Projet de Monte-Carlo & Simulation

Volatilité stochastique

Les modèles de volatilité en finance

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

Introduction du sujet

Y_t est le log-rendement d'un actif, de loi $N(0, \exp(X_t))$

$$X_t - \mu = \phi(X_{t-1} - \mu) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$$

Gibbs sampler

The Gibbs sampling algorithm for Bayesian statistics

The basic Gibbs sampling algorithm is:

1. Choose starting values for all of your components, i.e., $\theta^{(0)} = (\theta_1^{(0)}, \theta_2^{(0)}, \dots, \theta_d^{(0)})$.
2. Set $t = 1$.
3. Draw $\theta_j^{(t)}$ from the full conditional distribution $p(\theta_j | \theta_{-j}^{(t-1)}, y)$ for $j = 1, 2, \dots, d$.
4. Increment t.

loi à simuler :

Loi de μ :

Without prior

$$\pi_{\mu|X_{[0,T]},Y_{[0,T]},\sigma,\phi,X_{-1}}(.\,|x_{[0,T]},y_{[0,T]},\sigma,\phi,x_{-1}) \sim \mathcal{N}(C,D) \text{ with } \begin{cases} D = \frac{\sigma^2}{(T+1)(\phi-1)^2} \\ C = D\frac{(\phi-1)}{\sigma^2}\sum_{t=0}^{T}(\phi x_{t-1} - x_t) \end{cases}$$

Loi de σ :

Prior distribution : $\pi_0(\sigma^2) \sim \mathcal{IG}(\frac{\sigma_1}{2},\frac{S_1}{2})$

$$\pi_{\sigma^2|X_{[0,T]},Y_{[0,T]},\mu,\phi,X_{-1}}(.\,|x_{[0,T]},y_{[0,T]},\mu,\phi,x_{-1}) \sim \mathcal{IG}(A,B) \text{ with } \begin{cases} A = \frac{T+1+\sigma_1}{2} \\ B = \frac{\sum_{t=0}^{T}((x_t-\mu)-\phi(x_{t-1}-\mu))^2}{2} + \frac{S_1}{2} \end{cases}$$

Loi de ϕ :

Prior distribution : $\pi_0(\phi) \sim \mathcal{N}_{]-1;1[}(\mu_\phi,\sigma_\phi)$

$$\pi_{\phi|X_{[0,T]},Y_{[0,T]},\mu,\sigma,X_{-1}}(.\,|x_{[0,T]},y_{[0,T]},\mu,\sigma,x_{-1}) \sim \mathcal{N}_{]-1;1[}(E,F) \text{ with } \begin{cases} F = (\sum_{t=0}^{T}\frac{(x_{t-1}-\mu)^2}{\sigma^2} + \frac{1}{\sigma_\phi^2})^{-1} \\ E = F.[\sum_{t=0}^{T}\frac{(x_t-\mu)(x_{t-1}-\mu)}{\sigma^2} + \frac{\mu_\phi}{\sigma_\phi^2}] \end{cases}$$

Loi de X_t :

$\forall t \in [1, T-1], \pi_{X_t|X_{-t},Y_{[0,T]},\mu,\sigma,X_{-1},\phi}(k|x_{-t}, y_{[0,T]}, \mu, \sigma, x_{-1}, \phi) \propto \exp$

$(\frac{-1}{2\sigma^2}\{(k-mu)^2(1+\phi^2) - 2\phi(k-\mu)(x_{t-1}+x_{t+1}-2\mu)\} - \frac{y_t^2}{2e^k} - \frac{x}{2})$

Ce n'est pas une loi connue. Nous nous sommes inspirés des priors de l'article PARTICLE GIBBS METHODS IN STOCHASTIC VOLATILITY MODELS by Chen Gong, puis nous les avons adaptés à l'exercice.

L'algorithme de Gibbs :

$\mu^0 = 0 \text{ or } \mu^0 \sim \mathcal{N}(0, 10)$

$\phi^0 \sim \mathcal{N}(\mu_\phi, \sigma_\phi)$

$\sigma^0 \sim \mathcal{IG}(\frac{\sigma_1}{2}, \frac{S_1}{2})$

$$X^0_{[-1,T]} = \begin{bmatrix} \mu^0 \\ \mu^0 \\ \vdots \\ \mu^0 \end{bmatrix}$$

où $(\frac{\sigma_1}{2}, \frac{S_1}{2})$ et (μ_ϕ, σ_ϕ) sont des hyperparamètres trouvés dans l'article cité précedemment.

$\forall n \in [1, N]$:

$X^n_{-1} = \mu^{n-1}$

$X^n_{[0,T]} \sim \text{Metropolis}(\mu^{n-1}, \phi^{n-1}, \sigma^{n-1}, X^{n-1}_{[0,T]}, Y_{[0,T]}, X^n_{-1})$

$(\sigma^n)^2 \sim \pi_{\sigma^2|X_{[0,T]},Y_{[0,T]},\mu,\phi,X_{-1}}(.\,|X^n_{[0,T]}, Y_{[0,T]}, \mu^{n-1}, \phi^{n-1}, X^n_{-1})$

$\sigma^n = \sqrt{(\sigma^n)^2}$

$\phi^n \sim \pi_{\phi|X_{[0,T]},Y_{[0,T]},\mu,\sigma,X_{-1}}(.\,|X^n_{[0,T]}, Y_{[0,T]}, \mu^{n-1}, \sigma^n, X^n_{-1})$

$\mu^n \sim \pi_{\mu|X_{[0,T]},Y_{[0,T]},\sigma,\phi,X_{-1}}(.\,|X^n_{[0,T]}, Y_{[0,T]}, \sigma^n, \phi^n, X^n_{-1})$

Simulation de Xt

L'algorithme de Metropolis :

La loi invariante de X_t n'étant pas connue, on va utiliser un algo de Metropolis pour la simuler :

Pour cela on se sert du fait que : $X_t | X_{-t}, \phi, \mu, \sigma \sim \mathcal{N}(\mu + \phi(X_{t-1} - \mu), (\sigma)^2)$ indépdendant de la simulation précédente de X_t

Notons $f_{X_t | X_{t-1}, \phi, \mu, \sigma}(. \,| X_{t-1}, \phi, \mu, \sigma)$ sa densité.

$\forall n \in [1, N]$, Metropolis$(\mu^{n-1}, \phi^{n-1}, \sigma^{n-1}, X^{n-1}_{[0,T]}, Y_{[0,T]}, X^{n}_{-1})$

$\forall k \in nbsimu$:

$xsimu = X_t^{n-1}$

$\forall t \in [1, T-1]$:

$$X_{new} \sim \mathcal{N}(\mu^{n-1} + \phi^{n-1}(X^n_{t-1} - \mu^{n-1}), (\sigma^{n-1})^2)$$

$$U \sim \mathcal{U}(0,1)$$

$$r = \frac{\pi_{X_t | X_{-t}, Y_{[0,T]}, \mu, \sigma, X_{-1}, \phi}(X_{new} | (X^n_{[0,t-1]}, X^{n-1}_{[t+1,T]}), Y_{[0,T]}, \mu^{n-1}, \sigma^{n-1}, x^n_{-1}, \phi^{n-1})}{\pi_{Xsimu | X_{-t}, Y_{[0,T]}, \mu, \sigma, X_{-1}, \phi}(Xsimu | (X^n_{[0,t-1]}, X^{n-1}_{[t+1,T]}), Y_{[0,T]}, \mu^{n-1}, \sigma^{n-1}, x^n_{-1}, \phi^{n-1})} \cdot \frac{f_{X_t | X_{t-1}, \phi, \mu, \sigma}(Xsimu | X^n_{t-1}, \phi^{n-1}, \mu^{n-1}, \sigma^{n-1})}{f_{X_t | X_{t-1}, \phi, \mu, \sigma}(X_{new} | X^n_{t-1}, \phi^{n-1}, \mu^{n-1}, \sigma^{n-1})}$$

$Si\ U < min(1, r)$:

$$xsimu = X_{new}$$

$Sinon$:

$$xsimu = xsimu$$

$X_t^n = xsimu$

Simulation de Xt

L'algorithme d'acceptation rejet

Un autre moyen de simuler Xt est d'utiliser un algorithme d'acceptation rejet. D'après Kim & Al. 1998

On utilise le fait que

$\forall t \in [1, T-1], \pi_{X_t|X_{-t},Y_{[0,T]},\mu,\sigma,X_{-1}}(k|x_{-t}, y_{[0,T]}, \mu, \sigma, x_{-1}) \propto \exp\left(\frac{-1}{2\sigma^2}\{(k-mu)^2(1+\phi^2) - 2\phi(k-\mu)(x_{t-1}+x_{t+1}-2\mu)\}\right)\exp\left(-\frac{y_t^2}{2e^k} - \frac{x}{2}\right)$

$\propto f_{X_t|X_{-t},\mu,\sigma,X_{-1}}(k|x_{-t}, \mu, \sigma, x_{-1}))f_{Y_t|X_{-t},X_t,\mu,\sigma,X_{-1}}(y_t|x_{-t}, k, y_{[0,T]}, \mu, \sigma, x_{-1}))$

On va donc supposer pouvoir tirer $X_t \sim f_{X_t|X_{-t},\mu,\sigma,X_{-1},\phi}(k|x_{-t}, \mu, \sigma, x_{-1}, \phi))$ et utiliser l'acceptation rejet.

Or $f_{X_t|X_{-t},\mu,\sigma,X_{-1},\phi}(k|x_{-t}, \mu, \sigma, x_{-1}, \phi)) \propto \exp(-\frac{(k-G)^2}{2\eta^2})$ avec $\begin{cases} \eta = \sqrt{\frac{\sigma^2}{1+\phi^2}} \\ G_t = \mu + \frac{\phi(x_{t-1}+x_{t+1}-2\mu)}{1+\phi^2} \end{cases}$.

On va commencer par chercher une densité qui domine $\pi_{X_t|X_{-t},Y_{[0,T]},\mu,\sigma,X_{-1}}(k|x_{-t}, y_{[0,T]}, \mu, \sigma, x_{-1})$

On à $\log(f_{Y_t|X_{-t},X_t,\mu,\sigma,X_{-1}}(y_t|x_{-t}, k, y_{[0,T]}, \mu, \sigma, x_{-1})) = -\frac{k}{2} - \frac{y_t^2}{2\exp(k)} \leq -\frac{k}{2} - \frac{y_t^2}{2}(\exp(-G_t)(1+G_t) - k\exp(-G_t)) = \log(g(y_t, k, G_t, \mu, \phi, \sigma))$

donc on a va faire l'algorithme suivant :

$\forall t \in [1, T-1]$:

$X_t^n = 0$

tant que$X_t^n = 0$ et $i < iter$:

$X_{new} \sim \mathcal{N}(G_t + \frac{\eta^2}{2}(y_t^2\exp(-G_t) - 1), \eta^2)$

$U \sim \mathcal{U}(0,1)$

Si $U < \frac{f_{Y_t|X_{-t},X_t,\mu,\sigma,X_{-1}}(y_t|x_{-t}, X_{new}, y_{[0,T]}, \mu, \sigma, x_{-1})}{g(y_t, X_{new}, G_t, \mu, \phi, \sigma)}$:

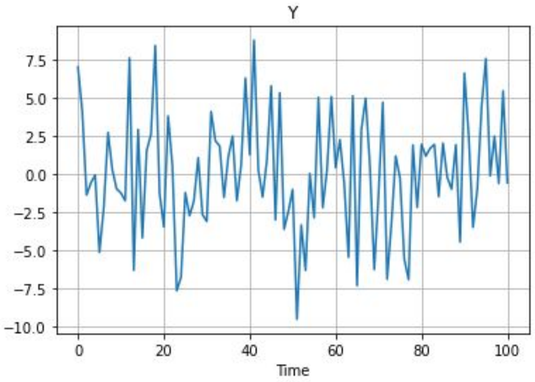
$X_t^n = X_{new}$

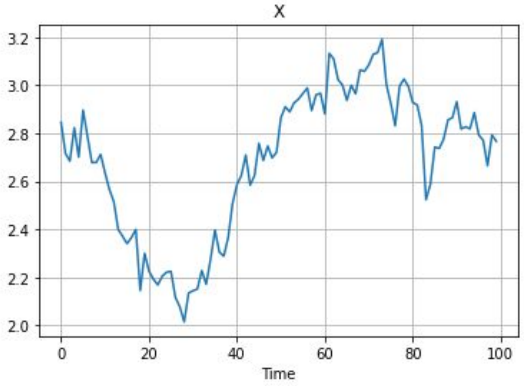
$i = i + 1$

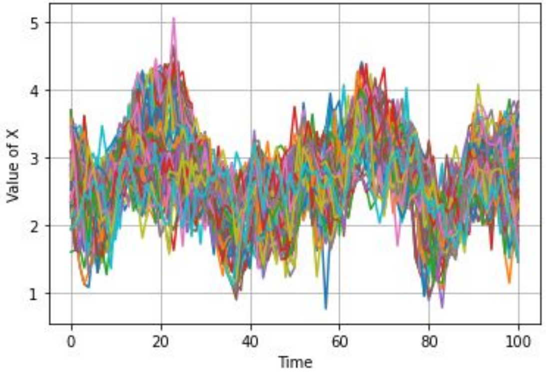
si $X_t^n = 0$:

$X_t^n = X_t^{n-1}$

Résultats avec Metropolis

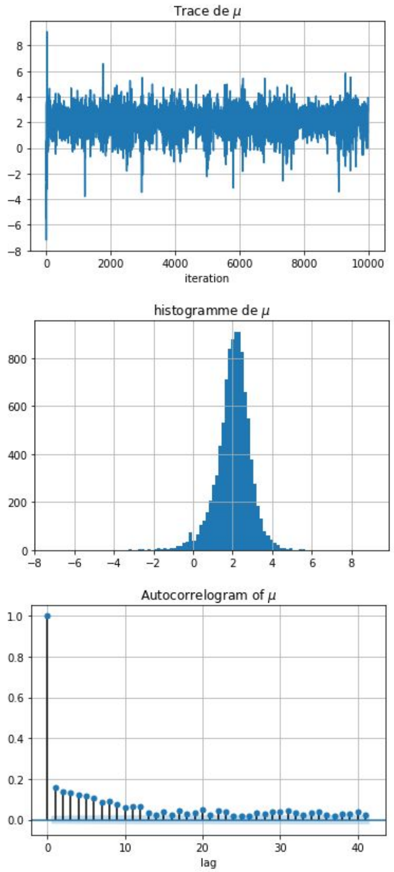




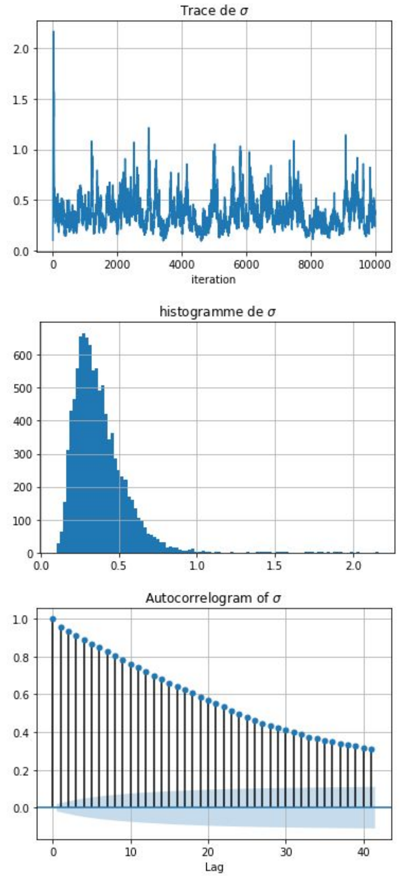


Simulé avec $\mu = 3$, $\sigma = 0.1$, $\varphi = 0.95$

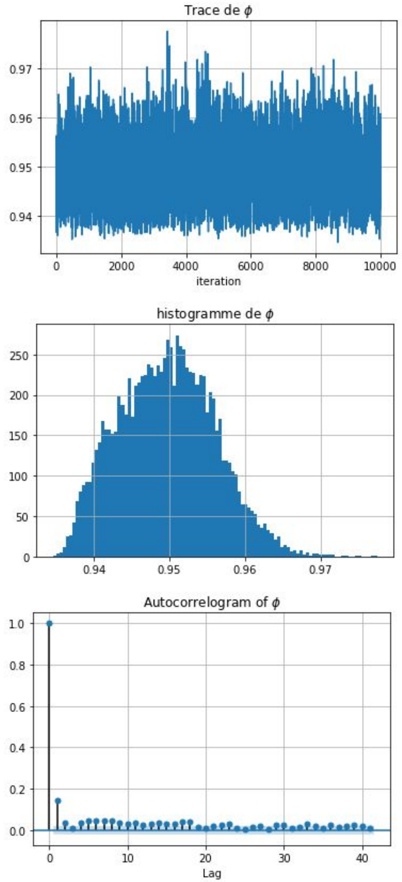
mean : 2.037
std : 0.879



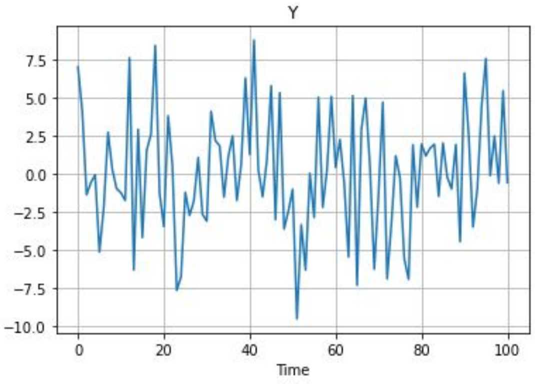
mean : 0.371
std : 0.165

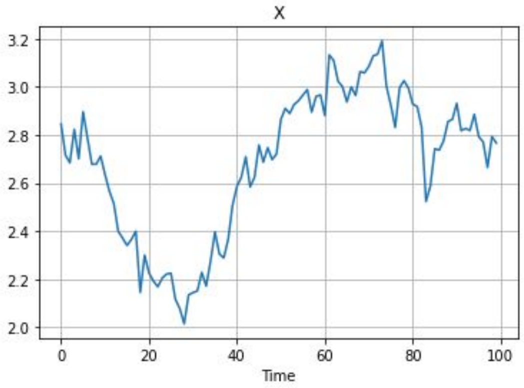


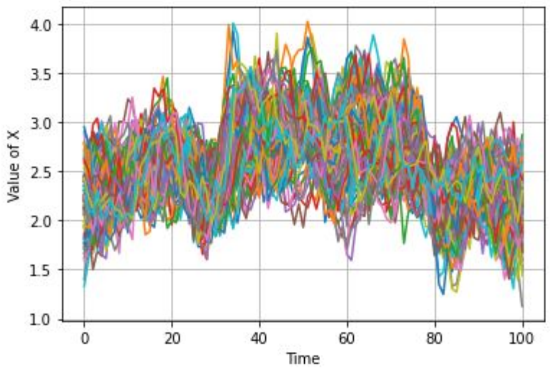
mean: 0.950
std : 0.006



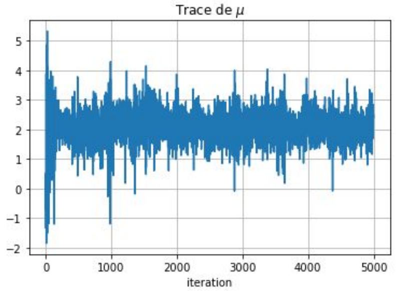
Résultats avec acceptation-rejet

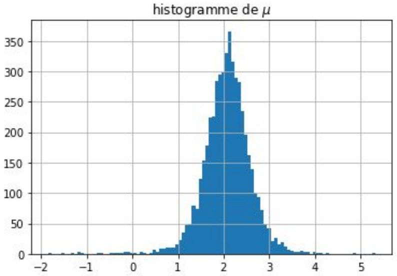


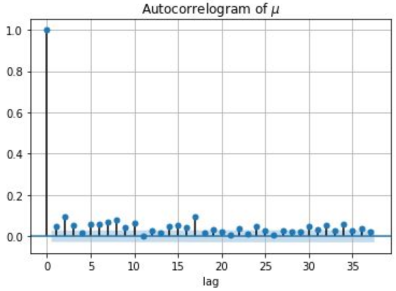




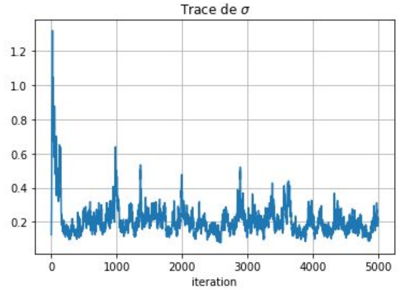
mean : 2.076
std : 0.515

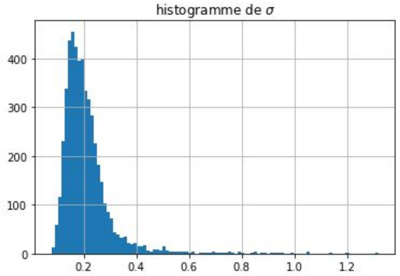


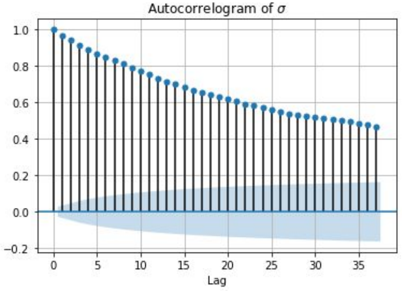




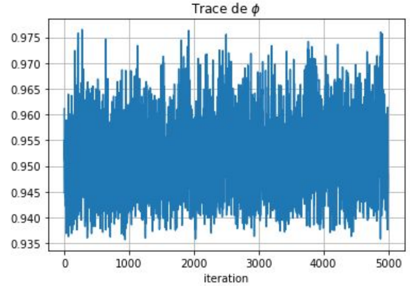
mean : 0.209
std : 0.097

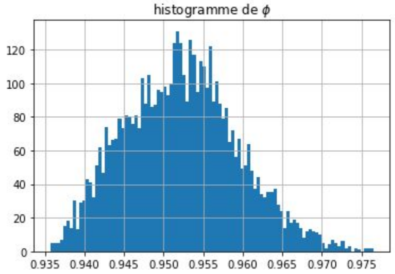


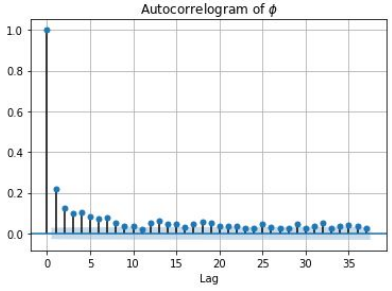




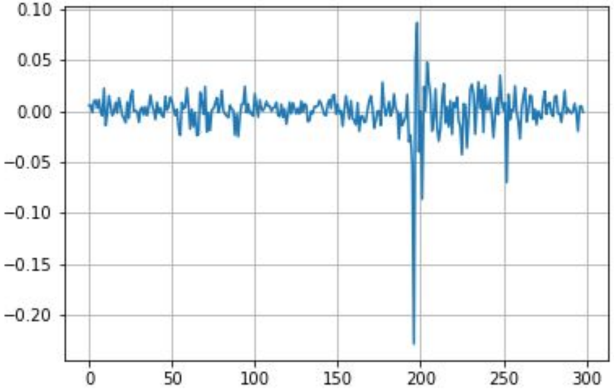
mean : 0.952
std : 0.007

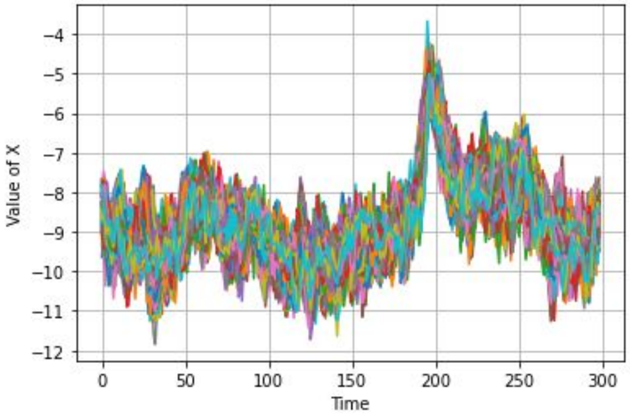




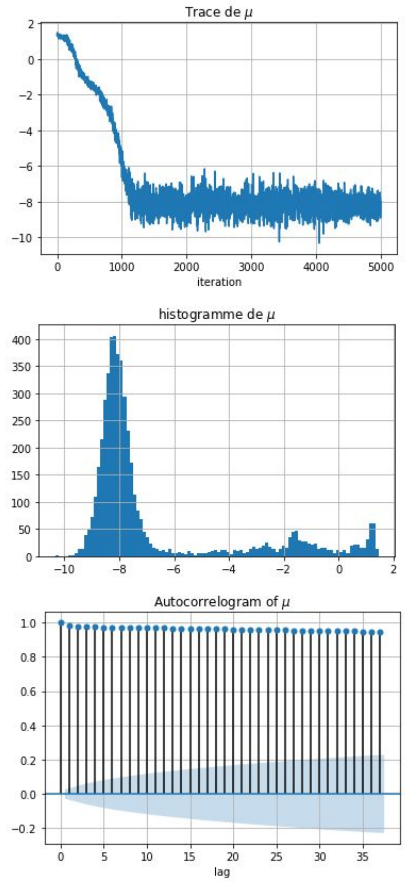


Test avec les données du S&P500

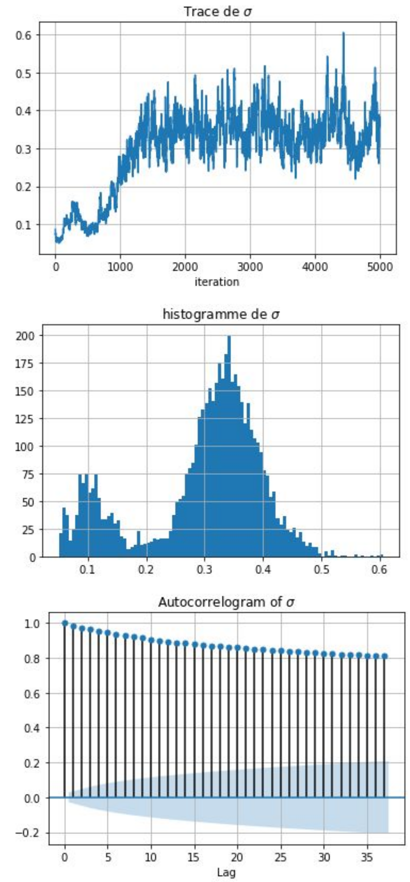




mean : -6.754
std : 2.867



mean : 0.298
std : 0.104



mean : 0.954
std : 0.008