1、 $\because$ $\hat\theta$ 是 θ 的 UE

$\therefore E(\hat\theta)=\theta$

$Var(\hat\theta)= E(\hat\theta^2) - E(\hat\theta)^2 > 0$

$E(\hat\theta^2) = \underbrace{Var(\hat\theta)}_{>0} + \underbrace{E(\hat\theta)^2}_{\theta^2} > \theta^2$

$\therefore E(\hat\theta^2) \neq \theta^2$

$\therefore$ $\hat\theta^2$ 不是 θ^2 的 UE

2、 $E(\bar X) = E(X) = \int_{\theta-\frac12}^{\theta+\frac12} 1\cdot x\, dx = \frac{x^2}{2}\Big|_{\theta-\frac12}^{\theta+\frac12} = \frac{\theta^2+\theta+\frac14}{2} - \frac{\theta^2-\theta+\frac14}{2}$

$= \frac{\cancel{\theta^2}+\theta+\cancel{\frac14}-\cancel{\theta^2}+\theta-\cancel{\frac14}}{2} = \theta$ $\therefore$ $\bar X$ 是 θ 的 UE

$Var(\bar X) = \frac{Var(X)}{n}$ $X \sim U(\theta-\frac12, \theta+\frac12)$

$\therefore Var(\bar X) = \frac{\frac{1^2}{12}}{n} = \frac{1}{12n}$

$X-(\theta-\frac12) \sim U(0,1)$

$\therefore X_i-(\theta-\frac12) \sim U(0,1)$ $i=1,2,\cdots n$

$Y_i = X_i-(\theta-\frac12) \sim U(0,1)$ $Y_{(i)} \sim Be(i, n-i+1)$

$\cancel{\frac12}[X_{(1)}-(\theta-\frac12)+X_{(n)}-(\theta-\frac12)] = \cancel{\frac12}(Y_{(1)}+(Y_{(n)})$

$\therefore \frac{X_{(1)}+X_{(n)}}{2} = \frac{Y_{(n)}+Y_{(1)}+2\theta-1}{2}$

$E(\frac12(X_{(1)}+X_{(n)}) = \frac12 E(X_{(1)}+X_{(n)}) = \frac12\ \cancel{\frac{n}{n+1}} + \cancel{\frac{1}{n+1}} + 2\theta \cancel{-1} = \theta$

$\therefore$ $\frac12(X_{(1)}+X_{(n)})$ 是 θ 的 UE

$Var(\frac12(X_{(1)}+X_{(n)}) = \frac14 Var(X_{(1)}+X_{(n)}) = \frac14 Var(Y_{(n)}+Y_{(1)}+\cancel{2\theta-1})$

$= \frac14 (\underbrace{Var(Y_{(n)})}_{\frac{2n}{(n+1)^2(n+2)}} + \underline{Var(Y_{(1)})} + 2cov(Y_{(n)}, Y_{(1)}))$

$Y_{(n)} - Y_{(1)} \sim Be(n-1, 2)$

$$Var(Y_{(n)} - Y_{(1)}) = \frac{2(n-1)}{(n+1)^2(n+2)} = \frac{2n}{(n+1)^2(n+2)} - 2Cov(Y_{(n)}, Y_{(1)})$$

$$2Cov(Y_{(n)}, Y_{(1)}) = \frac{2n-2(n-1)}{(n+1)^2(n+2)}$$

$$Cov(Y_{(n)}, Y_{(1)}) = \frac{1}{(n+1)^2(n+2)}$$

$$Var(\frac{1}{2}(X_{(1)}+X_{(n)})) = \frac{1}{4}\left(\frac{2n}{(n+1)^2(n+2)} + \frac{2}{(n+1)^2(n+2)}\right) = \frac{1}{2(n+1)(n+2)}$$

$12n$ vs $2(n+1)(n+2)$

在 $n>2$ 时 $2(n+1)(n+2) > 12n$

$\therefore Var(\bar{X}) > Var(\frac{1}{2}(X_{(1)} + X_{(n)}))$

$\therefore \frac{1}{2}(X_{(1)}+X_{(n)})$ 更有效

3. $X \sim \pi(\mu, \sigma^2)$

$$E(Y) = E(a\bar{x}_1 + b\bar{x}_2) = aE(\bar{x}_1) + bE(\bar{x}_2) = a\mu + b\mu = \mu\underbrace{(a+b)}_{1} = \mu$$

$\therefore$ 对 $\forall$ a,b s.t $a+b=1$, $Y = a\bar{x}_1 + b\bar{x}_2$ 都是 μ 的 UE

$$Var(Y) = Var(a\bar{x}_1 + b\bar{x}_2) = a^2 Var(\bar{x}_1) + b^2 Var(\bar{x}_2) = \frac{a^2\sigma^2}{n_1} + \frac{b^2\sigma^2}{n_2}$$

$$\frac{d\left(\frac{a^2\sigma^2}{n_1} + \frac{(1-a)^2\sigma^2}{n_2}\right)}{da} = \frac{2a\sigma^2}{n_1} - \frac{2(1-a)\sigma^2}{n_2} = 0$$

$$\frac{a}{n_1} = \frac{(1-a)}{n_2}$$

$$n_2 a = n_1 - a n_1$$

$$a = \frac{n_1}{n_1+n_2}, \quad b = 1-a = \frac{n_2}{n_1+n_2}$$

概统 hw-20

1、

$$L(\theta)=\prod_{i=1}^{n}\theta x_i^{\theta-1}$$

$$\ln L(\theta)=\sum_{i=1}^{n}(\ln\theta+(\theta-1)\ln x_i)=n\ln\theta+(\theta-1)\ln\prod_{i=1}^{n}x_i=n\ln\theta+(\theta-1)\sum_{i=1}^{n}\ln x_i$$

$$\frac{d\ln L(\theta)}{d\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\ln x_i=0$$

$$\hat\theta=\frac{n}{-\sum_{i=1}^{n}\ln x_i}$$

$$\therefore\ g(\hat\theta)=\frac{1}{\hat\theta}=\frac{-\sum_{i=1}^{n}\ln x_i}{n}$$

2、 $X\sim Exp(\frac{1}{\theta})\qquad p(x)=\frac{1}{\theta}e^{-\frac{1}{\theta}x}$

(a)

$$E(X)=\theta$$

$$\therefore\ \hat\theta_{矩}=\bar X$$

$$L(\theta)=\prod_{i=1}^{n}\frac{1}{\theta}e^{-\frac{1}{\theta}x_i}=\frac{1}{\theta^n}e^{-\frac{\sum_{i=1}^{n}x_i}{\theta}}$$

$$\ln L(\theta)=-n\ln\theta+-\frac{\sum_{i=1}^{n}x_i}{\theta}$$

$$\frac{d\ln L(\theta)}{d\theta}=\frac{-n}{\theta}+\frac{\sum_{i=1}^{n}x_i}{\theta^2}=0$$

$$-n\theta+\sum_{i=1}^{n}x_i=0$$

$$\sum_{i=1}^{n}x_i=n\theta$$

$$\hat\theta_{似}=\frac{\sum_{i=1}^{n}x_i}{n}=\bar X$$

$\therefore$ θ 的 $\hat\theta_{矩}$ 和 $\hat\theta_{似}$ 都是 $\bar X$

(b)

$$E(\bar X)=E(X)=\frac{1}{\frac{1}{\theta}}=\theta$$

$\therefore$ $\bar X$ 是 θ 的 UE

$\lim_{n\to\infty}E(\bar X)=\theta\qquad \lim_{n\to\infty}Var(\bar X)=\lim_{n\to\infty}\frac{n\,Var(X)}{n^2}=\lim_{n\to\infty}\frac{\theta^2}{n}=0$

$\therefore$ $\bar X$ 是 θ 的相合估计

(c)

$\hat{\theta}_n = a\bar{X}$

$MSE(\bar{X}) = \frac{\theta^2}{n} + 0 = \frac{\theta^2}{n}$

$MSE(\hat{\theta}_n) = a^2\frac{\theta^2}{n} + (a\theta-\theta)^2$

$$\frac{\partial(MSE(\hat{\theta}_n))}{\partial a} = 2a\frac{\theta^2}{n} + 2(a\theta-\theta)\theta = 0$$

$$\frac{\cancel{2}a\theta}{n} + \cancel{2}\theta(a-1) = 0$$

$$\frac{a}{n} = 1-a$$

$$a = n-na$$

$$a = \frac{n}{n+1}$$

$$MSE(\hat{\theta}_n) = \frac{n^2}{(n+1)^2}\frac{\theta^2}{n} + \theta^2\cdot\left(\frac{n}{n+1}-1\right)^2$$

$$= \frac{n\theta^2}{(n+1)^2} + \theta^2\frac{1}{(n+1)^2} = \frac{\theta^2\cancel{(n+1)}}{(n+1)^{\cancel{2}}} = \frac{\theta^2}{n+1} < \frac{\theta^2}{n}$$

$\therefore$ 当 $a=\frac{n}{n+1}$ 时 $a\bar{X}$ 优于 $\bar{X}$

3、

(a) $\hat{\sigma}_1^2 = S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$

(b) $\frac{(n-1)S_X^2}{\sigma^2} \sim \chi^2(n-1)$ $\qquad$ $\frac{(m-1)S_Y^2}{\sigma^2} \sim \chi^2(m-1)$

$$\frac{(n-1)\hat{\sigma}_1^2+(m-1)\hat{\sigma}_2^2}{\sigma^2} = \frac{(n-1)S_X^2+(m-1)S_Y^2}{\sigma^2} \sim \chi^2(m+n-2)$$

$$S_w^2 = \frac{(n-1)\hat{\sigma}_1^2}{m+n-2} + \frac{(m-1)\hat{\sigma}_2^2}{m+n-2}$$

$\hat{\sigma}^2 = S_w^2$ $\qquad$ $\hat{\sigma}_1^2 = S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$

$\hat{\sigma}_2^2 = S_Y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i-\bar{Y})^2$

(c) $E(\hat{\sigma}_1^2) = \sigma^2$ $\therefore$ $\hat{\sigma}_1^2$ 是 σ^2 的 UE

$Var(\hat{\sigma}_1^2) = \frac{1}{(n-1)^2}Var\left(\sum_{i=1}^{n}(X_i-\bar{X})^2\right)$

$E(S_w^2) = \frac{(n-1)\sigma^2}{m+n-2} + \frac{(m-1)\sigma^2}{m+n-2} = \sigma^2$ $\therefore$ S_w^2 是 σ^2 的 UE

$Var(S_w^2) = \frac{(n-1)^2}{(m+n-2)^2}Var(\hat{\sigma}_1^2) + \frac{(m-1)^2}{(m+n-2)^2}Var(\hat{\sigma}_2^2)$

$$Var(S_w^2)=\frac{Var(\sum_{i=1}^{n}(X_i-\bar{X})^2}{(m+n-2)^2}+\frac{Var(\sum_{i=1}^{m}(Y_i-\bar{Y})^2}{(m+n-2)^2}$$

$$Var(\hat{\sigma}_1^2)=\frac{Var(\sum_{i=1}^{n}X_i^2-n\bar{X}^2)}{(n-1)^2} \qquad Var(\hat{\sigma}_2^2)=\frac{Var(\sum_{i=1}^{m}Y_i^2-m\bar{Y}^2)}{(m-1)^2}$$

$$Var(S_w^2)<Var(\hat{\sigma}_1^2)$$

S_w^2 更有效

且 S_w^2 包含了 σ^2 的全部信息，而 $\hat{\sigma}_1^2$ 只包含了 $X_1 \cdots X_n$ 样本的信息，S_w^2 是充分统计量