1 Problem 3

1.1 Correctness

Reduction The problem is in NP as we can verify in polynomial time that a correct almost-clique $G = (V, E)$ is valid, where $|V| = n$. Assuming graph G is set as an adjacency list, we remove node y and all of its associated edges in $O(n)$. We then iterate through all pairs of nodes to see if there is another common neighbor (as defined in the original problem) $c \neq y$ that is a neighbor to all other nodes in V. If there is such a common node, then the almost-clique is valid. Because there are n^2 pairs and it take $O(n)$ time to check each pair, the total running time is bound by $O(n^3)$ which is polynomial.

Now, we will show that the CLIQUE problem can be reduced to the ALMOST CLIQUE problem. Note that the CLIQUE problem is NP-complete. Suppose the graph $G = (V, E)$ for the CLIQUE instance with m edges $e_1, ..., e_m$, and $e_j = (v_j, w_j)$ and n nodes $n_1, ..., n_n$. We construct an instance of ALMOST CLIQUE as follows. We first insert a node y into G. For all n nodes $n_1, ..., n_m$, insert edge $e_i = (y, n_i)$. Generate a subset of nodes V', such that $|V'| = |E|$. For each node in $v_i \in V'$, assign v_i to edge e_i.

Lemma 1. *In the instance constructed, the given graph G with special node y and a target size k contains a clique of size k if and only if G has an almost-clique among ys neighbors of size k.*

Proof of Lemma 1. Suppose G contains a clique of size k including node y. Then, there is a set S of nodes that form a connected graph. Because node y is included in the connected graph, nodes in set $S \subset V$ are all neighbors of node y, as for all pairs of nodes $v, w \in S$, $(v, w) \in E$. Hence, there exists an almost-clique among y's neighbors of size k.
Then, suppose G' contains an almost-clique among y's neighbors of size k. Then, each node is connected by one inserted neighbor node $n \in V'$. Thus if we remove all inserted neighbor nodes, the resulting subgraph consist of the original nodes and y which is connected to every node in $V \setminus V'$, which forms a connected graph. Thus, G contains a clique of size k. Therefore, the ALMOST CLIQUE problem is NP-complete.

1.2 Runtime and Space Complexity

The runtime of the reduction is $O(n)$ as adding y and each edge $e_i = (y, n_i)$ for nodes $n_1, ..., n_n$ takes constant time. Further, generating a subset of nodes V', such that $|V'| = |E|$ and for each node in $v_i \in V'$, assigning v_i to edge e_i is bounded by $O(n)$ as there are at most n nodes to insert into the adjacency list which takes constant time for each node.

Note, because the runtime is polynomial, the space complexity is polynomial as well.