1.

$$1.(1)\quad A^\top A=\begin{pmatrix}1&2&0\\1&2&0\end{pmatrix}\begin{pmatrix}1&1\\2&2\\0&0\end{pmatrix}=\begin{pmatrix}5&5\\5&5\end{pmatrix}$$

$\left|\lambda E-A^\top A\right|=\begin{vmatrix}\lambda-5&-5\\-5&\lambda-5\end{vmatrix}=(\lambda-5)^2-25=\lambda(\lambda-10)=0.$

$\lambda_1=10,\lambda_2=0.$

$\sigma_1=\sqrt{\lambda_1}=\sqrt{10},\sigma_2=\sqrt{\lambda_2}=0$

1.(2)　由 (1) 知 $\Sigma=\begin{pmatrix}\sqrt{10}&0\\0&0\\0&0\end{pmatrix}$

$(i)\lambda=10\ \lambda E-A^\top A=\begin{pmatrix}5&-5\\-5&5\end{pmatrix}\sim\begin{pmatrix}1&-1\\0&0\end{pmatrix}$

$y_1=\begin{pmatrix}1\\1\end{pmatrix}$

$(ii)\lambda=0\ ,\lambda E-A^\top A=\begin{pmatrix}-5&-5\\-5&-5\end{pmatrix}\sim\begin{pmatrix}1&1\\0&0\end{pmatrix}$

$y_2=\begin{pmatrix}-1\\1\end{pmatrix}$

$v_1=\frac{1}{\|y_1\|_2}y_1=\begin{pmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}$

$v_2=\frac{1}{\|y_2\|_2}y_2=\begin{pmatrix}\frac{-1}{\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}$

$V=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}.$

当 $\sigma\neq0$ 时即 $\sigma_1=\sqrt{10},\quad u_1=\frac{1}{\sigma_1}Av_1=\frac{1}{\sqrt{10}}\begin{pmatrix}1&1\\2&2\\0&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt5}\\\frac{2}{\sqrt5}\\0\end{pmatrix}$

u_2, u_3 为 Nul(A) 的一组标准正交基.

$$A^\top y = 0 \quad \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0. \quad A^\top \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$y_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, y_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad u_2 = \frac{1}{||y_2||_2} y_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}; u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U = UV^\top = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^\top$$

2.

$$\begin{aligned}
& Cov(x,x) = Var(x) = \tfrac{1}{2}\left[\left(\tfrac{1}{3}\right)^2 + \left(\tfrac{7}{3}\right)^2 + \left(\tfrac{8}{3}\right)^2\right] = \tfrac{57}{9} \\
& Cov(y,y) = Var(y) = \tfrac{1}{2}\left[0^2 + 1^2 + 1^2\right] = 1 \\
& Cov(z,z) = Var(z) = \tfrac{1}{2}\left[0^2 + 2^2 + 2^2\right] = 4 \\
& Cov(x,y) = Cov(y,x) = \tfrac{1}{2}\left[\tfrac{-1}{3} \times 0 + \tfrac{7}{3} \times 1 + \tfrac{8}{3} \times 1\right] = \tfrac{5}{2} \\
(1) \quad & Cov(x,z) = Cov(z,x) = \tfrac{1}{2}\left[\tfrac{-1}{3} \times 0 + \tfrac{-7}{3} \times 2 - \tfrac{8}{3} \times 2\right] = -5 \\
& Cov(y,z) = Cov(y,z) = \tfrac{1}{2}[0 - 1 \times 2 - 1 \times 2] = -2 \\
& \Sigma = \begin{pmatrix} \frac{57}{9} & \frac{5}{2} & -5 \\ \frac{5}{2} & 1 & -2 \\ -5 & -2 & 4 \end{pmatrix}
\end{aligned}$$

(2)$P^{-1}\Sigma P = P^\top \Sigma P = \Lambda$

$$P^\top = \begin{bmatrix} 0 & -0.664 & -0.748 \\ 0.894 & 0.334 & -0.297 \\ 0.447 & -0.669 & 0.594 \end{bmatrix}$$

$\Lambda = \begin{bmatrix} -0 & 0 & 0 \\ 0 & 0.037 & 0 \\ 0 & 0 & 11.296 \end{bmatrix}$ 取 k=1, 则第一主成分对应的特征向量为 [-0.748,-0.297, 0.594]

7.

A: $\lambda_1 = 0 \neq \lambda_2 = -2$, 因此 A 可对角化

B: $\lambda_1 = \lambda_2 = 2, \lambda_3 = 1$

$\dim(Nul(2I - B)) = 2, \dim(Nul(I - B)) = 1$
$\dim(Nul(2I - B)) + \dim(Nul(I - B)) = 3$ 因此 B 可对角化。