数字基础 9th

一、 极大似然估计

(1).
$$L(\theta, c)=\prod_{i=1}^{n}\left[\frac{1}{\theta}e^{-(x_i-c)/\theta}\right]=\left(\frac{1}{\theta}\right)^n e^{\frac{nc-\sum_{i=1}^{n}x_i}{\theta}}$$

$$\ln L(\theta, c)=n\ln\frac{1}{\theta}+\frac{nc-n\bar{x}}{\theta}=\frac{n(c-\bar{x})}{\theta}-n\ln\theta$$

$$\frac{\ln L(\theta, c)}{\partial\theta}=0=-\frac{n(c-\bar{x})}{\theta^2}-\frac{n}{\theta}$$

$$\cancel{\frac{n}{\theta}}=\frac{-\cancel{n}(c-\bar{x})}{\theta\cancel{2}}$$

$$\hat{\theta}=\bar{x}-c$$

当求 $\hat{c}$ 时则不能求导，已知 $c\le x_1\le x_2\le x_3\cdots\le x_n$

又需要 $\hat{c}$ s.t. $\left(\frac{1}{\theta}\right)^n e^{\frac{n\hat{c}-n\bar{x}}{\theta}}$ 最大，则 $\hat{c}$ 也越大越好，c 最大

只可以取到 x_1

$\therefore\quad \hat{c}=x_1$

$\therefore\quad \hat{\theta}=\bar{x}-c=\bar{x}-x_1$

矩估计

(2).
$$f(x_i)=\begin{cases}\frac{1}{\theta}e^{-(x_i-c)/\theta} & x_i\ge c\\ 0 & \text{otherwise}\end{cases}$$

$x_i\sim Exp(c,\theta)$

$x_i-c\sim Exp(0,\theta)$

$x_1-c,\ x_2-c,\ x_3-c,\ \cdots\ x_n-c$

$$f(\underbrace{x_i-c}_{h_i})=\begin{cases}\frac{1}{\theta}e^{-\frac{h_i}{\theta}} & h_i\ge 0\\ 0 & \text{otherwise}\end{cases}$$

$\therefore$ 新的 r.v $h_i\sim Exp(\frac{1}{\theta})$

$\therefore\quad E(H)=\theta=E(X-c)=E(X)-c=\bar{x}-c$

$Var(H)=\theta^2=Var(X-c)=Var(X)=S^2$

$\therefore\quad \hat{\theta}=S\qquad \hat{c}=\bar{x}-\hat{\theta}=\bar{x}-S$

2、

1、 $L(\theta) = \prod_{i=1}^{n} \left(\frac{1}{\theta} x_i^{\frac{(1-\theta)}{\theta}}\right) \quad 0 < x_i < 1 \quad 0 < \theta < +\infty$

$= \left(\frac{1}{\theta}\right)^n (x_1 x_2 \cdots x_n)^{\frac{1-\theta}{\theta}} \quad 0 < x_i < 1$

$\ln L(\theta) = n \ln \frac{1}{\theta} + \frac{1-\theta}{\theta} \ln x_1 x_2 \cdots x_n$

$= \frac{1-\theta}{\theta} \left(\sum_{i=1}^{n} \ln x_i\right) - n \ln \theta$

$\frac{\ln L(\theta)}{d\theta} = 0 = -\frac{1}{\theta^2} \sum_{i=1}^{n} \ln x_i - \frac{n}{\theta} = 0$

$\frac{n}{\cancel{\theta}} = -\frac{1}{\theta \cancel{2}} \sum_{i=1}^{n} \ln x_i$

$\hat{\theta} = \frac{-\sum_{i=1}^{n} \ln x_i}{n}$

$\therefore$ θ 的最大似然估计为 $-\frac{1}{n}\sum_{i=1}^{n} \ln x_i$ ，得证

2、 无偏估计，需证 $E(\hat{\theta}) = \theta$

$E(\hat{\theta}) = E\left(\frac{-\sum_{i=1}^{n} \ln x_i}{n}\right) = -\frac{1}{n} E\left(\sum_{i=1}^{n} \ln x_i\right)$ $\because$ $X_1, \cdots X_n$ 独立同分布

$\therefore$ $E(\ln X_i) = E(\ln X_1)$

$\therefore$ $E\left(\sum_{i=1}^{n} \ln x_i\right) = n E(\ln X_1)$

$\therefore$ $E(\hat{\theta}) = -\frac{n}{n} E(\ln X_1) = -E(\ln X_1)$

$\cancel{E(X) = \int_0^1 \frac{1}{\theta} x^{\frac{1}{\theta}} dx = \frac{1}{\theta} \int_0^1 x^{\frac{1}{\theta}-1} dx = \frac{1}{\theta} \cdot \frac{x^{\frac{1}{\theta}}}{\frac{1}{\theta}} \Big|_0^1}$

令 $Y = \ln X \quad x = e^Y$

$f(y;\theta) = \begin{cases} \frac{1}{\theta} e^{y\left(\frac{1-\theta}{\theta}\right)} e^y & y \in (-\infty, 0) \\ 0 & \text{otherwise} \end{cases}$

$E(Y) = E(\ln X) = \int_{-\infty}^{0} y \frac{1}{\theta} e^{\frac{y - \cancel{y\theta + \theta y}}{\theta}} dy = \frac{1}{\theta} \int_{-\infty}^{0} y e^{\frac{y}{\theta}} dy = \int_{-\infty}^{0} y \, d e^{\frac{y}{\theta}} = y \cdot e^{\frac{y}{\theta}} \Big|_{-\infty}^{0} -$

$\int_{-\infty}^{0} e^{\frac{y}{\theta}} dy = 0 - \theta \int_{-\infty}^{0} e^{\frac{y}{\theta}} d\frac{y}{\theta} = -\theta\left(e^{\frac{y}{\theta}} \Big|_{-\infty}^{0}\right) = -\theta(1-0) = -\theta$ $\therefore$ $E(\hat{\theta}) = -E(\ln X_1) = \theta$，得证

3、 $p(x_1 \cdots x_n \mid \mu) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$

$$\pi(\mu) = \frac{1}{\sqrt{2\pi\sigma_\mu^2}} \exp\left\{-\frac{1}{2\sigma_\mu^2}(\mu-\mu_0)^2\right\}$$

$$\pi(\mu \mid x) = \frac{h(x,\mu)}{m(x)} = \frac{p(x\mid\mu)\pi(\mu)}{\int_{-\infty}^{+\infty} p(x\mid\mu)\pi(\mu)\,d\mu}$$

$$= \frac{\cancel{(2\pi\sigma^2)^{-\frac{n}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \cancel{\frac{1}{\sqrt{2\pi\sigma_\mu^2}}} \exp\left\{-\frac{1}{2\sigma_\mu^2}(\mu-\mu_0)^2\right\}}{\int_{-\infty}^{+\infty} \cancel{(2\pi\sigma^2)^{-\frac{n}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \cancel{\frac{1}{\sqrt{2\pi\sigma_\mu^2}}} \exp\left\{-\frac{1}{2\sigma_\mu^2}(\mu-\mu_0)^2\right\} d\mu}$$

$$= \frac{\exp\left\{-\frac{1}{2\sigma^2}\left(\cancel{\sum_{i=1}^{n}x_i^2} - 2\mu\sum_{i=1}^{n}x_i + n\mu^2\right)\right\} \exp\left\{-\frac{1}{2\sigma_\mu^2}(\mu-\mu_0)^2\right\}}{\int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(\cancel{\sum_{i=1}^{n}x_i^2} - 2\mu\sum_{i=1}^{n}x_i + n\mu^2\right)\right\} \exp\left\{-\frac{1}{2\sigma_\mu^2}(\mu-\mu_0)^2\right\} d\mu}$$

$$= \frac{\exp\left\{-\frac{1}{2\sigma^2}\left(n\mu^2 - 2\mu\sum_{i=1}^{n}x_i\right) - \frac{1}{2\sigma_\mu^2}(\mu^2-2\mu\mu_0)\right\}}{\int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(n\mu^2 - 2\mu\sum_{i=1}^{n}x_i\right) - \frac{1}{2\sigma_\mu^2}(\mu^2-2\mu\mu_0)\right\} d\mu}$$

$$= \frac{\exp\left\{-\frac{n}{2\sigma^2}(\mu^2-2\mu\bar{x}) - \frac{1}{2\sigma_\mu^2}(\mu^2-2\mu\mu_0)\right\}}{\int_{-\infty}^{+\infty} \exp\left\{-\frac{n}{2\sigma^2}(\mu^2-2\mu\bar{x}) - \frac{1}{2\sigma_\mu^2}(\mu^2-2\mu\mu_0)\right\} d\mu}$$

$$= \frac{\exp\left\{-\frac{n\sigma_\mu^2(\mu^2-2\mu\bar{x}) + \sigma^2(\mu^2-2\mu\mu_0)}{2\sigma^2\sigma_\mu^2}\right\}}{\int_{-\infty}^{+\infty} \exp\left\{-\frac{n\sigma_\mu^2(\mu^2-2\mu\bar{x}) + \sigma^2(\mu^2-2\mu\mu_0)}{2\sigma^2\sigma_\mu^2}\right\} d\mu}$$

$$= \frac{\exp\left\{-\frac{\mu^2(n\sigma_\mu^2+\sigma^2) - 2\mu(\bar{x}n\sigma_\mu^2+\mu_0\sigma^2)}{2\sigma^2\sigma_\mu^2}\right\}}{\int_{-\infty}^{+\infty} \exp\left\{-\frac{\mu^2(n\sigma_\mu^2+\sigma^2) - 2\mu(\bar{x}n\sigma_\mu^2+\mu_0\sigma^2)}{2\sigma^2\sigma_\mu^2}\right\} d\mu}$$

$$= \frac{\exp\left\{\frac{1}{-2\left(\frac{\sigma^2\sigma_\mu^2}{n\sigma_\mu^2+\sigma^2}\right)}\left[\mu-\left(\frac{n\bar{x}\sigma_\mu^2+\mu_0\sigma^2}{n\sigma_\mu^2+\sigma^2}\right)\right]^2\right\}\frac{1}{\sqrt{2\pi\frac{\sigma^2\sigma_\mu^2}{n\sigma_\mu^2+\sigma^2}}}}{\underbrace{\int_{-\infty}^{+\infty}\exp\left\{\frac{1}{-2\left(\frac{\sigma^2\sigma_\mu^2}{n\sigma_\mu^2+\sigma^2}\right)}\left[\mu-\left(\frac{n\bar{x}\sigma_\mu^2+\mu_0\sigma^2}{n\sigma_\mu^2+\sigma^2}\right)\right]^2\right\}\frac{1}{\sqrt{2\pi\frac{\sigma^2\sigma_\mu^2}{n\sigma_\mu^2+\sigma^2}}}\,d\mu}}$$

$N\left(\frac{n\bar{x}\sigma_\mu^2+\mu_0\sigma^2}{n\sigma_\mu^2+\sigma^2}, \frac{\sigma^2\sigma_\mu^2}{n\sigma_\mu^2+\sigma^2}\right)$ 的密度函数在 $(-\infty, +\infty)$ 积分恒为 1

$$\therefore \pi(\mu|X) = \frac{1}{\sqrt{2\pi\frac{\sigma^2\sigma_\mu^2}{n\sigma_\mu^2+\sigma^2}}}\exp\left\{\frac{1}{-2\left(\frac{\sigma^2\sigma_\mu^2}{n\sigma_\mu^2+\sigma^2}\right)}\left[\mu-\left(\frac{n\bar{x}\sigma_\mu^2+\mu_0\sigma^2}{n\sigma_\mu^2+\sigma^2}\right)\right]^2\right\}$$

$$\therefore \mu|X \sim N\left(\frac{n\bar{x}\sigma_\mu^2+\mu_0\sigma^2}{n\sigma_\mu^2+\sigma^2}, \frac{\sigma^2\sigma_\mu^2}{n\sigma_\mu^2+\sigma^2}\right)$$

$$\therefore \mu|X \sim N\left(\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\sigma_\mu^2}}{\frac{n}{\sigma^2}+\frac{1}{\sigma_\mu^2}}, \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\sigma_\mu^2}}\right)$$

4, $\pi(\lambda) = \dfrac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \quad \lambda \geq 0$

$P(x_1 \cdots x_n | \lambda) = \dfrac{\lambda^{\sum_{i=1}^{n} x_i}}{x_1! x_2! \cdots x_n!} e^{-n\lambda} \qquad x_1 \cdots x_n = 0, 1, \sim$

$$\pi(\lambda | X) = \frac{\pi(\lambda) P(x_1 \cdots x_n | \lambda)}{\int_0^{+\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \frac{\lambda^{\sum_{i=1}^{n} x_i}}{x_1! x_2! \cdots x_n!} e^{-n\lambda} \, d\lambda}$$

$$= \frac{\cancel{\frac{\beta^{\alpha}}{\Gamma(\alpha)}} \lambda^{\alpha-1} e^{-\beta\lambda} \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\cancel{x_1! x_2! \cdots x_n!}} e^{-n\lambda}}{\int_0^{+\infty} \cancel{\frac{\beta^{\alpha}}{\Gamma(\alpha)}} \lambda^{\alpha-1} e^{-\beta\lambda} \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\cancel{x_1! x_2! \cdots x_n!}} e^{-n\lambda} \, d\lambda}$$

$$= \frac{\lambda^{\alpha-1} e^{-\beta\lambda} \lambda^{n\bar{x}} e^{-n\lambda}}{\int_0^{+\infty} \lambda^{\alpha-1} \lambda^{n\bar{x}} e^{-\beta\lambda} e^{-n\lambda} \, d\lambda} = \frac{\lambda^{\alpha-1+n\bar{x}} e^{-\beta\lambda-n\lambda}}{\int_0^{+\infty} \lambda^{\alpha-1+n\bar{x}} e^{-\beta\lambda-n\lambda} \, d\lambda}$$

$$= \frac{\lambda^{(\alpha+n\bar{x})-1} e^{-\lambda(\beta+n)}}{\int_0^{+\infty} \lambda^{\alpha+n\bar{x}-1} e^{-\lambda(\beta+n)} \, d\lambda} = \frac{\frac{(\beta+n)^{\alpha+n\bar{x}}}{\Gamma(\alpha+n\bar{x})} \lambda^{(\alpha+n\bar{x})-1} e^{-\lambda(\beta+n)}}{\underbrace{\int_0^{+\infty} \frac{(\beta+n)^{\alpha+n\bar{x}}}{\Gamma(\alpha+n\bar{x})} \lambda^{\alpha+n\bar{x}-1} e^{-\lambda(\beta+n)} \, d\lambda}_{Ga(\alpha+n\bar{x},\ \beta+n) \quad = 1}}$$

$$\therefore \pi(\lambda | X) = \frac{(\beta+n)^{\alpha+n\bar{x}}}{\Gamma(\alpha+n\bar{x})} \lambda^{(\alpha+n\bar{x}-1)} e^{-\lambda(\beta+n)}$$

$\therefore \quad \lambda | X \sim Ga(\alpha+n\bar{x},\ \beta+n)$

$\therefore \quad \hat{\lambda} = \dfrac{\alpha+n\bar{x}}{\beta+n}$ （后验分布求期望）