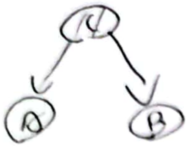
习题一

(i)



$$P(A,B,C)=P(A|C)P(B|C)\cdot P(C)$$

$$\because P(A,B|C)=\frac{P(A,B,C)}{P(C)}=P(A|C)\cdot P(B|C)$$

$\therefore$ A和B在给定C的条件下独立

$\therefore$ $A \perp\!\!\!\perp B \mid C$

(ii)

$$P(A,B,C)=P(A)P(C|A)P(B|C)$$

$$P(A,B|C)=\frac{P(A,B,C)}{P(C)}=\frac{P(A)P(C|A)P(B|C)}{P(C)}$$

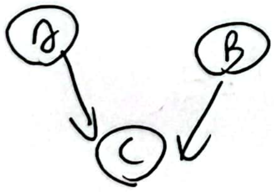
$$P(A)P(C|A)=P(A,C)=P(C)P(A|C)$$

$$\therefore P(A,B|C)=P(A|C)\cdot P(B|C)$$

$\therefore$ A和B在给定C的条件下独立

$\therefore$ $A \perp\!\!\!\perp B \mid C$

(iii)



$$P(A,B,C)=P(A)P(B)P(C|A,B)=P(A)P(B)\cdot\frac{P(A,B,C)}{P(A,B)}$$

$$\therefore P(A,B)=P(A)P(B)$$

$\therefore$ $A \perp\!\!\!\perp B \mid \varnothing$

二、

(1) e^x 为指数函数 ∵ 对任意 $a \in R$，函数 e^{ax} 在R上是凸的

令 $h(x) = e^x$，$h(x)'' = e^x$ 在 dom f 上恒大于0 ∴ h(x) 严格凸

$f(x) = e^x + 1$

∴ $f(x) = e^x + 1$ 为凸函数

(2) $A \in R^{m \times n}$，$x \in R^n$，$b \in R^m$

书 P553 所有的范数都是凸函数，∵ 范数满足三角不等式

∴ $\|\cdot\|_2$、$\|\cdot\|_1$ 都是凸函数

∴ $\|Ax+b\|_2$，$\|x^T x\|_1$ 是凸函数

∵ max 为保凸运算

∴ $f(x) = \max(\|Ax+b\|_2, \|x^T x\|_1)$ 是凸函数

(3) $f(x) = -\cos x$　$f(x)' = \sin x$　$f(x)'' = \cos x$ 在 $[-\frac{\pi}{2}, \frac{\pi}{2}]$ 上恒 ≥ 0

∴ f(x) 为凸函数

扫描全能王 创建

三、 习题3

$\Phi(x)>0$

要证 $\log(\Phi(x))$ 是凹函数

$\because \left(\log(\Phi(x))\right)' = \frac{1}{\Phi(x)}(\Phi'(x))$

$\left(\log(\Phi(x))\right)'' = -\frac{1}{\Phi^2(x)}(\Phi'(x))^2 + \frac{1}{\Phi(x)}(\Phi''(x)) < 0$

$\therefore \cancel{\frac{1}{\Phi(x)}}(\Phi''(x)) < \frac{1}{\Phi^{\cancel{2}}(x)}(\Phi'(x))^2$

即证 $\Phi(x)(\Phi''(x)) < (\Phi'(x))^2$

$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du$

$\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ $\quad \Phi''(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot -\frac{1}{\cancel{2}}\cdot 2x = \frac{-x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$(\Phi'(x))^2 = \frac{1}{2\pi}e^{-x^2}$ $\quad \Phi(x)\Phi''(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{u^2}{2}}du\cdot\frac{-x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$x>0$ 时 $(\Phi'(x))^2>0$ $\quad = \frac{-x}{2\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{x}e^{-\frac{u^2}{2}}du$

$\Phi(x)\Phi''(x) = -\frac{x}{2\pi}\cdot e^{-\frac{x^2}{2}}\int_{-\infty}^{x}e^{-\frac{u^2}{2}}du < 0$

$\therefore$ $\Phi(x)\Phi''(x) < (\Phi'(x))^2$ 成立

$x=0$ 时 $\Phi(x)\Phi''(x) = 0 < (\Phi'(x))^2 = \frac{1}{2\pi}$

$\therefore$ $\Phi(x)\Phi(x)'' < (\Phi'(x))^2$ 成立

$x<0$ 时 $\because \frac{u^2}{2}$ 是凸函数 $= f(u)$

$f(u) \geq f(x) + \nabla f(x)^T(u-x)$

$\frac{u^2}{2} \geq \frac{x^2}{2} + \cancel{\frac{1}{2}}\cdot\cancel{2}x(u-x) = \frac{x^2}{2} + ux - x^2 = ux - \frac{x^2}{2}$

$-\frac{u^2}{2} \leq \frac{x^2}{2} - ux$ $\quad \int_{-\infty}^{x} e^{-\frac{u^2}{2}}du \leq \int_{-\infty}^{x} e^{\frac{x^2}{2}-ux}du$

$-\frac{1}{x}e^{\frac{x^2}{2}}\cdot e^{-x^2} = -\frac{1}{x}e^{-\frac{x^2}{2}}$

$\therefore \frac{-x}{2\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{x}e^{-\frac{u^2}{2}}du \leq \frac{\cancel{-x}}{2\pi}e^{-\frac{x^2}{2}}\cdot\frac{1}{\cancel{-x}}e^{-\frac{x^2}{2}} = \frac{1}{2\pi}e^{-x^2} = (\Phi'(x))^2$

$\therefore \Phi(x)(\Phi''(x)) < (\Phi'(x))^2$ 在 $x\in R$ 上成立 $\therefore \log(\Phi(x))$ 是凹函数

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四、

(i) $f(x) = -\log x$，定义域为 $K = R_{++}$

$h(x) = \underline{yx - f(x)} = yx + \log x$

$y + \frac{1}{x} = 0$

$\frac{1}{x} = -y$

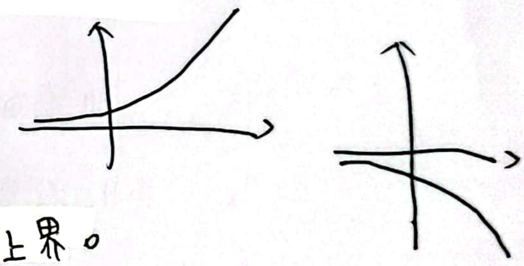
$x = -\frac{1}{y}$

当 $y \geq 0$ 时 $h(x)$ 无上界

当 $y<0$ 时，$x = -\frac{1}{y}$ 时，$h(x)$ 取到最大值

∴ 定义域为 $\mathrm{dom} f^* = \{y \mid y<0\}$

共轭函数为 $f^*(y) = -1 + \log(-y)^{-1} = -\log(-y) - 1 \quad (y<0)$



(ii) $f(x) = e^x$，定义域为 R

$h(x) = xy - f(x) = xy - e^x$

$y=0$ 时 有上界。

$y<0$ 无上界

$y>0$ $\frac{d h(x)}{dx} = y - e^x = 0$ >0 $=0$ <0

有上界

$e^x = y$

$x = \log y$

在 $x=\log y$ 处取到最大值

$f^*(y) = y\log y - y \quad (y>0)$

又∵ $0\log 0 - 0 = 0$ （规定 $0\log 0 = 0$）

∴ 定义域为 $\mathrm{dom} f^* = \{y \mid y\geq 0\}$

共轭函数为 $f^*(y) = y\log y - y$