# STAT 201

Week 7

# Lecture goals:

By the end of this lecture, the students are expected to be able to:

- Describe the Law of Large Numbers.
- Describe a Normal distribution.
- Explain the Central Limit Theorem and its role in constructing confidence intervals.
- Write a computer script to calculate confidence intervals based on the assumption of normality / the Central Limit Theorem.
- Discuss the potential limitations of these methods.
- Decide whether to use asymptotic theory or bootstrapping to compute estimator uncertainty.

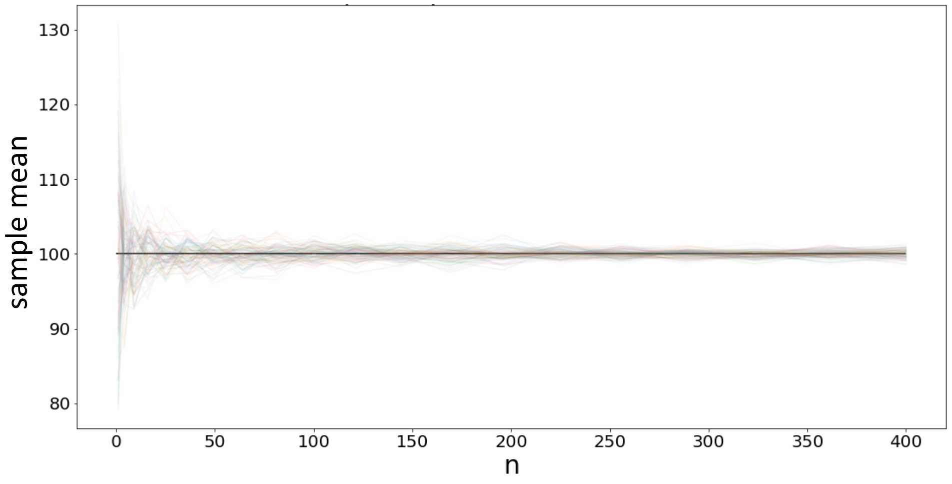
# Law of Large Numbers

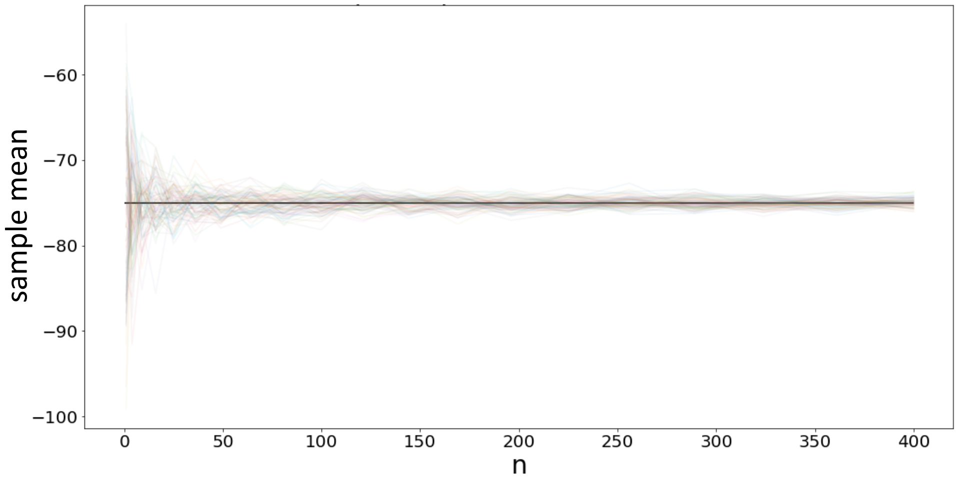
# (Strong) Law of Large Numbers

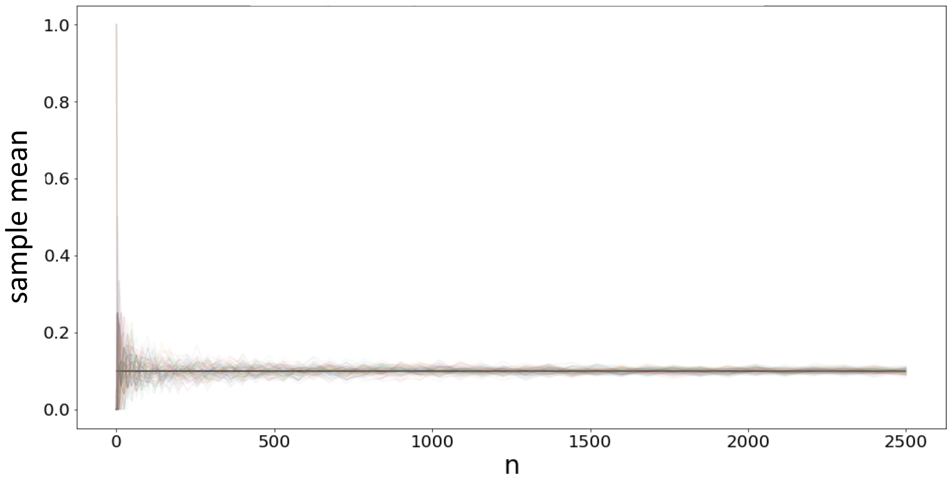
- The Law of Large Numbers (LLN) states that the sample average converges to the population mean.

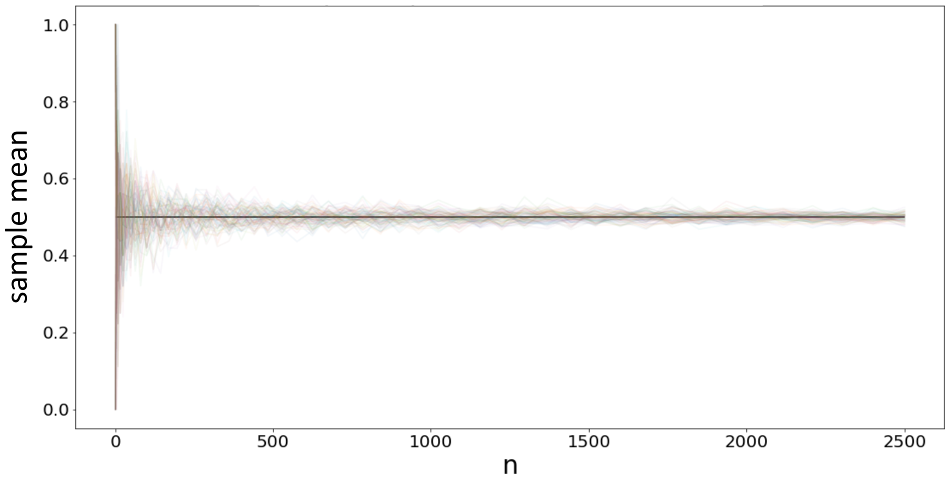
- In other words, as the sample size increases, the sample average gets closer and closer to the population mean with higher and higher probability.

# (Strong) Law of Large Numbers



# Normal Distribution

# Gaussian distribution

- The Gaussian (or Normal) distribution is one of the most (if not the most) important distribution in statistics.

- Many of the methods in statistics and data analysis assume Normality. Besides, the Central Limit Theorem assigns a central role for the Gaussian distribution.

- Today, we are going to explore the Normal distribution in more detail.

# Gaussian distribution

- Let $X \sim N(\mu, \sigma)$ be a random variable. The density of $X$ is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}, \qquad x \in \mathbb{R}$$

- Note that the term

$$\frac{(x-\mu)^2}{\sigma^2}$$

measures the square of the distance between $x$ and $\mu$ in terms of standard deviations.

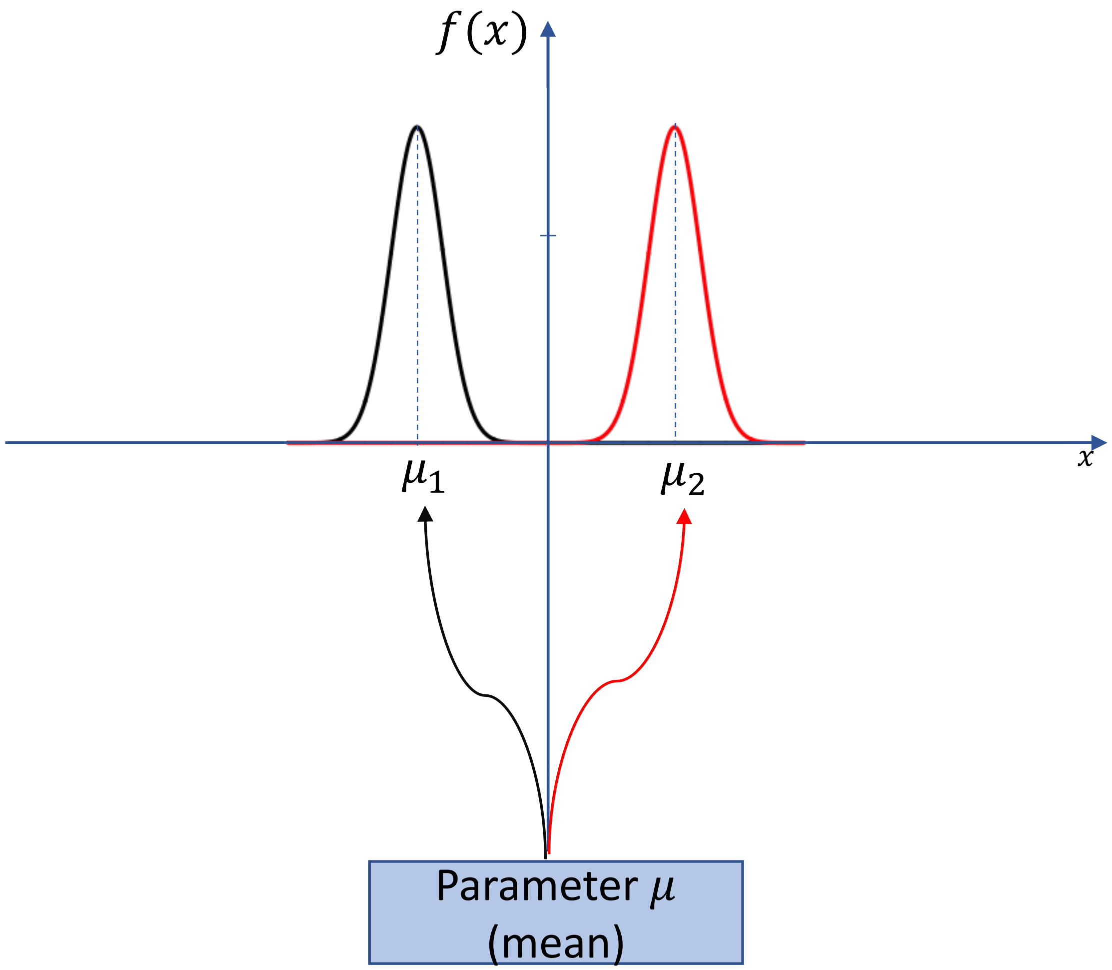
# Gaussian distribution

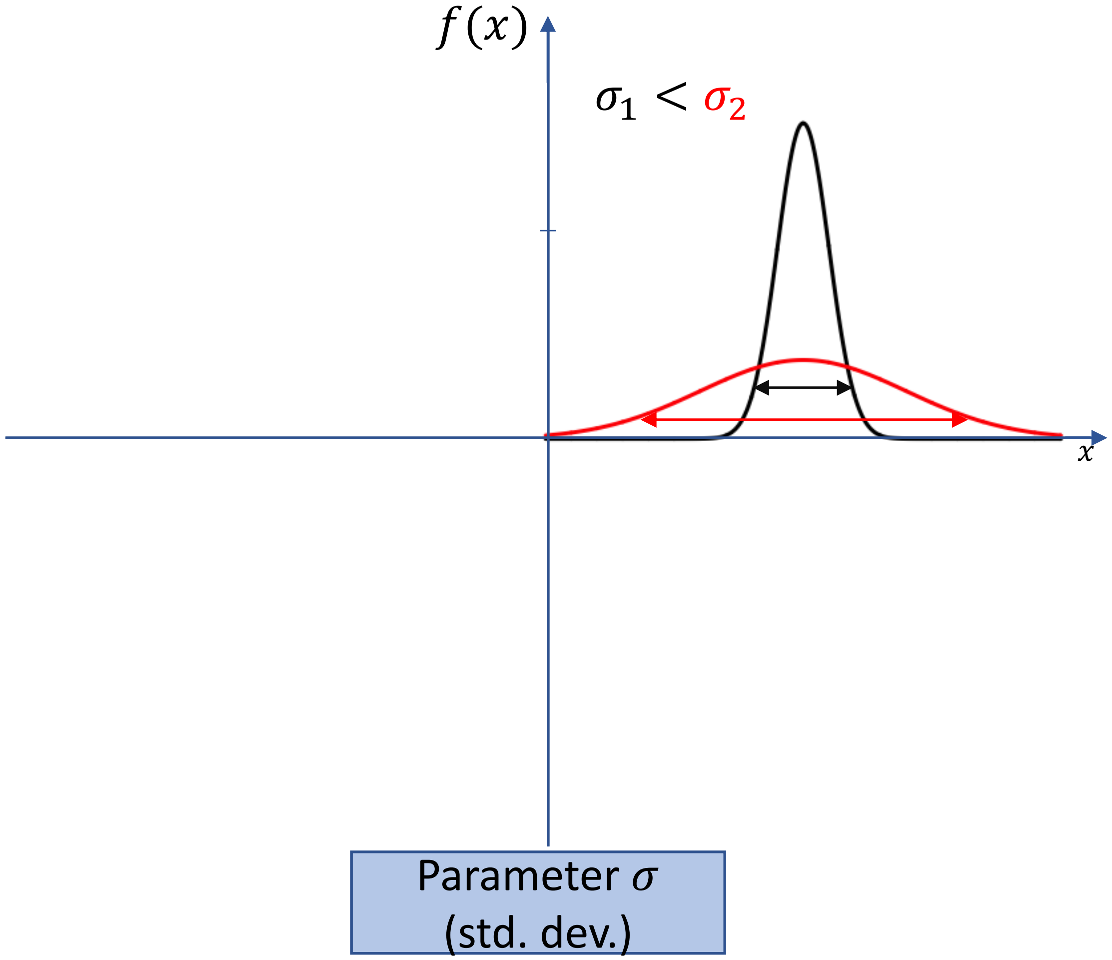
- Unimodal and bell-shaped;

- Symmetric around the mean, $\mu$;

- The standard deviation $\sigma$, controls the spread of the curve (wider or narrower);

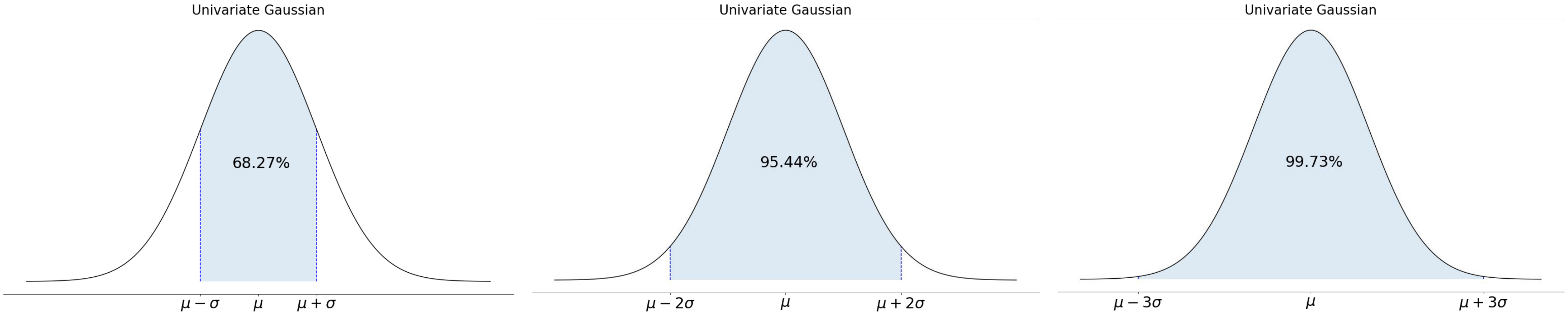
# Gaussian distribution



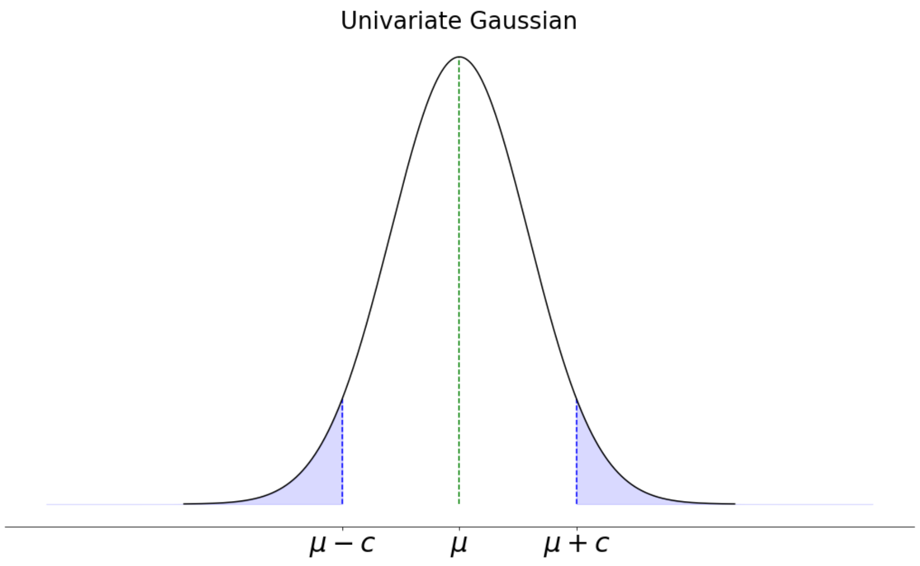
# Gaussian distribution

- Regardless of the values of $\mu$ and $\sigma$ we have that:



# Gaussian distribution

- Let $X \sim N(\mu, \sigma)$ be a random variable. Then,
  - $E[X] = \mu$
  - $Var(X) = \sigma^2$
  - $X$ is symmetric around the mean, which means:
  $$P(X \geq \mu + c) = P(X \leq \mu - c)$$
  for any constant c.




- Also, $\frac{X-\mu}{\sigma} \sim N(0,1)$. The $N(0,1)$ is known as *Standard Normal*.

# Gaussian distribution

- Unfortunately, the CDF of the Gaussian distribution does not have a closed-form.

- We need to use software packages to get the desired probability or quantiles.

**R:**

```
### Probabilities: e.g., P(X ≤ 3)
pnorm(3, μ, σ)

### Quantiles: e.g., P(X ≤ x) = 0.95
qnorm(0.95, μ, σ)
```

# Central Limit Theorem

# Central Limit Theorem

- Let $x_1, x_2, \dots, x_n$ be a random sample from a population.

- The CLT states that for large sample sizes (large n) the sampling distribution of the sample mean (or sample proportion) will converge to the Normal distribution.

- $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ $\qquad$ $\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$

# Central Limit Theorem: Assumptions

- The central limit theorem makes the following assumptions:
  - The sample is drawn in an independent fashion.

  - In general, if your sample size is greater than 10% of the population size, there will be a severe violation of independence.

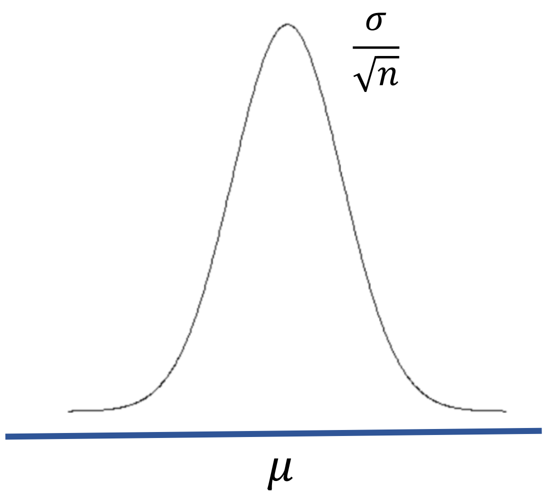
  - The sample size must be large enough.
    - For the proportion, you can check if $n \times p \geq 10$ and $n \times (1-p) \geq 10$.
    - For the sample mean, there is no universal guideline, and we might need a large sample size. Usually, however, sample sizes between 30 and 50 are enough to get a reasonable approximation (but it is not guaranteed).

# Confidence Intervals based on CLT

# Confidence intervals based on CLT: Mean

- Let $x_1, x_2, \dots, x_n$ be a random sample from a population with mean $\mu$ and standard deviation $\sigma$.

- Assuming the CLT conditions are satisfied, we have that:




$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

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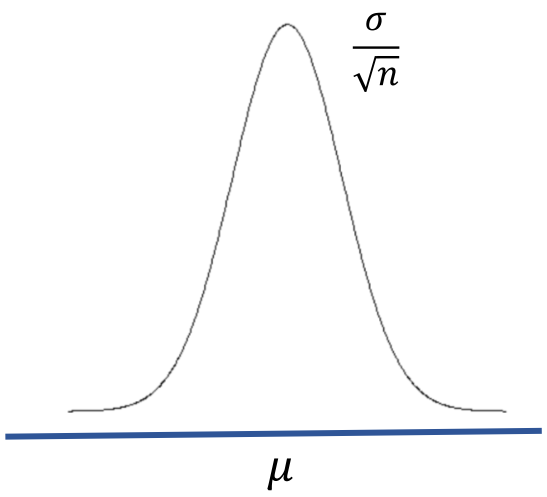



$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

Therefore,

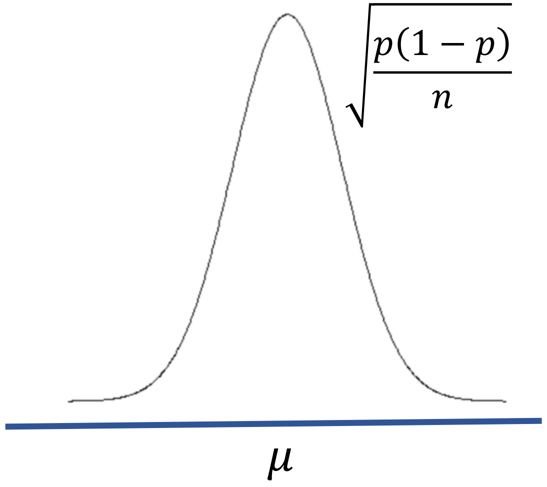
$$IC(\mu, \alpha) = \bar{x} \pm z^*_{1-\alpha} \times \frac{s}{\sqrt{n}}$$

**Note:** we could actually get a better approximation using $t$-distribution that you are going learn next week.
However, for large $n$ the Normal and $t$-distributions are quite close. In fact, for $n \geq 50$, both distributions are essentially the same.

# Confidence intervals based on CLT: Proportion

- Let $x_1, x_2, \dots, x_n$ be a random sample from a population with proportion $p$.

- Assuming the CLT conditions are satisfied, we have that:




$$\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

# Confidence intervals based on CLT: Proportion

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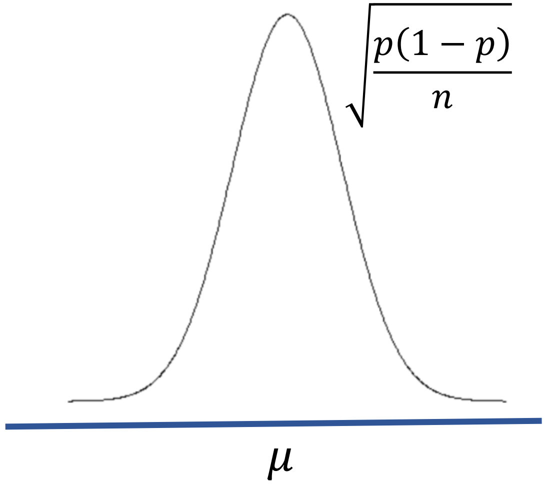


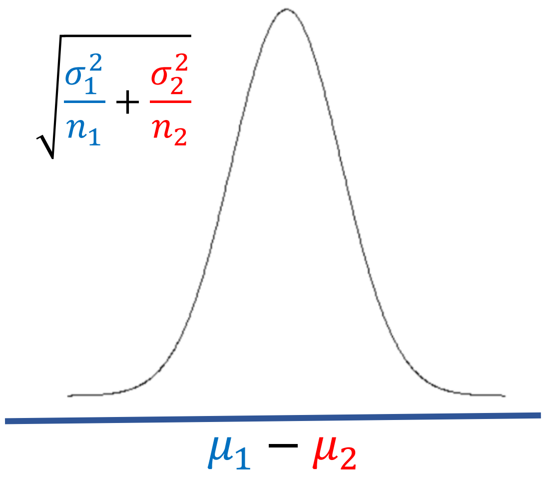

$$\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

Therefore,

$$IC(p, \alpha) = \hat{p} \pm z^*_{1-\alpha} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

# Confidence intervals based on CLT: Difference in means

- Let $x_1, x_2, \dots, x_n$ be a random sample from a population with mean $\mu_1$ and standard deviation $\sigma_1$.
- Let $y_1, y_2, \dots, y_n$ be a random sample from a population with mean $\mu_2$ and standard deviation $\sigma_2$.
- Assuming the CLT conditions are satisfied, we have that:




$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$$

# Confidence intervals based on CLT: Difference in means

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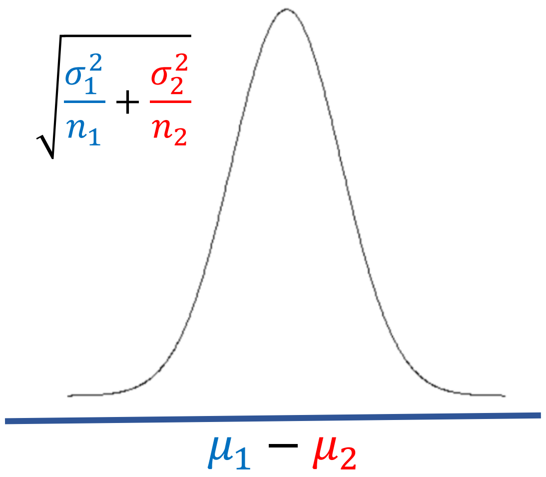



$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$$

Therefore,

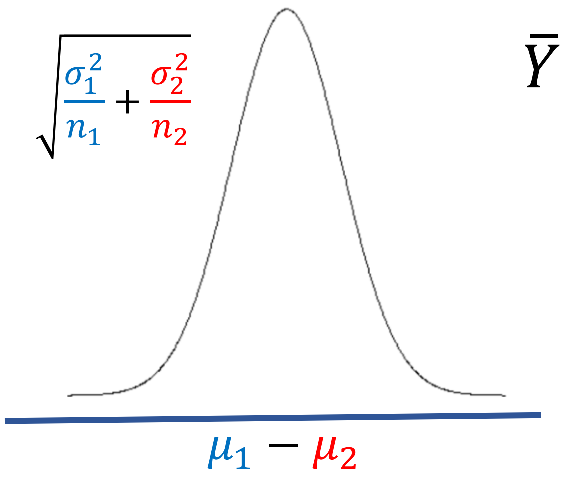
$$IC(\mu, \alpha) = \bar{x} - \bar{y} \pm z_{1-\alpha}^{*} \times \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

<u>**Note:**</u> we could actually get a better approximation using $t$-distribution that you are going learn next week.
However, for large $n$ the Normal and $t$-distributions are quite close. In fact, for $n \geq 50$, both distributions are essentially the same.

# Confidence intervals based on CLT: Difference in proportions

- Let $x_1, x_2, \ldots, x_n$ be a random sample from a population with proportion $p_1$.
- Let $y_1, y_2, \ldots, y_n$ be a random sample from a population with proportion $p_2$.

- Assuming the CLT conditions are satisfied, we have that:




$$\bar{Y} \sim N\left(p_1 - p_2, \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right)$$

# Confidence intervals based on CLT: Difference in proportions

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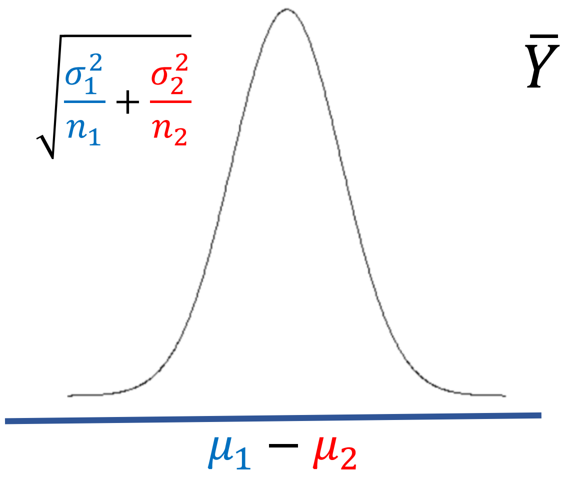


$$\bar{Y} \sim N\left(p_1 - p_2, \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right)$$

Therefore,

$$IC(\mu, \alpha) = \hat{p}_1 - \hat{p}_2 \pm z^*_{1-\alpha} \times \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$