DaSE算法 第二次作业

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1. $X^* = \frac{X-\mu}{\sigma}$ $E(X^*) = E(\frac{X-\mu}{\sigma}) = 0$

$Var(X^*) = \frac{Var(X)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1$

$\because$ X^* 的期望方差都存在

$\therefore$ 根据切比雪夫不等式

$P(|X^* - 0| \geq c) \leq \frac{Var(X^*)}{c^2}$

$\therefore$ $P(|X^*| \geq c) \leq \frac{1}{c^2}$，得证

2. $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $E(X_i) = \mu$ $Var(X_i) = \sigma^2 = E[(X_i-\mu)^2]$

$\because$ $X_1, X_2 \cdots X_n$ 独立同分布

$\therefore$ $E(\bar{x}) = \frac{1}{n} \cdot n \cdot E(X_i) = \mu$

$Var(\bar{x}) = \frac{1}{n^2} \cdot n \cdot Var(X_i) = \frac{\sigma^2}{n}$

$\because$ $\bar{x}$ 的期望、方差都存在

$\therefore$ 根据切比雪夫不等式

$P(|\bar{x} - E(\bar{x})| \geq \varepsilon) \leq \frac{Var(\bar{x})}{\varepsilon^2}$

即 $P(|\bar{x} - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$

3. $X \sim b(n, \frac{1}{2})$ $\mu = E(X) = np = \frac{n}{2}$ $Var(X) = npq = n \times \frac{1}{4} = \frac{n}{4}$

a. $p(X < \frac{n}{4}) < P(|X - \frac{n}{2}| > \frac{n}{4}) \leq \frac{\frac{n}{4}}{\frac{n^2}{16}} = \frac{n}{4} \cdot \frac{16}{n^2} = \frac{4}{n}$

b. chernoff $P(X < (1-\delta)\mu) < \exp(-\frac{\mu\delta^2}{2})$

$X < \frac{n}{4} = \frac{1}{2}\mu = (1-\delta)\mu$ $\delta = \frac{1}{2}$ $\frac{\frac{n}{2}\cdot\frac{1}{4}}{2}$ $\frac{\frac{n}{8}}{2}$

$\therefore$ $p(X < (1-\frac{1}{2})\mu) < \exp(-\frac{\mu \cdot \frac{1}{4}}{2})$

$\therefore$ $p(X < \frac{n}{4}) < \exp(-\frac{n}{16})$

4、(1) 对任意 $t>0$，我们有

$$X>(1+\delta)\mu \Rightarrow tX>t(1+\delta)\mu \Rightarrow e^{tX}>e^{t(1+\delta)\mu}$$

$$\therefore P(X>(1+\delta)\mu)=P(\underbrace{e^{tX}}_{\text{非负}}>\underbrace{e^{t(1+\delta)\mu}}_{>0})$$

$\therefore$ 满足 Markov 不等式使用条件

$$\therefore P(e^{tX}>e^{t(1+\delta)\mu})\le\frac{E(e^{tX})}{e^{t(1+\delta)\mu}}=\frac{E(\exp(tX))}{e^{t(1+\delta)\mu}}$$

$$X=\sum_{i=1}^{n}X_i$$

$$\therefore E(\exp(tX))=E(\prod_{i=1}^{n}e^{tX_i})=\prod_{i=1}^{n}E(e^{tX_i})$$

$$E(\exp(tX_i))=p_ie^t+(1-p_i)=p_i(e^t-1)+1=1-p_i(1-e^t)$$

$$\because 1-x<e^{-x}$$

$$\therefore 1-p_i(1-e^t)<e^{-p_i(1-e^t)}=\exp(p_i(e^t-1))$$

$$\therefore E(\exp(tX_i))=1-p_i(1-e^t)<\exp(p_i(e^t-1))$$

$$\therefore \prod_{i=1}^{n}E(\exp(tX_i))<\prod_{i=1}^{n}\exp(p_i(e^t-1))$$

$$\Downarrow \quad \because \mu=\sum_{i=1}^{n}p_i$$

$$\exp(\mu(e^t-1))$$

$$\therefore P(X>(1+\delta)\mu)\le\frac{\exp(\mu(e^t-1))}{e^{t(1+\delta)\mu}}=\exp(\mu(e^t-1-t-t\delta))$$

选择 t 值得到更紧的上界 $\quad \frac{d(e^t-1-t-t\delta)}{dt}=e^t-1-\delta=0$

$$\therefore e^t=1+\delta \quad \underline{t=\ln(1+\delta)}$$

代入

$$\therefore P(X>(1+\delta)\mu)<\exp(\mu(\cancel{1}+\delta-\cancel{1}-\ln(1+\delta)-\delta\ln(1+\delta)))$$

$$=\exp(\mu(\delta-(1+\delta)\ln(1+\delta)))$$

$$=\left(\frac{e^\delta}{e^{(1+\delta)\ln(1+\delta)}}\right)^\mu=\left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu \text{ 得证}$$

(2)、由第1问结论得

$$P(X>(1+\delta)\mu) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

$$\Downarrow$$

$$e^{\ln\left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}}$$

$$= e^{\mu[\ln(e^{\delta})-\ln(1+\delta)^{(1+\delta)}]}$$

$$= e^{\mu(\delta-(1+\delta)\ln(1+\delta))} = \exp(\mu(\delta-(1+\delta)\ln(1+\delta)))$$

构造函数 $F(\delta)=\ln(1+\delta)-\frac{\delta}{1+\frac{\delta}{2}} = \ln(1+\delta)-\frac{2\delta}{2+\delta}$

$F'(\delta)=\frac{1}{1+\delta}-2\left(\frac{2+\cancel{\delta}-\cancel{\delta}}{(2+\delta)^2}\right)=\frac{1}{1+\delta}-\frac{4}{(2+\delta)^2}=\frac{\cancel{4}+\cancel{4\delta}+\delta^2-\cancel{4}-\cancel{4\delta}}{(1+\delta)(2+\delta)^2}>0$，递增且 $F(0)=0$

$\therefore$ $\delta\in(0,1)$ 时

$F(\delta)$ 恒 $>0 \Rightarrow \ln(1+\delta)>\frac{2\delta}{2+\delta}$ $\therefore \delta-(1+\delta)\ln(1+\delta)<\delta-\frac{2\delta+2\delta^2}{2+\delta}=\frac{\cancel{2\delta}+\delta^2-\cancel{2\delta}-2\delta^2}{2+\delta}=\frac{-\delta^2}{2+\delta}$

$\therefore P(X>(1+\delta)\mu)<\exp\left(-\frac{\mu\delta^2}{2+\delta}\right)$ $\quad\because \delta\in(0,1)$

$\therefore P(X>(1+\delta)\mu)<\exp\left(-\frac{\mu\delta^2}{3}\right)$ 得证

运用 chernoff 不等式的下界和上界公式

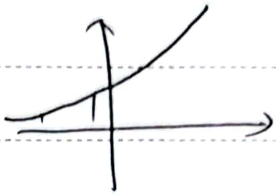
5. $P(|X-\mu|>\delta\mu)$

$=P(X>\mu+\delta\mu)+P(X<\mu-\delta\mu)$

$=P(X>\mu(1+\delta))+P(X<\mu(1-\delta))$

$\Downarrow$

$P(X>\mu(1+\delta))<\exp(-\frac{\mu\delta^2}{3})$



$P(X<\mu(1-\delta))<\exp(-\frac{\mu\delta^2}{2})<\exp(-\frac{\mu\delta^2}{3})$

$\therefore P(X>\mu(1+\delta))+P(X<\mu(1-\delta))<\exp(-\frac{\mu\delta^2}{3})+\exp(-\frac{\mu\delta^2}{2})$

$<2\exp(-\frac{\mu\delta^2}{3})$

$\therefore$ $P(|X-\mu|>\delta\mu)<2\exp(-\frac{\mu\delta^2}{3})$，得证

6. 要使 $P(|\bar{x}-p|\le\epsilon p)\ge 1-\delta$

即 $-P(|\bar{x}-p|\le\epsilon p)\le\delta-1$

$1-P(|\bar{x}-p|\le\epsilon p)\le\delta$

$P(|\bar{x}-p|\ge\epsilon p)\le\delta$

$P(\bar{x}\ge p+\epsilon p)+P(\bar{x}\le p-\epsilon p)\le\delta$ $\quad\delta\in(0,1)$

$\because$ $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}X_i$ $\quad$ 令 $X=\sum_{i=1}^{n}X_i=n\bar{x}$

$\therefore$ $P(X\ge p(1+\varepsilon)n)+P(X\le p(1-\varepsilon)n)\le\delta$

$\because$ X_i $i=1,\dots,n$ 独立同分布于 Bernoulli(p)

$\therefore np=E(X)=E(\sum_{i=1}^{n}X_i)$ $\quad\therefore$ 上式分解为两个chernoff不等式

$\therefore P(X\ge p(1+\varepsilon)n)+P(X\le p(1-\varepsilon)n)<2\exp(-\frac{np\varepsilon^2}{3})$ (第5题结论)

$\exp(\frac{-np\varepsilon^2}{3})\le\frac{\delta}{2}$

$\frac{-np\varepsilon^2}{3}\le\ln(\frac{\delta}{2})$

$np\varepsilon^2\ge-3\ln(\frac{\delta}{2})$

$n\ge\frac{-3\ln(\frac{\delta}{2})}{p\varepsilon^2}$