Introduction to Algorithms	Problem Set 5
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1 Problem 3 Part A

1.1 Algorithm

Assume the graph is constructed as an adjacency list.

1. Let there be a set U_1 of n nodes that represents each university. Then, let there also be a duplicate set U_2 (to map exchange agreements between universities). Let node s be the source and node t be the sink.
2. For each university $u_i \in U_1$, insert directed edge (s, u_i) with capacity g_i to represent the students who would like to participate in the exchange program from university u_i
3. For each university $u_j \in U_2$, insert directed edge (u_j, t) with capacity h_j that represents the number of students university u_j is willing to host
4. For each $u_i \in U_1$, loop through the associated list L_i
   - For each u_j on L_i, add directed edge from u_i to $u_j \in U_2$. Each edge will have capacity h_j.
5. Then, $d = \sum_{i=1}^{n} g_i$ is the demand placed on the sink and $-d$ on the source
6. Run Ford-Folkerson and return

1.2 Time and Space Complexity

To construct the graph, steps 1-3 is bounded by $O(n)$ as we create edges from s to every group and every group to t. Step 4 takes at most $O(n^2)$ iterations, with each iteration taking constant time to calculate edge capacity. Running Ford-Folkerson is given to take $O((n^2+2n)C) = O(n^2C)$, (proportional to the number of edges in the constructed graph) and C is a bound for the maximum possible flow value (edges out of s would be completely saturated with flow). Hence, total running time is polynomial. Because the time complexity is polynomial, the space complexity is polynomial as well.

1.3 Correctness

Lemma 1. *There is a way to as assign students to universities in a way that respects all constraints if and only if there is a feasible circulation in the flow network we have constructed.*

Proof of Lemma 1. First, if there is a way to assign students to universities days in a way that respects all constraints, then we can construct the following circulation. If there is an agreement between university $u_i \in U_1$ and $u_j \in U_2$, then we send one unit of flow along the path s, u_i, u_j, t; we do this for all such (u_i, u_j) pairs. Since the assignment of students satisfied all the constraints, the resulting circulation respects all capacities; and it sends d units of flow out of s and into t, so it meets the demands.

Conversely, suppose there is a feasible circulation. For this direction of the proof, we will show how to use the circulation to construct a schedule for all the doctors. First, by (7.52) from the textbook, there is a feasible circulation in which all flow values are integers. We now construct the following assignment: If the edge (u_i, u_j) carries a unit of flow, then we have a student from u_i studying at university u_j. Because of the capacities, the resulting assignment has at most g_i students from u_i distributed amongst in agreement universities, in which at most h_j students from u_i are studying at an in agreement university u_j.

2 Problem 3 Part B

2.1 Algorithm

Follow the same algorithm in part A, with the modification:

1. For each $u_i \in U_1$, loop through the associated list L_i
   - For each u_j on L_i, if u_j is in the same country as u_i, insert a gadget if not already created as follows. We include a new node w_i with an incoming edge with capacity $g_i/3$ from the university node u_i and with outgoing edge of capacity $min(g_i/3, h_j)$ to university node u_j. Note, if node w_i already exists for university u_i, only add an outgoing edge from w_i to university node u_j with capacity $min(g_i/3, h_j)$.

2.2 Time and Space Complexity

Let e be the number of edges in the graph in part A. We add at most $n^2 < e$ edges to the graph in part B (as each university's gadget can map to every other university, if every university was in the same country). Because the runtime of Ford-Folkerson is proportional to the number of edges in the constructed graph, the runtime is analogous to part A. Then, because the time complexity is polynomial, the space complexity is polynomial as well.

2.3 Correctness

Lemma 2. *There is a way to as assign students to universities in a way that respects all constraints if and only if there is a feasible circulation in the flow network we have constructed.*

Proof of Lemma 2. First, if there is a way to assign students to universities days in a way that respects all constraints, then we can construct the following circulation. If there is an agreement between university $u_i \in U_1$ and $u_j \in U_2$, then we send one unit of flow along the path s, u_i, w_i, u_j, t; we do this for all such (u_i, u_j) pairs. Since the assignment of students satisfied all the constraints, the resulting circulation respects all capacities; and it sends d units of flow out of s and into t, so it meets the demands.

Conversely, suppose there is a feasible circulation. For this direction of the proof, we will show how to use the circulation to construct a schedule for all the doctors. First, by (7.52) from the textbook, there is a feasible circulation in which all flow values are integers. We now construct the following assignment: If the edge (w_i, u_j) carries a unit of flow, then we have a student from u_i studying at u_j in the same country. If the edge (u_i, u_j) carries a unit of flow, then we have a student from u_i studying at u_j in a different country. Because of the capacities, the resulting assignment has at most g_i students from u_i distributed amongst in agreement universities, in which at most h_j students from u_i are studying at an in agreement university u_j. Further, at most $g_i/3$ students will study in in-agreement universities in the same country. Note, the total number of participating students is distributed between same country and out of country universities in an at most 2:1 ratio.