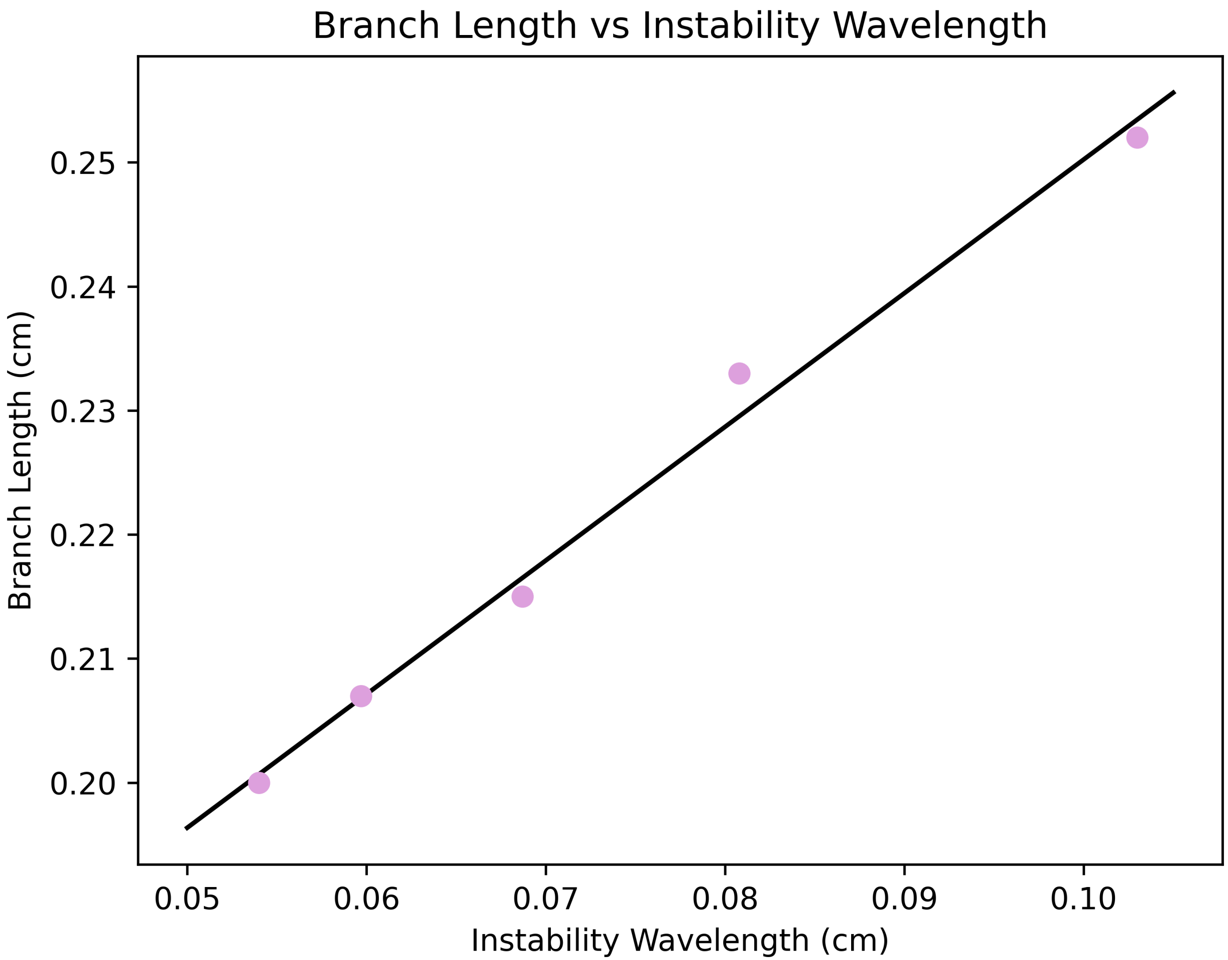
Wavelength of Instability



Branch Length



Linear Instability

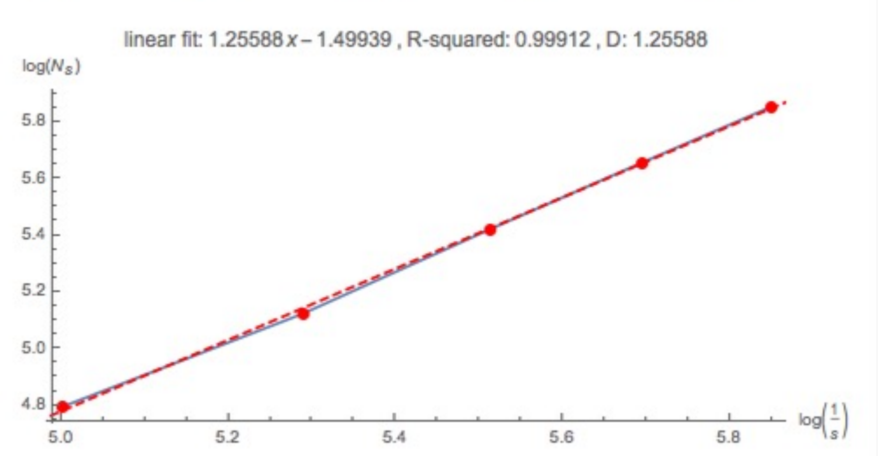


Fractal Dimension

- Fractals are defined shapes with a **Hausdorff Dimension** that is higher than the integer **topological dimension**

- Hausdorff Dimension can be calculated using a variety of different methods
  - Box Counting Dimension

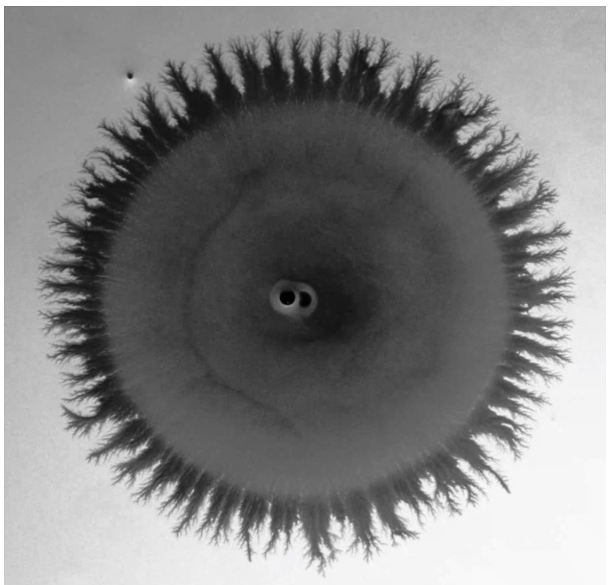
- Dimension = $\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$



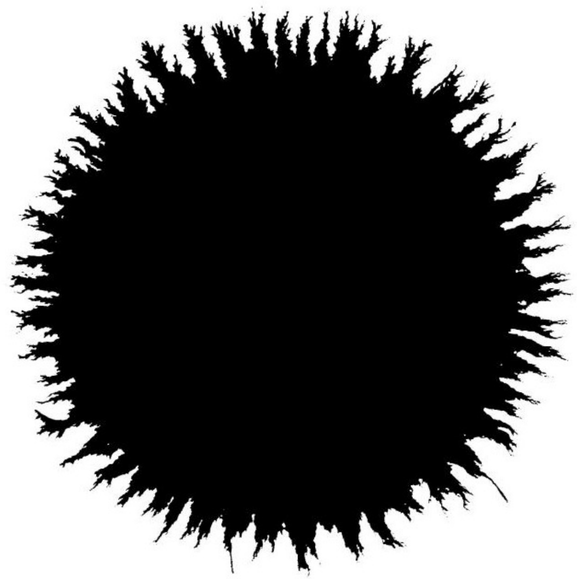
Examples from Wolfram Alpha's [Demonstration]

Fractal Dimension

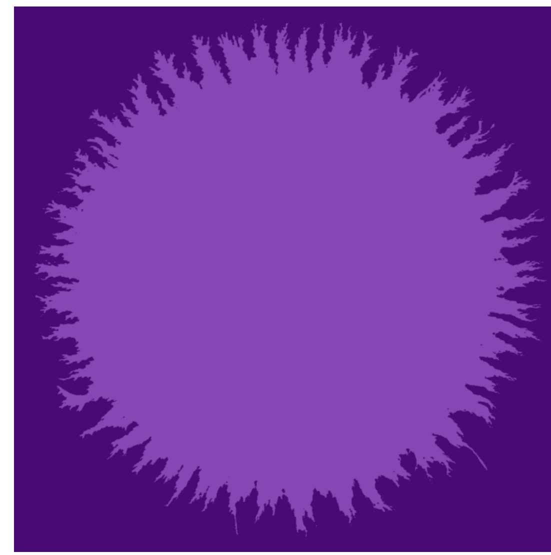
Raw Video



Grayscale

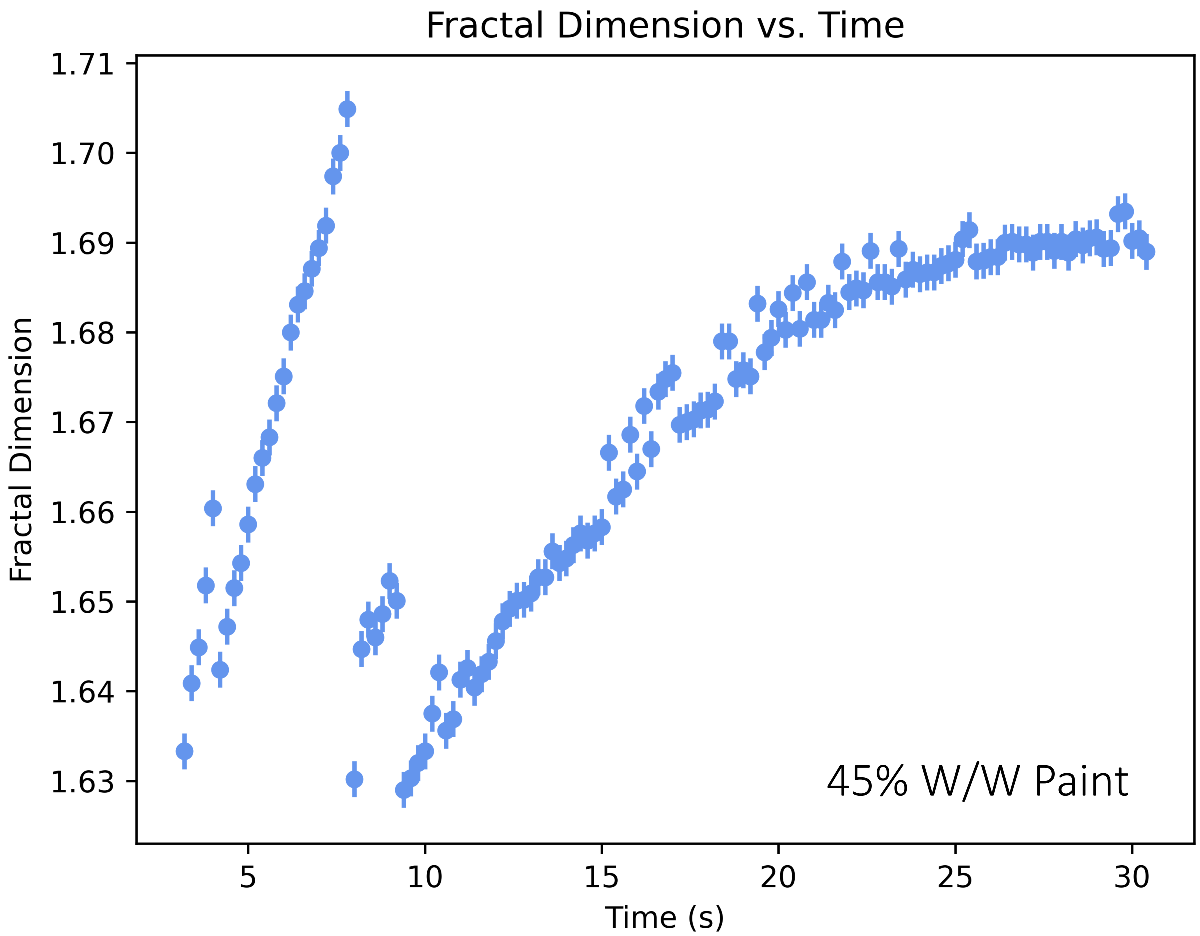


FracLac Box Counting

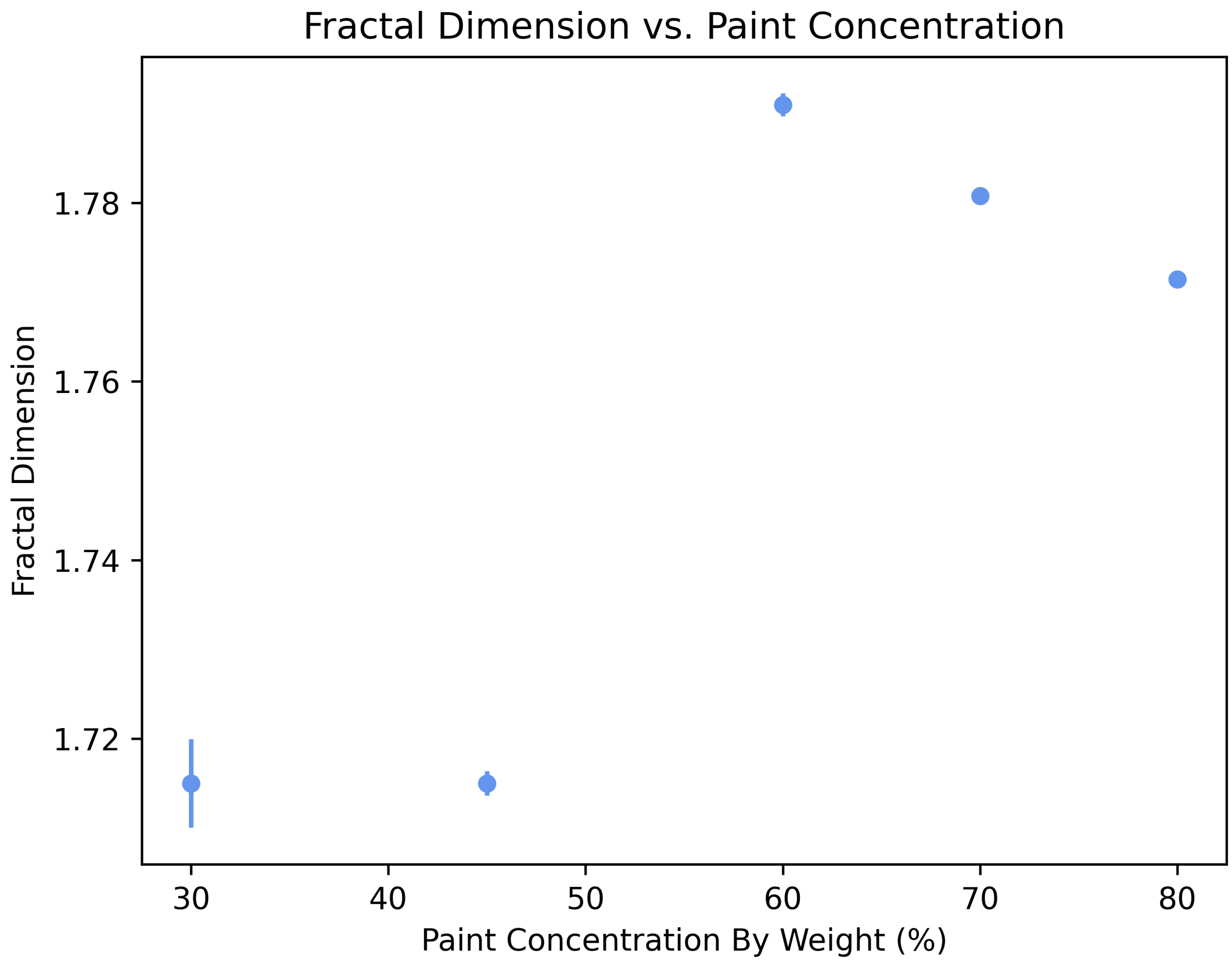


✓ *Able to accurately and efficiently track fractal dimension over time*

Experimental Data



Experimental Data



Insights

Varying Key Parameters

Flow Velocity

Negative Correlation

Paint Concentration

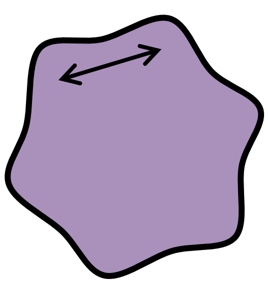
Positive Correlation

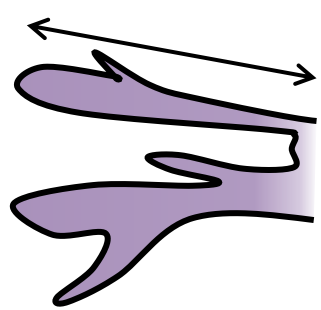
Ink Concentration

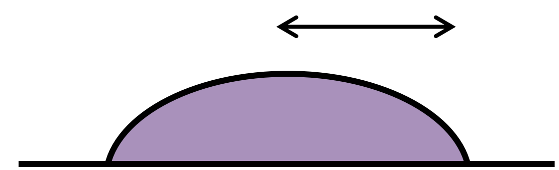
Instability Wavelength

Branch Length

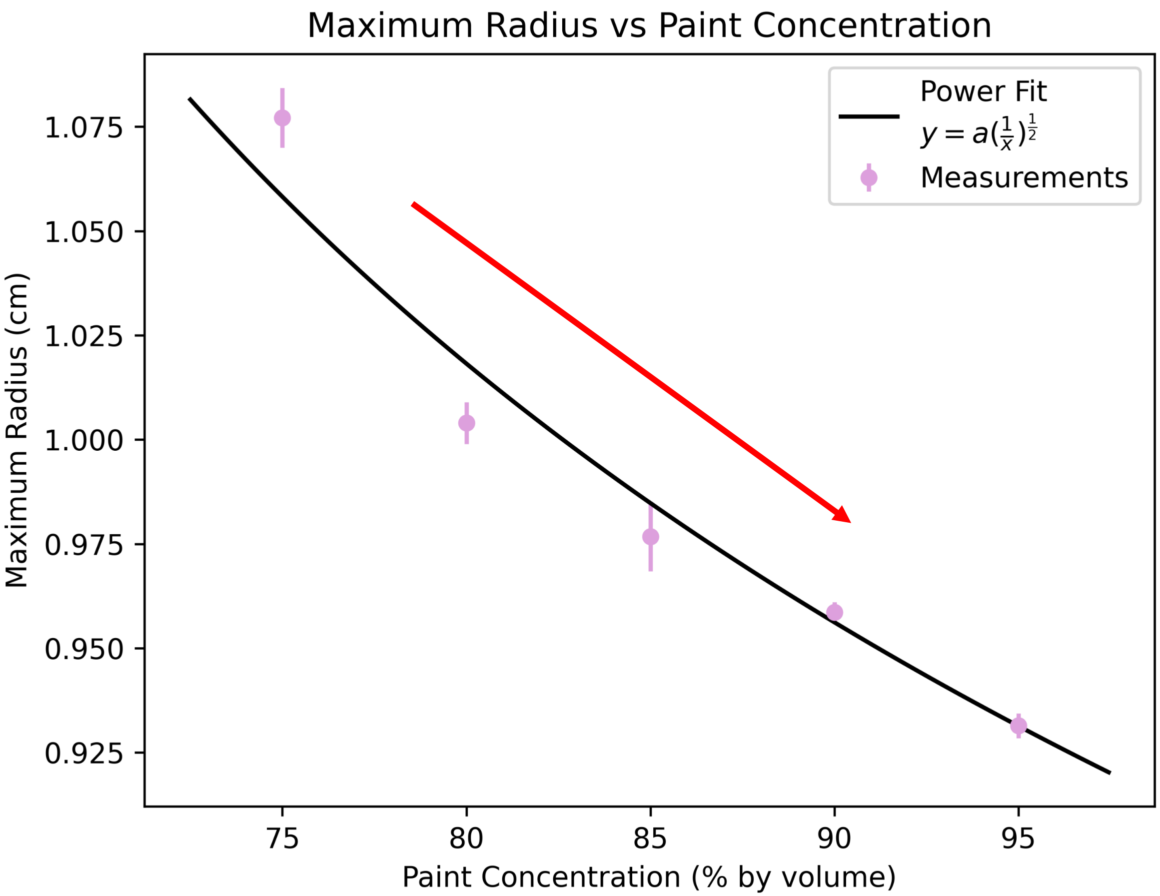
Maximum Radius







Varying Key Parameters

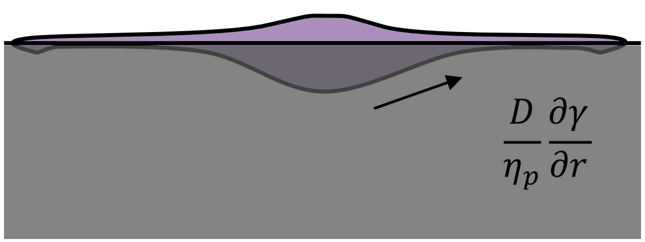


Paint Concentration

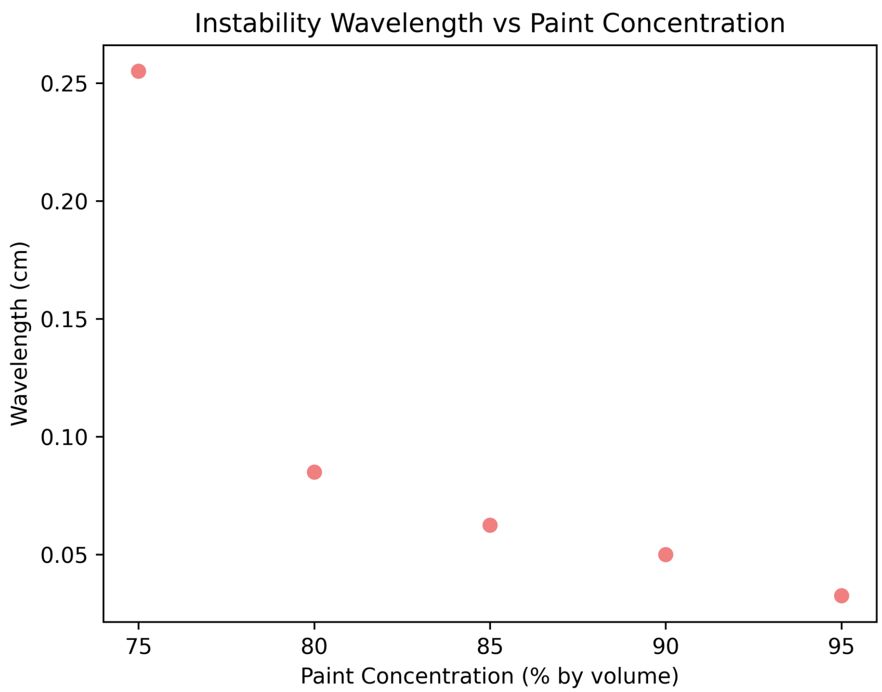
Maximum Radius

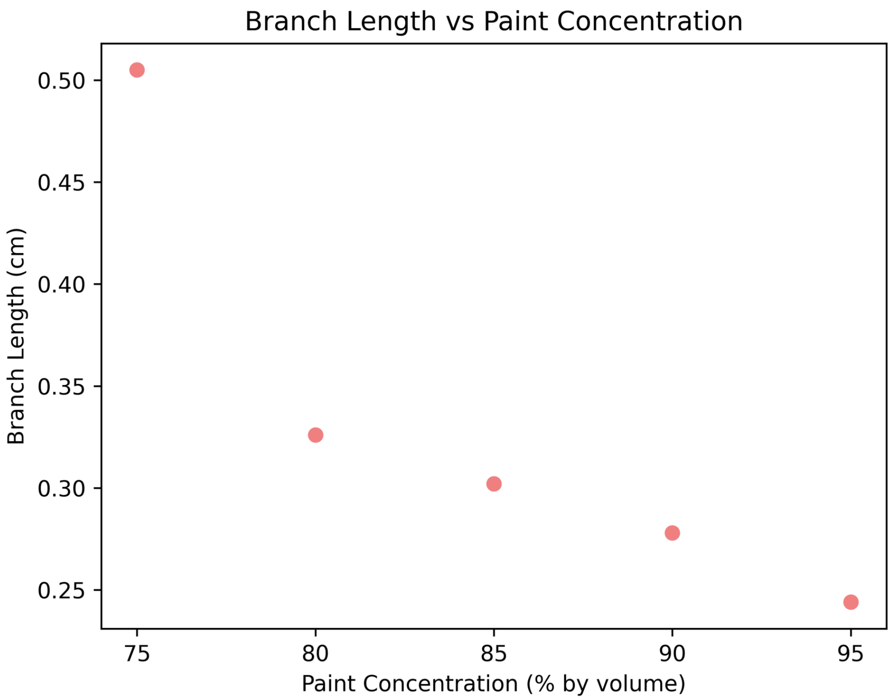
$$R \approx \left(\frac{(\phi_0 - \phi_f)\Delta\gamma V D}{(1-\phi_f) J \eta_p}\right)^{\frac{1}{4}}$$

More resistance to flow



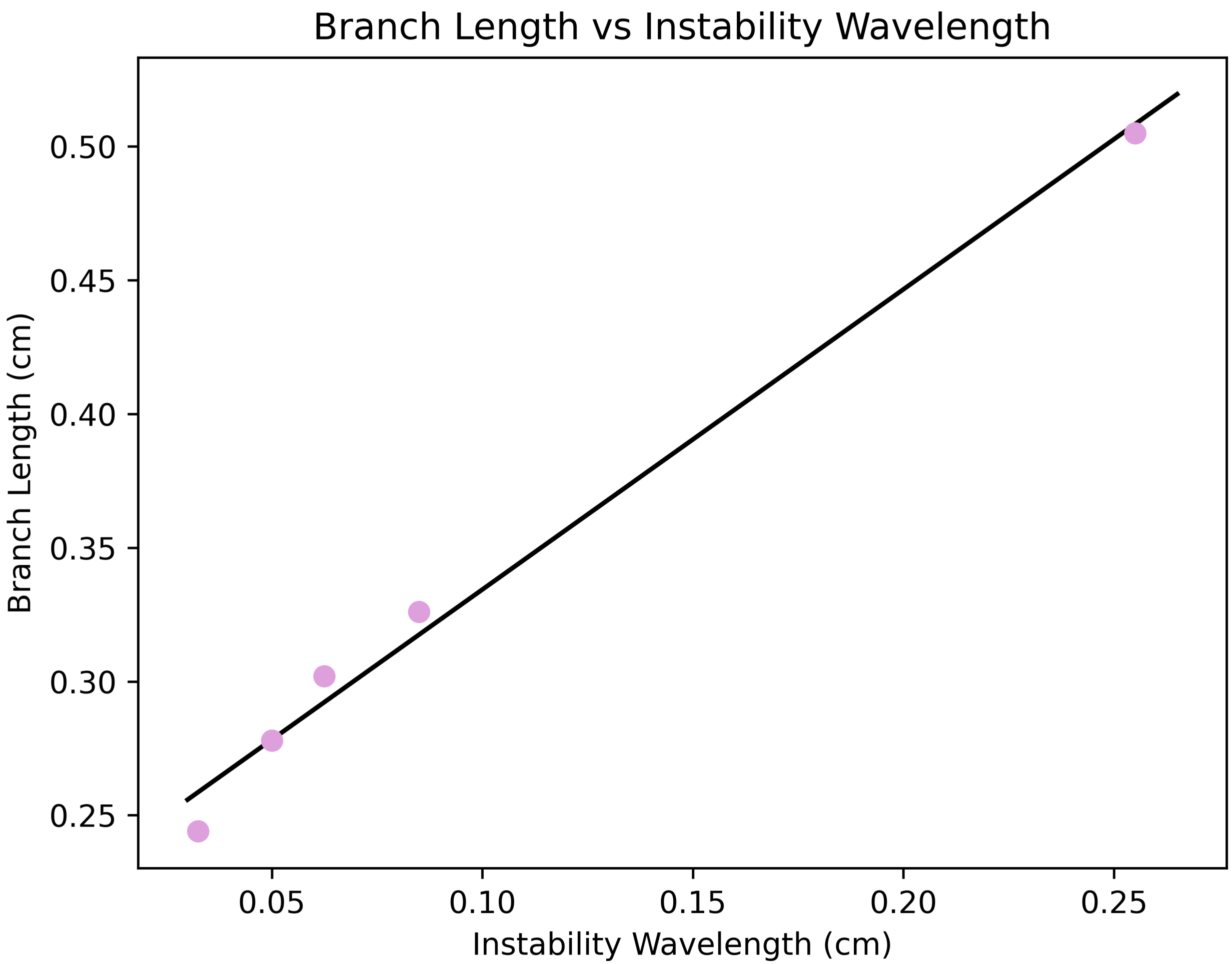
Varying Key Parameters



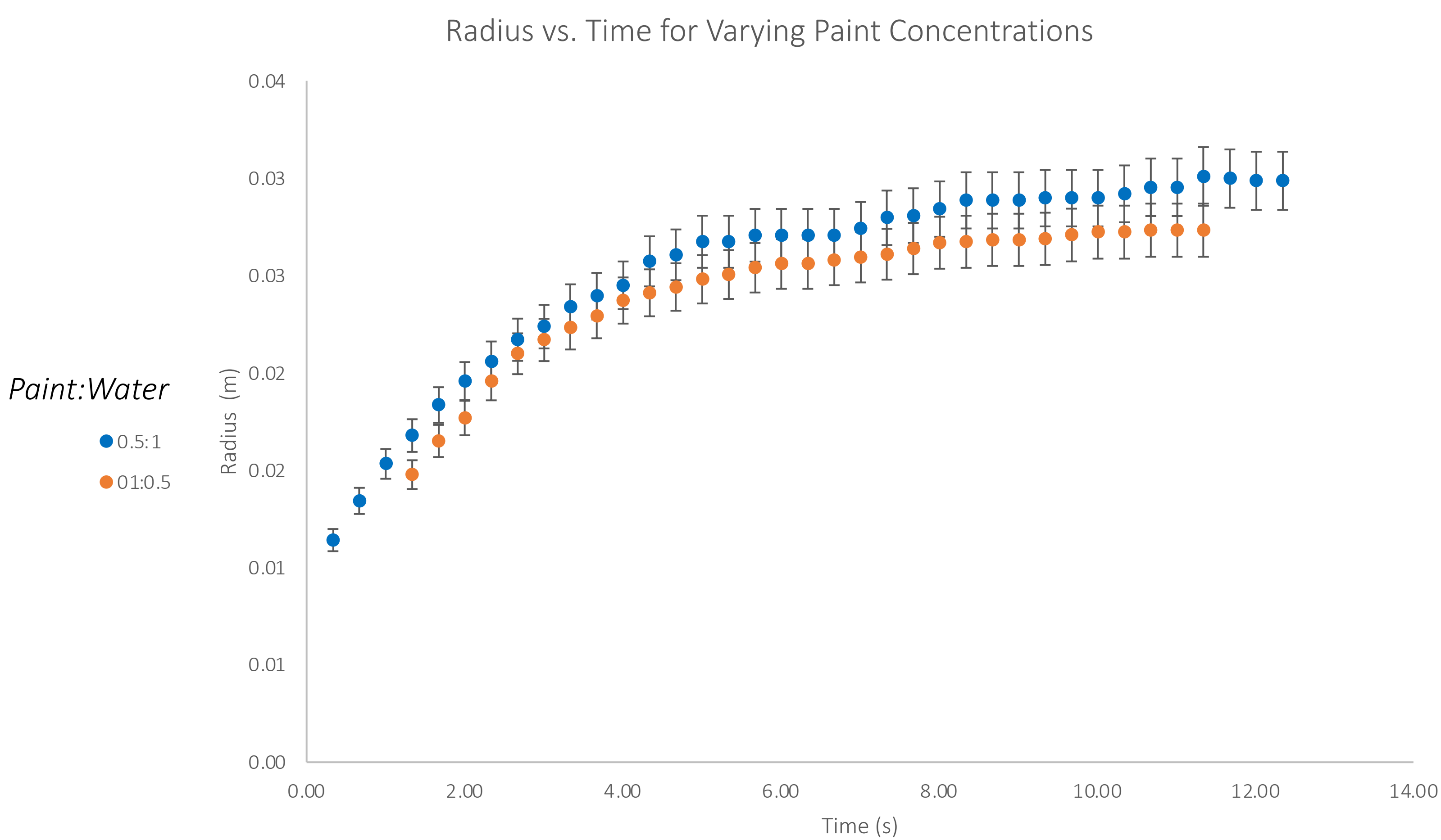


As flow velocity increases, the instability amplitude and wavelength increase

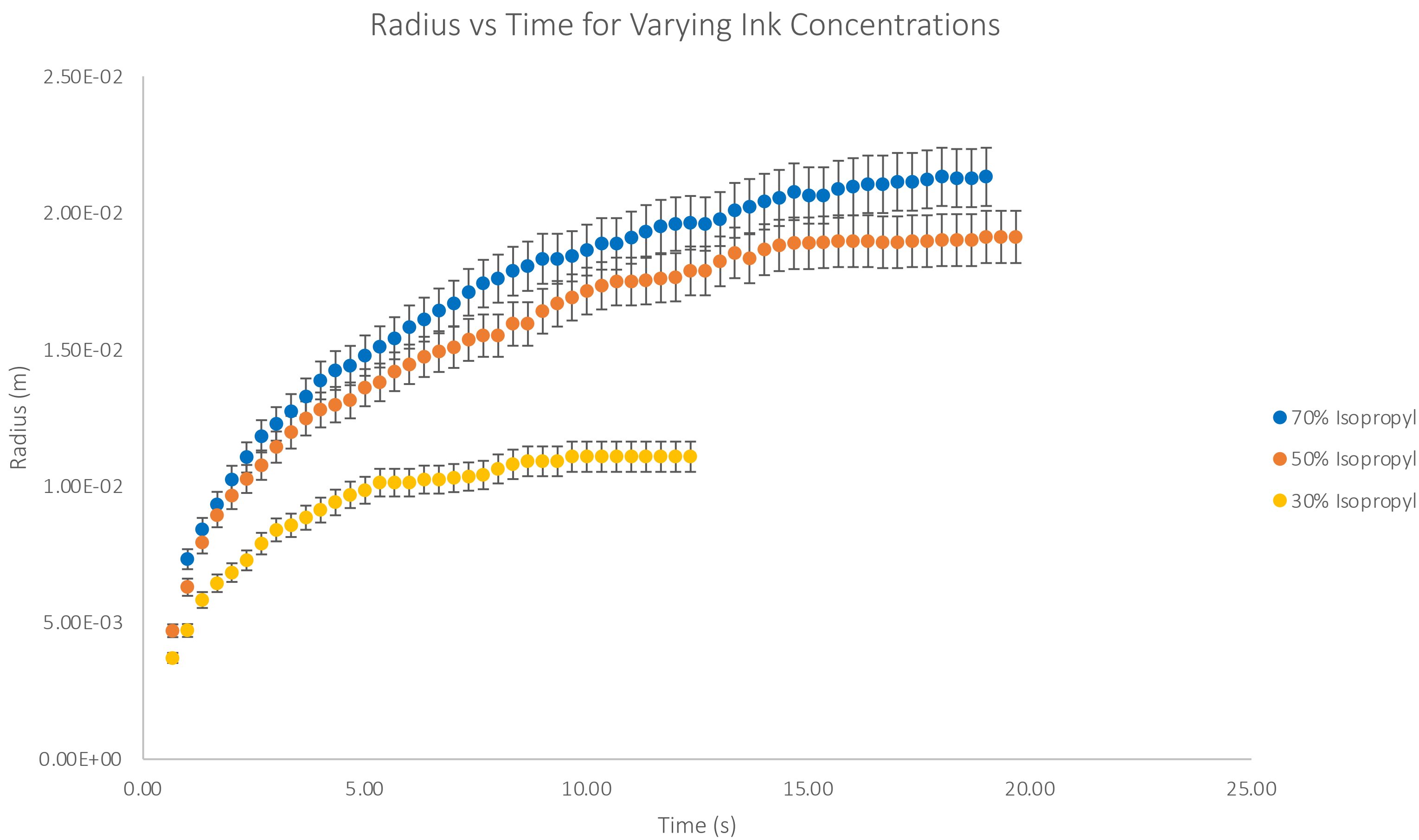
Varying Key Parameters



Varying Key Parameters



Varying Key Parameters



References

Gelderblom, H., Diddens, C., & Marin, A. (2022). Evaporation-driven liquid flow in sessile droplets. Soft Matter, 18(45), 8535–8553. https://doi.org/10.1039/d2sm00931e

Jia, F., Wang, T., Peng, X., & Sun, K. (2022). Three stages of Marangoni-driven film spreading for miscible fluids. Physics of Fluids, 34(12), 121705–121705. https://doi.org/10.1063/5.0132216

Keiser, L., Bense, H., Colinet, P., Bico, J., & Reyssat, E. (2017). Marangoni Bursting: Evaporation-Induced Emulsification of Binary Mixtures on a Liquid Layer. Physical Review Letters, 118(7). https://doi.org/10.1103/physrevlett.118.074504

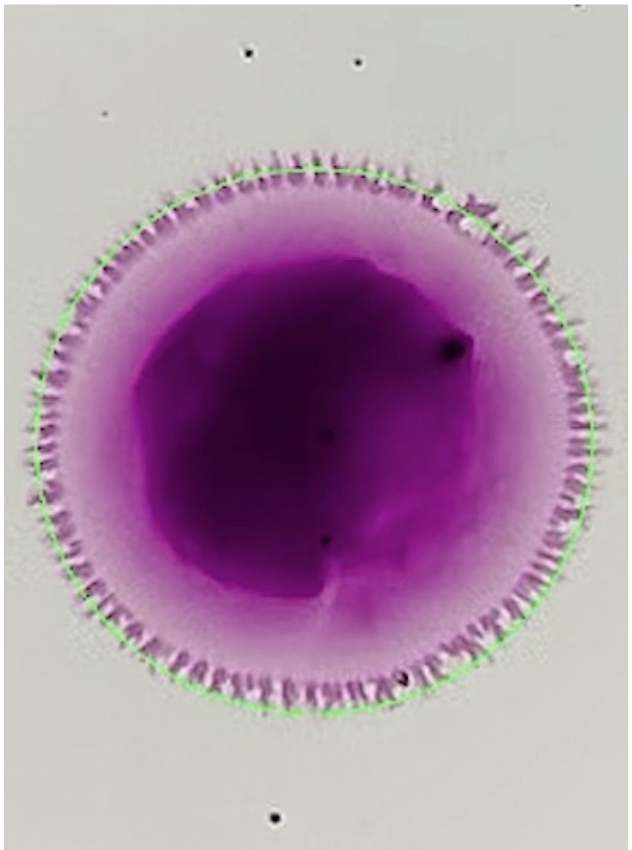
Kondic. (n.d.). Saffman-Taylor Instability Motivation. Retrieved January 20, 2023, from https://web.njit.edu/~kondic/capstone/2015/saffman-taylor.pdf

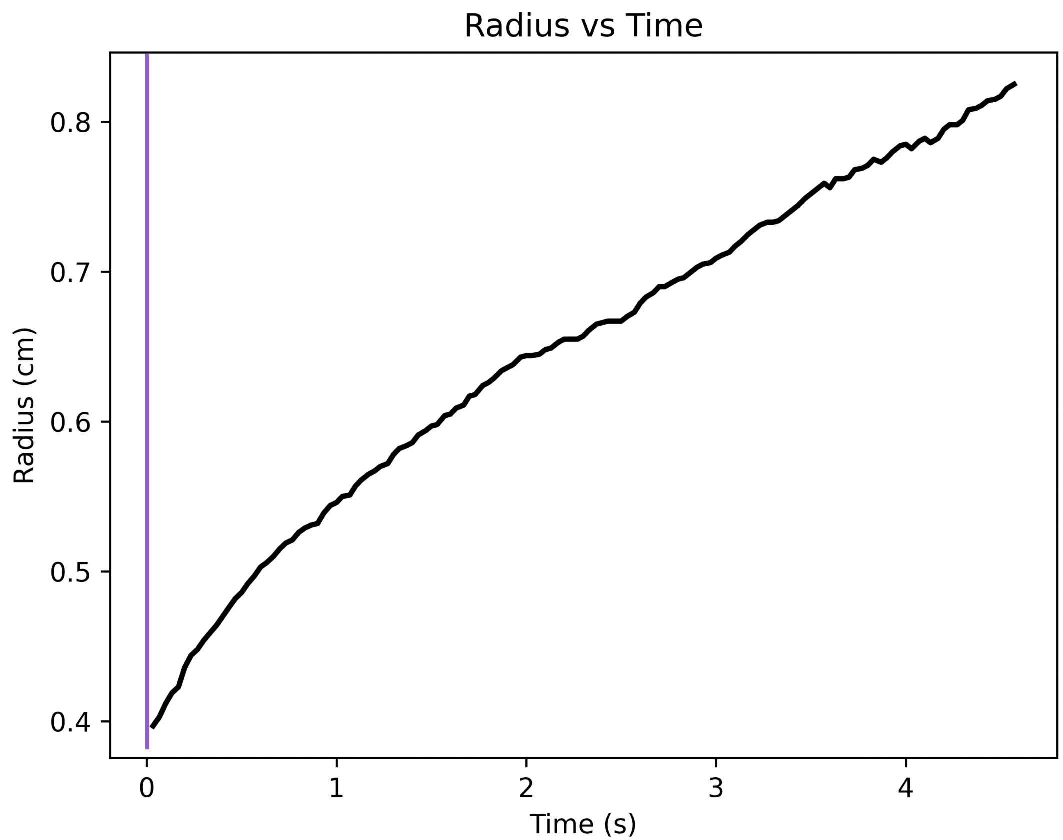
Lancaster, J. F. (1999). Introductory. Metallurgy of Welding, 1–39. https://doi.org/10.1533/9781845694869.1

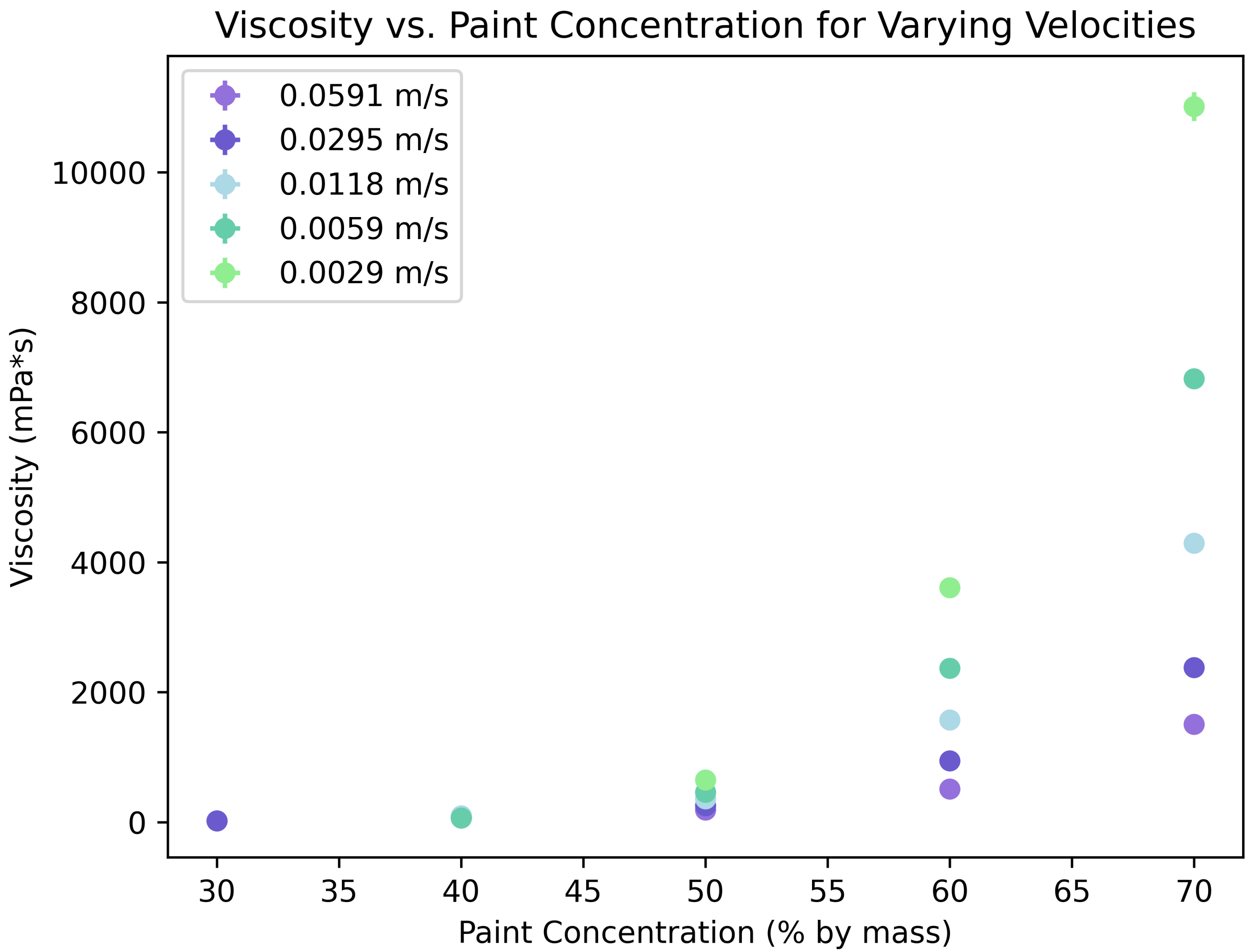
Thank you for listening!

Appendix

Radius Tracking



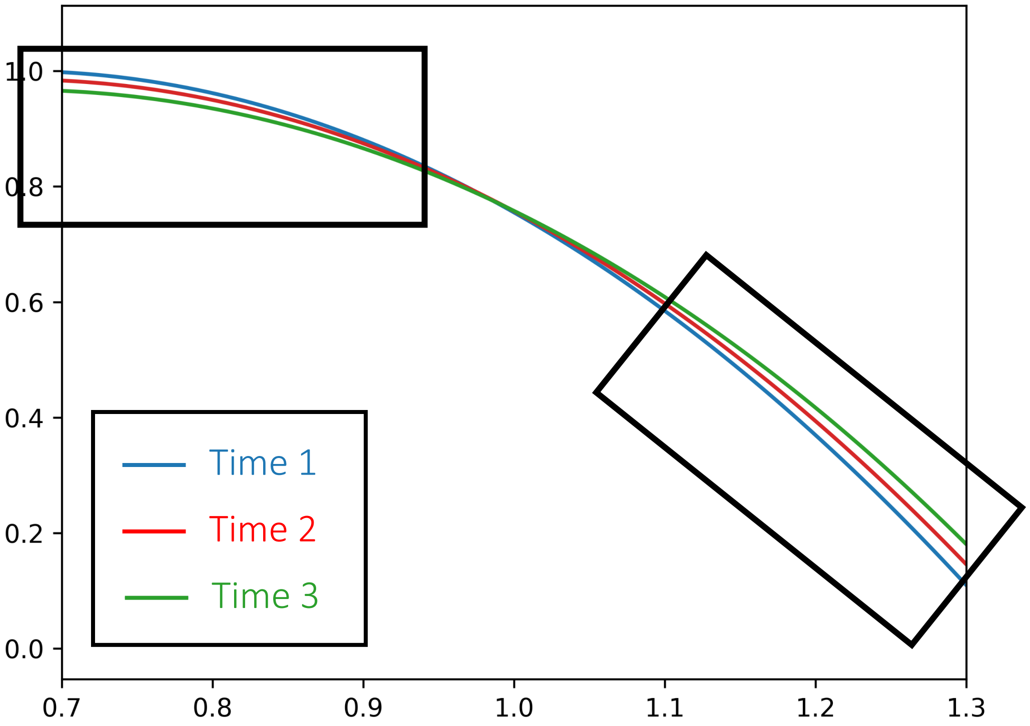



Viscosity vs. Paint Concentration for Varying Velocities
0.0591 m/s
0.0295 m/s
0.0118 m/s
0.0059 m/s
0.0029 m/s
Viscosity (mPa*s)
0
2000
4000
6000
8000
10000
30
35
40
45
50
55
60
65
70
Paint Concentration (% by mass)

Appendix

Numerical Solution

$$-\frac{\partial h}{\partial t}=\frac{1}{\rho}J-\frac{1}{r}\frac{\partial}{\partial r}\left(k\frac{(h^2+2Dh)}{\eta_p\eta_d}\frac{\partial}{\partial r}\left(-\frac{Jt}{h\rho}\right)\right) \quad \textit{where} \quad -\frac{Jt}{h\rho}=0 \quad \textit{if} \quad Jt\geq\phi_d h$$

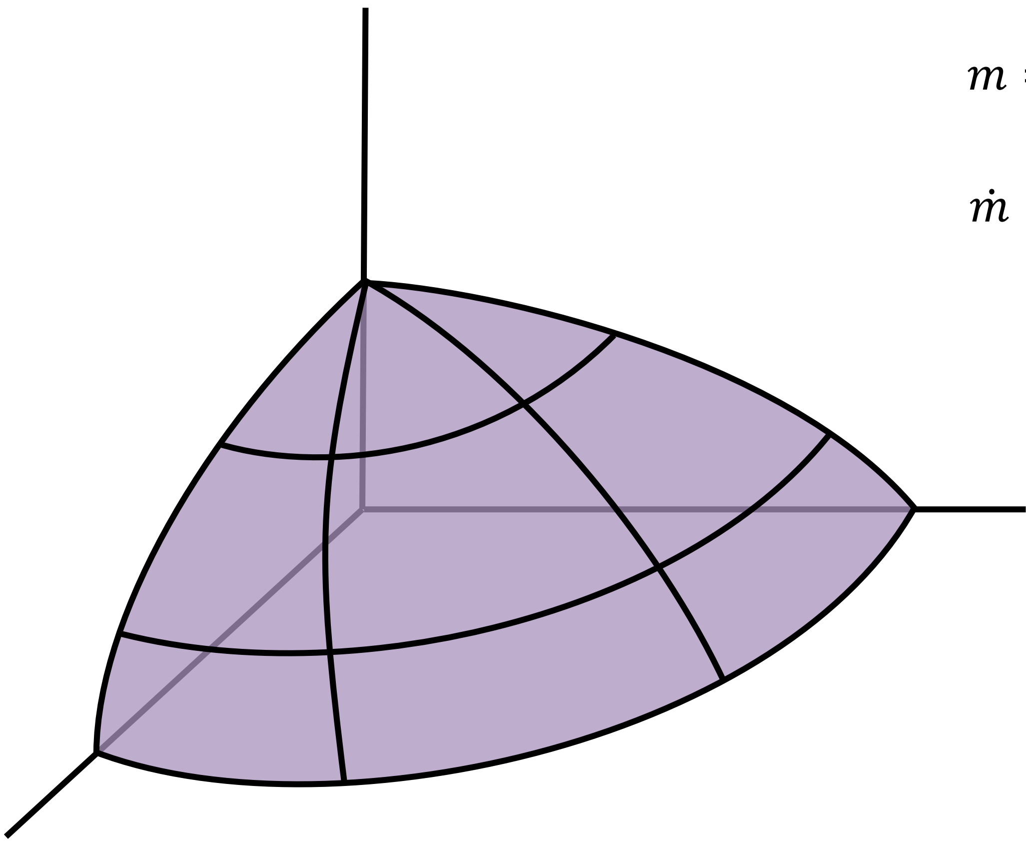


Height decrease at small r

Height increase at large r

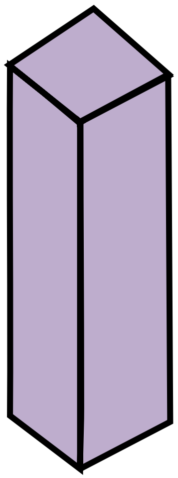
✓ *Flow as predicted by qualitative*

Continuity Derivation



$$m = \rho dr(rd\theta)h$$

$$\dot{m} = \rho dr(rd\theta)\frac{dh}{dt} = \sum \dot{m} = m_u + m_e$$



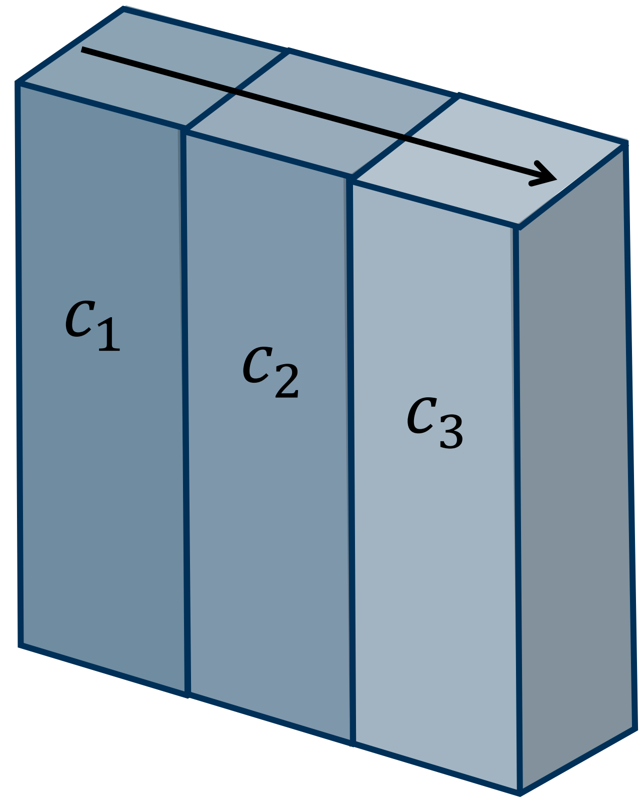
$$\rho dr(rd\theta)\frac{dh}{dt} = -Jrd\theta dr + \rho h(rd\theta)\left(u_r\Big|_{r+dr} - u_r\Big|_r\right) + (\rho hdr)\left(u_\theta\Big|_{\theta+d\theta} - u_\theta\Big|_\theta\right)$$

Concentration Gradient

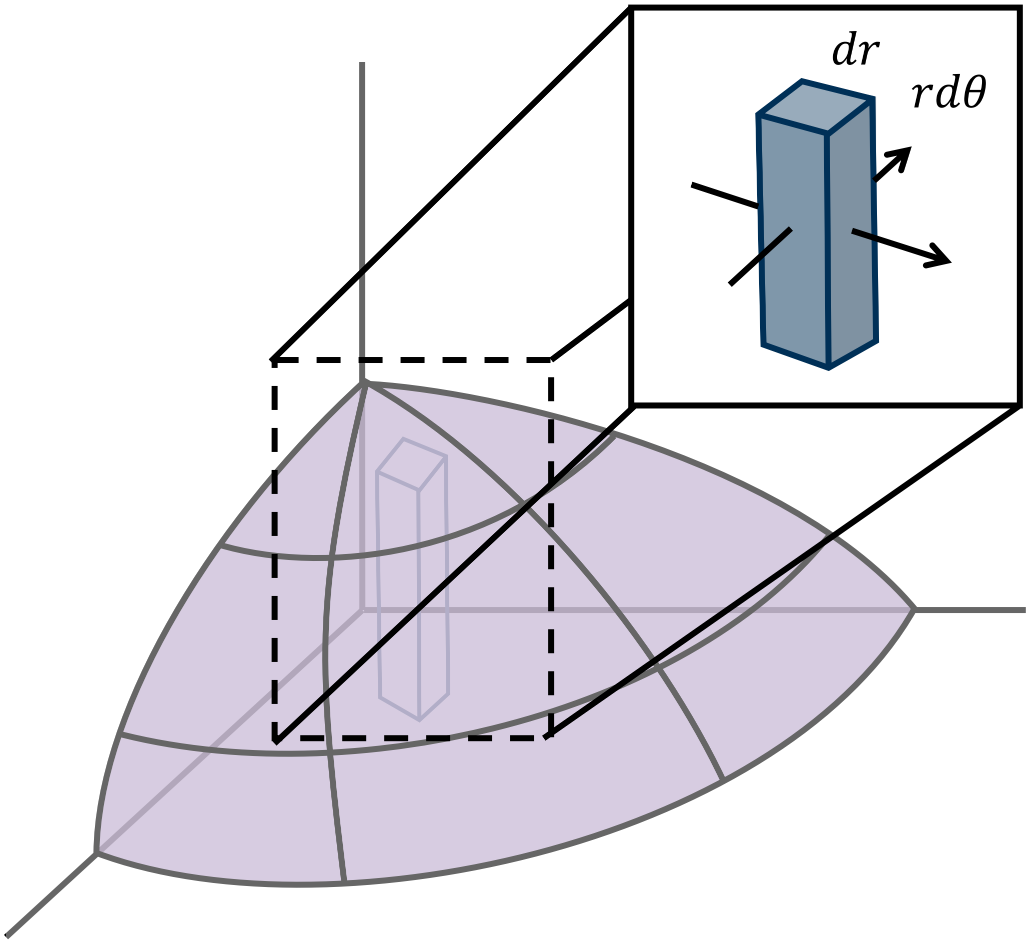
Previous Assumption

$$c(r,t) \approx \frac{\phi_d h - \frac{J}{\rho} t}{h}$$

Concentration over time is nonlinear



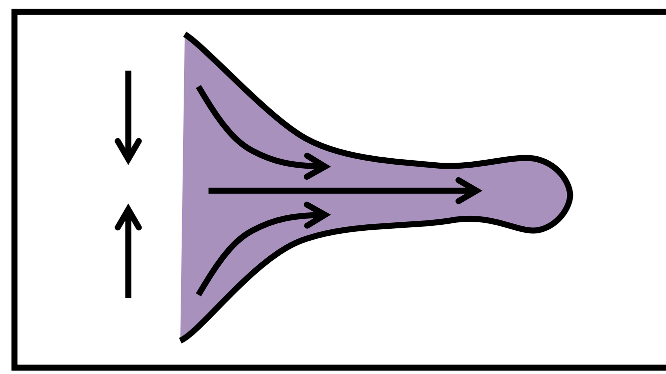
Three-Dimensional Model



$$\frac{dh}{dt} = -\frac{1}{\rho}J + \frac{\partial Q_r}{\partial r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta}$$

$$\begin{cases} u_r = \dfrac{h^2 + 2Dh}{\eta_p \eta_d}\dfrac{\partial \gamma}{\partial r} \\ u_\theta = \dfrac{h^2 + 2Dh}{\eta_p \eta_d}\dfrac{\partial \gamma}{\partial \theta} \end{cases}$$

Accounting for azimuthal flow



Governing Equations

Continuity for drop and alcohol with a restriction such that $Q = 0$ *if* $h_{alcohol} = 0$

$$\begin{cases} \dfrac{dh_{total}}{dt} = -\dfrac{1}{\rho}J + \dfrac{\partial Q_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial Q_\theta}{\partial \theta} \\ \dfrac{dh_{alcohol}}{dt} = -\dfrac{1}{\rho}J + \dfrac{\partial Q_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial Q_\theta}{\partial \theta} \end{cases}$$

Flow derived from shearing of drop interfaces

$$\begin{cases} u_r = \dfrac{h^2 + 2Dh}{\eta_p \eta_d}\dfrac{\partial \gamma}{\partial r} \\ u_\theta = \dfrac{h^2 + 2Dh}{\eta_p \eta_d}\dfrac{\partial \gamma}{\partial \theta} \end{cases}$$

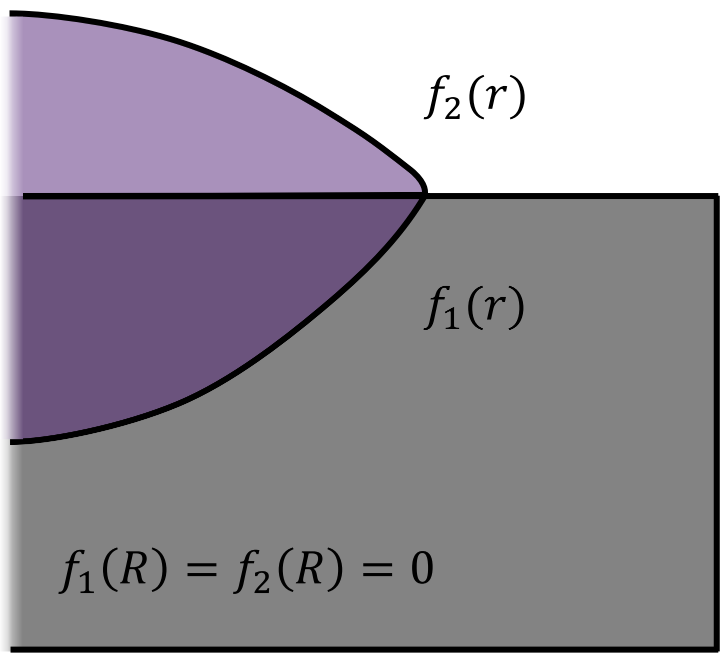
Surface tension gradient is proportional to concentration

$$\left\{ \dfrac{\partial \gamma}{\partial r} = k\dfrac{h_{alcohol}}{h_{total}} \right.$$

Initial Conditions

Minimize Total Energy of Drop:

$$U_d + U_p + E_{ad} + E_{dp}$$



$$U_d = \rho_d g\pi \int_0^R r([f_2(r)]^2 - [f_1(r)]^2)dr$$

$$U_p = \rho_p g\pi \int_0^R r([f_1(r)]^2)dr$$

$$E_{ad} = \gamma_{ad} 2\pi \int_0^R r\sqrt{1 + \left(\frac{df_2}{dr}\right)^2}\, dr$$

$$E_{dp} = \gamma_{dp} 2\pi \int_0^R r\sqrt{1 + \left(\frac{df_1}{dr}\right)^2}\, dr$$

$$V = 2\pi \int_0^R r(f_2(r) - f_1(r))dr$$

$$\boxed{\textit{Find } f_2(r) - f_1(r)}$$

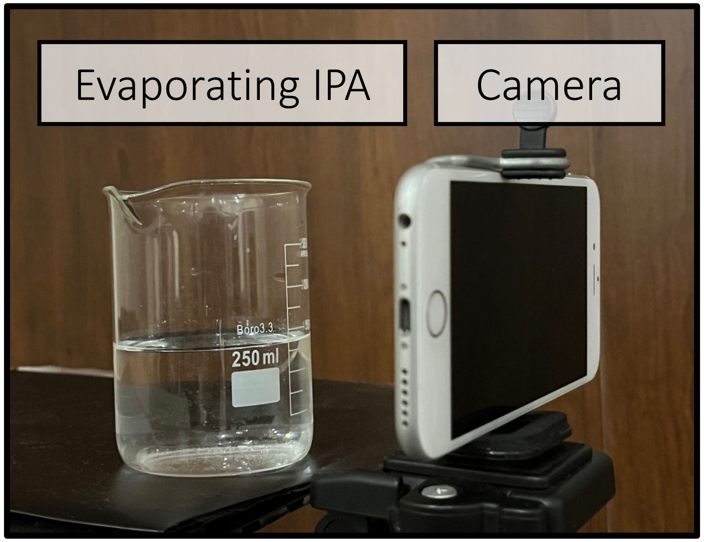
Continuity Derivation

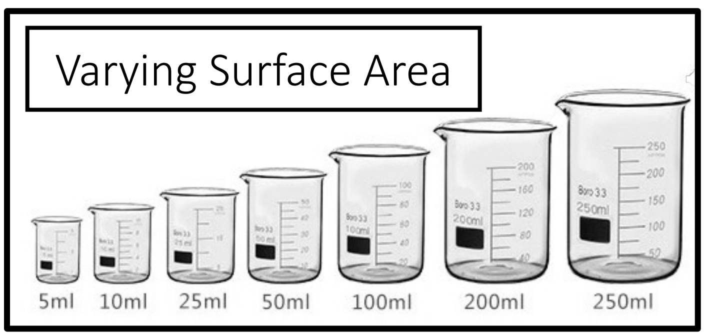
$$\rho dr(rd\theta)\frac{dh}{dt} = -Jrd\theta dr + \rho h(rd\theta)\left(u_r\Big|_{r+dr} - u_r\Big|_r\right) + (\rho h dr)\left(u_\theta\Big|_{\theta+d\theta} - u_\theta\Big|_\theta\right)$$

$$\frac{dh}{dt} = -\frac{J}{\rho} + h\frac{\partial u_r}{\partial r} + \frac{h}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$\boxed{\frac{dh}{dt} = -\frac{J}{\rho} + \nabla \cdot (h\vec{u}) = -\frac{J}{\rho} + \nabla \cdot \vec{Q}}$$

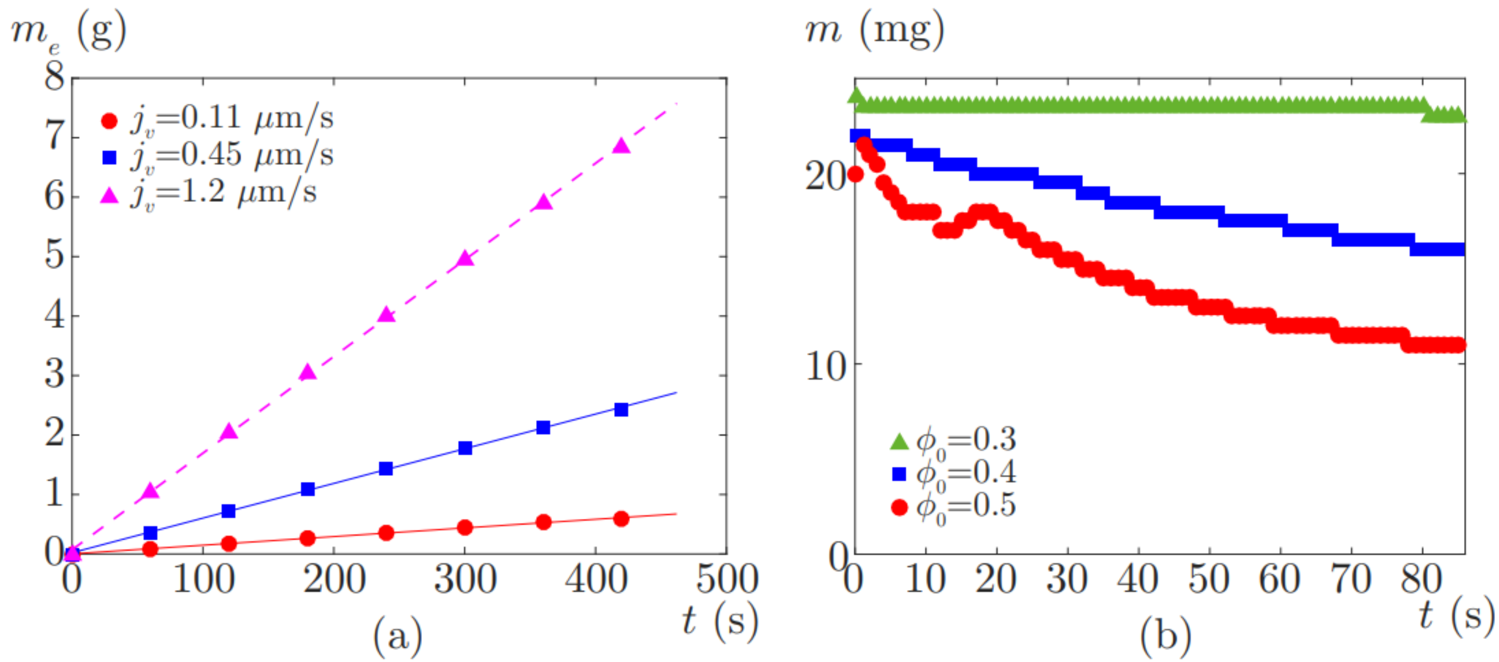
Evaporative Flux





! *Change in volume is too small to observe, analytical balance required to analyze mass change over time*

Concentration Dependence On J

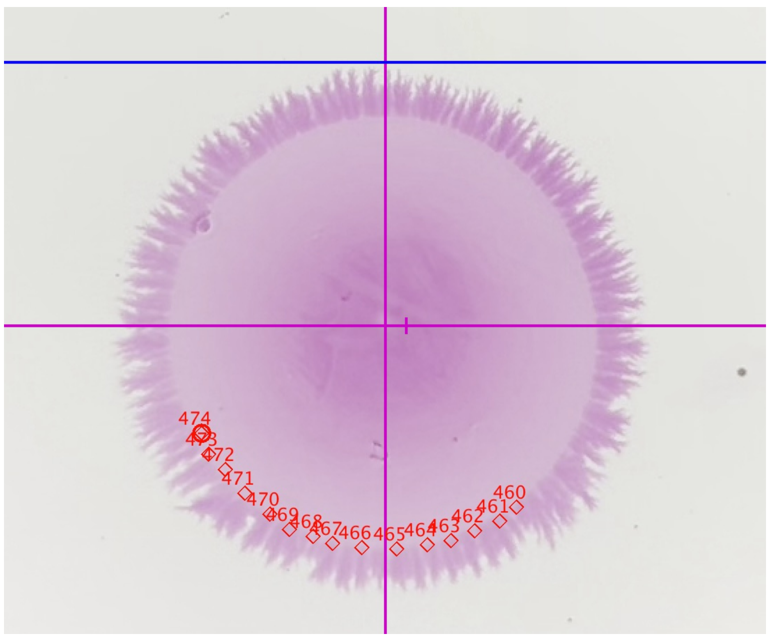


Evaporation rate of alcohol does not vary greatly with concentration

Keiser, L., et al. "Marangoni Bursting: Evaporation-Induced Emulsification of Binary Mixtures on a Liquid Layer." Physical Review Letters, vol. 118, no. 7, 17 Feb. 2017, 10.1103/physrevlett.118.074504.

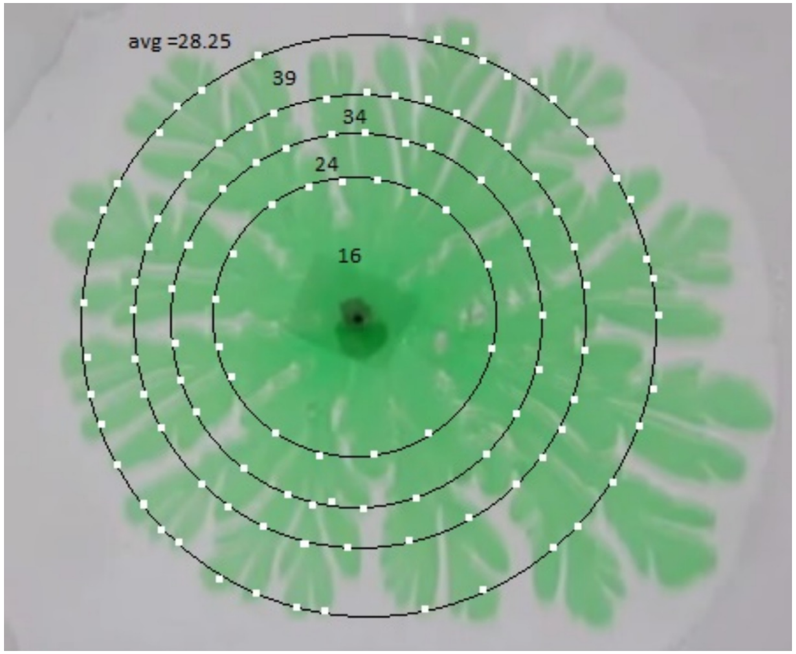
Radius Measurement: Tracker

Tracker
Video Analysis and Modeling Tool



1. *Measure **inner** and **outer** radius around the fractal*
2. *Determine the **average** of these values to find **radius** as defined (in the report)*
3. *Error bars correspond to the **standard error** of all tracked values*
4. ***Scale the error** by the derivative of the radius equation when scaling graph*

Instability Wavelength Measurement



Kondic, L., Lastra, Y., Moulton, L., & Anazco, R. (2014). (rep.). *The Hele-Shaw Cell/Saffman Taylor Instability: Theoretical and Experimental Comparison of Newtonian Fluids*.

1. *Determine **number of branches** touching circles of different radii*
2. *Calculate **arc length** between each branch*
3. *Divide arc length by radius to determine **angular distance** between branches*
4. *Divide total # of fingers by # of circles to get **average number of fingers***

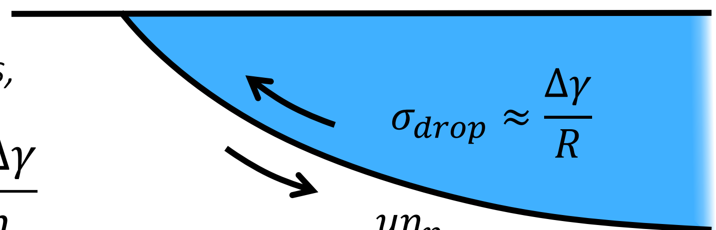
Other Slides

Literature

Viscous Stress

Equating the stresses,

$$\frac{\Delta\gamma}{R} \approx \frac{u\eta_p}{H} \qquad u \approx \frac{D\Delta\gamma}{R\eta_p}$$



$$\sigma_{drop} \approx \frac{\Delta\gamma}{R}$$

$$\sigma_{paint} \approx \frac{u\eta_p}{D}$$

Alcohol Volume

$$\phi_f V_f - \phi_0 V = JAT$$

$$\phi_f V_f - \phi_0 V \approx J(R)^2 T$$

Ink Volume

Conserved;

$$(1-\phi_0)V = (1-\phi_f)V_f$$

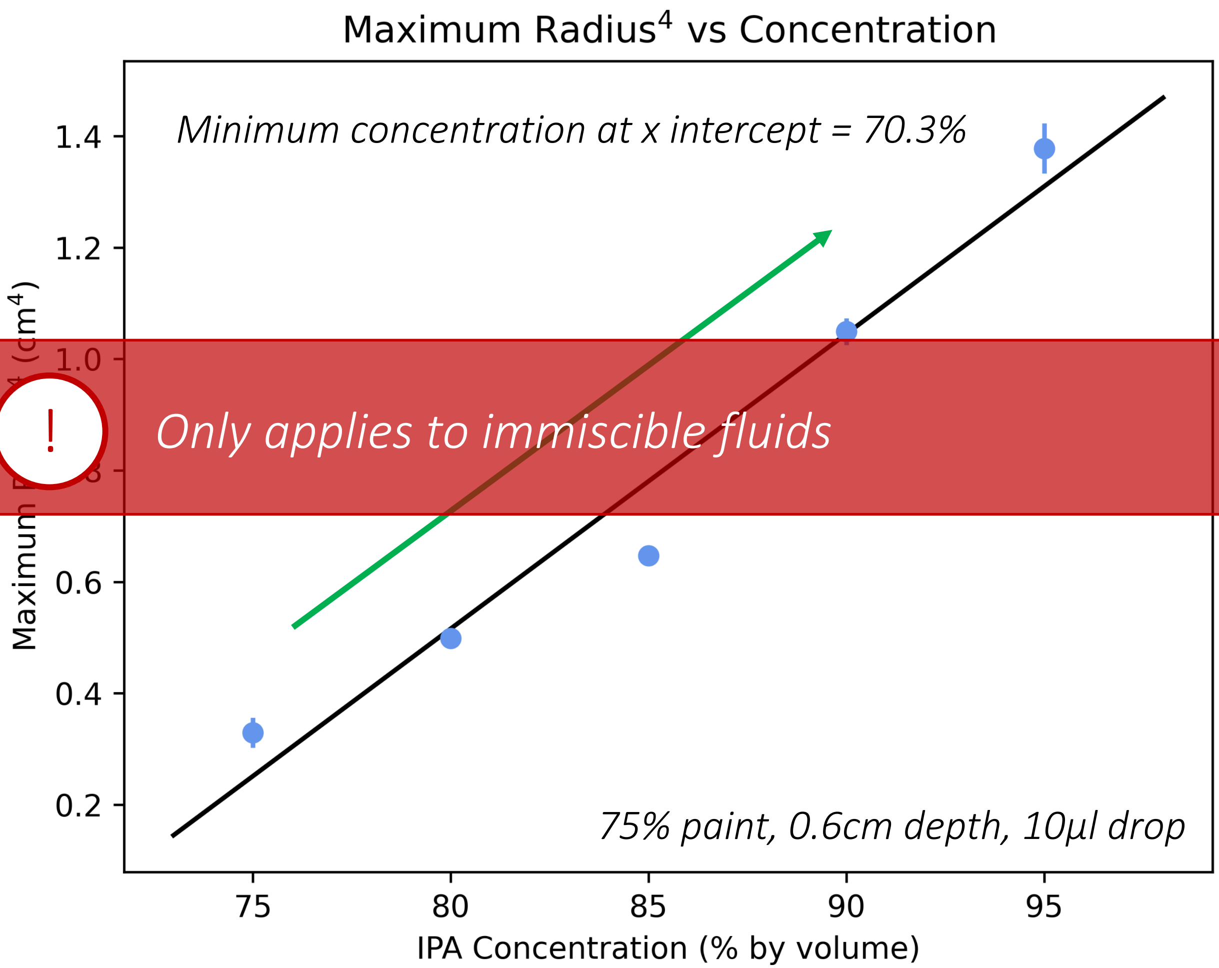
$$T \approx \frac{R}{u} = \frac{\eta_p R^2}{D\Delta\gamma}$$

$$R \approx \left(\frac{(\phi_0-\phi_f)\Delta\gamma V D}{(1-\phi_f)J\eta_p}\right)^{\frac{1}{4}}$$

$$T \approx \left(\frac{(\phi_0-\phi_f)V\eta_p}{(1-\phi_f)J\Delta\gamma D}\right)^{\frac{1}{2}}$$

(Keiser et al., 2017)

Literature Experimental Verification



$$R \approx \left(\frac{(\phi_0 - \phi_f)\Delta\gamma V D}{(1-\phi_f)J\eta_p}\right)^{\frac{1}{4}}$$

! Only applies to immiscible fluids

Slower IPA depletion at edges

Greater overall flow velocity

Miscible Viscous Fingering

Velocity Induced Fractal Fingering

$$\lambda_c = \pi b \sqrt{\frac{\sigma}{\Delta \mu V}}$$

σ = surface tension
μ= viscosity of diluted acrylic paint
b = space between parallel plates
V = interfacial velocity imposed by an external pressure gradient

Qualitatively, the Marangoni flow and alcohol evaporation induces the velocity in this problem, increasing the instability.

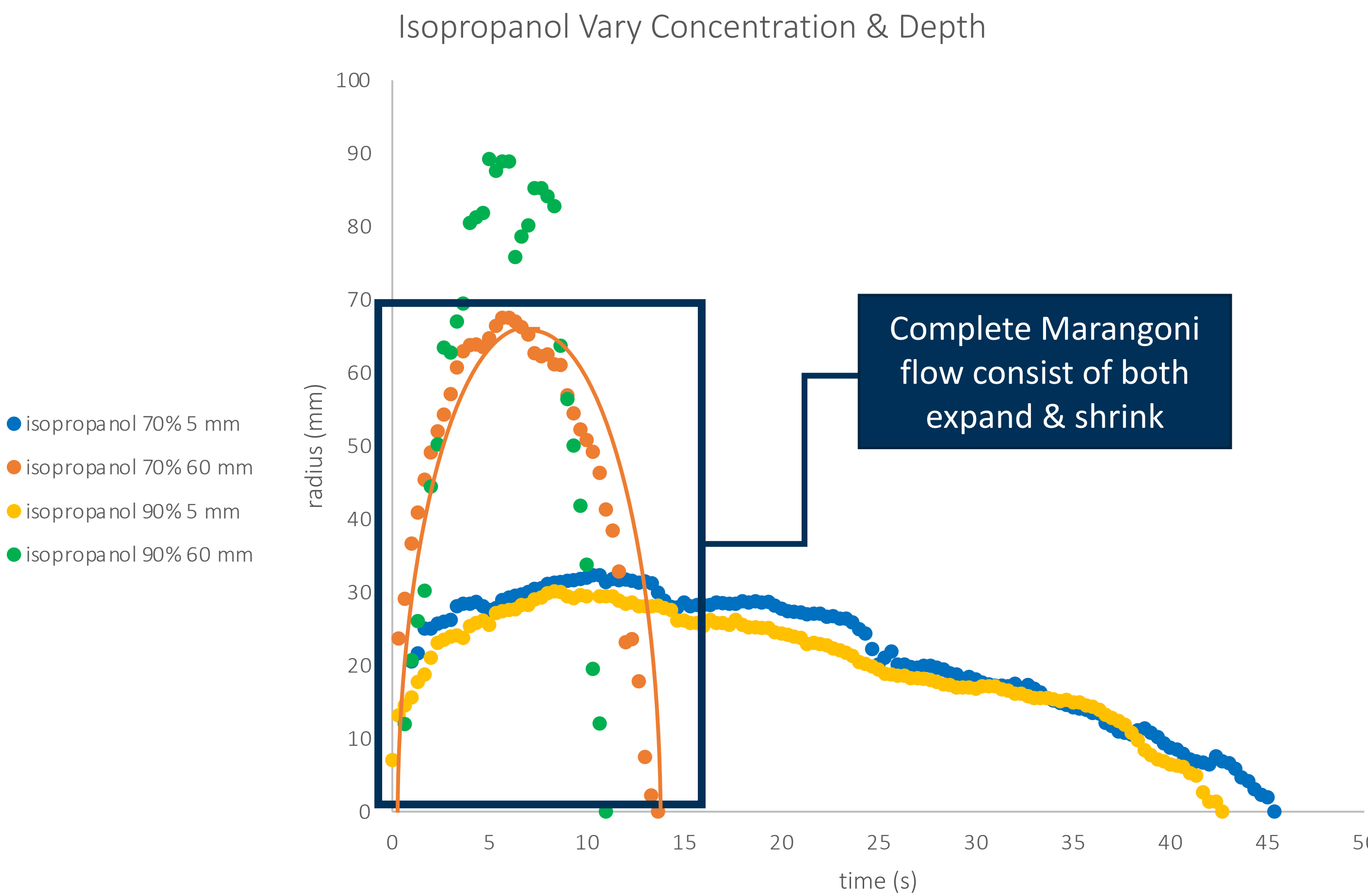
Viscosity Ratio Function

$$A \equiv \frac{\mu_a - \mu_d}{\mu_a + \mu_d}$$

A = dimensionless parameter

As A approaches 1, $\frac{Rf}{Ri}$ $increases\ rapidly\ within\ a\ very\ small\ range\ of\ A$.

Experimental Verification

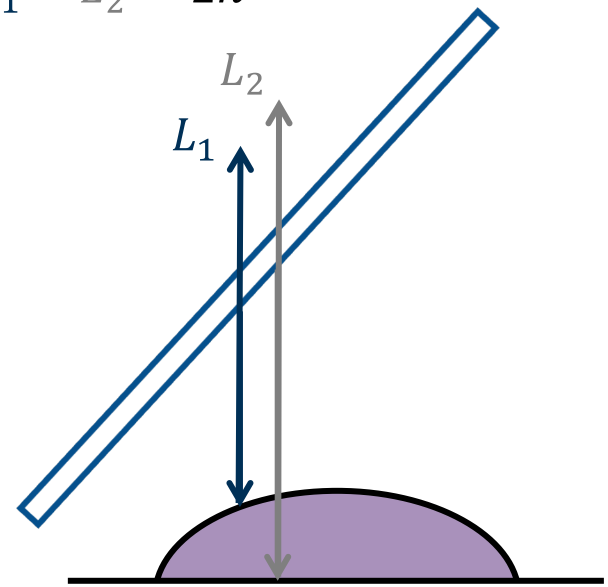


Non log log plot

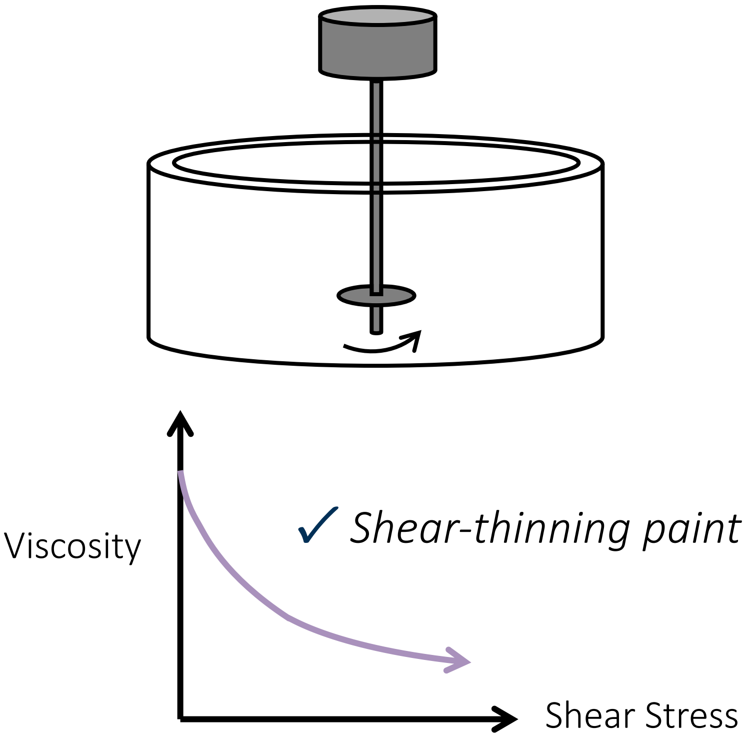
Further Insights

Interferometry

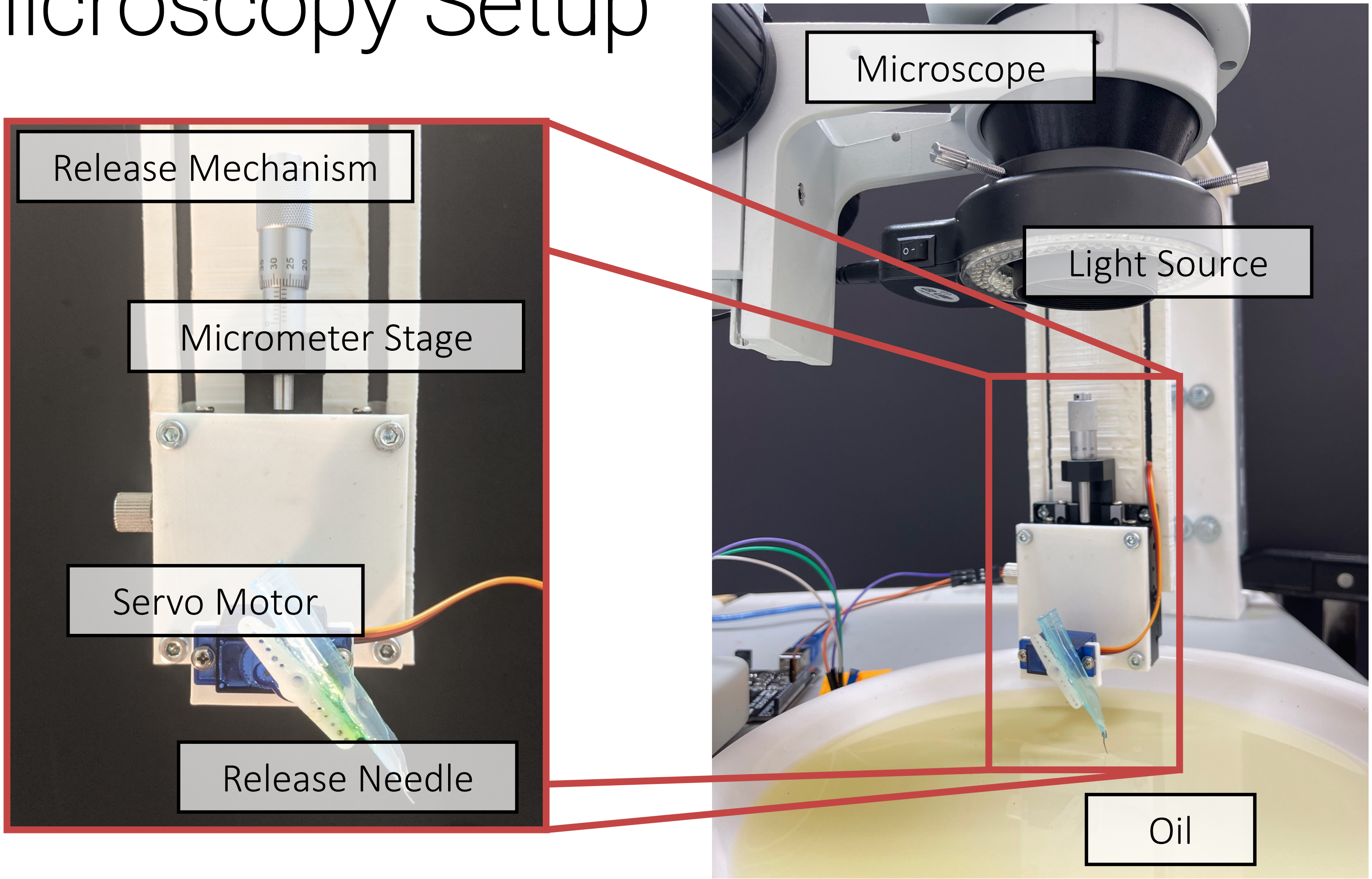
$L_1 - L_2 = 2h$



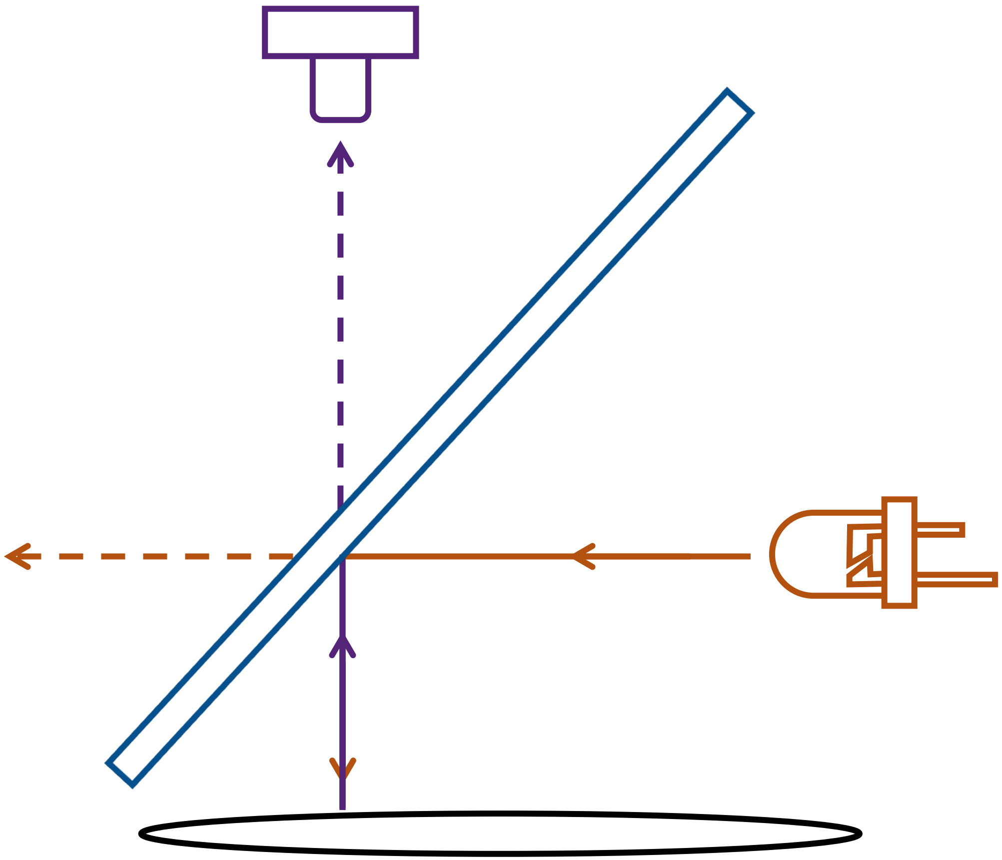
Viscometry

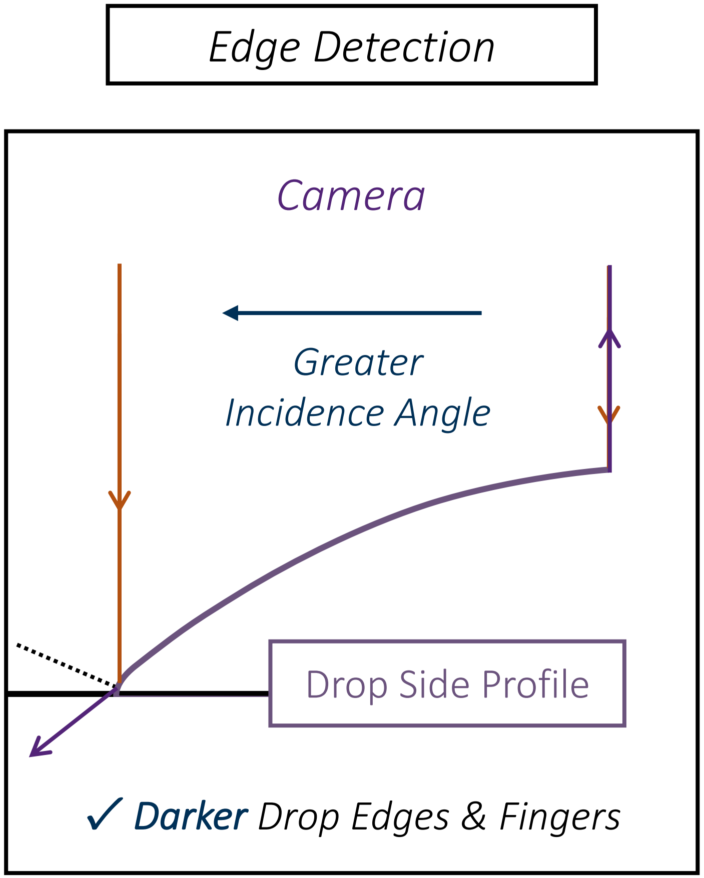


Microscopy Setup



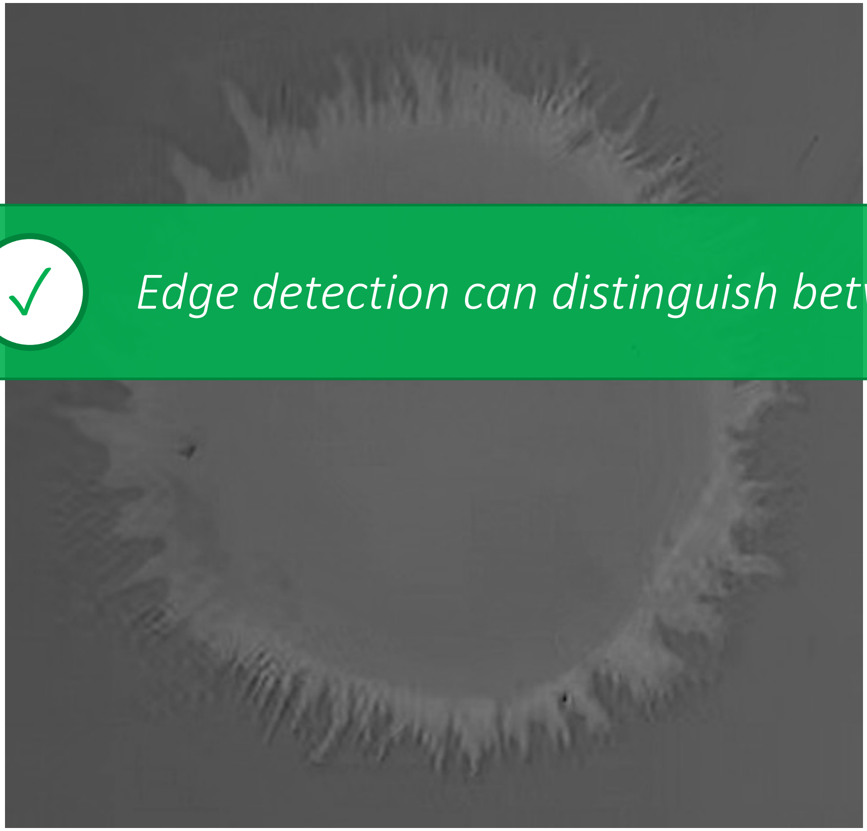
Lighting



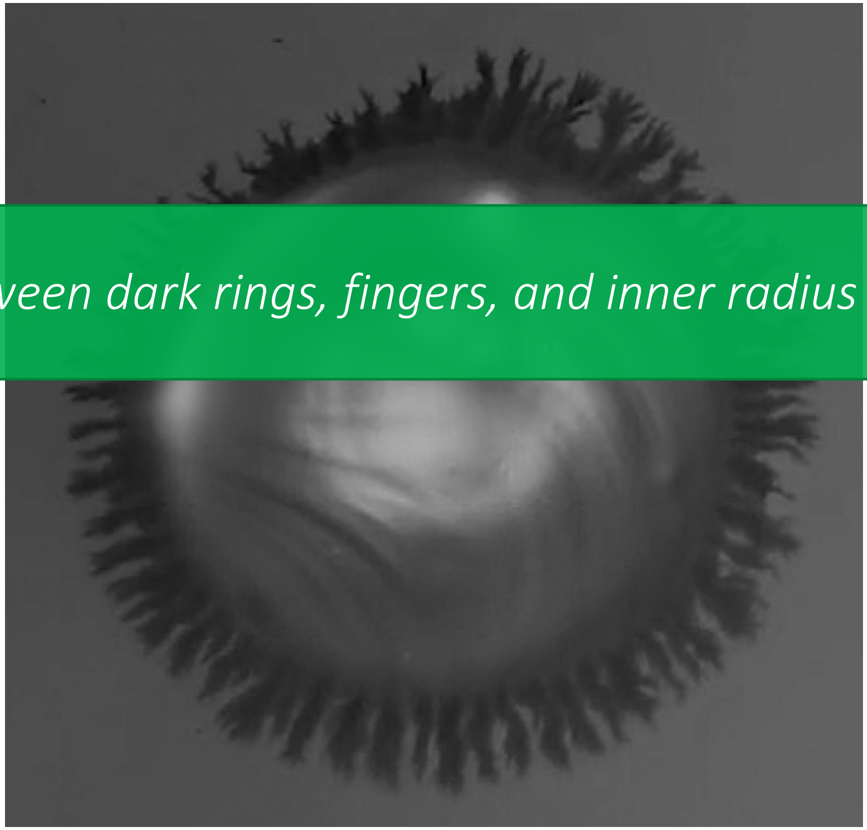

Edge Detection
Camera
Greater
Incidence Angle
Drop Side Profile
✓ Darker Drop Edges & Fingers

Lighting

Without Lighting

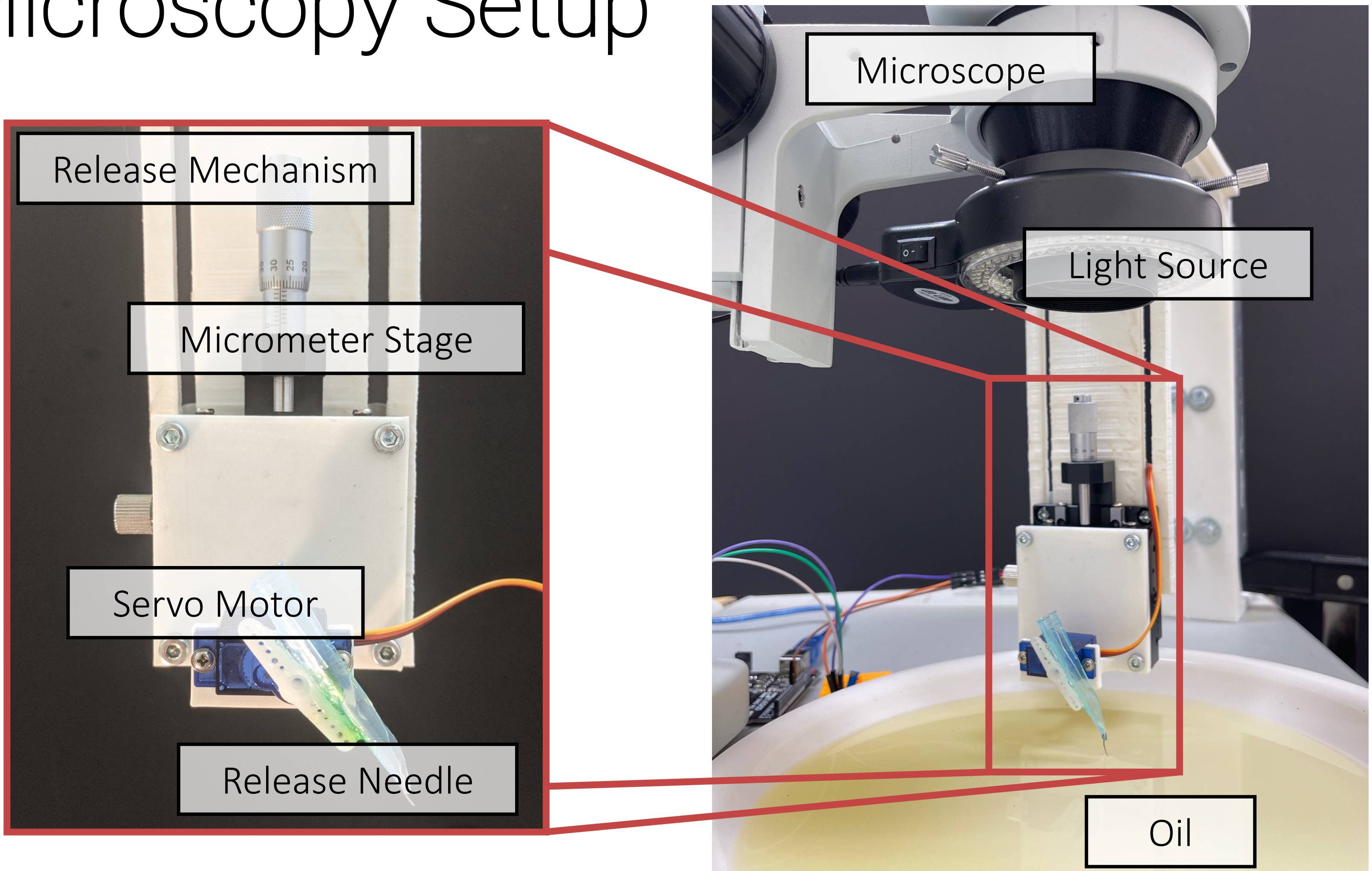


With Lighting

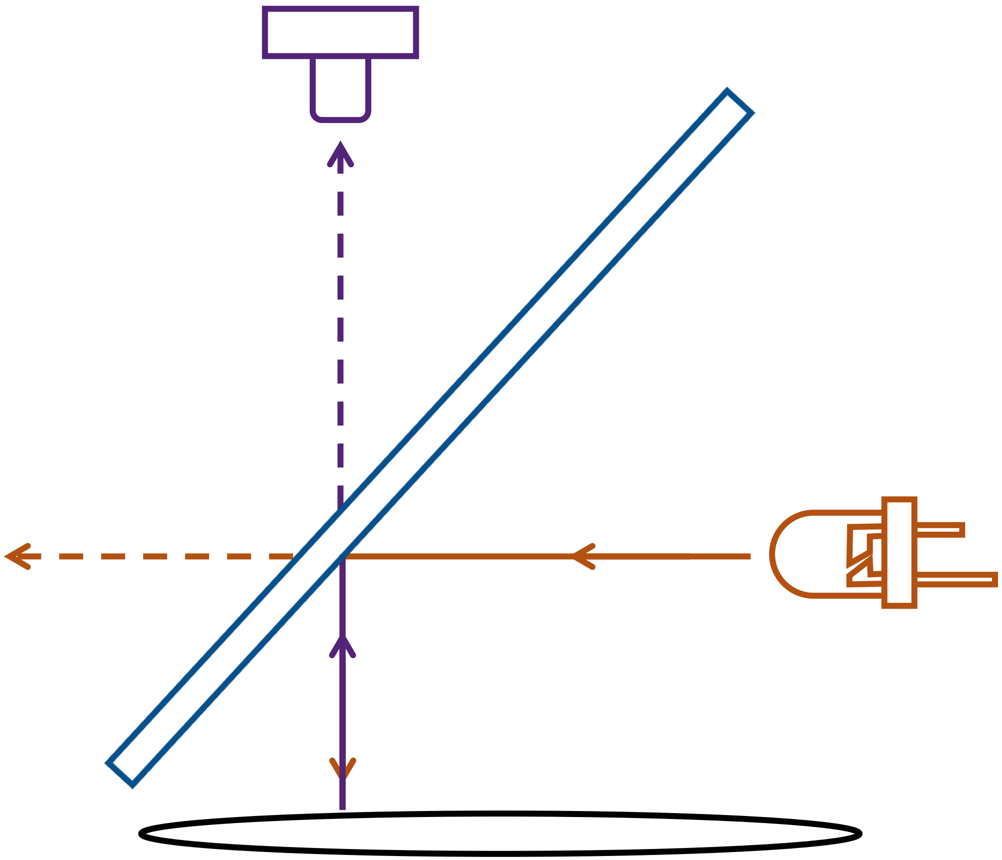


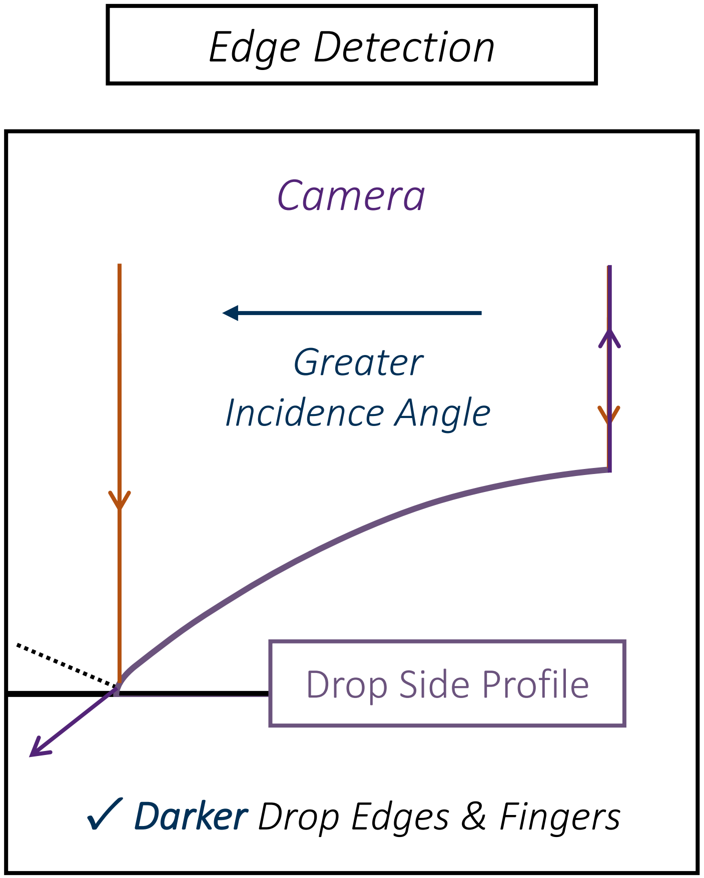
Edge detection can distinguish between dark rings, fingers, and inner radius

Microscopy Setup



Lighting

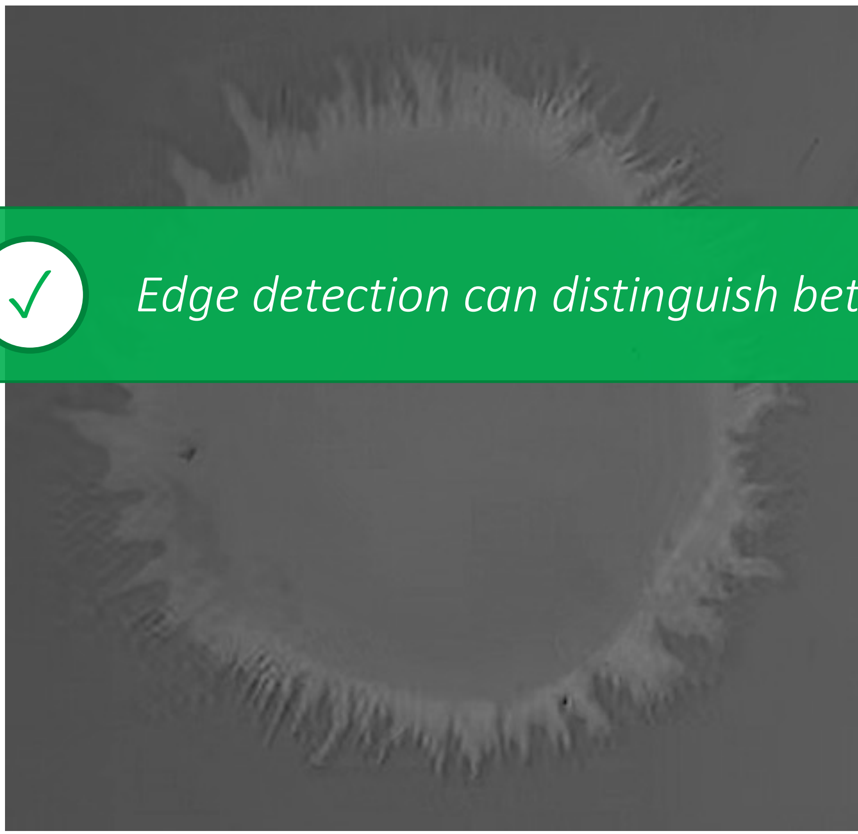


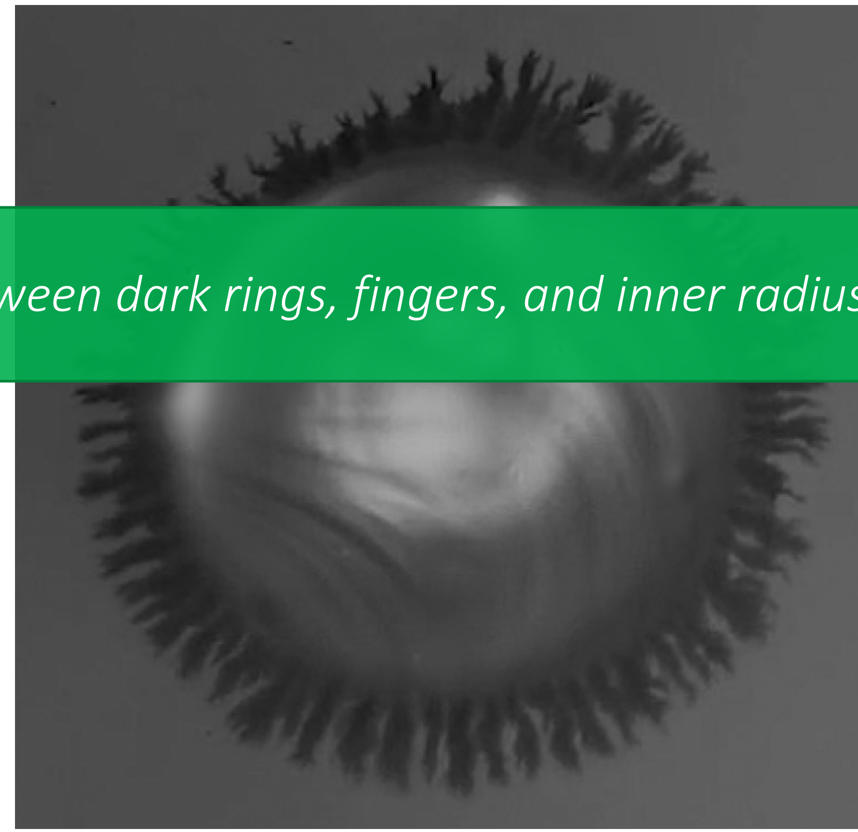

Edge Detection
Camera
Greater
Incidence Angle
Drop Side Profile
✓ **Darker** Drop Edges & Fingers

Lighting

Without Lighting

With Lighting





✓ *Edge detection can distinguish between dark rings, fingers, and inner radius*

[2]

Marangoni and VS Stress

Marangoni Stress $\sim \dfrac{\partial \gamma}{\partial r}$  Viscous Shear Stress $\sim \dfrac{\mu u}{l}$ 

Setting them both as equal:

$$\frac{\partial \gamma}{\partial r} \approx \mu(u/l) \rightarrow u \approx \frac{r^{\frac{1}{3}}}{(\mu p)^{\frac{1}{3}}}\left(\frac{\partial \gamma}{\partial r}\right)$$

$$u \approx \frac{(\gamma_+ - \gamma_-)^{2/3}}{(\mu \rho r)^{1/3}} \qquad u = \frac{\partial r}{\partial t} = \frac{(\gamma_+ - \gamma_-)^{2/3}}{(\mu \rho r)^{1/3}}$$

r = radius
u = expansion speed at r
l = depth into acrylic
γ_- = surface tension of alcohol-ink
γ_+ = surface tension of acrylic
μ = viscosity of acrylic
ρ = density of diluted acrylic

For $\gamma_-(t)$, surface tension of alcohol-ink,
dependent on alcohol concentration varying through time

Marangoni Flow

Marangoni Stress $\sim \dfrac{\partial \gamma}{\partial r}$

Viscous Shear Stress $\sim \dfrac{\mu u}{l}$

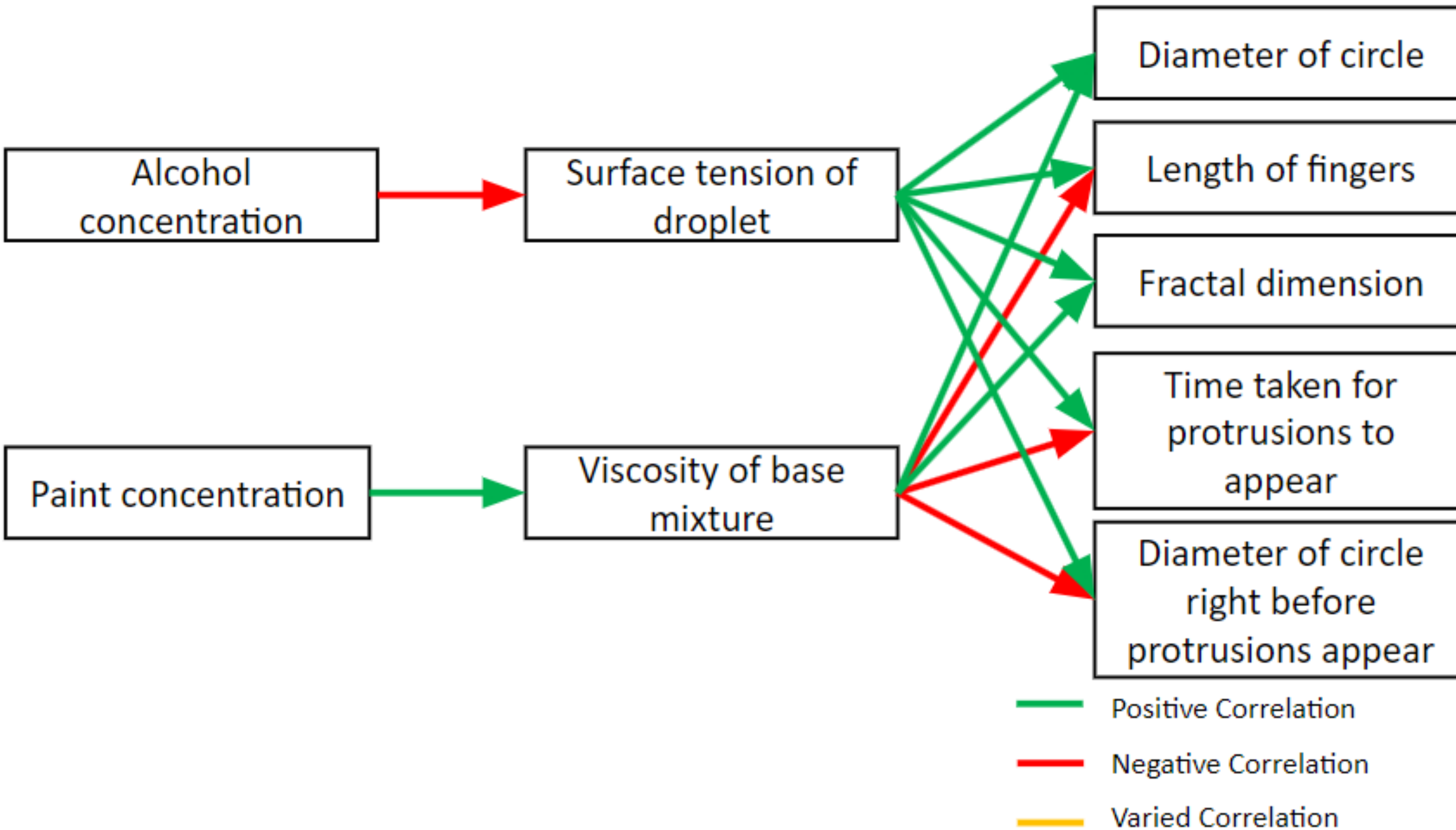
Equating Marangoni stress with opposing viscous stresses,

$$\frac{\partial \gamma}{\partial r} \approx \mu(u/l) \rightarrow u \approx \frac{r^{\frac{1}{3}}}{(\mu p)^{\frac{1}{3}}}\left(\frac{\partial \gamma}{\partial r}\right)$$

$$u \approx \frac{(\gamma_+ - \gamma_-)^{2/3}}{(\mu \rho r)^{1/3}} \qquad u = \frac{\partial r}{\partial t} = \frac{(\gamma_+ - \gamma_-)^{2/3}}{(\mu \rho r)^{1/3}}$$

r = radius
u = expansion speed at r
l = depth into acrylic
γ_- = surface tension of alcohol-ink
γ_+ = surface tension of acrylic

Key Parameters

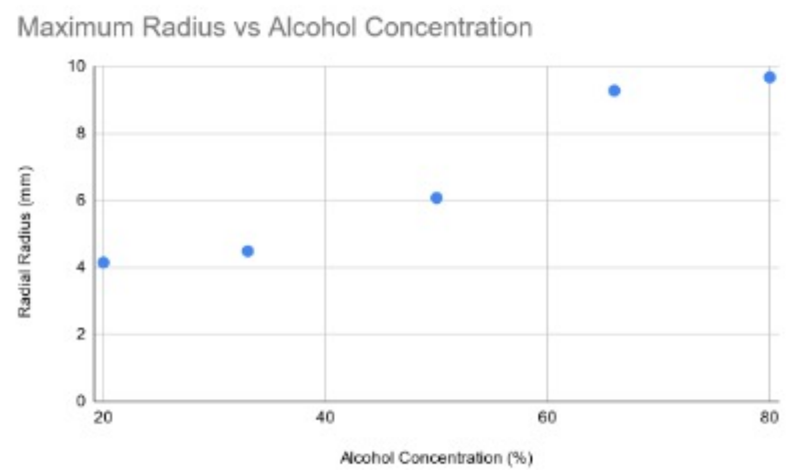


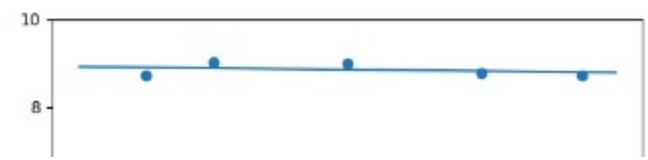
Fingering Results

Alcohol Concentration ϕ has obvious positive correlation with radius, supported by Marangoni Flow Theory

$$r = \left(\frac{4}{3} \int \frac{(\gamma_+ - \gamma_-)^{2/3}}{(\mu\rho)^{1/3}} dt \right)^{3/4}$$

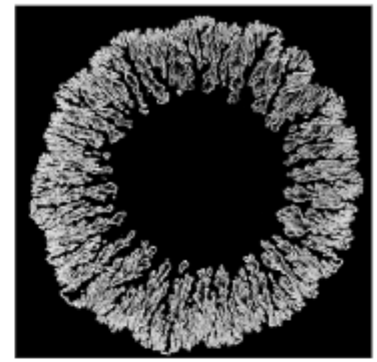
Concentration Φ increase implies γ_-





Fractal Dimension

The Minkowski Dimension, also called the box-counting Dimension, is a quantitative index for the (loosely speaking) *roughness* or *complexity* of a fractal. The definition of a fractal is that its detail stays constant at different zoom levels. If you zoom in a lot, it will still look as rough and complex as it originally did.



$$\dim_{\text{box}}(S) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log 1/\epsilon}$$

The box-counting dimension qualitatively measures this by seeing how much grid boxes on the plane the shape coincides with at a certain degree of zoom level, divided by the zoom level. The dimension is the limit of this ratio as the grid boxes become infinitely finer. The upper and lower box dimensions, counting the maximum and minimum box count at local ranges of zoom, are

Fractal Dimension

Numerical Dimension Approximation

As our images don't have infinite resolution,
We computationally approximate the box dimension
And verify the constant roughness by counting the grid boxes taken up at different box lengths (or zoom levels).

We plot their logarithms. The resulting graph being linear means that the shape instead has fractal properties, and the (negated) slope is thus the dimension.

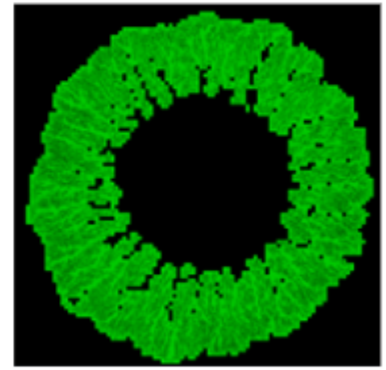
$$\frac{\log N(l)}{\log l_0/l}$$

l_0 = Base Box Length

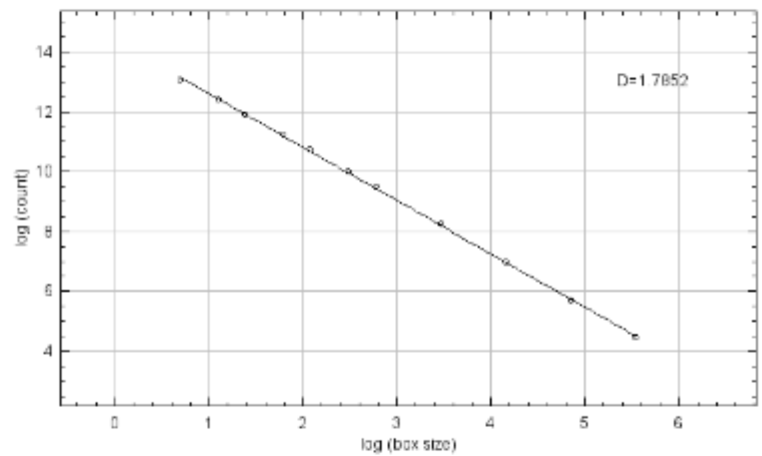
l = Box Length

$N(l)$ = Minimal Box Count

Green boxes show grid boxes being taken up, here we show a relatively small box length



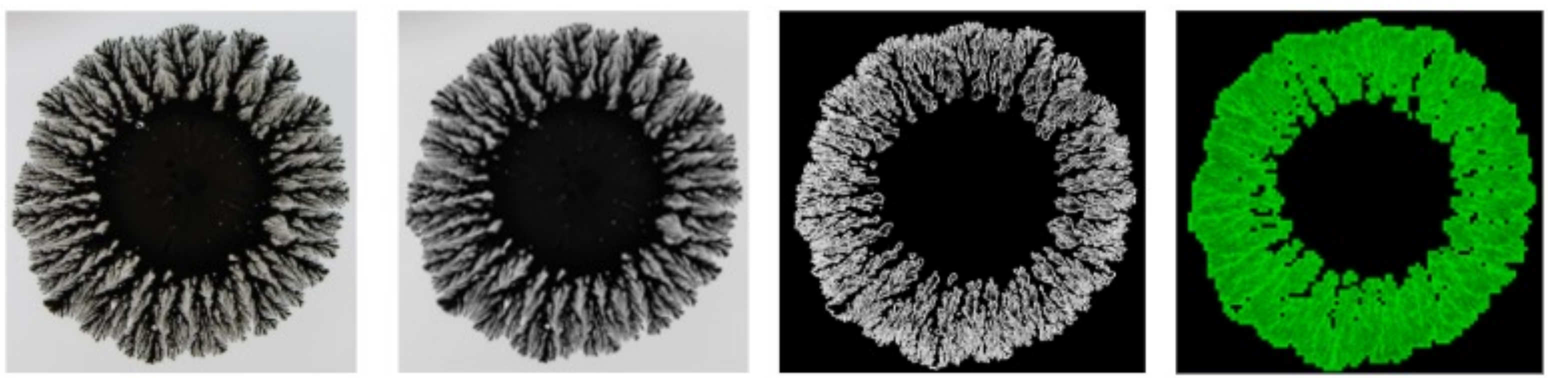
log(box_count) vs log(box_size) of the above image, note the linearity of the graph and the slope (negating slope gives the dimension)



Plot Shows Linear Relation, Verified Fractalness Of Fingering

Processing Pipeline

Julia Image Processing Pipeline. The original image is fed in, and grayscale edge detection is performed. Dimension analysis is run on the resulting contour.



Original | Grayscale | Edge Detection | Box Counting