一、 $X \sim Nb(2,\theta)$

(1) $E(X) = \frac{2}{\theta}$

$\therefore \hat{\theta} = \frac{2}{\bar{X}}$

(2)

$E(X) = \int_0^1 x(\theta+1)x^{\theta} dx = \int_0^1 (\theta+1)x^{\theta+1} dx = (\theta+1)\frac{x^{\theta+2}}{\theta+2}\Big|_0^1$

$= \frac{(\theta+1)}{\theta+2}$

$\hat{\theta} = \frac{1-2\bar{X}}{\bar{X}-1}$

二、

$\hat{\mu} = \bar{x}$ 假设 $X \sim N(\mu,1)$ 则 $y = \frac{X-\mu}{1} \sim N(0,1)$

$f = \frac{k}{n} = P(X>0)$

$\therefore$ $P(X<0) = 1-\frac{k}{n}$

$P(y<-\mu) = 1-\frac{k}{n} = \Phi(-\mu) = 1-\Phi(\mu)$

$\therefore \Phi(\hat{\mu}) = \frac{k}{n}$

$\therefore \hat{\mu} = u_{k/n}$

3、

$L(\theta) = (\frac{1}{2\theta})^n \exp\{-\frac{\sum_{i=1}^n |x_i|}{\theta}\}$ $\theta>0$

取对数

$\ln L(\theta) = n\ln\frac{1}{2\theta} + \frac{-\sum_{i=1}^n|x_i|}{\theta}$

$\frac{d\ln L(\theta)}{d\theta} = n\cdot -\frac{1}{\theta} + \cancel{-}\sum_{i=1}^n|x_i|(-1)\frac{1}{\theta^2} = -\frac{n}{\theta} + \frac{\sum_{i=1}^n|x_i|}{\theta^2} = 0$

$\therefore \hat{\theta} = \frac{\sum_{i=1}^n|x_i|}{n}$

验证 $\frac{d\ln^2 L(\theta)}{d\theta^2} = \frac{n}{\theta^2} + \frac{-2\sum_{i=1}^n|x_i|}{\theta^3}$ $(\theta=\hat{\theta})$ $= \frac{n^3}{(\sum_{i=1}^n|x_i|)^2} + \frac{-2n^3}{(\sum_{i=1}^n|x_i|)^2} = \frac{-n^3}{(\sum_{i=1}^n|x_i|)^2} < 0$

$\therefore \hat{\theta} = \frac{\sum_{i=1}^n|x_i|}{n}$

(b)

$$L(\theta) = \frac{1}{(\theta_2-\theta_1)^n} I\{\theta_1 < x_i < \theta_2,\ i=1,2\cdots n\}$$

$$= \frac{1}{(\theta_2-\theta_1)^n} I\{\theta_1 < x_{(1)} < x_{(n)} < \theta_2\} \uparrow$$

$\underline{\theta_2-\theta_1}$ 小 但 在 $x_{(1)}$ 和 $x_{(n)}$ 两边

∴ $\hat{\theta}_1 = x_{(1)}$ $\hat{\theta}_2 = x_{(n)}$

四、

$p(x) = e^{-(x-\theta)}$ $x>\theta$

$$F(x) = -\int_{-\infty}^{x} e^{-x+\theta}\, d(-x+\theta)$$

$$= -(e^{-x+\theta}\Big|_{\theta}^{x})$$

$$= -(e^{-x+\theta} - e^{0})$$

$$= -e^{-x+\theta} + 1$$

$L(\theta) = e^{-\sum_{i=1}^{n} x_i + n\theta}$ $x>\theta$

$\cancel{\ln L(\theta) = -\sum_{i=1}^{n} x_i + n\theta \quad x>\theta}$

$L(\theta)\uparrow$ $\theta\uparrow$ ∴ $\hat{\theta}_1 = x_{(1)}$

UE

$$E(\hat{\theta}_1) = E(x_{(1)})$$

$$P_1(x) = n\cdot(\cancel{1+}e^{-x+\theta}\cancel{-1})^{n-1} e^{-x+\theta}$$

$$= n(e^{-x+\theta})^{n-1}\cdot e^{-x+\theta}$$

$$= n(e^{-x+\theta})^{n} \quad x>\theta$$

$$E(x_{(1)}) = n\int_{\theta}^{+\infty} x\, e^{-nx+n\theta}\, dx = -\int_{\theta}^{+\infty} x\, e^{-nx+n\theta}\, d(-nx+n\theta)$$

$$= -\int_{\theta}^{+\infty} x\, d\, e^{-nx+n\theta} = -\left(x\cdot e^{-nx+n\theta}\Big|_{\theta}^{+\infty} - \int_{\theta}^{+\infty} e^{-nx+n\theta}\, dx\right)$$

$$= -(0-\theta\cdot \underset{1}{\cancel{e^{-n\theta+n\theta}}} - \tfrac{1}{n}) = -(-\tfrac{1}{n}-\theta) = \theta + \tfrac{1}{n}$$

∴ $E(\hat{\theta}_1) = \theta + \frac{1}{n} \neq \theta$ ∴ $\hat{\theta}_1$ 不是 θ 的 UE

$$\lim_{n\to\infty} E(\hat{\theta}_1) = \lim_{n\to\infty}(\theta + \tfrac{1}{n}) = \theta$$

$$\lim_{n\to\infty} Var(\hat{\theta}_1) = \lim_{n\to\infty} E(\hat{\theta}_1^2) - \theta^2 = \lim_{n\to\infty} E(x_{(1)}^2) - \theta^2$$

$$E(x_{(1)}^2) = n\int_{\theta}^{+\infty} x^2 e^{-nx+n\theta}\, dx = -\left(x^2\cdot e^{-nx+n\theta}\Big|_{\theta}^{+\infty} - 2\int_{\theta}^{+\infty} x\cdot e^{-nx+n\theta}\, dx\right)$$

$$= -(\cancel{0} - \theta^2\cdot \cancel{e^{0}} - 2\,\frac{\theta+\frac{1}{n}}{n}) = \theta^2 + \frac{2\theta+\frac{2}{n}}{n} = \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2}$$

∴ $\lim_{n\to\infty} Var(\hat{\theta}_1) = \theta^2 - \theta^2 = 0$

∴ $\hat{\theta}_1$ 是 θ 的相合估计

(b)

$$E(X) = \int_{\theta}^{+\infty} x e^{-(x-\theta)} dx = -\int_{\theta}^{+\infty} x e^{-x+\theta} d(-x) = -\int_{\theta}^{+\infty} x \, d e^{-x+\theta}$$

$$= -\left(x \cdot e^{-x+\theta} \Big|_{\theta}^{+\infty} - \int_{\theta}^{+\infty} e^{-x+\theta} dx \right) = -(0 - \theta - 1) = 1+\theta$$

$\therefore \ \hat{\theta}_2 = \bar{X} - 1$

UE: $E(\hat{\theta}_2) = E(\bar{X}-1) = E(\bar{X}) - 1 = E(X) - 1 = 1+\theta-1 = \theta$

$\therefore$ $\hat{\theta}_2$ 是 θ 的UE

$$\lim_{n\to\infty} E(\hat{\theta}_2) = \theta$$

$$\lim_{n\to\infty} Var(\hat{\theta}_2) = \lim_{n\to\infty} Var(\bar{X}-1) = \lim_{n\to\infty} Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \lim_{n\to\infty} \frac{nVar(X)}{n^2}$$

$$= \lim_{n\to\infty} \frac{Var(X)}{n}$$

$$Var(X) = E(X^2) - (1+\theta)^2 = -(0-\theta^2-(2+2\theta)) - 1 - 2\theta - \theta^2$$

$$= \cancel{\theta^2} + 2 + \cancel{2\theta} - 1 - \cancel{2\theta} - \cancel{\theta^2} = 1$$

$$\therefore \lim_{n\to\infty} Var(\hat{\theta}_2) = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore$ $\hat{\theta}_2$ 是 θ 的相合估计

5、

同		异		
p		$1-p$		
同男	同女	同男	同女	一男一女
$\frac{p}{2}$	$\frac{p}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}-p$
z_1	z_2	n_1-z_1	n_2-z_2	n_3

$* \ n_1+n_2+n_3=n$

$$L(p; n_1,n_2,n_3,z_1,z_2) = \left(\frac{p}{2}\right)^{z_1} \left(\frac{p}{2}\right)^{z_2} \cdot \left(\frac{1}{4}\right)^{n_1-z_1} \cdot \left(\frac{1}{4}\right)^{n_2-z_2} \left(\frac{1}{2}-p\right)^{n_3}$$

$$l(p; n_1,n_2,n_3,z_1,z_2) = (z_1+z_2)\ln\frac{p}{2} + (n_1+n_2-z_1-z_2)\ln\frac{1}{4} + n_3\ln\left(\frac{1}{2}-p\right)$$

$$\frac{\partial l(p; n_1,n_2,n_3,z_1,z_2)}{\partial p} = \frac{z_1+z_2}{p} + \frac{2n_3}{2p-1} = 0$$

$E(z_1|p) = \frac{p}{2}$ $\qquad$ $E(z_2|p) =$