数子基础 7^{th}

1、

$$f(A) = \frac{1}{2} \|Ax+b-y\|_2^2$$
$$= \frac{1}{2}\left(\sqrt{(Ax+b-y)^T(Ax+b-y)}\right)^2$$
$$= \frac{1}{2}(x^TA^T + (b-y)^T)(Ax+b-y)$$
$$= \frac{1}{2}(x^TA^TAx + x^TA^T(b-y) + (b-y)^TAx + (b-y)^T(b-y))$$

$$\underset{1\times n}{x^T} \cdot \underset{n\times m}{A^T} \underset{m\times 1}{(b-y)} = 1\times 1$$

$$\therefore \quad x^T \cdot A^T(b-y) = (x^TA^T(b-y))^T = (b-y)^TAx$$

$$= \frac{1}{2}(x^TA^TAx + 2(b-y)^TAx + (b-y)^T(b-y))$$

$$f(A) = \frac{1}{2}x^TA^TAx + (b-y)^TAx + \frac{1}{2}(b-y)^T(b-y)$$

$$\frac{\partial f}{\partial A} = \frac{1}{2}(Axx^T + Axx^T) + (b-y)x^T = Axx^T + (b-y)x^T$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}A^TAx + \frac{1}{2}A^TAx + \underset{\substack{\| \\ ((b-y)^TA)^T}}{A^T(b-y)} = A^TAx + A^T(b-y)$$

2、

$$\frac{\partial x^TAx}{\partial x} = Ax + \frac{\partial (Ax)^T}{\partial x}x = Ax + \frac{\partial x^TA^T}{\partial x}x = Ax + A^Tx = (A+A^T)x$$

$$\frac{\partial x^TAx}{\partial A} = xx^T \qquad \because \quad \frac{\partial x^TAx}{\partial A} = (x_ix_j)_{m\times m}$$

$$x^TAx = \sum_{j=1}^{m}\sum_{i=1}^{m} a_{ij}x_ix_j$$

3、 $0 = dI = d(ww^{-1}) = dw\,w^{-1} + w\,dw^{-1}$

$$w\,dw^{-1} = -dw\,w^{-1}$$

$$dw^{-1} = -w^{-1}dw\,w^{-1}$$

$\dfrac{\partial Tr(w^{-1})}{\partial w}$

$$dTr(w^{-1}) = Tr(dw^{-1}) = Tr(-w^{-1}dw\,w^{-1})$$

$$= Tr(-(w^{-1})^2 dw)$$

$$\frac{\partial Tr(w^{-1})}{\partial w} = \frac{\partial Tr(-(w^{-1})^2 dw)}{\partial w} = -(w^{-T})^2$$

4、(1) $f(z) = \dfrac{\exp(z)}{1^T \exp(z)}$ $\quad q = f(z)$, $J = -p^T \log(q) = -p^T \log(f(z))$

$$f(z) = \left(\frac{e^{z_1}}{\sum_{i=1}^{n} e^{z_i}}, \dots, \frac{e^{z_n}}{\sum_{i=1}^{n} e^{z_i}} \right)^T$$

$$\log(f(z)) = \log(\exp(z)) - \log(1^T \exp(z))$$

$$J = -p^T \left[\log(\exp(z)) - 1 \cdot \log(1^T \exp(z)) \right] = -p^T \left((z) - 1 \cdot \log(1^T \exp(z)) \right)$$

$$J = -p^T z + p^T \cdot 1 \cdot \log(1^T \exp(z))$$

$$\frac{dJ}{dz} = -p + p^T \cdot 1 \cdot \frac{\exp(z)}{1^T \exp(z)}$$ $\quad p^T \cdot 1 =$ 概率求和 $= 1$

$$= -p + \frac{\exp(z)}{1^T \exp(z)} = -p + f(z) = -p + q$$

(2) $$dJ = tr\left(\frac{dJ}{dz^T} dz\right) = tr\left((q-p)^T dz\right)$$

$$z = wx \quad dz = d(wx) = dw\,x + w\,dx = dw \cdot x$$

$$dJ = tr\left((q-p)^T dw \cdot x\right) = tr\left(x(q-p)^T dw\right)$$

$$\frac{dJ}{dw} = (q-p)x^T$$

5、

(1) 求 $\frac{\partial L}{\partial \mu}$ L中与μ有关的项 $-\frac{1}{2}\sum_t (X_t-\mu)^T\Sigma^{-1}(X_t-\mu)$

$$\frac{\partial L}{\partial \mu} = -\frac{1}{2}\sum_t \frac{\partial (X_t-\mu)^T\Sigma^{-1}(X_t-\mu)}{\partial \mu}$$

$$(X_t-\mu)^T\Sigma^{-1}(X_t-\mu) = X_t^T\Sigma^{-1}X_t - X_t^T\Sigma^{-1}\mu - \mu^T\Sigma^{-1}X_t + \mu^T\Sigma^{-1}\mu$$

$$= X_t^T\Sigma^{-1}X_t - 2X_t^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}\mu$$

$$\frac{\partial L}{\partial \mu} = -\frac{1}{2}\sum_t \frac{\partial\left[\mu^T\Sigma^{-1}\mu - 2X_t^T\Sigma^{-1}\mu\right]}{\partial \mu}$$

$$= -\frac{1}{2}\sum_t \left((\Sigma^{-1}+\Sigma^{-T})\mu - 2\Sigma^{-T}X_t\right)$$ Σ^{-1}是对称的

$$= -\frac{1}{2}\sum_t (2\Sigma^{-1}\mu - 2\Sigma^{-1}X_t)$$

$$= \sum_t (\Sigma^{-1}X_t - \Sigma^{-1}\mu)$$

$$= \sum_t \Sigma^{-1}(X_t-\mu)$$

(2) $\frac{\partial L}{\partial \Sigma}$ $d\left(\frac{N}{2}\ln|\Sigma|\right) = \frac{N}{2}d(\ln|\Sigma|) = \frac{N}{2}\frac{1}{|\Sigma|}d|\Sigma| = \frac{N}{2}\frac{1}{|\Sigma|}\Sigma^* d\Sigma$

$$\Sigma^{-1} = \frac{\Sigma^*}{|\Sigma|} \qquad |\Sigma| = \frac{\Sigma^*}{\Sigma^{-1}}$$

$$\therefore d\left(\frac{N}{2}\ln|\Sigma|\right) = \frac{N}{2}\cancel{\frac{1}{|\Sigma|}}\cancel{|\Sigma|}\Sigma^{-1}d\Sigma = \frac{N}{2}\Sigma^{-1}d\Sigma$$

$$d\left(-\frac{1}{2}\sum_t (X_t-\mu)^T\Sigma^{-1}(X_t-\mu)\right) = -\frac{1}{2}\sum_t d\left((X_t-\mu)^T\Sigma^{-1}(X_t-\mu)\right)$$

$$= -\frac{1}{2}\sum_t (X_t-\mu)(X_t-\mu)^T d\Sigma^{-1})$$

$$d\Sigma^{-1} = -\Sigma^{-1}d\Sigma\,\Sigma^{-1}$$

$$\therefore \frac{\partial L}{\partial \Sigma} = -\frac{N}{2}\Sigma^{-1} + \frac{1}{2}\sum_t (\Sigma^{-1})(X_t-\mu)(X_t-\mu)^T(\Sigma^{-1})$$

$$\frac{\partial L}{\partial \Sigma} = 0 \qquad -\frac{N}{2}\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}\sum_t (x_t-\mu)(x_t-\mu)^T\Sigma^{-1} = 0$$

$$\cancel{\tfrac{1}{2}}\cancel{\Sigma^{-1}}\sum_t (x_t-\mu)(x_t-\mu)^T\Sigma^{-1} = \cancel{\tfrac{N}{2}}\cancel{\Sigma^{-1}}$$

$$\sum_t (x_t-\mu)(x_t-\mu)^T\Sigma^{-1} = N$$

$$\sum_t (x_t-\mu)(x_t-\mu)^T = N\Sigma$$

$$\Sigma = \frac{\sum_t (x_t-\mu)(x_t-\mu)^T}{N}$$

$$\mu = \frac{1}{N}\sum_t x_t$$

$$\Sigma = \frac{\sum_t \left(x_t - \frac{\sum_t x_t}{N}\right)\left(x_t - \frac{\sum_t x_t}{N}\right)^T}{N}$$

6. $\dfrac{d|X^k|}{dX}$ $X \in R^{m \times m}$ $|X| \neq 0$

$|X^k| = |X|^k$ $\dfrac{\partial |X|}{\partial X} = |X|(X^{-1})^T$

$$d|X^k| = Tr(|X^k|(X^k)^{-1} d(X^k)) = Tr(|X^k|(X^k)^{-1} k(X^{k-1}) dX)$$

$$= Tr(k|X^k|(X^{1-k} \cdot X^k)^{-1} dX)$$

$$= Tr(k|X^k|(X)^{-1} dX)$$

$$\frac{\partial |X^k|}{\partial X} = k|X^k|(X^{-1})^T$$

7、 $\dfrac{\partial Tr(AXBX^TC)}{\partial X}$ $\quad d\,Tr(AXBX^TC) = d\,Tr(C^TXB^TX^TA^T)$

$$= Tr(d(C^TXB^TX^TA^T)) = Tr(C^T\,dX\,B^TX^TA^T + C^TXB^T\,dX^TA^T)$$

$$= Tr(C^T\,dX\,B^TX^TA^T) + Tr(C^TXB^T\,dX^TA^T)$$

$$= Tr(B^TX^TA^TC^T\,dX) + Tr((dX)^TA^TC^TXB^T)$$

$$= Tr(B^TX^TA^TC^T\,dX) + Tr(BX^TCA\,dX)$$

$$= Tr((B^TX^TA^TC^T + BX^TCA)\,dX)$$

$$\therefore \frac{\partial Tr(AXBX^TC)}{\partial X} = CAXB + A^TC^TXB^T$$