习题 9　10215501435　杨茜雅　DaSE 算法

一、

(1). $AA^T = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$|AA^T - \lambda I| = \begin{vmatrix} 2-\lambda & 4 & 0 \\ 4 & 8-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

$$(2-\lambda)(8-\lambda)(-\lambda) + 16\lambda = 0$$
$$(16 - 10\lambda + \lambda^2)(-\lambda) + 16\lambda = 0$$
$$\cancel{-16\lambda} + 10\lambda^2 - \lambda^3 \cancel{+16\lambda} = 0$$
$$\lambda^3 - 10\lambda^2 = 0$$
$$\lambda^2(\lambda - 10) = 0$$
$$\lambda_1 = 10 \quad \lambda_2 = \lambda_3 = 0$$

$\therefore$ 矩阵A的奇异值为 $\sqrt{10}$

(2). $AA^T = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$　$\lambda_1 = 10$, $\lambda_2 = \lambda_3 = 0$

$\begin{pmatrix} \lambda-2 & -4 & 0 \\ -4 & \lambda-8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\lambda_1 = 10$　$\begin{pmatrix} 8 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\lambda_2 = \lambda_3 = 0$　$\begin{pmatrix} -2 & -4 & 0 \\ -4 & -8 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & -2 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$　$u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ $U = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$|\lambda I - A^T A| = \begin{vmatrix} \lambda-5 & -5 \\ -5 & \lambda-5 \end{vmatrix} = 0$$

$$(\lambda-5)^2 - 25 = 0$$

$$(\lambda-5)^2 = 25$$

$$\lambda^2 - 10\lambda + \cancel{25 = 25}$$

$$\lambda_1 = 0 \quad \lambda_2 = 10$$

$\lambda_1 = 0$ $\quad \begin{pmatrix} -5 & -5 \\ -5 & -5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 10$ $\quad \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} = U \Sigma V^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2.

(1)

$$\Sigma = \begin{pmatrix} \mathrm{cov}(X,X), & \mathrm{cov}(X,Y), & \mathrm{cov}(X,Z) \\ \mathrm{cov}(Y,X), & \mathrm{cov}(Y,Y), & \mathrm{cov}(Y,Z) \\ \mathrm{cov}(Z,X), & \mathrm{cov}(Z,Y), & \mathrm{cov}(Z,Z) \end{pmatrix}$$

$$\bar{x} = \frac{4}{3} \qquad \bar{y} = 2 \qquad \bar{z} = 1$$

$$\mathrm{cov}(x,x) = \frac{(1-\frac{4}{3})^2 + (-1-\frac{4}{3})^2 + (4-\frac{4}{3})^2}{2} = \frac{19}{3}$$

$$\mathrm{cov}(X,Y) = \frac{(1-\frac{4}{3})\times\cancel{(2-2)} + (-1-\frac{4}{3})\times(1-2) + (4-\frac{4}{3})\times(3-2)}{2} = \frac{5}{2}$$

$$\mathrm{cov}(X,Z) = \frac{(1-\frac{4}{3})\times\cancel{(1-1)} + (-1-\frac{4}{3})\times(3-1) + (4-\frac{4}{3})\times(-1-1)}{2} = -5$$

$$\mathrm{cov}(Y,Y) = \frac{\cancel{(2-2)^2} + (1-2)^2 + (3-2)^2}{2} = 1$$

$$\mathrm{cov}(Z,Z) = \frac{\cancel{(1-1)^2} + (3-1)^2 + (-1-1)^2}{2} = 4$$

$$\mathrm{cov}(Z,Y) = \frac{\cancel{(2-2)\times(1-1)} + (1-2)\times(3-1) + (3-2)\times(-1-1)}{2} = -2$$

$$\Sigma = \begin{pmatrix} \frac{19}{3} & \frac{5}{2} & -5 \\ \frac{5}{2} & 1 & -2 \\ -5 & -2 & 4 \end{pmatrix}$$

7、

A: $\begin{vmatrix} 1-\lambda & -1 \\ 3 & -3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-3-\lambda)+3=0$

$(1-\lambda)(3+\lambda)-3=0$

$\cancel{3}+\lambda-3\lambda-\lambda^2=\cancel{3}$

$-\lambda^2-2\lambda=0$

$\lambda^2+2\lambda=0$

$\lambda_1=0 \quad \lambda_2=-2$

(1) $\begin{pmatrix} \lambda-1 & 1 \\ -3 & \lambda+3 \end{pmatrix}$

$\lambda_1=0$时 $\begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0 \quad \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

∴特征向量为$c_1(1,1)^T$

$\lambda_2=-2$时 $\begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}$ 特征向量为$c_2(1,3)^T$

每一个特征值的几何重数和代数重数都为1

∴ $A=\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$ 可对角化

(2) $B=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & 6 & -2 \end{pmatrix}$

$|\lambda I-B| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-5 & 2 \\ 0 & -6 & \lambda+2 \end{vmatrix} = 0$

$(\lambda-2)(\lambda-5)(\lambda+2)+2\times6\times(\lambda-2)=0$

$(\lambda^2-3\lambda-10)(\lambda-2)+12(\lambda-2)=0$

$(\lambda^2-3\lambda+2)(\lambda-2)=(\lambda-2)(\lambda-1)(\lambda-2)=0$

$\lambda_1=1 \quad \lambda_2=\lambda_3=2$

$\lambda_1=1$ $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & -6 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ 特征向量为$c_1(0,1,2)^T$

$\lambda_2=2$时 $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & -6 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ 特征向量为 $c_2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+c_3\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

每一个特征值的几何重数和代数重数都相等

∴ B矩阵可对角化