概统 hw 15

1、

$\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i \qquad \bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$

(a) $\sum_{i=1}^{n}y_i = a\left(\sum_{i=1}^{n}x_i\right)+nb$

$\bar{y} = a\bar{x}+b$

(b)

$S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$S_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2$

$\bar{y}^2 = a^2\bar{x}^2+2ab\bar{x}+b^2$

$y_i^2 = a^2x_i^2+2abx_i+b^2$

$a^2\left(\sum_{i=1}^{n}x_i^2\right)+2ab\sum_{i=1}^{n}x_i+nb^2$

$\sum_{i=1}^{n}x_i = n\bar{x}$

$\frac{S_x^2}{S_y^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(ax_i+\cancel{b}-a\bar{x}-\cancel{b})^2} = \frac{\sum_{i=1}^{n}x_i^2-n\bar{x}^2}{\sum_{i=1}^{n}y_i^2-n\bar{y}^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{a^2\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{1}{a^2}$

$= \frac{\sum_{i=1}^{n}x_i^2-n\bar{x}^2}{a^2\left(\sum_{i=1}^{n}x_i^2\right)+\cancel{2abn\bar{x}}+\cancel{nb^2}-a^2n\bar{x}^2-\cancel{2abn\bar{x}}-\cancel{b^2n}}$

$= \frac{\sum_{i=1}^{n}x_i^2-n\bar{x}^2}{a^2\left(\sum_{i=1}^{n}x_i^2\right)-a^2n\bar{x}^2} = \frac{1}{a^2} = \frac{S_x^2}{S_y^2}$

2、

$Cov(\bar{x},S^2) = E\left((\bar{x}-E(\bar{x}))(S^2-E(S^2))\right)$

$E(\bar{x}) = E\left(\frac{X_1+X_2+\cdots X_n}{n}\right) = E(X_1) = E(X)$　X_1 与总体同分布

$E(S^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right) = \frac{1}{n-1}E\left(\sum_{i=1}^{n}x_i^2-2\bar{x}\sum_{i=1}^{n}x_i+n\bar{x}^2\right)$　$-2n\bar{x}^2$

$= \frac{1}{n-1}E\left(\sum_{i=1}^{n}x_i^2-n\bar{x}^2\right) = \frac{1}{n-1}\left(nVar(X)+n\cancel{E(X)^2}-Var(X)-\cancel{nE(X)^2}\right)$

$=Var(X)$

$E(X_1^2) = Var(X_1)+(E(X_1))^2$

$E(n\bar{x}^2) = nE(\bar{x}^2) = n\left(Var(\bar{x})+E(\bar{x})^2\right)$

$= n\left(\frac{nVar(X)}{n^2}+E(X)^2\right)$

$\therefore\ Cov(\bar{x},S^2) = E\left((\bar{x}-E(X))(S^2-Var(X))\right)$

$= E(\bar{x}\cdot S^2)-Var(X)E(\bar{x})-E(X)\cancel{E(S^2)}+\cancel{E(X)}Var(X)$

$= E(\bar{x}\cdot S^2)-E(S^2)E(\bar{x})$　设 $E(\bar{x})=E(X_i)=E(X)=\mu$

$= E[(\bar{x}-\mu)S^2]$

扫描全能王 创建

$$(\bar{x}-\mu)s^2 = (\bar{x}-\mu)\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{\bar{x}-\mu}{n-1}\sum_{i=1}^{n}(x_i^2-2x_i\bar{x}+\bar{x}^2)$$

$$= \frac{\bar{x}-\mu}{n-1}\left(\sum_{i=1}^{n}x_i^2-2n\bar{x}^2+n\bar{x}^2\right) = \frac{\bar{x}-\mu}{n-1}\left(\sum_{i=1}^{n}x_i^2-n\bar{x}^2\right)$$

$$= \frac{\bar{x}-\mu}{n-1}\left(\sum_{i=1}^{n}x_i^2-2\mu\sum_{i=1}^{\infty}x_i+n\mu^2-n\bar{x}^2+2\mu\underbrace{\sum_{i=1}^{\infty}x_i}_{n\bar{x}}-n\mu^2\right)$$

$$= \frac{\bar{x}-\mu}{n-1}\left[\sum_{i=1}^{n}(x_i-\mu)^2-(n\bar{x}^2-2n\mu\bar{x}+\mu^2 n)\right]$$

$$= \frac{\bar{x}-\mu}{n-1}\left[\sum_{i=1}^{n}(x_i-\mu)^2-n(\bar{x}-\mu)^2\right]$$

$$E[(\bar{x}-\mu)s^2] = E\left(\frac{\bar{x}-\mu}{n-1}\sum_{i=1}^{n}(x_i-\mu)^2-\frac{n(\bar{x}-\mu)}{n-1}(\bar{x}-\mu)^2\right)$$

$$= \frac{1}{n-1}E\left[(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\mu)^2-n(\bar{x}-\mu)^3\right]$$

$$= \frac{1}{n-1}\left[\underbrace{E\left[(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\mu)^2\right]}_{①}-\underbrace{E\left[n(\bar{x}-\mu)^3\right]}_{②}\right]$$

② : $E[n(\bar{x}-\mu)^3] = nE((\bar{x}-\mu)^3)$

$$E(\bar{x}-\mu)^3 = E\left(\frac{\sum_{i=1}^{n}x_i}{n}-\frac{n\mu}{n}\right)^3 = \frac{1}{n^3}E\left(\sum_{i=1}^{n}x_i-n\mu\right)^3$$

$$= \frac{1}{n^3}E(x_1-\mu+x_2-\mu+x_3-\mu+\cdots x_n-\mu)^3$$

assume 只有两项 $E(X_1-\mu+X_2-\mu)^3 = E\left((X_1-\mu)^3+3(X_1-\mu)^2(X_2-\mu)+3(X_1-\mu)(X_2-\mu)^2\right.$

$$= E(X_1-\mu)^3+3E\,\underline{(X_1-\mu)^2(X_2-\mu)}+3E\,\underline{(X_1-\mu)(X_2-\mu)^2}+E(X_2-\mu)^3+(X_2-\mu)^3)$$

$X_1-\mu$ 与 $X_2-\mu$ 独立

$$\therefore = E(X_1-\mu)^3+3E(X_1-\mu)^2\cancel{E(X_2-\mu)}+3E\cancel{(X_1-\mu)}E(X_2-\mu)^2+E(X_2-\mu)^3$$

$$E(X_2-\mu) = E(X_2)-\mu = \mu-\mu = 0 = E(X_1-\mu)$$

$\therefore$ 两项的情况是 $E(X_1-\mu)^3+E(X_2-\mu)^3$

$\therefore$ 推至 n 项

$$\therefore \frac{1}{n^3}E(X_1-\mu+X_2-\mu+X_3-\mu+\cdots X_n-\mu)^3 = \frac{1}{n^3}\sum_{i=1}^{n}E(X_i-\mu)^3$$

$$\therefore nE((\bar{x}-\mu)^3) = \frac{n}{n^3}\sum_{i=1}^{n}V_3 = \frac{V_3}{n}$$

①: $E\left((\bar{x}-\mu)\sum_{i=1}^{n}(X_i-\mu)^2\right)=E\left((\bar{x}-\mu)\left[(X_1-\mu)^2+(X_2-\mu)^2+\cdots(X_n-\mu)^2\right]\right)$

$E(\bar{x}-\mu)(X_1-\mu)^2=E\left(\frac{\sum_{i=1}^{n}X_i}{n}-\frac{n\mu}{n}\right)(X_1-\mu)^2$

$\frac{1}{n}E(X_1-\mu+X_2-\mu+X_3-\mu+\cdots X_n-\mu)(X_1-\mu)^2$

$=\frac{1}{n}E\left((X_1-\mu)(X_1-\mu)^2+(X_2-\mu)(X_1-\mu)^2+\cdots(X_n-\mu)(X_1-\mu)^2\right)$

$E(X_n-\mu)(X_1-\mu)^2$ 当 $n\neq1$ 时 $X_n-\mu$ 与 $(X_1-\mu)^2$ 独立

$\therefore=E(X_n-\mu)E(X_1-\mu)^2$

$=\underbrace{(E(X_n)-\mu)}_{=0}E(X_1-\mu)^2$

$=0$

$\therefore\quad E(\bar{x}-\mu)(X_1-\mu)^2=\frac{1}{n}E(X_1-\mu)^3$

$\therefore\quad E\left((\bar{x}-\mu)\sum_{i=1}^{n}(X_i-\mu)^2\right)=\frac{1}{n}\sum_{i=1}^{n}E(X_i-\mu)^3=\frac{1}{n}\cdot n\cdot V_3=V_3$

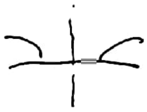
$\therefore$ 原式$=\frac{1}{n-1}V_3-\frac{V_3}{n(n-1)}=\frac{nV_3-V_3}{n(n-1)}=\frac{V_3}{n}$

3. $\bar{X}_1 \sim N(\mu, \frac{\sigma^2}{n})$ $\bar{X}_2 \sim N(\mu, \frac{\sigma^2}{n})$

$Z = \bar{X}_1 - \bar{X}_2 \sim N(0, \frac{2\sigma^2}{n})$

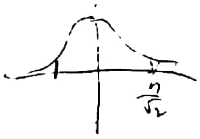
$P(|\bar{X}_1 - \bar{X}_2| > \sigma) \le 0.01$

$P(Z > \sigma) + P(Z < -\sigma) \le 0.01$



$2P(Z < -\sigma) \le 0.01$

$P(Z < -\sigma) \le \frac{0.01}{2}$



$P\left(\frac{Z-0}{\sqrt{\frac{2\sigma^2}{n}}} < \frac{-\sigma}{\sqrt{\frac{2\sigma^2}{n}}}\right) \le \frac{0.01}{2}$

$\Phi\left(\frac{-\sqrt{n}}{\sqrt{2}}\right) \le 0.005$

$1 - \Phi\left(\frac{\sqrt{n}}{\sqrt{2}}\right) \le 0.005$

$\Phi\left(\frac{\sqrt{n}}{\sqrt{2}}\right) \ge 0.995$

$\frac{\sqrt{n}}{\sqrt{2}} \ge 2.575$

$n \ge 13.261$

$\therefore$ n 至少为14

4. X为总体

$E(X) = \frac{1}{\frac{1}{\theta}} = \theta$ $Var(X) = \frac{1}{\frac{1}{\theta^2}} = \theta^2$

$\therefore \bar{X} \sim N(\theta, \frac{\theta^2}{40})$

5. (a) $y_i = F(X_{(i)})$ $y_1 = F(X_{(1)})$ $y_2 = F(X_{(2)}) \cdots$ $y_n = F(X_{(n)})$

$\because F(x)$ 为分布函数 $\therefore$ 其单调非减

又 $\because x_1 \le x_2 \cdots \le x_n$ $\therefore F(X_{(1)}) \le F(X_{(2)}) \le \cdots F(X_{(n)})$

$\therefore y_1 \le y_2 \cdots \le y_n$

$\because$ 分布函数有界

$P(y_i = x) = P(F(X_{(i)}) = x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$\therefore y_i \sim U(0,1)$

$\therefore y_i$ 是来自 $U(0,1)$ 总体的次序统计量

(b)

$\because$ y_i 为是来自均匀分布 $U(0,1)$ 总体的次序统计量

$\therefore$ $y_i \sim Be(i, n-i+1)$

$\therefore$ $E(y_i) = \frac{i}{n+1}$ $\quad Var(y_i) = \frac{i(n-i+1)}{(n+1)^2(n+2)}$ $\quad i=1,2\cdots n$

(3)

$$Cov(y_i, y_j) = E[(y_i - E(y_i))(y_j - E(y_j))] = E(y_i y_j) - E(y_i)E(y_j)$$

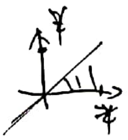
$$E(y_i)E(y_j) = \frac{i}{n+1}\cdot\frac{j}{n+1} = \frac{ij}{(n+1)^2}$$

$$P_{ij}(y,z) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F(y)]^{i-1}\,p(y)\,[F(z)-F(y)]^{j-i-1}[1-F(z)]^{n-j}\,p(y)p(z) \quad 0<y\le z\le 1$$

$$p(y) = p_i(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{otherwise} \end{cases} \qquad p(z) = p_j(z) = \begin{cases} 1 & 0<z<1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(y) = \begin{cases} y & 0<y<1 \\ 0 & y\le 0 \\ 1 & y\ge 1 \end{cases} \qquad F(z) = \begin{cases} z & 0<z<1 \\ 0 & z\le 0 \\ 1 & z\ge 1 \end{cases}$$

$$\therefore P_{ij}(y,z) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}y^{i-1}(z-y)^{j-i-1}(1-z)^{n-j} \quad 0<y\le z\le 1$$



$$E(y_i y_j) = \int_0^1 z\,dz\int_0^z y\frac{n!}{(i-1)!(j-i-1)!(n-j)!}y^{i-1}(z-y)^{j-i-1}(1-z)^{n-j}\,dy$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_0^1 z\,dz\int_0^z y^i(z-y)^{j-i-1}(1-z)^{n-j}\,dy$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_0^1 z^{j-i}(1-z)^{n-j}\,dz\int_0^z y^i\Big(1-\underbrace{\tfrac{y}{z}}_{x}\Big)^{j-i-1}dy$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_0^1 z^{j-i}(1-z)^{n-j}\,dz\int_0^1 z^i x^i(1-x)^{j-i-1}z\,dx$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_0^1 z^{j+1}(1-z)^{n-j}\,dz\int_0^1 x^i(1-x)^{j-i-1}\,dx$$

$$= \frac{ij+i}{(n+1)(n+2)} \qquad cov(y_i, y_j) = \frac{a_1(1-a_2)}{(n+2)}$$

$$\begin{pmatrix} Var(y_i) & Cov(y_i,y_j) \\ Cov(y_j,y_i) & Var(y_j) \end{pmatrix} \qquad Cov(y_j,y_i) = Cov(y_i,y_j)$$

$$Var(y_i) = \frac{i(n-i+1)}{(n+1)^2(n+2)} = \underbrace{\frac{i}{n+1}}_{a_1}\cdot\frac{n-a_1(n+1)+1}{(n+1)(n+2)} = \frac{a_1(1-a_1)}{(n+2)}$$

$$Var(y_j) = \frac{j(n-j+1)}{(n+1)^2(n+2)} = \frac{a_2(1-a_2)}{(n+2)}$$

$$\Rightarrow \begin{pmatrix} \frac{a_1(1-a_1)}{(n+2)} & \frac{a_1(1-a_2)}{(n+2)} \\ \frac{a_1(1-a_2)}{(n+2)} & \frac{a_2(1-a_2)}{n+2} \end{pmatrix}$$

6、 $m_{0.5} \sim N\left(X_{0.5}, \dfrac{0.5(1-0.5)}{n \cdot p^2(0.5)}\right)$

$p^2(0.5) = (6 \times \frac{1}{2} \times \frac{1}{2})^2 = \frac{9}{4}$

$\therefore \quad m_{0.5} \sim N(0.5, \frac{1}{9n})$ 渐近分布

概统 hw_16

1、 $X_1, X_2 \cdots X_n \sim N(\mu, 1)$

$\therefore \bar{X} \sim N(\mu, \frac{1}{n})$

$P(|\bar{X}| < c) = P(-c < \bar{X} < c) = P(\bar{X} < c) - P(\bar{X} < -c)$

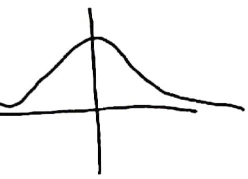
$= P\left(\frac{\bar{X}-\mu}{\sqrt{\frac{1}{n}}} < \frac{c-\mu}{\sqrt{\frac{1}{n}}}\right) - P\left(\frac{\bar{X}-\mu}{\sqrt{\frac{1}{n}}} < \frac{-c-\mu}{\sqrt{\frac{1}{n}}}\right)$

$= \underline{\Phi\left(\frac{c-\mu}{\sqrt{\frac{1}{n}}}\right) - \Phi\left(\frac{-c-\mu}{\sqrt{\frac{1}{n}}}\right)} \le \alpha$

$\Phi(\sqrt{n}(c-\mu)) - \Phi(\sqrt{n}(-c-\mu)) \le \alpha$

$\varphi(\sqrt{n}(c-\mu)) \cdot (-\sqrt{n}) - \varphi(\sqrt{n}(-c-\mu)) \cdot (-\sqrt{n}) \le \alpha$

$-\sqrt{n}(\varphi(\sqrt{n}(c-\mu)) - \varphi(\sqrt{n}(-c-\mu))) \le \alpha$



$\varphi(\sqrt{n}(-c-\mu)) \le \varphi(\sqrt{n}(c-\mu))$

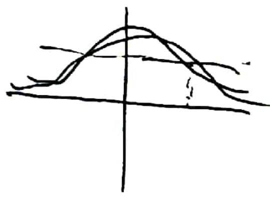
$\therefore -\sqrt{n}(\varphi(\sqrt{n}(c-\mu)) - \varphi(\sqrt{n}(-c-\mu))) \le 0$

$\therefore$ 其最大值为 $-c-\mu+c-\mu=0$ 时 $\mu = 0$

$\Phi(\sqrt{n}c) - \Phi(-\sqrt{n}c) \le \alpha$

$2\Phi(\sqrt{n}c) - 1 \le \alpha$

$\Phi(\sqrt{n}c) \le \frac{\alpha+1}{2}$



$\therefore 0 \le \sqrt{n}c \le u_{\frac{\alpha+1}{2}}$

$0 \le c \le \frac{u_{\frac{\alpha+1}{2}}}{\sqrt{n}}$

2. $X \sim F(n, m)$

$$\therefore p_X(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{n}{m}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \cdot x^{\frac{n}{2}-1}\left(1+\frac{n}{m}x\right)^{-\left(\frac{m+n}{2}\right)} \quad x \geq 0$$

$$z = \frac{\frac{n}{m}x}{1+\frac{n}{m}x} \qquad \left(1+\frac{n}{m}x\right)z = \frac{n}{m}x \qquad x = \frac{mz}{n-nz} \qquad \frac{d\left(\frac{m}{n-nz}\right)}{dz} = \frac{mn}{(n-nz)^2} = \frac{m}{n(1-z)^2}$$

$$p_z(z) = \frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{n}{m}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \cdot \left(\frac{mz}{n-nz}\right)^{\frac{n}{2}-1}\left(1+\cancel{\frac{n}{m}}\frac{\cancel{m}z}{\cancel{n}-\cancel{n}z}\right)^{-\frac{m+n}{2}} \cdot \frac{m}{n(1-z)^2} \quad 0<z<1$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right)\cancel{\left(\frac{n}{m}\right)^{\frac{n}{2}}}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\cancel{\left(\frac{m}{n}\right)^{\frac{n}{2}-1}}\left(\frac{z}{1-z}\right)^{\frac{n}{2}-1}\left(\frac{1}{1-z}\right)^{-\left(\frac{m+n}{2}\right)} \cdot \frac{\cancel{m}}{\cancel{n}(1-z)^2}$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{1}{1-z}\right)^{\frac{n}{2}-1}(z)^{\frac{n}{2}-1}\left(\frac{1}{1-z}\right)^{-\left(\frac{m+n}{2}\right)}\frac{1}{(1-z)^2}$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}(z)^{\frac{n}{2}-1}(1-z)^{\left(\frac{m}{2}-1\right)} \quad 0<z<1$$

$\therefore z \sim Be\left(\frac{n}{2}, \frac{m}{2}\right)$ 参数为F分布的分子分母自由度的$\frac{1}{2}$

hw-16

3、 $S_w^2 = \frac{(n-1)S_x^2+(m-1)S_y^2}{n+m-2}$ ， $S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$S_y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(y_i-\bar{y})^2$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2+\sum_{i=1}^{m}(y_i-\bar{y})^2}{n+m-2}$$

$$= \frac{\sum_{i=1}^{n}x_i^2-n\bar{x}^2+\sum_{i=1}^{m}y_i^2-m\bar{y}^2}{m+n-2}$$

$\bar{x} \sim (\mu_1, \frac{\sigma^2}{n})$ $\bar{y} \sim (\mu_2, \frac{\sigma^2}{m})$

$c\bar{x} \sim (c\mu_1, \frac{c^2\sigma^2}{n})$ $d\bar{y} \sim (d\mu_2, \frac{d^2\sigma^2}{m})$

$c\bar{x}+d\bar{y} \sim (c\mu_1+d\mu_2, \frac{c^2\sigma^2}{n}+\frac{d^2\sigma^2}{m})$

$$\frac{c\bar{x}+d\bar{y}-c\mu_1-d\mu_2}{\sqrt{\frac{c^2\sigma^2}{n}+\frac{d^2\sigma^2}{m}}} \sim N(0,1)$$

$\frac{(n-1)S_x^2}{\sigma^2} \sim X(n-1)$ $\frac{(m-1)S_y^2}{\sigma^2} \sim X(m-1)$

$\frac{(n-1)S_x^2}{\sigma^2}$ 与 $\frac{(m-1)S_y^2}{\sigma^2}$ 独立

$$\therefore \frac{(n-1)S_x^2+(m-1)S_y^2}{\sigma^2} \sim x^2(m+n-2) = \frac{(n+m-2)S_w^2}{\sigma^2}$$

$\because S_w^2$ 与 $\bar{x}+\bar{y}$ 独立 $\frac{(c\bar{x}+d\bar{y}-c\mu_1+d\mu_2)\sigma}{\sqrt{\frac{c^2\sigma^2}{n}+\frac{d^2\sigma^2}{m}}\ S_w} \sim t(m+n-2)$

$$\therefore \frac{c(\bar{x}-\mu_1)+d(\bar{y}-\mu_2)}{S_w\sqrt{\frac{c^2}{n}+\frac{d^2}{m}}} \sim t(m+n-2)$$