Introduction to Algorithms	Problem Set 4
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1 Problem 2

1.1 Part A

Let X be the random variable of the number of different coupons you have. Further, let

$$X_i = \begin{cases} 1 & 1 \text{ if you have the } i\text{th kind of coupon} \\ 0 & \text{otherwise,} \end{cases}$$

Hence,

$$X = \sum_i X_i.$$

Then, the probability that you didn't get the $i$the coupon after n rounds is $(1 - \frac{1}{n})^n$. Hence, the probability that you did get the ith coupon after n rounds is $1 - (1 - \frac{1}{n})^n$. Thus, the expected number of different coupons after n rounds using linearity of expectation is

$$\begin{aligned} X &= \sum_{i=1}^{n} X_i \\ E(X) &= E(\sum_{i=1}^{n} X_i) \\ &= \sum_{i=1}^{n} E(X_i) \\ &= n \cdot (1 - (1 - \frac{1}{n})^n) \\ \frac{E(X)}{n} &= 1 - (1 - \frac{1}{n})^n \\ E(X) &= n(1 - (1 - \frac{1}{n})^n) \\ \lim_{n\to\infty} \frac{E(X)}{n} &= 1 - \frac{1}{e}. \end{aligned}$$

1.2 Part B

Let Y_i be the number of coupons of type i you received after n rounds. Further, let

$$Y_{ik} = \begin{cases} 1 & 1 \text{ if you get the } i\text{th kind of coupon at round k} \\ 0 & \text{otherwise,} \end{cases}$$

Then, the probability that you get the ith coupon on round k is $\frac{1}{n}$. Thus, the expected number of the same coupon i after n rounds is

$$\begin{aligned} E(Y_i) &= E(\sum_{k=1}^{n} Y_{ik}) \\ &= \sum_{k=1}^{n} E(Y_{ik}) \\ &= n \cdot \frac{1}{n} \\ &= 1. \end{aligned}$$

1.3 Part C

We see $E(max_i Y_i)$ represents the expected number of the most common coupon of type i you received after n rounds. However, $max_i E(Y_i)$ is the maximum value of the expected number of the same coupon i after n rounds. However, by part B $E(Y_i)$ for any coupon i is 1. Thus,

$$\begin{aligned} max_i E(Y_i) &= max_i(1) \\ &= 1. \end{aligned}$$

Hence, $max_i E(Y_i) = 1$ for all i, while $max_i Y_i$ may vary from 1 to n. Note, $E(max_i Y_i) \neq 1$ as $E(max_i Y_i) = \sum_{j=1}^{n} jP(max_i Y_i = j) > 1$, as the $\sum_{j=1}^{n} P(max_i Y_i = j) = 1$.