Introduction to Algorithms	Problem Set 7
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1 Problem 1

1.1 Correctness

The Remote Traveler's Problem is in NP as verifying a valid Remote Traveler's Problem instance of the Remote Traveler's Problem takes polynomial time. Given directed graph $G = (V, E)$ at each node i in the path check that $min(gas(i) + g(i), W) \geq g(i, j)$ in which $gas(i)$ represents the total gas left in the car when just reaching node i and j is the next node in the path. Note, $gas(s) = g(s)$. There is at most $|E| = n$ edges in a path, which implies n total iterations. Each iteration takes constant time, hence overall time to check is polynomial.

We prove that the problem is NP-complete by a reduction from Hamiltonian Path which is know to be NP-complete.

Consider a Hamiltonian Path instance, whose input is a directed graph $G = (V, E)$. We need to show that we can decide if there is a simple path in G from s to t going through each node of the graph by running an algorithm for the Remote Traveler's Problem. To do this we set W to infinity, to ensure we are not constricted by the car tank constraint, for each edge $e_{ij} = (i, j)$ in E, assign edge weight $g(i, j) = 0$ and for each node n_i in V assign node weight $g(i) = 1$. Now, add a new edge $e_{ts} = (t, s)$ to graph G. Given, $|V| = n$ assign $g(t, s) = n$. Assuming the graph is set as an adjacency list, BFS will run in polynomial time. As we iterate through all the nodes and edges, and each iteration takes constant time, the transformation is done in polynomial time.

Claim 1. *There is a way to travel from s through the graph to t and then go directly back to s using edge (t, s) while never running out of gas if an only if the graph G contains a* Hamiltonian Path*, which establishes that* Hamiltonian Path $\leq$ Remote Traveler's Problem.

Proof. If there is a way to travel from s through the graph to t and then go directly back to s using edge (t, s) while never running out of gas, then the amount of gas in the car before taking edge $e_{ts} \geq n$. This can only be satisfied by visiting all n nodes to collect gas. Thus, G contains a valid Hamiltonian Path instance.

If there is a valid Hamiltonian Path instance, then the path visits all nodes n. Hence, the amount of gas in the car before taking edge $e_{ts} = n$. Thus, there is a way to travel from s through the graph to t and then go directly back to s using edge (t, s) while never running out of gas.