Introduction to Algorithms	Problem Set 7
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1 Problem 2

1.1 Correctness

Note that the MONOTONE SAT problem is in NP, as the input is given as a SAT formula, you verify if the formula is true in polynomial time.

We prove that the problem is NP-complete by a reduction from VERTEX COVER which is know to be NP-complete.

Consider an input graph $G = (V, E)$ for VERTEX COVER, and do the following construction. For each node n_i in V, have n_i correspond to variable x_{n_i}. Then for each edge $e_j = (v_j, w_j)$ in E, have e_j correspond to clause $L_j = x_{v_j} \vee x_{w_j}$. Suppose, G has m edges $e_1, ..., e_m$. Then, call the resulting formula $\phi = L_1 \wedge ... \wedge L_m$. Note, elements in vertex cover C, will correspond to the the variables to be set to true in ϕ.

Claim 1. *ϕ is satisfiable by setting at most k variables true, if and only if the graph $G = (V, E)$ contains a vertex cover of size at most k, which establishes that* VERTEX COVER $\leq$ MONOTONE SAT.

Proof. First, suppose that G contains a vertex cover C of size at most k and graph G has m edges $e_1, ..., e_m$, and $e_j = (v_j, w_j)$. Then, consider the corresponding formula ϕ in the instance of MONOTONE SAT. For each t from 1 to m, at least one element of C is an end of the edge e_t and so also in the corresponding clause L_t. By setting this element to true, L_t is true. Hence, if the variables that correspond to all elements of C are set to true, ϕ is satisfied.

Conversely, suppose there is a MONOTONE SAT instance that is satisfied by setting at most k variables true, and consider the corresponding set of nodes C in G. For each t from 1 to m, at least one variable in L_t is set to true, which corresponds to one node in C which is an end of the edge e_t. Because there is a clause that corresponds to each edge, the set C is a vertex cover.