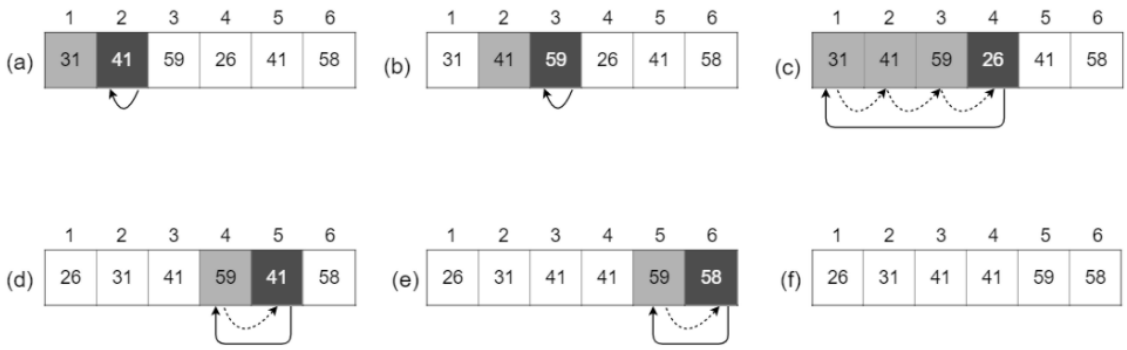
第一次作业答案

2.1-1



2.3-1

$$[3] \quad [41] \quad [52] \quad [26] \quad [38] \quad [57] \quad [9] \quad [49]$$

$$\downarrow$$

$$[3|41] \quad [26|52] \quad [38|57] \quad [9|49]$$

$$\downarrow$$

$$[3|26|41|52] \quad [9|38|49|57]$$

$$\downarrow$$

$$[3|9|26|38|41|49|52|57]$$

3.1-1

For asymptotically nonnegative functions $f(n)$ and $g(n)$, we know that

$$\begin{aligned}\exists n_1, n_2 : f(n) &\geq 0 \quad \text{for } n > n_1 \\ g(n) &\geq 0 \quad \text{for } n > n_2.\end{aligned}$$

Let $n_0 = \max(n_1, n_2)$ and we know the equations below would be true for $n > n_0$:

$$\begin{aligned}f(n) &\leq \max(f(n), g(n)) \\ g(n) &\leq \max(f(n), g(n)) \\ (f(n) + g(n))/2 &\leq \max(f(n), g(n)) \\ \max(f(n), g(n)) &\leq (f(n) + g(n)).\end{aligned}$$

Then we can combine last two inequalities:

$$0 \leq \frac{f(n) + g(n)}{2} \leq \max\left(f(n), g(n)\right) \leq f(n) + g(n).$$

Which is the definition of $\Theta(f(n) + g(n))$ with $c_1 = \frac{1}{2}$ and $c_2 = 1$

3.3

$2^{2^{n+1}}$
2^{2^n}
$(n+1)!$
$n!$
e^n
$n \cdot 2^n$
2^n
$(3/2)^n$
$(\lg n)^{\lg n} = n^{\lg \lg n}$
$(\lg n)!$
n^3
$n^2 = 4^{\lg n}$
$n \lg n$ and $\lg(n!)$
$n = 2^{\lg n}$
$(\sqrt{2})^{\lg n} (= \sqrt{n})$
$2^{\sqrt{2 \lg n}}$
$\lg^2 n$
$\ln n$
$\sqrt{\lg n}$
$\ln \ln n$
$2^{\lg^* n}$
$\lg^* n$ and $\lg^*(\lg n)$
$\lg(\lg^* n)$
$n^{1/\lg n} (= 2)$ and 1

$$f(n) = \begin{cases} 2^{2^{n+2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

3.4

a. Disprove, $n = O(n^2)$, but $n^2 \neq O(n)$.

b. Disprove, $n^2 + n \neq \Theta(\min(n^2, n)) = \Theta(n)$.

c. Prove, because $f(n) \geq 1$ after a certain $n \geq n_0$.

$$\begin{aligned} \exists c, n_0 : \forall n \geq n_0, 0 \leq f(n) \leq cg(n) \\ \Rightarrow 0 \leq \lg f(n) \leq \lg(cg(n)) = \lg c + \lg g(n). \end{aligned}$$

We need to prove that

$$\lg f(n) \leq d \lg g(n).$$

We can find d,

$$d = \frac{\lg c + \lg g(n)}{\lg g(n)} = \frac{\lg c}{\lg g(n)} + 1 \leq \lg c + 1,$$

where the last step is valid, because $\lg g(n) \geq 1$.

d. Disprove, because $2n = O(n)$, but $2^{2n} = 4^n \neq O(2^n)$.

e. Prove, $0 \leq f(n) \leq cf^2(n)$ is trivial when $f(n) \geq 1$, but if $f(n) < 1$ for all n, it's not correct. However, we don't care this case.

f. Prove, from the first, we know that $0 \leq f(n) \leq cg(n)$ and we need to prove that $0 \leq df(n) \leq g(n)$, which is straightforward with $d = 1/c$.

g. Disprove, let's pick $f(n) = 2^n$. We will need to prove that

$$\exists c_1, c_2, n_0 : \forall n \geq n_0, 0 \leq c_1 \cdot 2^{n/2} \leq 2^n \leq c_2 \cdot 2^{n/2},$$

which is obviously untrue.

h. Prove, let $g(n) = o(f(n))$. Then

$$\exists c, n_0 : \forall n \geq n_0, 0 \leq g(n) < cf(n).$$

We need to prove that

$$\exists c_1, c_2, n_0 : \forall n \geq n_0, 0 \leq c_1 f(n) \leq f(n) + g(n) \leq c_2 f(n).$$

Thus, if we pick $c_1 = 1$ and $c_2 = c + 1$, it holds.

4.2-6

- $(kn \times n)(n \times kn)$ produces a $kn \times kn$ matrix. This produces k^2 multiplications of $n \times n$ matrices.
- $(n \times kn)(kn \times n)$ produces an $n \times n$ matrix. This produces k multiplications and $k - 1$ additions.