1 Problem 2

1.1 Correctness

Lemma 1. *G is isomorphic to G' if and only if the resulting formula ∅ is satisfiable.*

Assume $|V| = |V'| = n$. Define the variables of ∅ as follows:

$$x_{ij} = \begin{cases} 1 & \pi \text{ creates an isomorphic graph that maps node } v_i \in V \text{ to } v_j \in V' \\ 0 & \text{otherwise.} \end{cases}$$

Hence formula will consist of the following types of clauses:

1. For each node $a \in [0, n-1]$, $C_i = (x_{a0} \vee x_{a1} \vee ... \vee x_{an-1})$. Hence, node $v_a \in V$ is mapped to some vertex in V'
2. For all nodes $a, b, c \in [0, n-1]$ in which $a \neq b$, $C_b = (\overline{x_{ac}} \vee \overline{x_{bc}})$
3. For all nodes $a, b, c, d \in [0, n-1]$ where $a \neq b, c \neq d$, edge $e_{ab} \in E$ if and only if edge $e_{cd} \notin E'$, $C_c = (\overline{x_{ac}} \vee \overline{x_{bd}})$.

Proof of Lemma 1. We will first prove that if ∅ is satisfiable, G is isomorphic to G'. Assuming ∅ is satisfiable; this implies that all clauses of ∅ are true. Note, the three types of clauses imply the conditions (1) node $v_a \in V$ is mapped to some vertex in V' for each $a \in [0, n-1]$, (2) no two distinct nodes are mapped to the same node in V', and (3) there a mapping of $V \to V'$, so that for all pairs of nodes v, w in V, the edge $(v, w) \in E$ if and only if $(\pi(v), \pi(w)) \in E'$. Because the assignment encodes all conditions of an isomorphism and as all clauses are true, the conditions of an isomorphism are met. Thus, function π exists and G is isomorphic to G'.

We will now prove that if G is isomorphic to G', ∅ is satisfiable. Assuming G is isomorphic to G' implies that conditions (1), (2), and (3) are met for graphs G' and G. As conditions (1), (2), and (3) represent type 1, 2, and 3 clauses, all clauses of type 1, 2, and 3 are true. Thus, the assignment of all the clauses ∅ is true and the instance of SAT is satisfiable.

Thus, Graph Isomorphism $\leq$ SAT.

1.2 Runtime and Space Complexity

The runtime of the above reduction algorithm is bounded by $O(n^4)$. Assume G, G' are set as adjacency matrices. Determining the output for a clause of type 1 takes $O(n)$ as for each node, we must iterate through all n nodes in graph G' to check for some map pairing. Because graph G has n nodes, the total time to calculate clause 1 for all n nodes in G is $O(n^2)$. The total time to calculate clause 2 for each pair (i, j) in G is $O(n^2)$, the number of pairs of nodes is bounded by n^2. Hence, checking that the mapping for each adjacent pair of nodes doesn't map to the same node in V' take $O(n^2)$. The total time to calculate clause 3 for each pair of edges $e_{ab} \in E$ and edge $e_{cd} \notin E'$ is $O(n^4)$, as the number of pairs of edges is bounded by n^4. Hence, checking that the mapping for each pair of edges maps accordingly takes $O(n^4)$.

Note, because the runtime is polynomial, the space complexity is polynomial as well.