习题 1

(1) $B_{m\times n}$ $\quad r(B)=m-1 \quad Null(B^T)=m-r(B)=m-m+1=1$

$$Null(B^T)+C(B) = Null(B^T)+m-1=R^m$$

$$\therefore \quad Null(B^T)=span\{1\}$$

(2) $$Col(B^T)=r(B)=m-1$$

$$Col(B^T)+Null(B)=R^n$$

(3) $$Col(B)=r(B)=m-1$$

(4) $$Null(B)=n-r(B)=n-m+1$$

习题 2

$$P_\pi = B(B^T B)^{-1} B^T \qquad B = \text{span}\{(1,-1,1)^T\}$$

$$B^T B = (1,-1,1)\begin{pmatrix}1\\-1\\1\end{pmatrix} = 3$$

$$= (3)^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix}(1,-1,1)$$

$$= \frac{1}{3}\begin{pmatrix}1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & 1\end{pmatrix}$$

$$P_\pi x = P_\pi\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

习题 4

记 $\tilde{A}^{(0)}=A$，令 u_1 是 A 的第1行，l_1 是 A 的第1列除以 u_{11}

$$l_1u_1^T=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}\begin{pmatrix}5 & 3 & -1 & 3\end{pmatrix}=\begin{pmatrix}5 & 3 & -1 & 3\\0 & 0 & 0 & 0\\-5 & -3 & 1 & -3\\0 & 0 & 0 & 0\end{pmatrix}$$

$$\tilde{A}^{(1)}=A-l_1u_1^T=\begin{pmatrix}5 & 3 & -1 & 3\\0 & 1 & 1 & -2\\-5 & -3 & 4 & -4\\0 & 1 & 1 & 0\end{pmatrix}-\begin{pmatrix}5 & 3 & -1 & 3\\0 & 0 & 0 & 0\\-5 & -3 & 1 & -3\\0 & 0 & 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 1 & -2\\0 & 0 & 3 & -1\\0 & 1 & 1 & 0\end{pmatrix}$$

令 u_2 是 $\tilde{A}^{(1)}$ 的第2行，l_2 是 $\tilde{A}^{(1)}$ 的第2列除以 u_{22}

$$l_2u_2^T=\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & -2\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 1 & -2\\0 & 0 & 0 & 0\\0 & 1 & 1 & -2\end{pmatrix}$$

$$\tilde{A}^{(2)}=\tilde{A}^{(1)}-l_2u_2^T=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 1 & -2\\0 & 0 & 3 & -1\\0 & 1 & 1 & 0\end{pmatrix}-\begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 1 & -2\\0 & 0 & 0 & 0\\0 & 1 & 1 & -2\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 3 & -1\\0 & 0 & 0 & 2\end{pmatrix}$$

令 u_3 是 $\tilde{A}^{(2)}$ 的第3行，l_3 是 $\tilde{A}^{(2)}$ 的第3列除以 u_{33}

$$l_3u_3^T=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 3 & -1\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 3 & -1\\0 & 0 & 0 & 0\end{pmatrix}$$

$$\tilde{A}^{(3)}=\tilde{A}^{(2)}-l_3u_3^T=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 3 & -1\\0 & 0 & 0 & 2\end{pmatrix}-\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 3 & -1\\0 & 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 2\end{pmatrix}$$

令 u_4 是 $\tilde{A}^{(3)}$ 的第4行，l_4 是 $\tilde{A}^{(3)}$ 的第4列除以 u_{44}

$$\tilde{A}^{(4)}=\tilde{A}^{(3)}-l_4u_4^T=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 2\end{pmatrix}-\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 2\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

$$\therefore A=LU=\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\-1 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}5 & 3 & -1 & 3\\0 & 1 & 1 & -2\\0 & 0 & 3 & -1\\0 & 0 & 0 & 2\end{pmatrix}$$

习题 5

$A^T = A$ $\quad a_{11} \neq 0$ $\quad$ A在经一步Gauss消去后

得 $\begin{pmatrix} a_{11} & a_1^T \\ 0 & A_2 \end{pmatrix}$ $\quad$ 令 $A_2 = B$ $\quad$ (Gauss消去不改变A的第一列，所以可写

为 $\begin{pmatrix} a_{11} & \frac{a_1^T}{A_2} \\ 0 & \end{pmatrix}$)

$$A_2 = B = (b_{ij}) \in B^{(n-1)\times(n-1)}$$

$$b_{ij} = a_{i+1,j+1} - \frac{a_{i+1,1}}{a_{11}} a_{1,j+1}$$

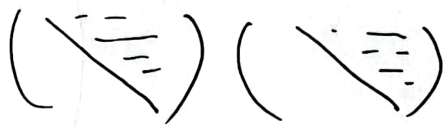
$\because$ $A^T = A$ $\quad \therefore$ $a_{ij} = a_{j,i}$ $\quad a_{i+1,j+1} = a_{j+1,i+1}$ $\quad a_{i+1,1} = a_{1,i+1}$

$a_{1,j+1} = a_{j+1,1}$

$$b_{ji} = a_{j+1,i+1} - \frac{a_{j+1,1}}{a_{11}} a_{1,i+1} = a_{i+1,j+1} - \frac{a_{i+1,1}}{a_{11}} a_{j+1,1} = b_{ij}$$

$\therefore$ A_2 仍为对称矩阵

习题6

设$A = a_{ij}$　$B = b_{ij}$ 都是n阶上三角矩阵 

$C = AB_{(ml)} = \sum_{i=1}^{n} a_{mi} b_{il}$，若上三角矩阵，则 $m > l$ 时 $C_{ml} = 0$

当 $m > l$ 时

$AB =$ 

$m > i$ 时　$a_{mi} = 0$

$l < m = i$ 时　$a_{mi} \neq 0$，$b_{il} = 0$

$l < m < i$ 时　$a_{mi} \neq 0$，$b_{il} = 0$

$\therefore$ $m > l$ 时　当 $\forall i \in [1, n]$　$a_{mi} b_{il} = 0$

$\therefore$ $m > l$ 时　$\sum_{i=1}^{n} a_{mi} b_{il} = 0 = C_{ml}$

当 $m = l$ 时　$m > i$ 时　$a_{mi} = 0$，$b_{il} \neq 0$

$l = m = i$ 时，$a_{mi} \neq 0$，$b_{il} \neq 0$

$l = m < i$ 时，$a_{mi} \neq 0$，$b_{il} = 0$

$\therefore$ $m = l$ 时　$\sum_{i=1}^{n} a_{mi} b_{il} \neq 0$

当 $m < l$ 时

$l > m > i$ 时　$a_{mi} = 0$，$b_{il} \neq 0$

$i = m < l$ 时　$a_{mi} \neq 0$，$b_{il} \neq 0$

$m < i$ 时，$a_{mi} \neq 0$

$\therefore$ $m < l$ 时　$\sum_{i=1}^{n} a_{mi} b_{il} \neq 0 = C_{ml}$

$\therefore$ $C = AB$ 仍为上三角矩阵

习题 7

$\alpha_1 = (1, 2, 2)^T$，记 $a_1 = \|\alpha_1\|_2 = \sqrt{1^2+2^2+2^2} = 3$

令 $w_1 = \dfrac{\alpha_1 - a_1 e_1}{\|\alpha_1 - a_1 e_1\|_2}$　　$\alpha_1 - a_1 e_1 = \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}3\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\2\\2\end{pmatrix}$

$\|\alpha_1 - a_1 e_1\|_2 = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$

$w_1 = \frac{1}{\sqrt{3}}(-1, 1, 1)^T$

$H_1 = I - 2w_1 w_1^T = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - 2\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}(-1\ \ 1\ \ 1)$

$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1&-1&-1\\-1&1&1\\-1&1&1\end{pmatrix} = \begin{pmatrix}\frac13&\frac23&\frac23\\\frac23&\frac13&-\frac23\\\frac23&-\frac23&\frac13\end{pmatrix}$

$H_1 A = \frac13\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}\begin{pmatrix}1&1\\2&0\\2&1\end{pmatrix} = \frac13\begin{pmatrix}9&3\\0&0\\0&3\end{pmatrix} = \begin{pmatrix}3&1\\0&0\\0&1\end{pmatrix}$

3×3　　3×2

$\beta_2 = (0, 1)^T$　　$b_2 = \|\beta_2\|_2 = \sqrt{1^2} = 1$

$w_2 = \dfrac{\beta_2 - b_2 e_1}{\|\beta_2 - b_2 e_1\|_2}$　　$\beta_2 - b_2 e_1 = \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

$\|\beta_2 - b_2 e_1\|_2 = \sqrt{1^2+1^2} = \sqrt{2}$

$w_2 = \frac{1}{\sqrt{2}}(-1, 1)^T$

$\tilde{H}_2 = \begin{pmatrix}1&0\\0&1\end{pmatrix} - \cancel{2}\times\cancel{\frac{1}{\sqrt{2}}}\times\cancel{\frac{1}{\sqrt{2}}}\begin{pmatrix}-1\\1\end{pmatrix}(-1, 1) = \begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$

$H_2 = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$

$H_2 H_1 A = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}3&1\\0&0\\0&1\end{pmatrix} = \begin{pmatrix}3&1\\0&1\\0&0\end{pmatrix}$　$\begin{pmatrix}R\\0\end{pmatrix} = \begin{pmatrix}3&1\\0&1\\0&0\end{pmatrix}$

$Q = H_1 H_2 = \frac13\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} = \frac13\begin{pmatrix}1&2&2\\2&-2&1\\2&1&-2\end{pmatrix}$

$A = Q\begin{pmatrix}R\\0\end{pmatrix} = \begin{pmatrix}\frac13&\frac23&\frac23\\\frac23&-\frac23&\frac13\\\frac23&\frac13&-\frac23\end{pmatrix}\begin{pmatrix}3&1\\0&1\\0&0\end{pmatrix} = \begin{pmatrix}1&1\\2&0\\2&1\end{pmatrix}$