数学基础 4^{th}

习题1

(i)

令 $H = A^T A = \begin{pmatrix} 14 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 3 \end{pmatrix}$ $\because (A^TA)^T = A^TA$ 对称

$\lambda_1 = 0.132 \quad \lambda_2 = 14.56 \quad \lambda_3 = 8.31 > 0$

$\therefore$ A^TA 为对称正定矩阵

$\therefore$ 存在 $A^TA = GG^T$

$g_{11} = \sqrt{a_{11}} = \sqrt{14} \quad g_{21} = \frac{a_{21}}{g_{11}} = \frac{-2}{\sqrt{14}} \quad g_{31} = \frac{a_{31}}{g_{11}} = 0$

$g_{22} = \sqrt{a_{22} - g_{21}^2} = \sqrt{6 - \frac{4}{14}} = \sqrt{\frac{80}{14}}$

$g_{32} = \dfrac{a_{32} - g_{31}g_{21}}{g_{22}} = \dfrac{-4}{\sqrt{\frac{80}{14}}} = -\frac{\sqrt{70}}{5}$

$g_{33} = \sqrt{a_{33} - g_{31}^2 - g_{32}^2} = \sqrt{3 - 0 - \frac{70}{25}} = \frac{\sqrt{5}}{5}$

$G = \begin{pmatrix} \sqrt{14} & 0 & 0 \\ \frac{-2}{\sqrt{14}} & \sqrt{\frac{80}{14}} & 0 \\ 0 & -\frac{\sqrt{70}}{5} & \frac{\sqrt{5}}{5} \end{pmatrix}$ $\quad GG^T = A^TA$

(ii)

$\|A^TA\|_2 = \|H\|_2 = \sqrt{\max_{\lambda_i}(eig(H^TH))}$ $eig(H)$ 为求方阵H特征值的函数

$eig(H) = (\lambda_1, \lambda_2, \ldots \lambda_n)^T$

$\|A\|_2 = \sqrt{\max_{\lambda_i}(eig(A^TA))}$ $\quad \because A^TA\alpha = eig(A^TA)\alpha \quad \alpha \neq 0$

$A^TA \cdot A^TA\alpha = A^TA\, eig(A^TA)\alpha = eig(A^TA)A^TA\alpha$

$= eig(A^TA) \cdot eig(A^TA)\alpha = (eig(A^TA))^2\alpha$

$= (A^TA)^T \cdot (A^TA)\alpha = H^T \cdot H\alpha$

$\therefore \quad eig(H^TH) = (eig(A^TA))^2$

$\therefore \quad \max_{\lambda_i} eig(H^TH) = (\max_{\lambda_i}(eig(A^TA)))^2$

$\therefore \sqrt{\max_{\lambda_i} eig(H^TH)} = \|A^TA\|_2 = \max_{\lambda_i}(eig(A^TA)) = \|A\|_2^2$

$G^TG = \underline{G^{-1}}GG^T\underline{G} = G^{-1}A^TAG \quad \therefore A^TA \sim G^TG \quad \therefore eig(A^TA) = eig(G^TG)$

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$$\|G\|_2 = \sqrt{\max_{\lambda_i}(\mathrm{eig}(G^T G))} = \sqrt{\max_{\lambda_i}(\mathrm{eig}(A^T A))} = \|A\|_2$$

$$\therefore \quad \|A^T A\|_2 = \|A\|_2^2 = \|G\|_2^2$$

习题 2:

(i) $A^TA = \begin{pmatrix} 160 & -136 & -56 \\ 136 & 148 & 80 \\ -56 & 80 & 52 \end{pmatrix}$ $\lambda_1 = 324$ $\lambda_2 = 36$ $\lambda_3 = 0$

$\underset{4\times3}{A} = \overset{4\times4}{U}\,\underset{4\times3}{\Sigma}\,\underset{3\times3}{V^T}$

$\Sigma = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\lambda_1 = 324$ $\vec{x}_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

$\lambda_2 = 36$ $\vec{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

$\lambda_3 = 0$ $\vec{x}_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$

$V = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$

$u_1 = \frac{1}{18} A \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ $u_2 = \frac{1}{6} A \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

2U为Hadamard矩阵 U中元素应为 $\pm\frac{1}{2}$

$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 18 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}^T$

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(ii) $\text{rank}(A) = 2$ $\|A\|_2 = \sqrt{\max_{\lambda_i} \text{eig}(A^T A)} = 18$

$\|A\|_F \sqrt{180 + 180} = \sqrt{360} = 6\sqrt{10}$

$A = U \Sigma V^T$

$R(A) = \text{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}$ $N(A) = \text{span}\left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \right\}$

(U的前r列)

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习题3

(i) $\underset{n\times n}{A} = U\Sigma V^T$ $\quad U = (u_1, u_2, \cdots u_n)$ $\quad UU^T = I$

$V = (v_1, v_2, \cdots v_n)$ $\quad VV^T = I$

$\Sigma = \operatorname{diag}(\sigma_1, \sigma^2, \cdots \sigma_n)$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$

$\because$ A可逆

$\therefore \operatorname{rank}(A) = n =$ 非0 特征值个数

$\therefore \sigma_n > 0$

$\because \forall i \in \{1, \cdots n\}$

$Av_i = \sigma_i u_i$

$A^{-1}Av_i = A^{-1}\sigma_i u_i = \sigma_i A^{-1} u_i = v_i$

$\therefore A^{-1}u_i = \frac{1}{\sigma_i} v_i$

$\therefore A^{-1} = (v_1, v_2, \cdots v_n) \operatorname{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots \frac{1}{\sigma_n})(u_1, u_2, \cdots u_n)^T$

$= V \operatorname{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots \frac{1}{\sigma_n}) U^T$

(ii)

Q 为正交阵

$Q^TQ = I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = \operatorname{diag}(1, \cdots 1)$

$\sigma_i = 1, \ i \in [0, n]$

$\therefore \Sigma = \operatorname{diag}(1, \cdots 1) = I_n$

$Q^TQ\alpha_i = \lambda_i \alpha_i = \alpha_i = I_n \alpha_i$

$\therefore (\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n) = I_n$

$\therefore V = I_n$

$\therefore Q = U I_n (I_n)^T \quad \therefore U = Q$

$\therefore$ Q 的 svd分解为 $Q = U I_n (I_n)^T$

$\sigma_i = 1 \quad \forall i \in \{1, \cdots n\}$

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习题3

(iii)

设 B 的 svd 分解为 $B=U\Sigma V^T$，U 和 V 都是 n 阶正交矩阵

$$A=QBQ^T=QU\Sigma V^TQ^T=QU\Sigma(QV)^T$$

Q 是正交矩阵

$\therefore$ QU、QV 都是正交矩阵

$\therefore$ svd 分解中 $\Sigma_A=\Sigma_B$

$\therefore$ A 和 B 有相同的奇异值

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