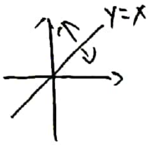
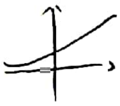
概统 hw 9

(1) $P(Z=1)=P(X\le Y)$ $\because X,Y$ 独立 $\therefore P(x,y)=\begin{cases}\lambda e^{-\lambda x}\cdot\mu\cdot e^{-\mu y} & x,y\ge 0\\ 0 & \text{else}\end{cases}$



$$=\int_{-\infty}^{+\infty}dx\int_x^{+\infty}\lambda e^{-\lambda x}\cdot\mu\cdot e^{-\mu y}\,dy=\lambda\mu\int_0^{+\infty}\cancel{\lambda}e^{-\lambda x}dx\int_x^{+\infty}\cancel{\mu}e^{-\mu y}\,dy$$



$$=-\lambda\frac{\mu}{\mu}\int_0^{+\infty}e^{-\lambda x}dx\int_x^{+\infty}e^{-\mu y}\,d(-\mu y)$$

$$=-\lambda\int_0^{+\infty}e^{-\lambda x}\cdot e^{-\mu y}\Big|_x^{+\infty}dx=-\lambda\int_0^{+\infty}e^{-\lambda x}\cdot(0-e^{-\mu x})\,dx$$

$$=\lambda\int_0^{+\infty}e^{-\lambda x-\mu x}dx=\frac{-\lambda}{\mu+\lambda}\int_0^{+\infty}e^{-(\lambda+\mu)x}\,d-(\lambda+\mu)x$$

$$=\frac{-\lambda}{\mu+\lambda}\cdot e^{-(\lambda+\mu)x}\Big|_0^{+\infty}=\frac{-\lambda}{\mu+\lambda}(0-1)=\frac{\lambda}{\mu+\lambda}$$

$$P(Z=0)=P(X>Y)=\int_0^{+\infty}dx\int_0^x\cancel{\lambda}e^{-\lambda x}\cancel{\mu}\cdot e^{-\mu y}\,dy$$

$$=-\lambda\mu\int_0^{+\infty}e^{-\lambda x}dx\int_0^x e^{-\mu y}\,dy\mu=-\lambda\int_0^{+\infty}e^{-\lambda x}\cdot(e^{-\mu y}\Big|_0^x)\,dx$$

$$=-\lambda\int_0^{+\infty}e^{-\lambda x}(e^{-\mu x}-1)\,dx=-\lambda\int_0^{+\infty}e^{-(\lambda+\mu)x}dx+\lambda\int_0^{+\infty}e^{-\lambda x}\,dx$$

$$=\frac{+\lambda}{\lambda+\mu}e^{-(\lambda+\mu)x}\Big|_0^{+\infty}+\frac{-\lambda}{\lambda}e^{-\lambda x}\Big|_0^{+\infty}=\frac{\lambda}{\lambda+\mu}(0-1)+-1(0-1)$$

$$=1-\frac{\lambda}{\lambda+\mu}=\frac{\mu}{\lambda+\mu}$$

(2)
(a) $P(x,y)=\begin{cases}e^{-(x+y)} & x,y>0\\ 0 & \text{else}\end{cases}$ $\quad U=X+Y,\ V=\frac{X}{X+Y}$ $\quad u=x+y,\ v=\frac{x}{x+y}$ $\quad X=uv,\ Y=u-uv$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\begin{vmatrix}1 & 1\\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2}\end{vmatrix}}=-(x+y)$$

$$P(u,v)=\begin{cases}e^{-(\cancel{uv}+u-\cancel{uv})}|-u|=ue^{-u} & 0<u,\ 0<v<1\\ 0 & \text{else}\end{cases}$$

(b)

$$P_U(u)=\int_0^1 ue^{-u}\,dv=ue^{-u}\qquad P_V(v)=\int_0^{+\infty}ue^{-u}\,du$$

$$=-\int_0^{+\infty}u\,de^{-u}$$

$$=\cancel{-}(\cancel{ue^{-u}\Big|_0^{+\infty}}\cancel{-}\int_0^{+\infty}e^{-u}\,du)$$

$$=-e^{-u}\Big|_0^{+\infty}=-(0-1)=1$$

$\therefore P_{(U,V)}(u,v)=ue^{-u}=P_U(u)\cdot P_V(v)$

$\therefore U$ 和 V 独立

(3)

$$P_i(X_i = x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \ge 0 \\ 0 & \text{else} \end{cases} \qquad F_i(x) = \begin{cases} 1 - e^{-\lambda_i x} & x \ge 0 \\ 0 & \text{else} \end{cases}$$

独立

$$P_{(X_1, X_2 \cdots X_n)}(x_1, x_2 \cdots x_n) = \left(\prod_{j=1}^{n} \lambda_j\right) \cdot e^{-\sum_{i=1}^{n} \lambda_i x_i}$$

$$P(X_i = \min\{X_1, X_2 \cdots X_n) = P(\{X_n \ge x_i, X_{n-1} \ge x_i \cdots X_1 \ge x_i\})$$

$$= \int_0^{+\infty} \int_{x_i}^{\infty} \cdots \int_{x_i}^{+\infty} \left(\prod_{j=1}^{n} \lambda_j\right) e^{-\sum_{i=1}^{n} \lambda_i x_i} \, dx_1 \cdots dx_i$$

$$= \int_0^{+\infty} \lambda_i \, e^{-(\lambda_1 + \cdots \lambda_n) x_i} \, dx_i$$

$$= \frac{\lambda_i}{\lambda_1 + \cdots \lambda_n} \cdot \left(-e^{-(\lambda_1 + \cdots \lambda_n) x_i} \Big|_0^{+\infty} \right)$$

$$= \frac{\lambda_i}{\lambda_1 + \cdots \lambda_n} \left[-(0 - 1) \right] = \frac{\lambda_i}{\lambda_1 + \cdots \lambda_n}$$

(4) "X_i" 第i天的得分

(a) $P_i(X_i = x) = 0.1 \quad x = 101, 102, 103, \cdots 110$.

$$F_i(x) = \begin{cases} 0 & x < 101 \\ 0.1x - 10 & 101 \le x < 110 \\ 1 & x \ge 110 \end{cases}$$

$X = \min\{X_1, X_2, X_3\}$

$$\begin{aligned} F_X(x) &= P(\min\{X_1, X_2, X_3\} \le x) \\ &= 1 - P(\min\{X_1, X_2, X_3\} > x) \\ &= 1 - P(X_1 > x) \cdot P(X_2 > x)\, P(X_3 > x) \\ &= 1 - (1 - F_1(x))(1 - F_2(x))(1 - F_3(x)) \quad \because X_1, X_2, X_3 \sim F_i(x) \\ &= 1 - [1 - F_1(x)]^3 \end{aligned}$$

X	101	102	103	104	105	106	107	108	109	110
P	0.271	0.217	0.169	0.127	0.091	0.061	0.037	0.019	0.007	0.001

(b) $E(X_1) = E(X_2) = E(X_3) = 105.5$

$E(X) = 103.025$

改变了 2.475

概统 10

$Y \sim Nb(2,p)$ $P(Y=k) = C_{k-1}^{1}(1-p)^{k-2}p^2$, $k=2,3,\cdots$

1、 $X \sim Ge(p)$ $P(X=k) = (1-p)^{k-1}p$, $k=1,2,3\cdots$

$P_{(X,Y)}(x,y)$ $X=x$ 第一次命中目标需射 x 次

$Y=y$ 第二次命中时总共射了 y 次 $\therefore y \geq x+1$

$P_{(X,Y)}(x,y) = (1-p)^{x-1}p(1-p)^{y-x-1}p = p^2(1-p)^{y-2}$, $x=1,2,3,\cdots,y-1$

$y=2,3,4\cdots$

① $\underbrace{\quad}_{}$ $x-1$次失败 $\circ$ 成功 $\underbrace{\quad}_{}$ $y-x-1$次失败 $\circ$ 成功 $\Rightarrow (1-p)^{x-1}\cdot p\cdot(1-p)^{y-x-1}\cdot p$

$P(X=x|Y=y) = \dfrac{P(X=x,Y=y)}{P(Y=y)}$ $P(Y=y|X=x) = \dfrac{P(X=x,Y=y)}{P(X=x)}$

$P(Y=y) = \sum\limits_{x=1}^{y-1} p^2(1-p)^{y-2} = (y-1)p^2(1-p)^{y-2}$

$P(X=x) = \sum\limits_{y=x+1}^{+\infty} p^2(1-p)^{y-2} = p^2(1-p)^{x-1} + p^2(1-p)^x + \cdots$ p^2 $\cdots$

$= p^2\dfrac{(1-p)^{x-1}}{1-\cancel{(1-p)}\,p} = p(1-p)^{x-1}$

$\therefore P(X=x|Y=y) = \dfrac{p^2\cancel{(1-p)^{y-2}}}{(y-1)p^2\cancel{(1-p)^{y-2}}} = \dfrac{1}{y-1}$ $P(Y=y|X=x) = \dfrac{p^2(1-p)^{y-2}}{p(1-p)^{x-1}} = p(1-p)^{y-x-1}$

X与$Y-X$独立 $y=2,3\cdots$ 有条件概率=无条件概率(独立) $x=1,2\cdots3,\cdots y-1$

$y=2,3,4\cdots$

② $P(X=x, Y=y) = P(Y=y|X=x)P(X=x) = P(X=x)P(Y-X=y-x|X=x)$

$= P(X=x)P(Y-X=y-x) = (1-p)^{x-1}\cdot p\cdot(1-p)^{y-x-1}p = p^2(1-p)^{y-2}$

2、 $P(x) = \begin{cases} 1 & 1<x<2 \\ 0 & \text{otherwise} \end{cases}$ $Y|X=x \sim Exp(x)$ $y \geq 0$

令 $U=XY$ $V=X$ $X=V$ $Y=\dfrac{U}{V}$ $J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{u}{-v^2} \end{vmatrix} = \dfrac{-1}{v}$

$P(x,y) = P_X(x)\cdot P(y|x) = \begin{cases} x\cdot e^{-xy} & 1<x<2, y\geq 0 \\ 0 & \text{otherwise} \end{cases}$

$P(u,v) = \begin{cases} \cancel{v}\cdot e^{-u}\cdot\cancel{\frac{1}{v}} & 1<v<2, u\geq 0 \\ 0 & \text{otherwise} \end{cases}$

$P_U(u) = P_{XY}(xy) = \int_1^2 e^{-u}dv = e^{-u} = \begin{cases} e^{-xy} & xy \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$\therefore$ XY 服从 $Exp(1)$

扫描全能王 创建

3、(a)

$$E[g(X)Y\mid X=x] = \int_{-\infty}^{+\infty} g(x)\, y \cdot p(y\mid x)\,dy = g(x)\int_{-\infty}^{+\infty} y\,p(y\mid x)\,dy = g(x)\,E(Y\mid X)$$

$$\therefore\ E[g(X)Y\mid X] = g(x)\,E(Y\mid X)$$

(b) $$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\,p(x,y)\,dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\,p(y\mid x)\,p_X(x)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} y\,p(y\mid x)\,dy\right) x\,p_X(x)\,dx = \int_{-\infty}^{+\infty} \underbrace{E(Y\mid X=x)}_{g(x)}\,p_X(x)\cdot x\,dx = \int_{-\infty}^{+\infty} \underline{g(x)\cdot x\,p_X(x)}\,dx$$

$$= E(g(X)\cdot X) = E(E(Y\mid X)\cdot X)$$

(c)

$$\mathrm{Cov}[X,Y] = E(XY) - E(X)E(Y) = E(X\cdot E(Y\mid X)) - E(X)E(E(X\mid Y))$$

$$= \mathrm{Cov}(X, E(Y\mid X))$$

(4)

(a) k为Pat到达的时间

$$P(k) = \begin{cases} \frac{1}{2} & 8<k<10 \\ 0 & \text{otherwise} \end{cases}$$

$k \geq 9$

$$P(k) = \begin{cases} 1 & 9<k<10 \\ 0 & \text{otherwise} \end{cases}$$

$E(k-9) = E(k) - 9 \quad (k \geq 9) = 9.5 - 9 = 0.5$ 小时

(b) z为约会持续时间

~~$z = \begin{cases} 3 & 0<x<1 \\ & 1<x<2 \end{cases}$~~

~~$P(z,x) = \begin{cases} \frac{1}{3-x} & 0<z<3-x,\ 1<x<2 \\ z=3 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$~~

"H"为Pat到达的时刻

$$P(h) = \begin{cases} \frac{1}{2} & 8<h<10 \\ 0 & \text{otherwise} \end{cases}$$

$x = h-8$

$$P(x) = \begin{cases} \frac{1}{2} & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$$

$$E(z\mid X=x) = \int_0^1 3\cdot\frac{1}{2}\,dx + \int_1^2 \frac{1}{3-x}\cdot\frac{1}{2}\,dx = \frac{3}{2} + \frac{\ln 2}{2}$$

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(c) "K" Pat 是否迟到超过 45 min

$K=1$

$P(K=1)=P(X\geq 1.75)=\int_{1.75}^{2}\frac{1}{2}\,dx=\frac{1}{2}\times 0.25=0.125$

$P(K=0)=P(0<X<1.75)=\int_{0}^{1.75}\frac{1}{2}\,dx=0.875$

$\therefore P(K)=\begin{cases}0.125 & K=1\\ 0.875 & K=0\end{cases}$

"M" 分手前的约会次数

$E(M|K=0)=1+E(M)$