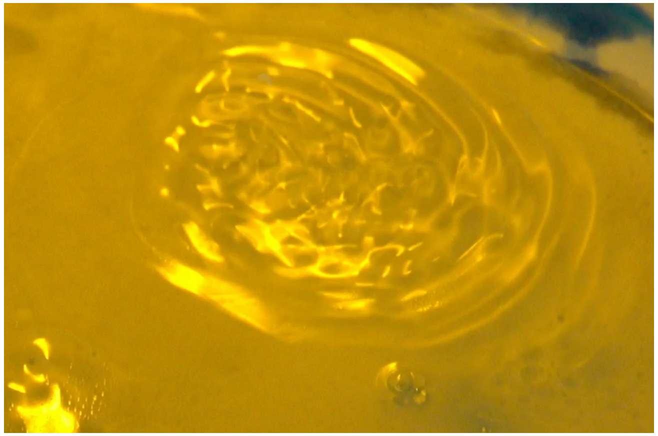
F. Faraday Waves

"A droplet of less viscous liquid floating in a bath of a more viscous liquid develops surprising wave-like patterns when the entire system is set to vertical oscillation. Investigate this phenomenon and the parameters relevant to the production of stable patterns."

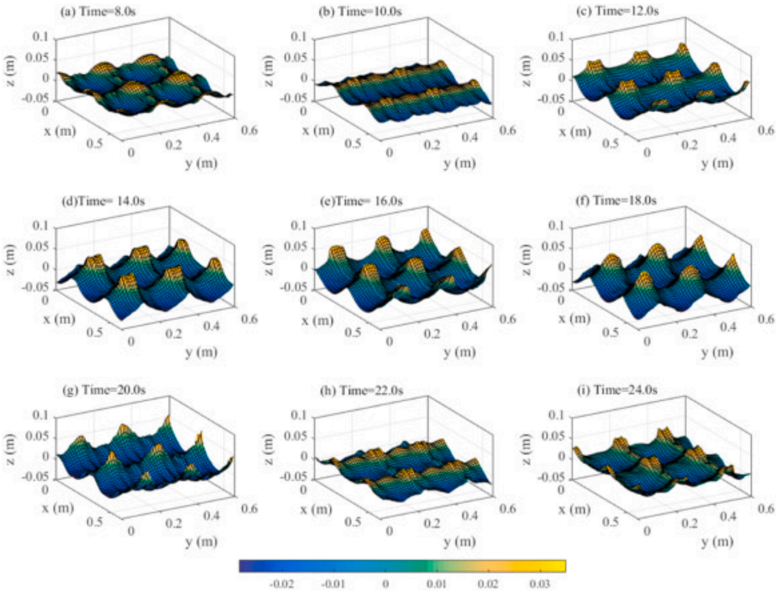


Problem Statement

"A droplet of less viscous liquid floating in a bath of a more viscous liquid develops surprising wave-like patterns when the entire system is set to vertical oscillation. Investigate this phenomenon and the parameters relevant to the production of stable patterns."

Parameters:

1. *Frequency of vertical oscillation*
2. *Amplitude of vertical oscillation*
3. *Viscosities of two liquids*
4. *Cavity shape*
5. *Initial droplet location*
6. *Droplet volume*
7. *Bath depth*



Overview

1 ***Introduction***
Introduction to Relevant Parameters

2 ***Experimental Setup***
Experimental Setup and Preliminary Observations

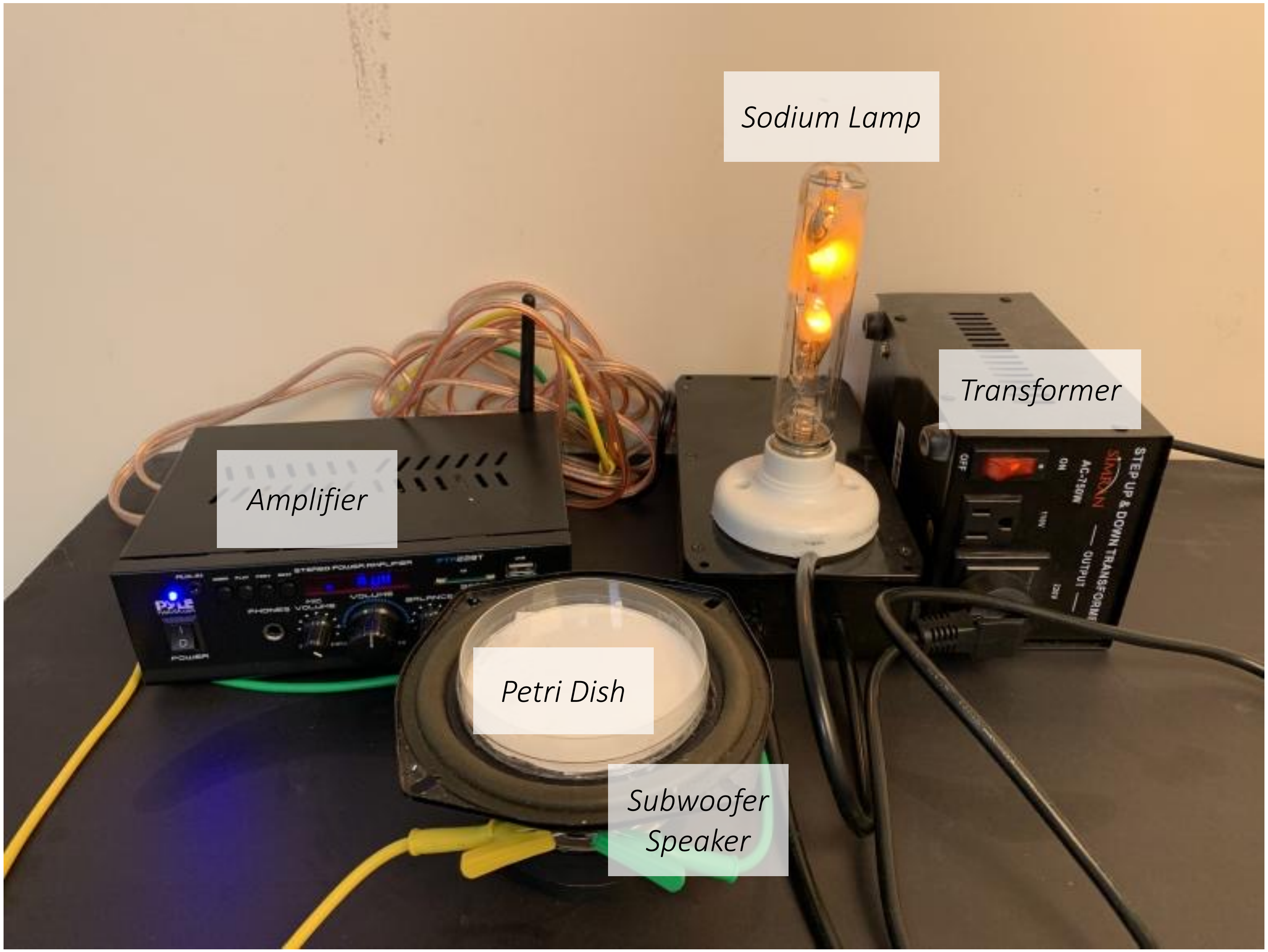
3 ***Theoretical Model***
Droplet Shapes, Surface Waves

4 ***Model Verification***
Experimental Agreement

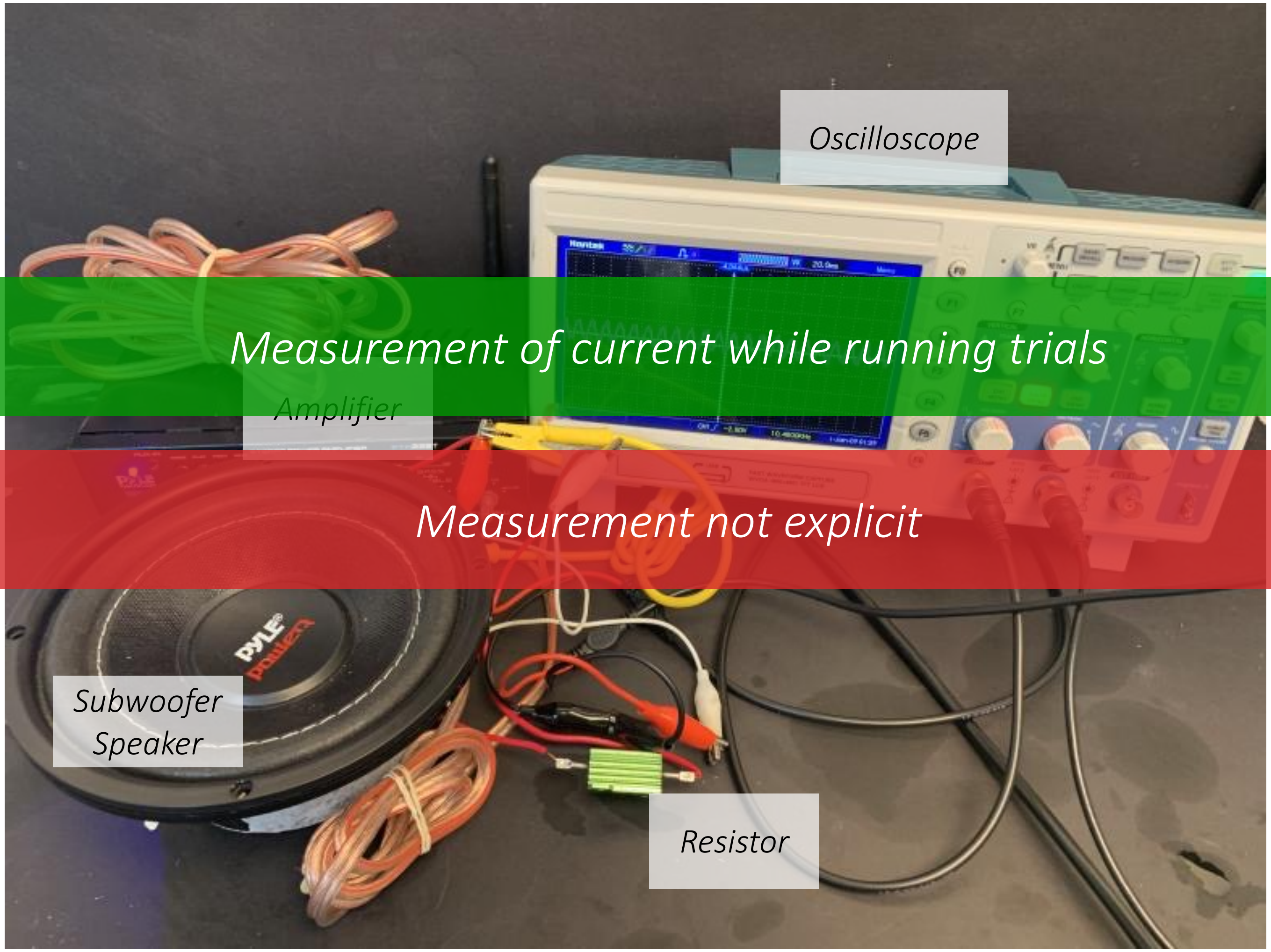
5 ***Conclusion***
Further Insights and General Improvements

Experimental Setup

Experimental Setup



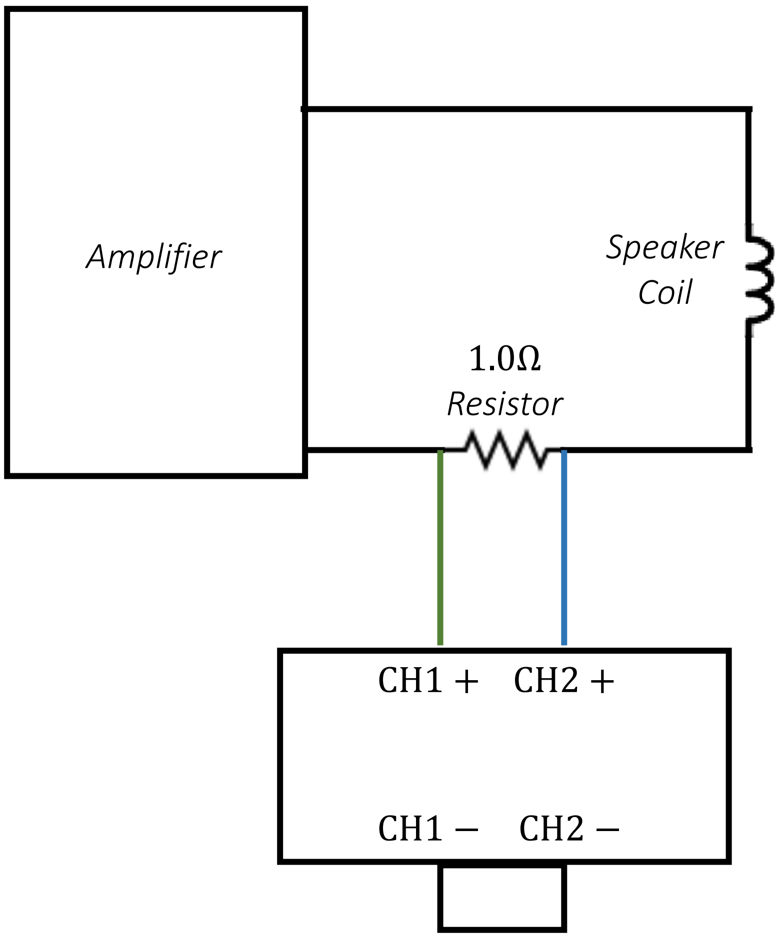
Amplitude Measurement I



Measurement of current while running trials

Measurement not explicit

Amplitude Measurement I



$$I \propto B \propto F \propto k\Delta x$$

$$V = IR$$

$$V(t) = I(t)R$$

$$A(t) = A_0 \cos \omega t$$

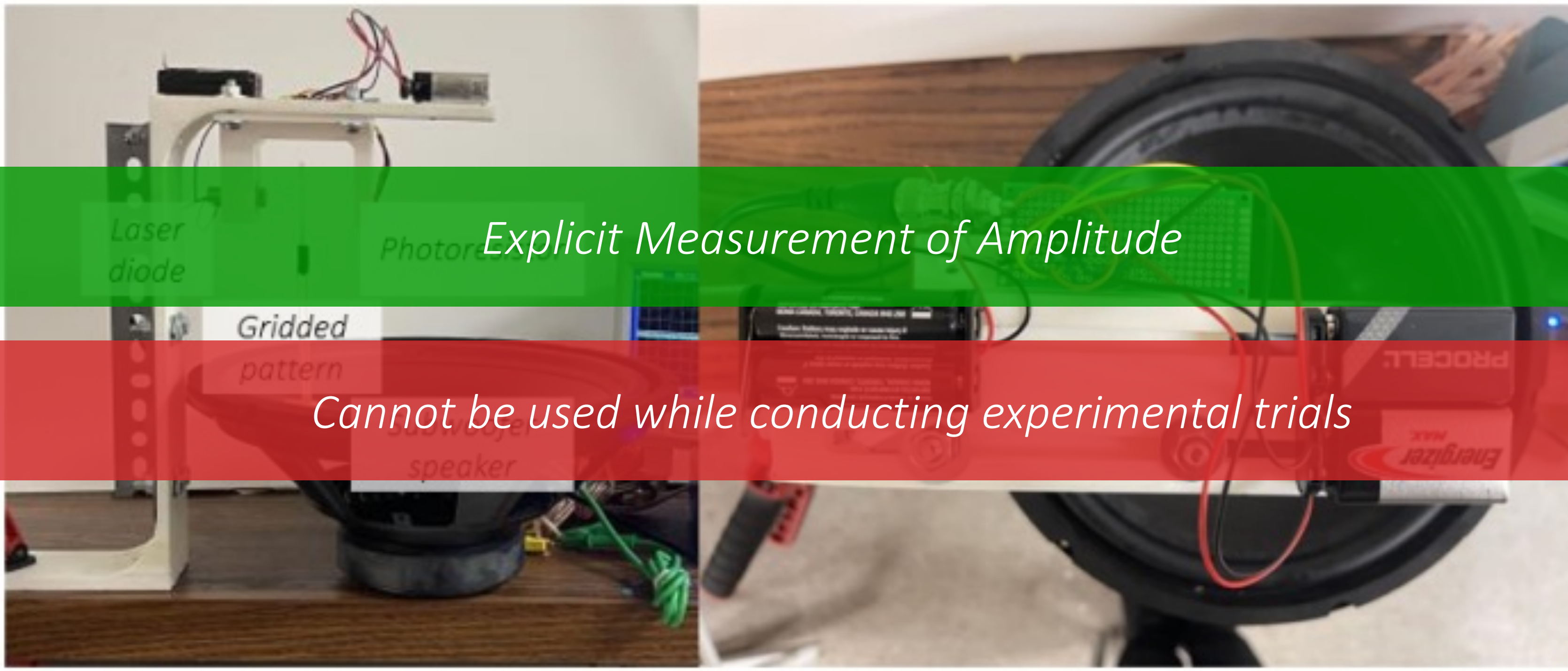
$$\ddot{A}(t) = -\omega^2 A_0 \cos \omega t \equiv \gamma_m$$

$$= -\omega^2 \beta I_0 \cos \omega t = \gamma_m$$

$$\frac{V_{max}}{R} = I_{max}$$

$$I_{max}\omega^2 = \beta\gamma_m$$

Amplitude Measurement II



Explicit Measurement of Amplitude

Cannot be used while conducting experimental trials

Observation

Liquid 1: vegetable oil
$\rho_1 = 920\ \mathrm{kg/m^3}$
$\mu_1 = 0.069\ \mathrm{Pa \cdot s}$
$\sigma_1 = 22.0\mathrm{mN/m}$

Liquid 2: glycerin
$\rho_2 = 1260\ \mathrm{kg/m^3}$
$\mu_2 = 1.142\ \mathrm{Pa \cdot s}$
$\sigma_2 = 63.4\ \mathrm{mN/m}$

Liquid 1: isopropyl
$\rho_1 = 786\ \mathrm{kg/m^3}$
$\mu_1 = 0.0024\ \mathrm{Pa \cdot s}$
$\sigma_2 = 23.0\mathrm{mN/m}$

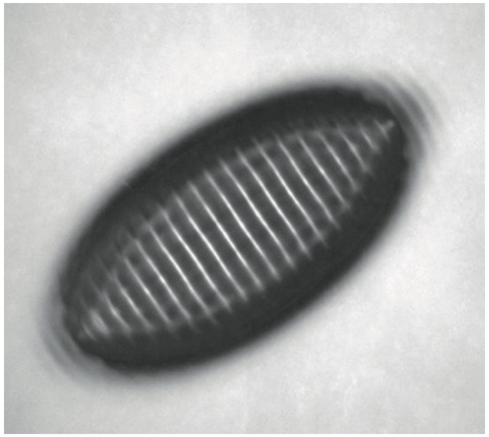
Liquid 2: vegetable oil
$\rho_2 = 920\ \mathrm{kg/m^3}$
$\mu_2 = 0.069\ \mathrm{Pa \cdot s}$
$\sigma_2 = 22.0\mathrm{mN/m}$

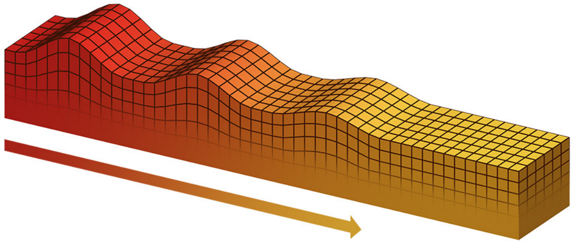
✓ *Two liquids must be immiscible*

✓ *In order to obtain floating droplets* $\rho_2 > \rho_1$

✓ *In order to trigger Faraday waves in the droplet only and avoid significant waves in the bath* $\mu_2 \gg \mu_1$

Theoretical Model





Droplet Shapes

Surface Waves

Deformed Regime

A droplet is in circular shape when it is initially dropped into a vertically oscillating bath:

1. Stable states are possible when ***eigenmode of Faraday wave "fits" in the size of circular droplets***
2. Stability of ***non-symmetric*** nodes is due to ***curved Faraday paths***
3. Possible ***rotation of the stable circular droplets***, ascribed to the wave radiation pressure exerted on the droplet by surrounding liquids

Elongated Regime

The circular droplet later elongates because its global modes are more easily excited due to a low ratio between wavelength and droplet size:

1. ***Standing Faraday waves*** on the elongated droplet surface
2. As forcing ***amplitudes*** increase, the ***elongation*** increases

Control Parameter

ω : wave angular frequency
ζ_0 : wave amplitude
R_0 : droplet radius at rest

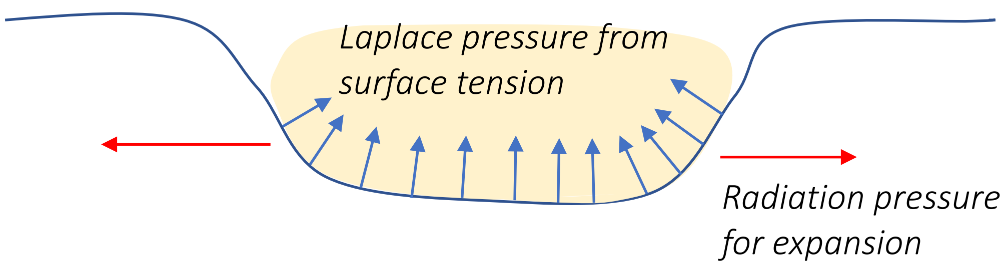
*The equilibrium shape for both deformed and elongated shapes result from a competition between destabilizing factor (**radiation pressure**) and restoring factor (**Laplace pressure**):*

A ratio between these two effects is expressed by a_0:

$$a_0 = \frac{\rho_1 \omega^2 \zeta_0^2}{4(\sigma_1 / R_0)}$$

When $a_0 \cong 1$, ***radiation pressure*** is of the order of magnitude of the ***surface tension response***, resulting in ***equilibrium shape***

Increase in wave amplitude ζ_0 (increase in radiation pressure) → *Larger ratio a_0* → *Further elongation of droplet* → *Increase in horizontal curvature at the tips (decrease in R_0)* → *Increase in restoring force due to surface tension*



Elongated Droplet Shape

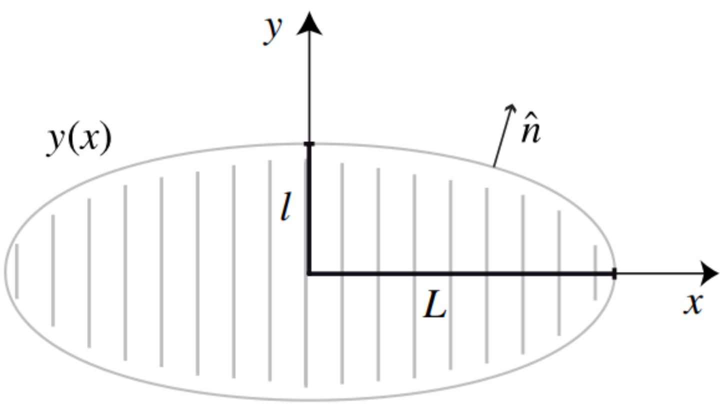
The shape of elongated droplet can be solved by a ***two-dimensional Laplace law*** modified by the ***radiation pressure along the normal***:

$$\boxed{P_h}+\boxed{P_r}\frac{y'^2(x)}{1+y'^2(x)}=-\sigma_1 f(\boxed{\theta})\frac{d}{dx}\frac{y'(x)}{\sqrt{1+y'^2(x)}}$$

Hydrostatic Pressure

Radiation Pressure

Wetting Angle



$$\boxed{b}+\boxed{a}\frac{y'^2(x)}{1+y'^2(x)}=\frac{d}{dx}\frac{y'(x)}{\sqrt{1+y'^2(x)}}$$

Unknown Pressure

Strength of waves with respect to surface tension effects $a=\frac{\rho_1\omega^2\zeta_0^2\sin^2\theta_s}{4p^{(1)}(\frac{\sigma_0}{R_0})}$

Solutions:

$$y(x)=\pm\sqrt{\frac{1}{b_0^2-x^2}}$$

where,

$$b_0=\frac{d}{dx}\,q(x)=\frac{d}{dx}\frac{y'(x)}{\sqrt{1+y'^2(x)}}$$

Elongated Droplet Shape

Elongated Droplet Equations

Riccati equation: $b + a[q(x)]^2 = q'(x)$

Boundary conditions: $y'(0) = q(x) = 0 \text{ for } x = 0$

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Solution to elongated droplet shape $A = \frac{a}{b}$*:*

$$y(x) = \pm\frac{1}{b\sqrt{A(1+A)}}\log\left\{\sqrt{1+A}\left[\cos\left(b\sqrt{A}x\right) + \sqrt{\frac{A}{1-A} - \sin^2\left(b\sqrt{A}x\right)}\right]\right\}$$

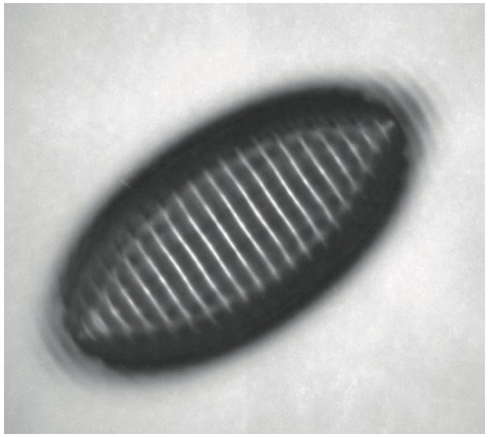
Aspect Ratio

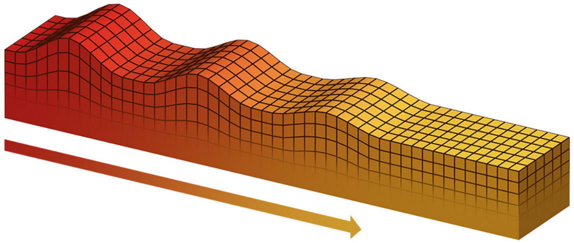
$$L = \frac{2}{b\sqrt{A}}\arctan\sqrt{A}$$

$$l = \frac{2}{b\sqrt{A(1+A)}}\log\left(\sqrt{1+A} + \sqrt{A}\right)$$

$$R = \frac{\log\left[\sqrt{A} + \sqrt{1+A}\right]}{\sqrt{1+A}\arctan\sqrt{A}}$$

Theoretical Model





Droplet Shapes

Surface Waves

Assumptions

Solutions to surface waves on droplet surface and droplet-bath interface is extremely difficult to find because the droplet shape modifies Faraday flow itself. Thus, the following assumptions:

Surface waves organize into standing waves

Radiation pressure only depends on the horizontal length scale of droplet

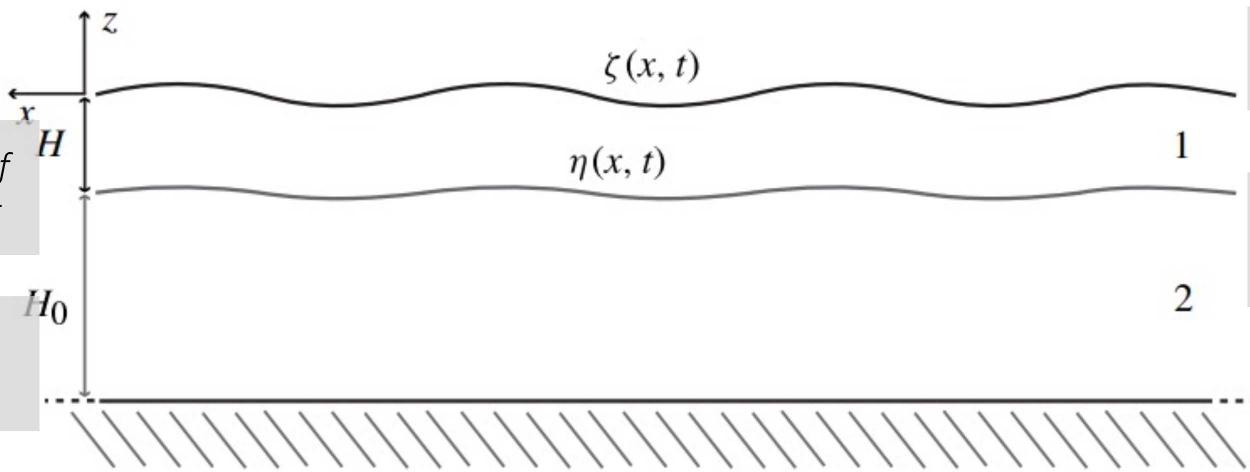
Droplet thickness is small compared with the horizontal scale of initial radius

Fluid is incompressible and irrotational

Geometry I: Surface Waves

Thickness of the droplet

Depth of liquid bath



$\zeta(x,t)$: surface wave on the droplet upper surface

$\eta(x,t)$: surface wave on the interface between droplet and bath

Unidirectional standing wave described by surface displacement function:

$$\zeta(x,t) = \zeta_0 \cos kx \cos \omega t$$
$$\eta(x,t) = \eta_0 \cos kx \cos \omega t$$

$$\frac{\zeta_0}{\lambda} \ll 1, \frac{\zeta_0}{H_0} \ll 1 \qquad \frac{\eta_0}{\lambda} \ll 1, \frac{\eta_0}{H_0} \ll 1$$

Assume droplet-bath interface perturbed by surface displacement

Geometry I: Surface Waves

Assume the following potential functions in the droplet (liquid 1) and the bath (liquid 2):

$$\begin{cases} \phi_1 = \left(Ae^{kz} + Be^{-kz}\right)\cos kx \sin \omega t\,, & -H \le z \le 0 \\ \phi_2 = \left(Ce^{kz} + De^{-kz}\right)\cos kx \sin \omega t\,, & z \le -H \end{cases}$$

Laplace equations $\nabla^2\phi_1 = 0, \nabla^2\phi_2 = 0$ and *boundary conditions*:

$$\begin{cases} \dfrac{\partial \phi_2}{\partial z} = 0 & \text{at } z = -H - H_0 \\ \dfrac{\partial \phi_1}{\partial z} = \dfrac{\partial \zeta}{\partial t} & \text{at } z = 0 \\ \dfrac{\partial \phi_1}{\partial z} = \dfrac{\partial \phi_2}{\partial z} = \dfrac{\partial \eta}{\partial t} & \text{at } z = -H \\ \dfrac{\partial \phi_1}{\partial t} + g\zeta - \dfrac{\sigma_1}{\rho_1}\dfrac{\partial^2 \zeta}{\partial x^2} = 0 & \text{at } z = 0 \\ \rho_1 \dfrac{\partial \phi_1}{\partial t} + \rho_1 g\eta - \rho_2 \dfrac{\partial \phi_2}{\partial t} - \rho_2 g\eta = \sigma_{12}\dfrac{\partial^2 \eta}{\partial x^2} & \text{at } z = -H \end{cases}$$

No-slip velocity at the bottom of the container

Expressions of surface wave velocities are equal at the droplet surface and droplet-bath interface

Bernoulli pressure equations at the droplet surface and droplet-bath interface

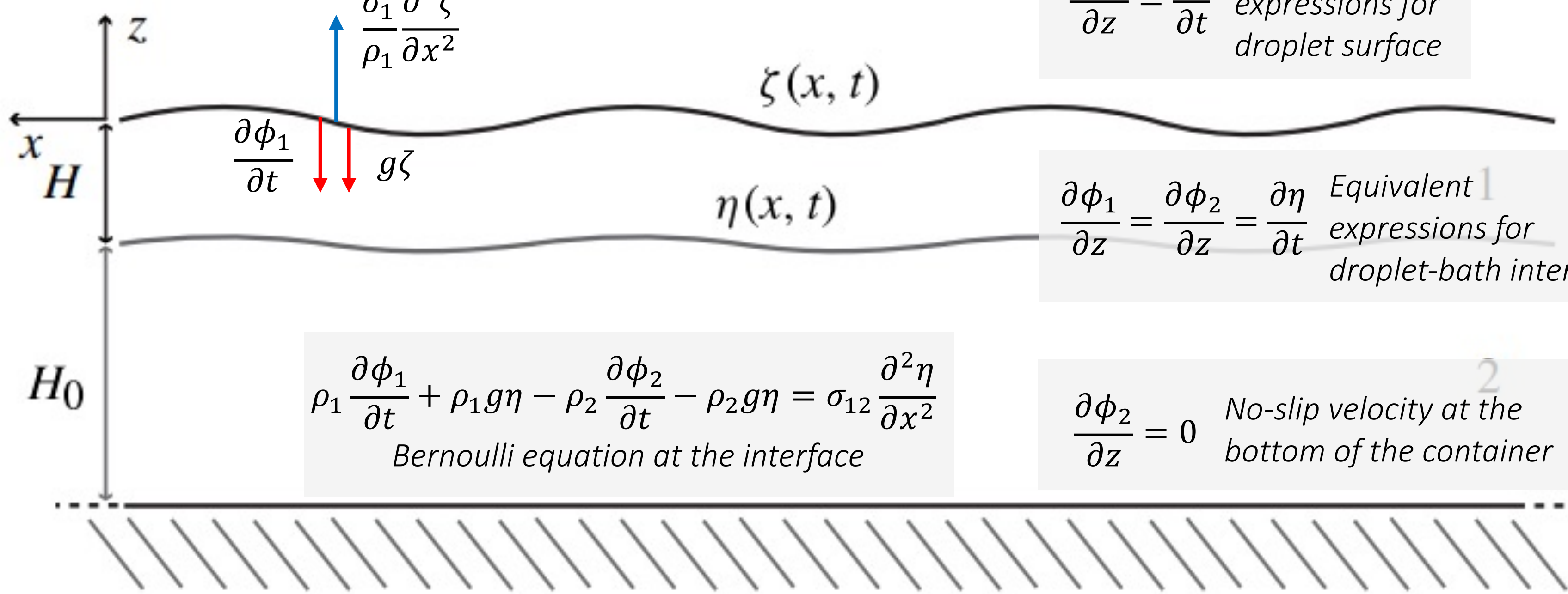
Geometry I: Surface Waves

$$\frac{\partial \phi_1}{\partial t} + g\zeta - \frac{\sigma_1}{\rho_1}\frac{\partial^2 \zeta}{\partial x^2} = 0$$

Bernoulli equation at the droplet surface

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \zeta}{\partial t}$$

Equivalent expressions for droplet surface



$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} = \frac{\partial \eta}{\partial t}$$

Equivalent expressions for droplet-bath interface

$$\rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g\eta - \rho_2 \frac{\partial \phi_2}{\partial t} - \rho_2 g\eta = \sigma_{12}\frac{\partial^2 \eta}{\partial x^2}$$

Bernoulli equation at the interface

$$\frac{\partial \phi_2}{\partial z} = 0$$

No-slip velocity at the bottom of the container

Geometry I: Surface Waves

From boundary conditions, the ***velocity potentials*** can be found:

$$\begin{cases} \phi_1 = \left[\frac{\omega}{2k}\zeta_0 R\left(\mathrm{e}^{kz} + \mathrm{e}^{-kz}\right) - \frac{\omega}{k}\zeta_0 \mathrm{e}^{kz}\right] \cos kx \sin \omega t\,, & -H \leq z \leq 0 \\ \phi_2 = \frac{\omega}{k}\frac{\eta_0}{\mathrm{e}^{k(2H_0-H)} - \mathrm{e}^{kH}}\left(\mathrm{e}^{k(z+2H_0)} + \mathrm{e}^{-kz}\right) \cos kx \sin \omega t & z \leq -H \end{cases}$$

Relationship between magnitudes of ***interface surface wave*** and ***droplet surface wave***:

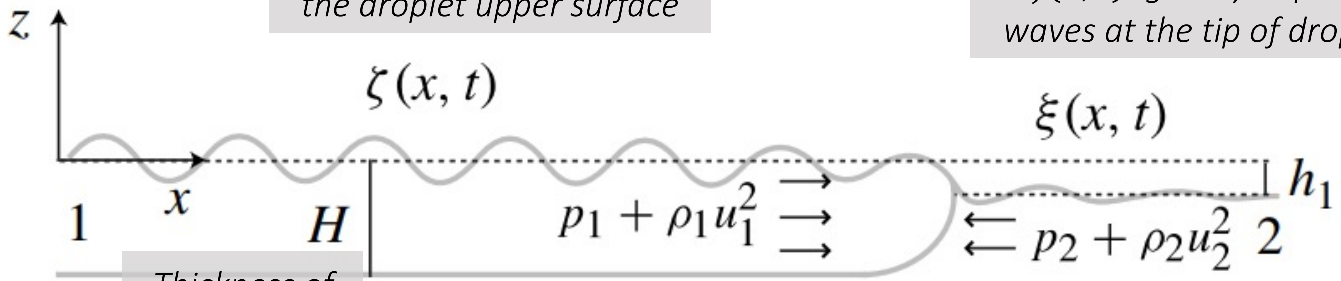
$$\eta_0 = \zeta_0\left(\mathrm{e}^{-kH} + F\sinh kH\right) = \zeta_0\left(\mathrm{e}^{-\frac{2\pi}{\lambda}H} + F\sinh\frac{2\pi}{\lambda}H\right)$$

in which, $F = 1 - \left(gk + \frac{\sigma_1}{\rho_1}k^2\right)/\omega^2 = 1 - \left(\frac{2\pi g}{\lambda} + \frac{\sigma_1}{\rho_1}\frac{4\pi^2}{\lambda^2}\right)/\omega^2$

Geometry II: Gravity-Capillary Waves

$\zeta(x,t)$: surface wave on the droplet upper surface

$\xi(x,t)$: gravity-capillary waves at the tip of droplets



Depth of gravity-capillary waves beneath the surface wave

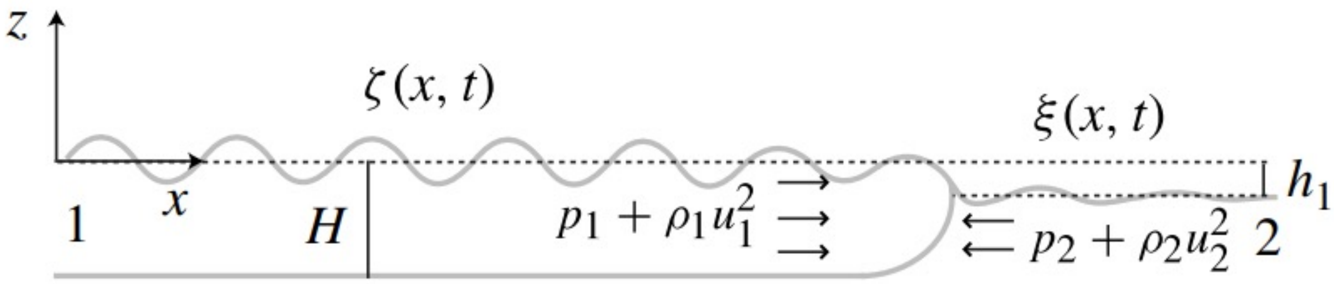
Thickness of the droplet

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{at } z = -H$$

becomes $$\frac{\partial \phi_1}{\partial z} = 0 \qquad \text{at } z = -H$$

$$\phi_1 = -\frac{\omega}{k}\zeta_0 \mathrm{e}^{kz} \cos kx \sin \omega t$$

Geometry II: Gravity-Capillary Waves

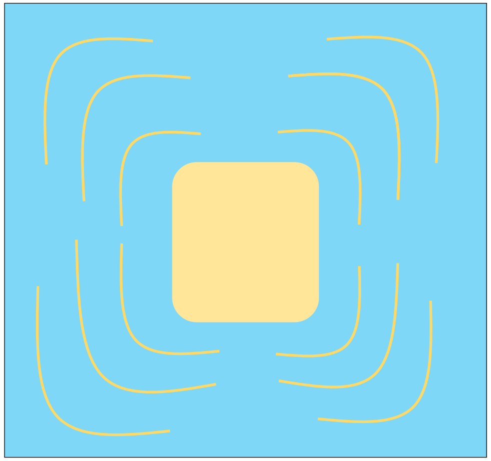


$$\phi_2 = -\frac{\omega}{k}\zeta_0 e^{kz} \cos kx \sin \omega t \quad \text{at } -H \leq z \leq -h_1$$

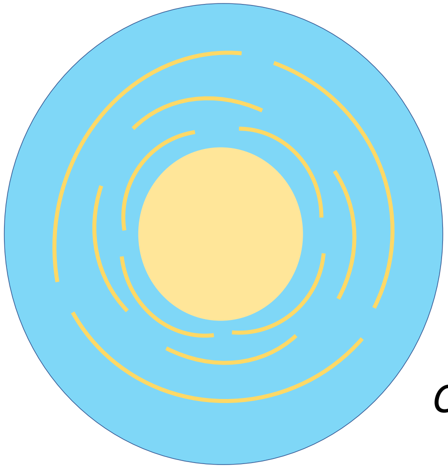
becomes $$\frac{\partial \xi}{\partial t} = \frac{\partial \phi_2}{\partial z} \quad \text{at } z = -h_1$$

$$\xi_0 = \frac{\zeta_0}{2} e^{-kh_1} \quad \text{at } z = -h_1$$

Geometry III: Circular Cavity



Square cavity



Circular cavity

Switching from ***square cavity*** to ***circular cavity***, switch the coordinate system to ***cylindrical coordinates***:

$$\frac{\partial \phi}{\partial r} = 0 \qquad \text{at } r = R$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = 0$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{at } z = -H$$

Bernoulli equation for pressure fields within bath because of oscillating interface:

$$p = -\rho \frac{\partial \phi}{\partial t} - \rho g \eta$$

Young-Laplace equation:

$$\frac{p}{\sigma} = -\frac{1}{R^2}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2}\right)$$

Volume conservation:

$$\int_0^{2\pi}\int_0^R r\xi(r,\theta)\,dr\,d\theta = 0$$

Geometry III: Circular Cavity

The general solution of the ***velocity potential***:

$$\phi(r,z) = \mathrm{i}\lambda \sum_{n=1}^{\infty} \frac{1}{k_n l} \frac{\cosh(k_{nl} z)}{\sinh(k_{nl} h)} \frac{\langle y, J_l\langle k_{nl} r\rangle\rangle}{\langle J_l\langle k_{nl} r\rangle, J_l\langle k_{nl} r\rangle\rangle} J_l\langle k_{nl} r\rangle$$

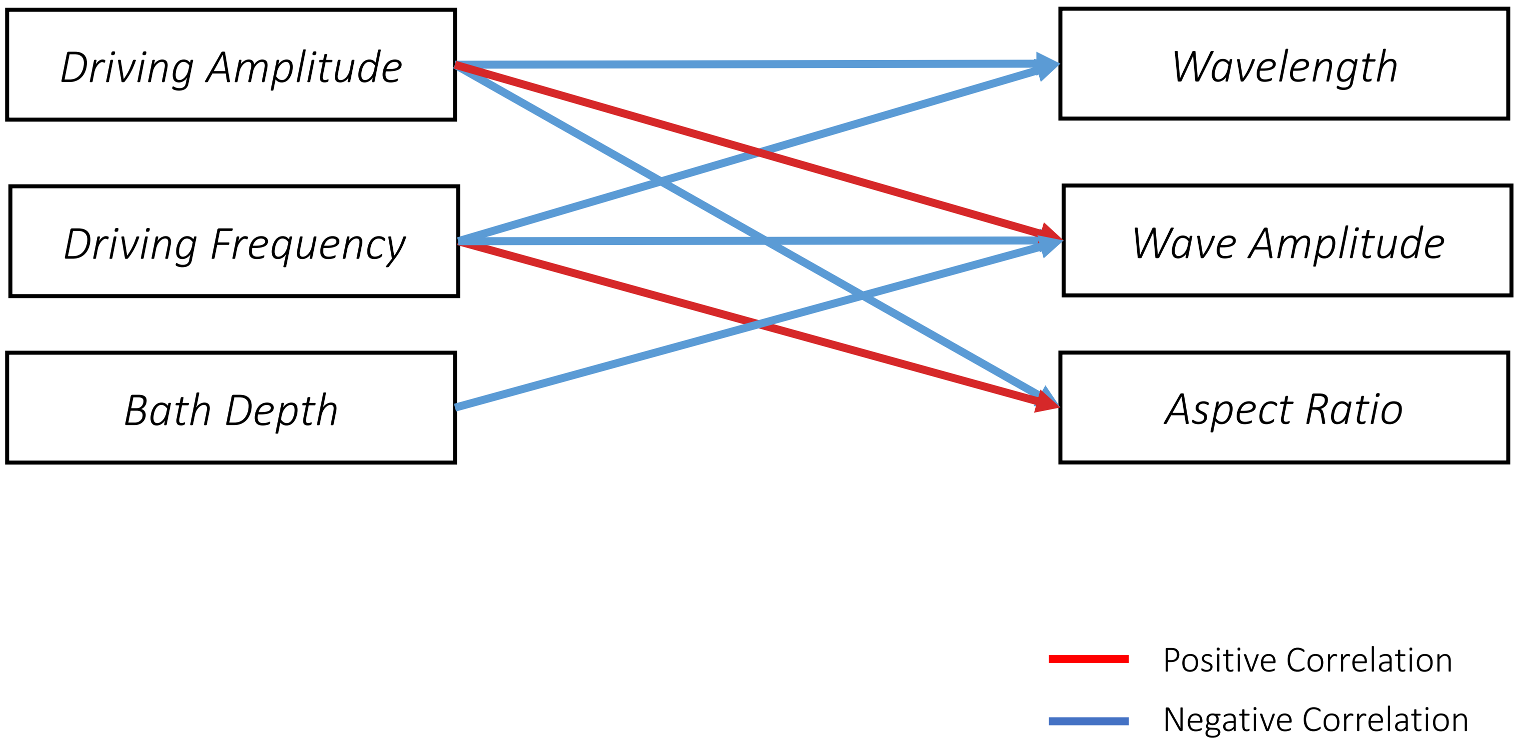
Velocity potential solution is written implicitly through y, apply this solution to ***Young-Laplace*** equation:

$$\lambda^2 \sum_{n=1}^{\infty} \frac{\coth(k_{nl} h)}{k_n l} \frac{\langle y, J_l\langle k_{nl} r\rangle\rangle}{\langle J_l\langle k_{nl} r\rangle, J_l\langle k_{nl} r\rangle\rangle} J_l\langle k_{nl} r\rangle - Bo\, y + \left[\frac{d^2 y}{dr^2} + \frac{1}{r}\frac{dy}{dr} - \frac{l^2}{r^2} y\right] = 0$$

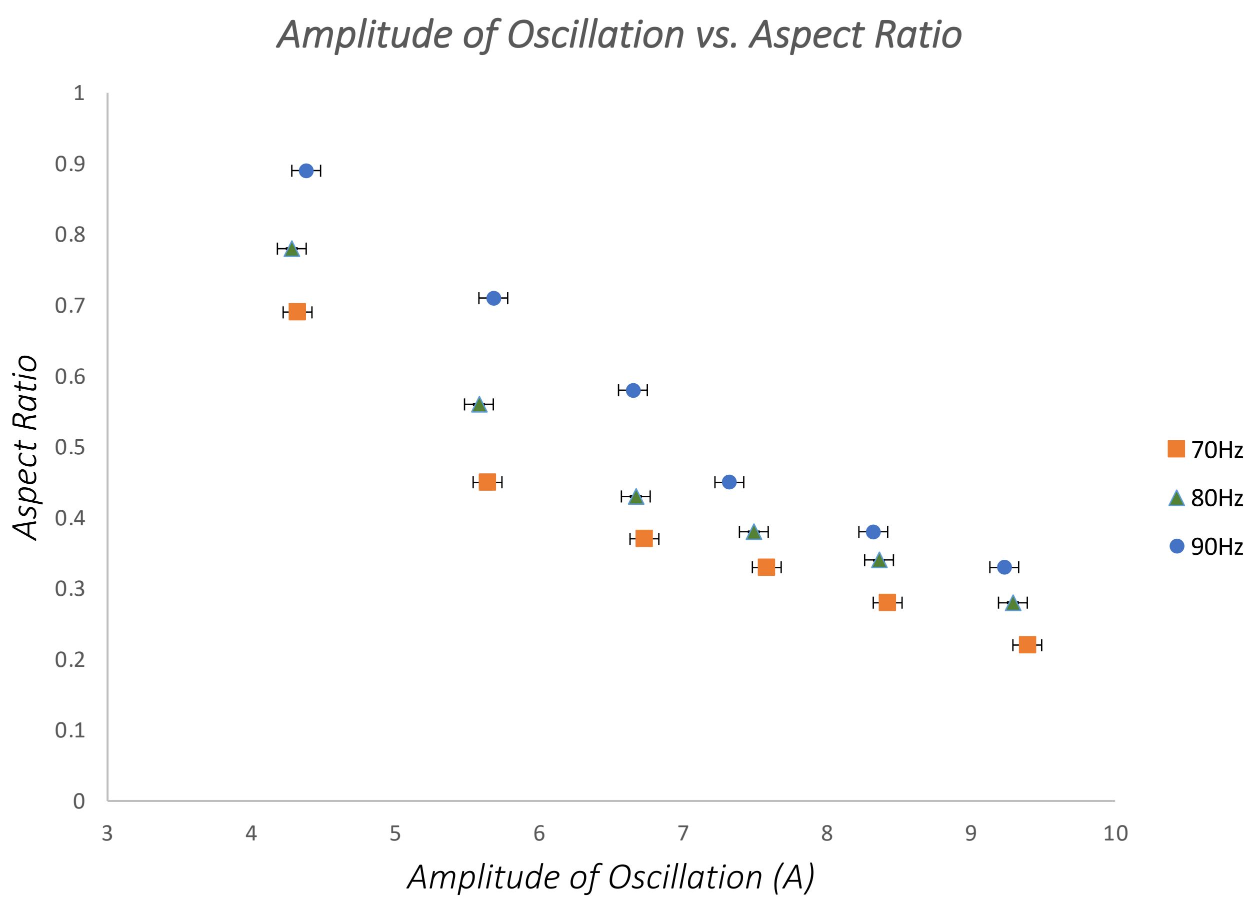
The inner product is defined as: $\langle f(r), g(r)\rangle = \int_0^1 r f(r) g(r) dr$

Key Parameters

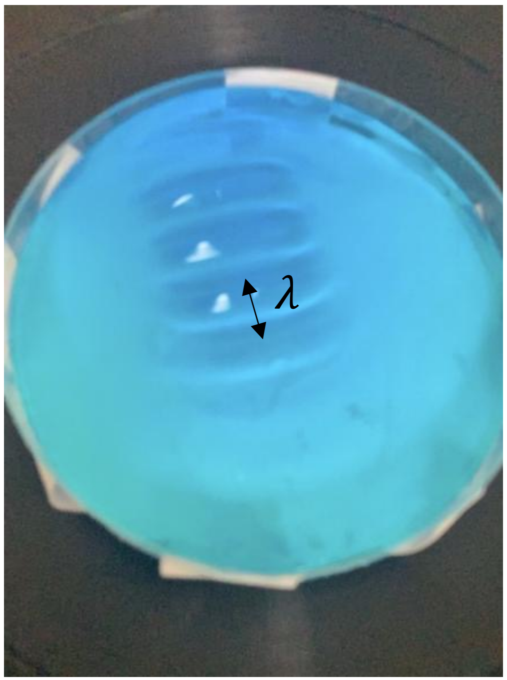
Key Parameters

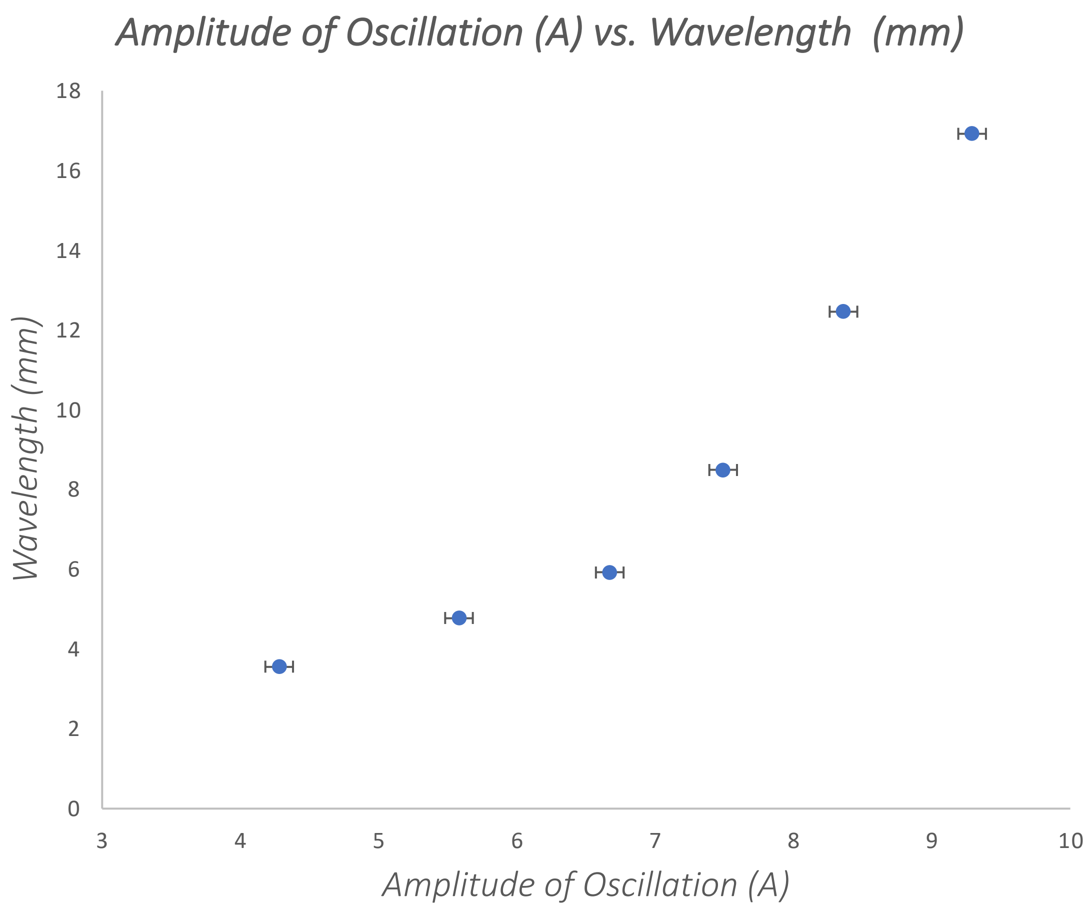


Driving Amplitude vs. Aspect Ratio

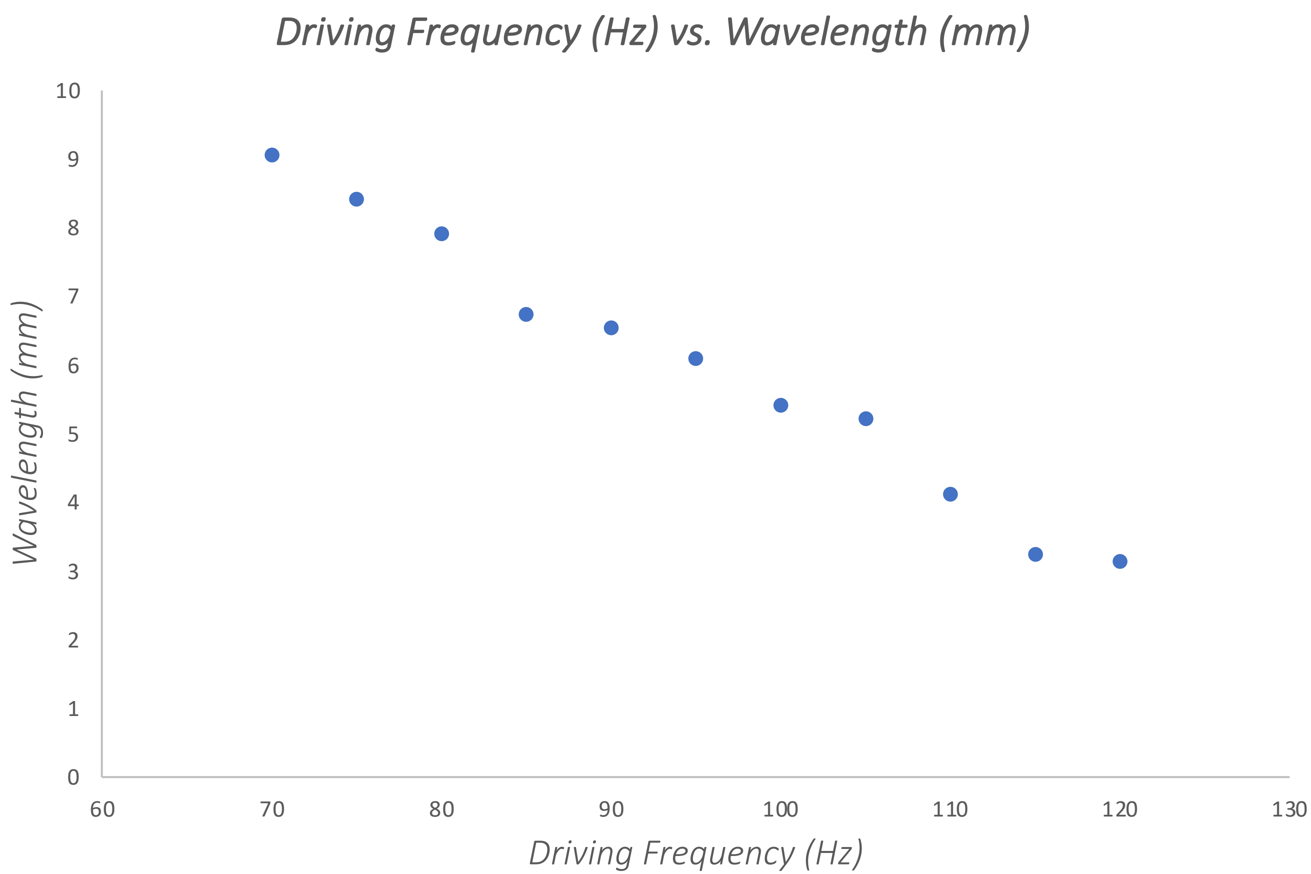


Driving Amplitude vs. Wavelength





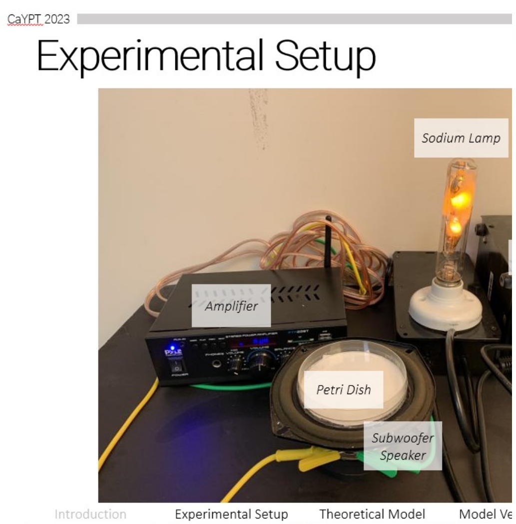
Driving Frequency vs. Wavelength

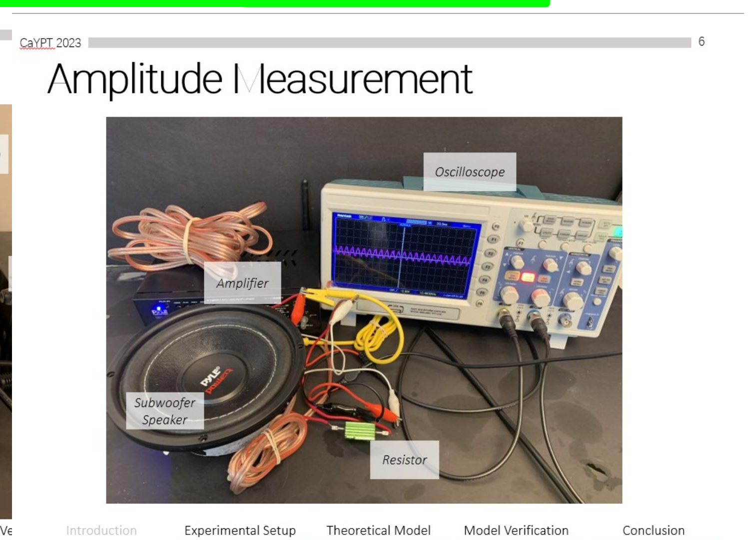


Conclusion

"A droplet of ***less viscous liquid*** *floating in a bath of a* ***more viscous liquid*** *develops surprising wave-like patterns when the entire system is set to* ***vertical oscillation****. Investigate this phenomenon and the parameters relevant to the* ***production of stable patterns****."*

Controlled Experimental Setup

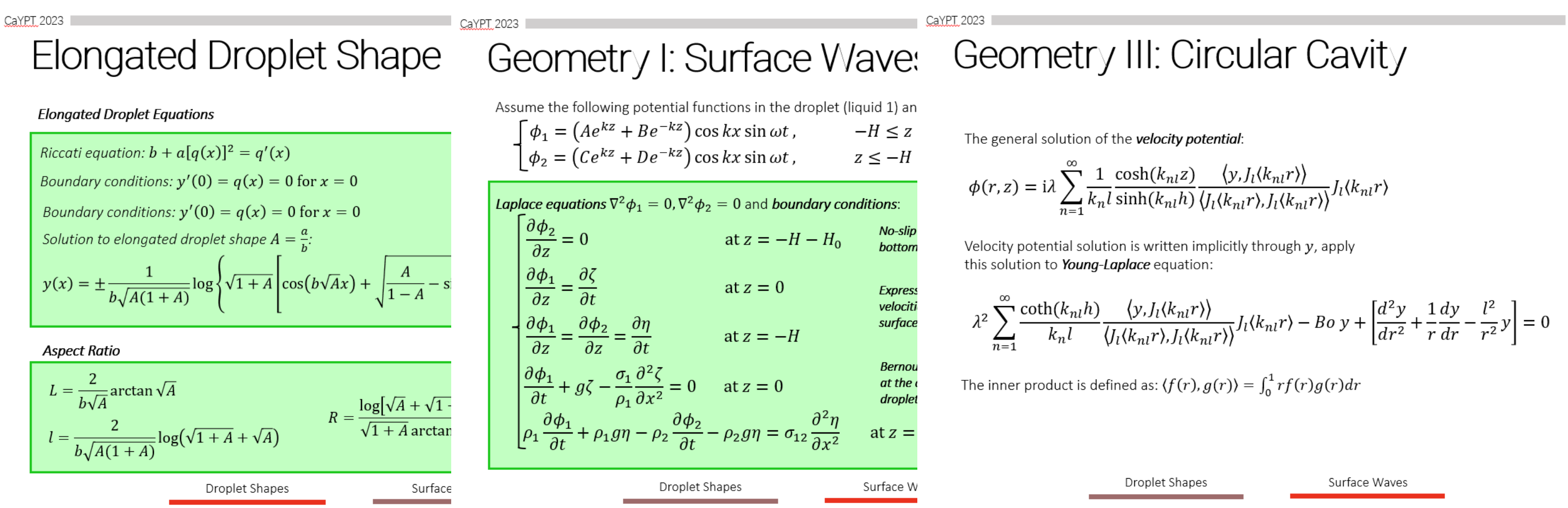




Conclusion

*“A droplet of **less viscous liquid** floating in a bath of a **more viscous liquid** develops surprising wave-like patterns when the entire system is set to **vertical oscillation**. Investigate this phenomenon and the parameters relevant to the **production of stable patterns**.”*

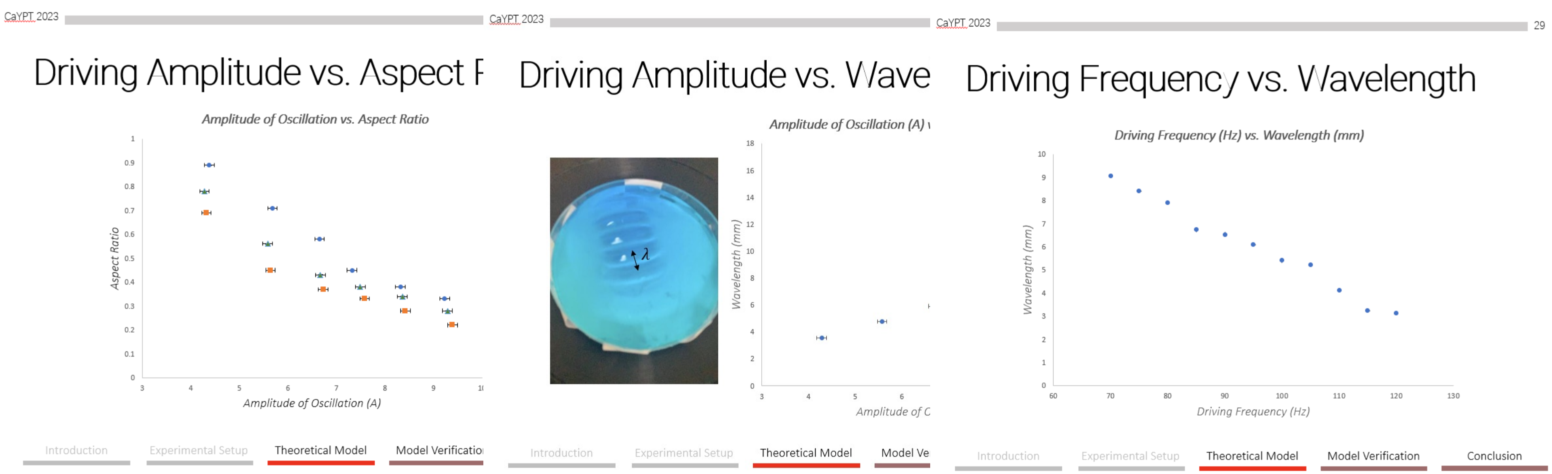
Thorough Theoretical Model



Conclusion

*"A droplet of **less viscous liquid** floating in a bath of a **more viscous liquid** develops surprising wave-like patterns when the entire system is set to **vertical oscillation**. Investigate this phenomenon and the parameters relevant to the **production of stable patterns**."*

Varied Parameters With Experimental Verification



Thank you for listening

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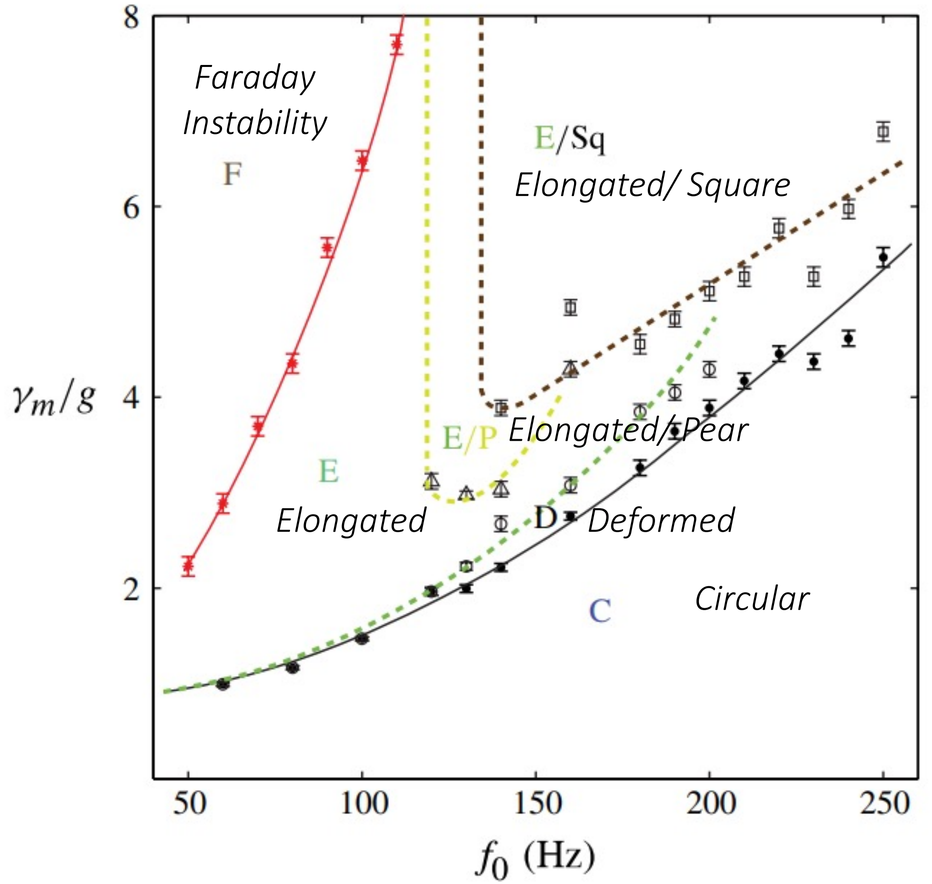
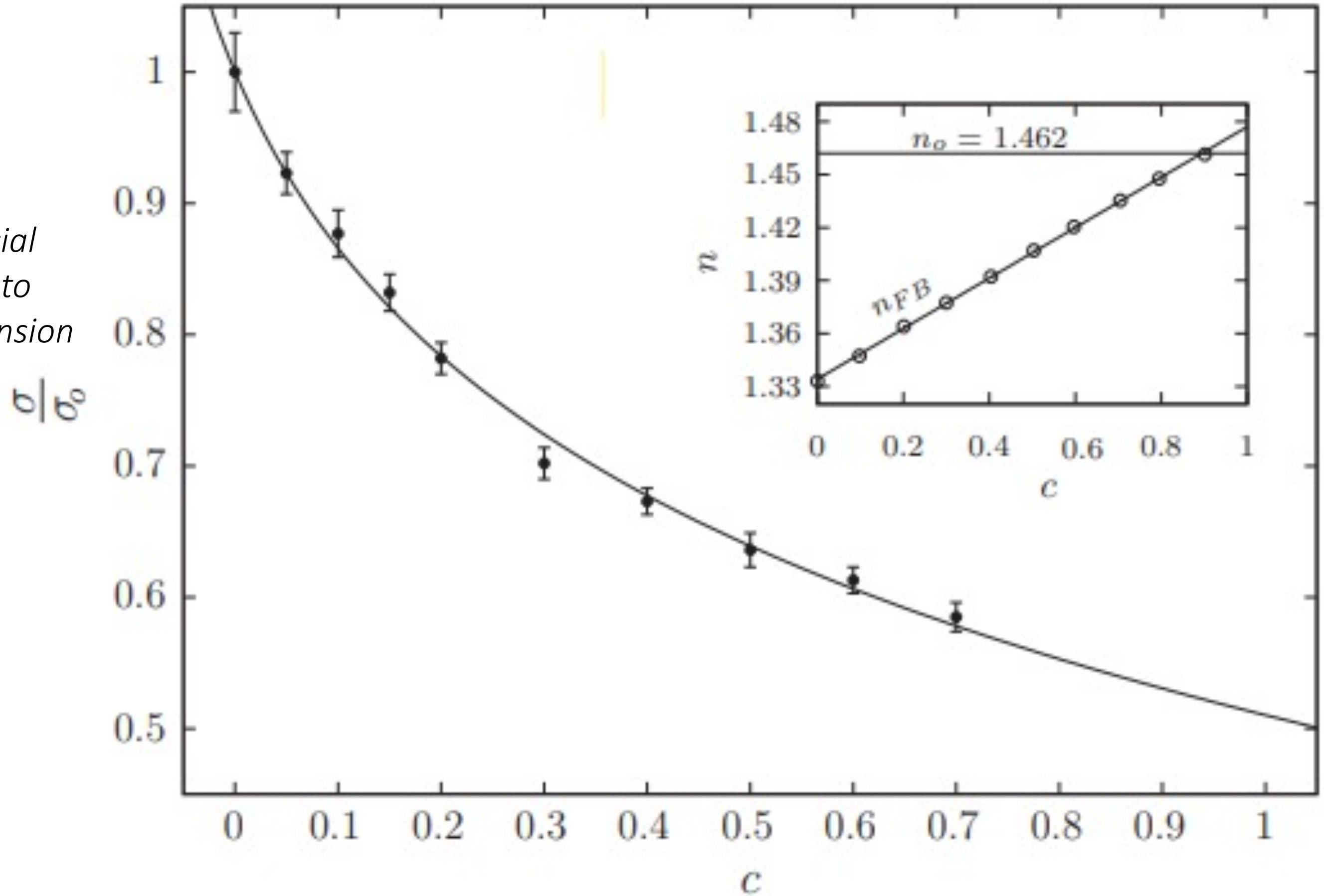
Appendix

Appendix A: Other Shapes



Appendix B: Surface Tension σ_{12}

Ratio of interfacial surface tension to glycerin surface tension



Concentration of the glycerin in the oil-glycerin interface

Appendix C: Mathieu Equation

$$z'' + 2\mu z' + \omega_0^2[1+\alpha(t)]z = 0$$

$$g(t) = g + A\omega^2\cos(\omega t) = g\,[1 + \Gamma\cos(\omega t)]$$

$$\omega_0^2(t) = k\left[g(t) + \frac{\gamma}{\rho}k^2\right]\tanh(kh)$$

$$-\frac{\Gamma\omega_0}{2} < \delta < \frac{\Gamma\omega_0}{2},$$

or

$$2\omega_0 - \frac{\Gamma\omega_0}{2} < \omega < 2\omega_0 + \frac{\Gamma\omega_0}{2}$$

$$-\sqrt{\left(\frac{\Gamma\omega_0}{2}\right)^2 - 4\mu^2} < \delta < \sqrt{\left(\frac{\Gamma\omega_0}{2}\right)^2 - 4\mu^2}$$

or

$$2\omega_0 - \sqrt{\left(\frac{\Gamma\omega_0}{2}\right)^2 - 4\mu^2} < \omega < 2\omega_0 + \sqrt{\left(\frac{\Gamma\omega_0}{2}\right)^2 - 4\mu^2}$$