On a fundamental polyhedron of a hyperbolic cone-manifold

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Abstract

In this paper we propose to consider two different solutions having a common problem of establishing hyperbolic structure on a 3-manifold. Under the consideration will be a cone 3-manifold with underlying space as a 3-sphere and a singular set nested in it. Furthermore, this paper is divided into two cases: a singular set as the 3_1 knot with a bridge and a singular set as the 6_1^3 link. The hyperbolic space $\mathbb{H}^3$ for the analytical examination in the first case will be a hyperboloid model, in the second using the upper-half space model. To show that the cone-manifold admits a hyperbolic structure, its fundamental set was constructed in the space of each case and the conditions of its existence were provided. In addition, for the first case we will present the analytical formula to calculate the hyperbolic volume of its manifold.

1 Introduction

The proof of Thurston's Geometrization Conjecture [21] by Grigory Perelman (ссылка на Перельмана) in 2003 completes the picture of understanding the nature of 3-manifolds. So 3-manifolds can be decomposed on components each of them are associated with one of the 8 geometries [21]. However, just 7 manifold types out of 8 are classified and well understood. The problematic one is hyperbolic 3-manifolds. For this reason, it is good to practice using different methods to accomplish this problem. They allow us to discover and to investigate new invariants or simplify calculations of ones already known. On the other hand, it is also just another way of looking at the problem. For instance, a connection between knot theory and hyperbolic 3-manifolds (for instance [17], [18], [21]) provides many tools for 3-manifold analysis (for instance [2], [8], [6], [24]).

We'll be dealing with a cone 3-manifold [7]. It is a special case of the orbifolds where cone angles are $\frac{2\pi}{m}$, for $m \geq 2$, m is integer.

In the present work, we consider two cone-manifolds (Sections 3, 4). Their underlying space is the 3-sphere S^3, and the singular set is the trefoil knot with a bridge which is denoted by 3_1 by Rolfsen's knot tabulation and the link 6_1^3. In the first case, the singular set has the cone angles α along the knot component and γ along the bridge. Hence, we denote the corresponding cone-manifold by

$3_1(\alpha, \gamma)$. In the second case, the angles around each component are equal to α and the cone-manifold will be denoted by $6_1^3(\alpha)$.

Our goal is to show that the manifolds under consideration $3_1(\alpha, \gamma)$ and $6_1^3(\alpha)$ admit the hyperbolic structure (Sections 3, 4). We will accomplish it in a few steps. The first step will be focusing on the geometrical construction of the fundamental polyhedron F of cone-manifold considered in suitable model of hyperbolic space (Section 3, Theorem 1 and Subsection **??**). In general, the construction of a fundamental polyhedron of the cone-manifold provides analytical geometrical relations, which could be useful for the knot itself. It allows to build "the geometry of a knot" to investigate or to find a new knot invariants. The conditions of the existence of such a polyhedra are exactly the same conditions providing the realization of hyperbolic geometry on the manifold. The last step is to find an analytic formula calculating hyperbolic volume for cone-manifold $3_1(\alpha, \gamma)$ which will be based on the resulting fundamental polyhedron and its existance conditions (Section 3, Theorem 2).

2 Preliminaries

In this section, we will provide the basic terms and definitions of some of the concepts used in the work.

A *knot* $K \subset S^3$ is a subset of points homeomorphic to a circle S^1 under a piecewise linear (PL) homeomorphism [11]. We may also think of a knot as a PL embedding $K : S^1 \to S^3$. We will use the same symbol K to refer to the map and its image $K(S^1)$. Generally speaking, a *link* is a subset of S^3 PL homeomorphic to a disjoint union of copies of S^1. Alternatively, we may think of a link as a PL embedding of a disjoint union of copies of S^1 into S^3.

For a knot K, the *knot complement* is the open manifold $S^3 - K$. Similarly, if L is a link, the *link complement* is $S^3 - L$.

The complement of a knot is *hyperbolic* (Euclidian, spherical) if and only if it admits a complete metric with all sectional curvatures equal to -1 (0, 1 respectively). Briefly, we may say a knot is hyperbolic if its complement is hyperbolic.

The *group* of a knot K is the fundamental group of the complement $S^3 - K$.

2.1 Models of hyperbolic space $\mathbb{H}^3$

Let $R^{1,n}$ be the real vector space R^{n+1} of dimension $(n+1)$ equipped with the bilinear form of signature $(1, n)$

$$\begin{aligned} &\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n, \\ &\forall \mathbf{x} = (x_0 \ldots x_n), \mathbf{y} = (y_0 \ldots y_n) \in R^{1,n}. \end{aligned} \tag{1}$$

And let

$$H_- = \{\mathbf{x} \in R^{1,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 > 0\}$$

be the upper half sheet of the two sheeted hyperboloid. The quadratic form induced by (1) on $R^{1,n}$ gives a Riemannian metric on H_- producing *hyperboloid model* of n - dimensional hyperbolic space $\mathbb{H}^n$. Under this metric, the hyperbolic distance between two points $\mathbf{x}$ and $\mathbf{y}$ from $H_- \subset R^{1,n}$ can be calculated by the following formula

$$\langle \mathbf{x}, \mathbf{y} \rangle = -\cosh d. \tag{2}$$

Denote by P the radial projection from $\{\mathbf{x} \in R^{1,n} \,|\, x_0 \neq 0\}$ to the affine hyperplane

$$\mathbb{P}^n_1 = \{\mathbf{x} \in R^{1,n} \mid x_0 = 1\},$$

along the rays through the origin. The projection P is a homeomorphism from H_- onto the n-dimensional ball $\mathbb{B}^n$ in $\mathbb{P}^n_1$ centered at $(1, 0, \dots, 0) \in \mathbb{P}^n_1$, which gives the projective model of $\mathbb{H}^n$. In other words, above builded projection gives *Klein model* in the ball $\mathbb{B}^n$, where the boundary $\partial\mathbb{B}^n$ is the absolute of this model. For more details one may read [12].

Now let us consider another model of the hyperbolic space $\mathbb{H}^3$. The *upper half-space model* can be described as a set of points $\{z + tj \,:\, z \in \mathbb{C}, t > 0\}$, which can be easily associated with the points in Euclidian $\mathbb{R}^3$ space, but in a basis $\{i,\, j,\, k\}$. The space is endowed by Riemannian metric

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2}$$

which is invariant under the group $PSL(2, \mathbb{C})$, whereas the elements of this group are the isometries of the model under consideration.

The well known fact $PSL(2, \mathbb{C}) \cong SL(2, \mathbb{C})/\{\pm\mathbf{1}\}$ gives the most convinient way to deal with this model [[26], p. 142], so that for $z \in \mathbb{C},\ t > 0$ its points can be represented in the following way

$$z + tj \to \begin{pmatrix} \dfrac{z}{t} & \dfrac{z\overline{z} + t^2}{t} \\ \dfrac{1}{t} & -\dfrac{\overline{z}}{t} \end{pmatrix}. \tag{3}$$

Let f be an isometry of upper-half space model $\mathbb{H}^3$ and P be a point (3) of $\mathbb{H}^3$. Then f acts on the point P in following way $f(P) = fPf^*$, where operation "$*$" means standard conjugation. The composition of two isometries f and g is carried out according to the rule $f \circ g = fgf^{-1}$.

There are a lot of helpful representations in $SL(2, \mathbb{C})$ for a geometric elements living in the upper half-space model, for this reason we recommend to see [26]. To avoid the confusion between these two models, we will use special notations: $\mathbb{H}^3_h$ – for hyperboloid model and $\mathbb{H}^3_u$ – for upper-half space model.

In the end, the fact must be noted that the above mentioned models are isometrically equivalent [10]. So one can use any model depending on the complexity of the calculations.

3 Cone-manifold $3_1(\alpha, \gamma)$

Let us consider the knot $K = 3_1$ [Fig. 1] with a bridge and its complement $S^3 - 3_1$. Its fundamental group $\pi_1(S^3 - 3_1)$ can be found with help of *Wirtinger presentation*. Thus, the fundamental group consists of two generators s and t:

$$\pi_1(S^3 - 3_1) = \langle s, t \mid stst^{-1}s^{-1}t^{-1} = 1 \rangle.$$

The holonomy map $\varphi : \pi_1(S^3 - 3_1) \to \mathrm{Isom}(\mathbb{H}^3_h)$ sends generators s and t of the fundamental group $\pi_1(S^3 - 3_1)$ into the isometries $\mathcal{S}$ and $\mathcal{T}$

$$\mathcal{S}(\mathbf{x}) = \mathbf{R}^{-1}\,\mathbf{S}\,\mathbf{R}\,\mathbf{x},\ \ \mathcal{T}(\mathbf{x}) = \mathbf{R}\,\mathbf{T}\,\mathbf{R}^{-1}\,\mathbf{x},\ \ \forall \mathbf{x} \in \mathbb{H}^3 \tag{4}$$

of the hyperbolic space $\mathbb{H}^3_h$, which we defined in section 2.1. The isometries $\mathcal{S}$ and $\mathcal{T}$ represent the rotation by angle α around the singular component, which corresponds to the knot 3_1. At the same time the holonomy map of relation $stst^{-1}s^{-1}t^{-1}$ is a rotation by angle γ around the singular component, which corresponds to the bridge of the knot 3_1.

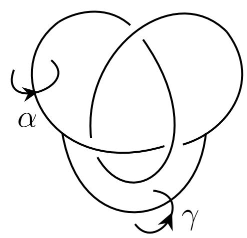

Fig. 1: *The 3_1 knot with a bridge.*

The matrices $\mathbf{S}$ and $\mathbf{T}$ of the rotations $\mathcal{S}$ and $\mathcal{T}$ respectively, have the following structure

$$\mathbf{S} = \frac{1}{M^2+1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & M^2+X^2-Y^2 & 2XY & -2MY \\ 0 & 2XY & M^2-X^2+Y^2 & 2MX \\ 0 & 2MY & -2MX & -1+M^2 \end{pmatrix},$$

$$\mathbf{T} = \frac{1}{M^2+1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & M^2+X^2-Y^2 & -2XY & 2MY \\ 0 & -2XY & M^2-X^2+Y^2 & 2MX \\ 0 & -2MY & -2MX & -1+M^2 \end{pmatrix},$$

where $M = \cot\frac{\alpha}{2}, X = \cos\frac{\theta}{2}, Y = \sin\frac{\theta}{2}$, and θ — is the angle of relative rotation between the singular arcs with a holonomy map images $\mathcal{S}$ and $\mathcal{T}$. The translation isometry of $\mathbb{H}^3_h$ space has the following matrix form

$$\mathbf{R} = \begin{pmatrix} \cosh\dfrac{a}{2} & 0 & 0 & \sinh\dfrac{a}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\dfrac{a}{2} & 0 & 0 & \cosh\dfrac{a}{2} \end{pmatrix},$$

where a is a distance between singular components with homology map images $\mathcal{S}$ and $\mathcal{T}$.

Let us consider a non-convex 12-faced polyhedron F with vertices P_i, N, S, $i = 1, ..., 6$. Let F consist of 6 tetrahedrons T_i, $i = 1, ..., 6$, with common edge NS. The tetrahedrons could be orientated by their own vertices with respect to the following rules

$$\begin{aligned} &T_1 : NP_1SP_2, \;\; T_4 : NP_4SP_5, \\ &T_2 : SP_3NP_2, \;\; T_5 : SP_6NP_5, \\ &T_3 : NP_3SP_4, \;\; T_6 : NP_6SP_1. \end{aligned} \tag{5}$$

At last we will present the defenition, wich will help us to identify fundamental polyhedron of the cone-manifold

Definition 1. *The non-convex polyhedron F is a fundamental polyhedron of the cone-manifold $3_1(\alpha, \gamma)$, if the following conditions are satisfied:*

(1) The isometries $\mathcal{S}$ and $\mathcal{T}$ of hyperbolic space $\mathbb{H}^3_h$, "gluing" curvilinear faces by the following rules:

$$\begin{aligned} &\mathcal{S} : P_1P_6P_5P_4 \to P_1P_2P_3P_4, \\ &\mathcal{T} : P_6P_1P_2P_3 \to P_6P_5P_4P_3. \end{aligned} \tag{6}$$

(2) The sum of interior dihedral angle along the edges $P_1 ... P_6$ is equal to the cone angle γ.

(3) Interior dihedral angles along the edges NP_1, NP_4, SP_3, SP_6 are equal to the cone angle α.

Theorem 1. *Let $\frac{\pi}{3} < \alpha < \pi$ and γ satisfies the following equation*

$$\cos^2\frac{\gamma}{2} = \frac{(C^3 - A\,B)^2}{C^6},$$

where $A = (4\,X^2 + 3\,X + 1)\,Y - 3\,X^2 - X$, $B = (4\,X + 3)\,Y^2 + (-3\,X - 1)\,Y + X$, $C = Y - X + 2\,X\,Y$, $X = \cosh a$, $Y = \cos\theta$. *Furthermore, if $X > 1$ then the following inequality holds $\frac{1}{2} < \frac{X}{X+1} < Y < 1$. Thus, the cone-manifold $3_1(\alpha, \gamma)$ admits a hyperbolic structure.*

Proof. We will use the fact that the cone-manifold $3_1(\alpha, \gamma)$ admits hyperbolic structure if its fundamental polyhedron exists in $\mathbb{H}^3_h$.

Isometries (4) identify the faces of polyhedron by the following rules

$$\begin{aligned} &\mathcal{S} : P_1P_6P_5P_4 \to P_1P_2P_3P_4, \\ &\mathcal{T} : P_6P_1P_2P_3 \to P_6P_5P_4P_3. \end{aligned} \tag{7}$$

Let edges P_1P_4 and P_3P_6 of a non-convex polyhedron F lie on a fixed axes of isometries $\mathcal{S}$ and $\mathcal{T}$ respectively. Thus, θ is the angle of relative rotation between edges P_1P_4 and P_3P_6. For the isometry map $\mathcal{G}$ (22) which is the rotation by angle γ, its fixed axis of rotation will be identified with edge P_1P_2.

In the space $\mathbb{H}^3_h$ the vertices $P_i,\ \ i = 1, 2, 3, 4, 5, 6$ of polyhedron F have the following structure

$$P_i = \frac{p_i}{\sqrt{|\langle p_i, p_i\rangle|}}, \tag{8}$$

they satisfy the conditions $\langle P_i, P_i\rangle = -1$. Additionally, we have to add two new vertices. Denote by N and S midpoints of edges P_1P_4 and P_3P_6 respectively

$$N = \frac{p_1 + p_4}{\sqrt{|\langle p_1 + p_4,\ \ p_1 + p_4\rangle|}},\ \ S = \frac{p_3 + p_6}{\sqrt{|\langle p_3 + p_6,\ \ p_3 + p_6\rangle|}}, \tag{9}$$

$\langle N, N\rangle = -1,\ \ \langle S, S\rangle = -1$. Let a be a hyperbolic distance between two vertices N and S

$$\langle N,\ S\rangle = -\cosh a. \tag{10}$$

By (8) and (9), the elements $p_i,\ \ i = 1, 2, 3, 4, 5, 6$ are coordinates defined parametrically

$$\begin{aligned}
&p_1 = (1, t\cos\frac{\theta}{2}, t\sin\frac{\theta}{2}, \tanh\frac{\theta}{2}),\ p_2 = (1, 0, y, 0),\\
&p_3 = (1, -t\cos\frac{\theta}{2}, t\sin\frac{\theta}{2}, -\tanh\frac{\theta}{2}),\ p_4 = (1, -t\cos\frac{\theta}{2}, -t\sin\frac{\theta}{2}, \tanh\frac{\theta}{2}),\\
&p_5 = (1, 0, -y, 0),\ p_6 = (1, t\cos\frac{\theta}{2}, -t\sin\frac{\theta}{2}, -\tanh\frac{\theta}{2}).
\end{aligned}$$

From the geometric point of view, coefficients of dilation/contraction $\lambda, \mu, \delta, \rho$ that appear under the action of isometric maps $\mathcal{S}$ and $\mathcal{T}$ on the coordinates of corresponding vertices were put under consideration. Then, by solving the system of 4 equations for each pair of vertices

$$\lambda\mathcal{S}(P_6) = P_2,\ \ \mu\mathcal{S}(P_5) = P_3,\ \ \delta\mathcal{S}(P_1) = P_5,\ \ \rho\mathcal{S}(P_2) = P_4, \tag{11}$$

we will get coefficient values satisfying all the vertices at once

$$\begin{aligned}
t &= \frac{\sqrt{1 + 2\cosh a}\ \ \operatorname{sech}\frac{a}{2}\tanh\frac{a}{2}\tan\frac{\theta}{2}}{\sqrt{-1 + 2\cos\theta}},\\
y &= \frac{\sqrt{1 + 2\cosh a}\ \ \cos\theta\sec\frac{\theta}{2}\operatorname{sech} a\sinh\frac{a}{2}}{\sqrt{-1 + 2\cos\theta}}.
\end{aligned}$$

Moreover, from (11), one can get new relations for the parameter $M = \cot\frac{\alpha}{2}$

$$M^2 = (-1 + 2\cos\theta)(1 + 2\cosh a), \tag{12}$$

showing the connections between angle α, hyperbolic distance a between edges P_1P_4 and P_3P_6, and angle θ which is the angle of relative rotation between them. Thus, we can conclude that the geometrical construction of polyhedron F depends on parameter variations with respect to their existence conditions and without loss of generalisation we will define it as $F_{a,\theta}$.

At last, from the necessarily condition

$$-\langle P_i, P_j \rangle > 1, \;\; i,j = 1,2,3,4,5,6, \;\; i \neq j$$

we get the existence conditions for the polyhedron $F_{a,\theta}$

$$\frac{1}{2} < \frac{\cosh a}{\cosh a + 1} < \cos\theta < 1, \tag{13}$$

and with use of (12), one can obtain

$$\frac{\pi}{3} < \alpha < \pi.$$

In conclusion, if $X = \cosh a$, $Y = \cos\theta$ and with the use of standard trigonometry identities, the representation for angle γ can be rewritten in the equivalent form

$$\cos^2 \frac{\gamma}{2}.$$

Remark 1. To check definition 1 for the polyhedron $F_{a,\theta}$, one can calculate the interior dihedral angles of orientated tetrahedrons (5) with use of the algorithm described in [4].

Therefore, polyhedron $F_{a,\theta}$ is a fundamental polyhedron of cone-manifold $3_1(\alpha,\gamma)$.

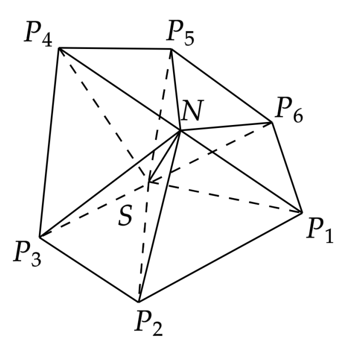

Fig. 2: *Polyhedron $F_{a,\theta}$ in the Klein model.*

Remark 2. The conditions (13) provide the information about polyhedron's $F_{a,\theta}$ vertices position in the space $\mathbb{H}^3_h$ Let $\cosh a = X$, $\cos\theta = Y$, then

if $Y = \frac{X}{X+1}$, then vertices $P_1 ... P_6$ lie on the absolute;

if $Y < \frac{X}{X+1}$, then vertices $P_1 ... P_6$ lie beyond the absolute.

Finally, we have shown that the cone-manifold $3_1(\alpha, \gamma)$ admits a hyperbolic structure. $\square$

The next step is to find hyperbolic volume of the cone-manifold $3_1(\alpha, \gamma)$, which coincides with a volume of fundamental polyhedron $F_{a,\theta}$. Further, all the elements of the fundamental polyhedron $F_{a,\theta}$ will be expressed in the terms of $X = \cosh a$ and $Y = \cos\theta$.

Theorem 2. *Let $X = \cosh a, Y = \cos\theta$, then the hyperbolic volume of a cone-manifold $3_1(\alpha, \gamma)$ can be calculated by the formula*

$$Vol(3_1(\alpha, \gamma)) = -\frac{1}{2}\int_1^{X_0} L_\gamma(X)d(\gamma(X)),$$

where if $X > 1$, X_0 is the only solution of the system of equations

$$\begin{cases} \dfrac{\sinh^2 L_\gamma}{\tan^2 \frac{\gamma}{4}} \dfrac{\operatorname{csch}^2 L_\alpha}{\tan^2 \frac{\alpha}{2}} = \dfrac{(X-Y)^2}{4}, \\ \cot^2 \dfrac{\alpha}{2} = (2X+1)(2Y-1). \end{cases}$$

Proof. Suppose that the conditions of Theorem 1 are satisfied and let $F_{a,\theta}$ be a fundamental polyhedron of the cone-manifold $3_1(\alpha, \gamma)$ as in proof of Theorem 1.

The hyperbolic length (1) for edges P_1P_4 and P_1P_2 of $F_{a,\theta}$ will be denoted as $\cosh L_\alpha$ and $\cosh L_\gamma$ respectively

$$\begin{aligned} \cosh L_\alpha &= \frac{(4X^2+X-1)Y^2 + (-X^2+2X+1)Y - X^2 - X}{2(X+1)Y^2 + 2X^2Y - 2X^2}, \\ \cosh L_\gamma &= \frac{(X+1)Y^2 + (-X^2+X+1)Y + X^2 - X - 1}{(X+1)Y^2 + X^2Y - X^2}. \end{aligned} \tag{14}$$

Then by differential Schläfli formula [23] we have

$$dV = -\frac{1}{2}(L_\alpha d\alpha + L_\gamma d\gamma), \tag{15}$$

where by the use of relation (12), the angle α has the following form

$$\sin^2\frac{\alpha}{2} = \frac{1}{-X + (2X+1)Y}, \tag{16}$$

for angle γ, we use the representation from Theorem 1

$$\cos\frac{\gamma}{2} = -\frac{C^3 - A\,B}{C^3}, \tag{17}$$

here $A = (4\,X^2 + 3\,X + 1)\,Y - 3\,X^2 - X$, $B = (4\,X + 3)\,Y^2 + (-3\,X - 1)\,Y + X$, $C = Y - X + 2\,X\,Y$.

Assuming that the angle α is fixed. Then, the differential form (15) has the following form

$$dV = -\frac{1}{2} L_\gamma d\gamma. \tag{18}$$

Now, it will be natural to represent the dependence of L_γ and the cone angle γ on α. To do this, we express the representation (16) in terms of the parameter Y and apply the resulting equality for the hyperbolic length L_γ expression (14) and for the angle γ (17). Thus, if the cone angle α is chosen, then L_γ and γ depend only on the parameter X, and the expression (18) becomes

$$dV = -\frac{1}{2} L_\gamma(X) d(\gamma(X)). \tag{19}$$

Now with the conclusions above in mind, we need to set the integration limits for the expression (19). First, it can be noted that if we let the distance a (10) between the edges P_1P_4 and P_3P_6 tends to 0 or same, X tend to 1, the hyperbolic distance L_γ will converge to zero. Then, according to (19), the volume function $V(X)$ at the point $X = 1$ will be equal to zero. On the other hand, from the relations (14), (16) and (17) the expression

$$\frac{\sinh^2 L_\gamma}{\tan^2 \frac{\gamma}{4}} \frac{\operatorname{csch}^2 L_\alpha}{\tan^2 \frac{\alpha}{2}} = \frac{(X-Y)^2}{4},$$

can be achieved in a unique way.

Now, let the cone angles α and γ be fixed. Then, together with the relation (12), where $M = \cot \frac{\alpha}{2}$, we get the system of equations

$$\begin{cases} \dfrac{\sinh^2 L_\gamma}{\tan^2 \frac{\gamma}{4}} \dfrac{\operatorname{csch}^2 L_\alpha}{\tan^2 \frac{\alpha}{2}} = \dfrac{(X-Y)^2}{4}, \\ \cot^2 \dfrac{\alpha}{2} = (2X+1)(2Y-1), \end{cases}$$

which for $X > 1$ provides a unique solution $X = X_0$, $Y = Y_0$ for the given angles α and γ.

Thus, integrating the expression (19) over the interval $[1, X_0]$ we obtain the volume of the cone manifold $3_1(\alpha, \gamma)$. □

In the table below, we present volumes found of the three-dimensional cone manifold $3_1(\alpha, \gamma)$

$\alpha,\ \gamma$	X	$Volume$
$2\pi/3,\ 2\pi/2$	1.11124	0.264774
$2\pi/4,\ 2\pi/2$	1.61803	1.83193
$2\pi/5,\ 2\pi/2$	2.32419	2.97947
$2\pi/6,\ 2\pi/2$	3.22138	3.46256
$2\pi/7,\ 2\pi/2$	4.30426	3.73303
$2\pi/3,\ 2\pi/5$	1.26927	2.2983

$2\pi/4,\ 2\pi/5$	1.86962	3.53971
$2\pi/5,\ 2\pi/5$	2.69991	4.12611
...		

These results were tested on the software Orb, which is based on numerical methods for finding the volume of three-dimensional cone manifolds.

4 Cone-manifold $6_1^3(\alpha)$

For this case, we consider a spesific setting for cone angles, where all of them are equal. So it will be reasonable to call it a "symmetrical" case. As the link 6_1^3 has three components [Fig. 3], then a cone-manifold $6_1^3(\alpha)$ with underlying space as the 3-sphere, has three singular components and each of them has a cone angle $\alpha = \beta = \gamma$.

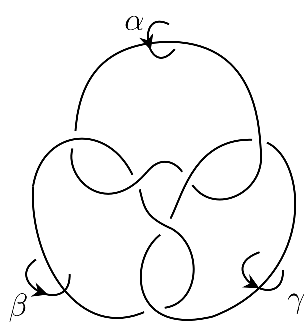

Fig. 3: The 6_1^3 link.

The fundamental group for link 6_1^3 has the following structure

$$\pi_1(\mathbb{S}^3 \backslash 6_1^3) = < a, b, c \,|\, a l_a = l_a a, b l_b = l_b b, c l_c = l_c c >, \tag{20}$$

where generators a, b and c are the *meridians* of the link.

To prove the main statment in this section, from this point, we will be dealing with the construction of the fundamental polyhedron of the cone-manifold $6_1^3(\alpha)$.

By using the geometrical structures of hyperbolic upper-half space model $\mathbb{H}_u^3$ [26] and properties of a exponential map [5], one may check that the holonomy map $\varphi : \ \pi_1(\mathbb{S}^3 \backslash 6_1^3) \to \mathrm{Isom}(\mathbb{H}_u^3)$ sends generators a, b and c into the isometries A, B and C, which have the following representations in the projective special linear group $\mathbb{PSL}(2, \mathbb{C})$

$$A = \begin{pmatrix} X + i\ c_0 Y & (c_0 - 1) Y \\ (1 + c_0) Y & X - i\ c_0 Y \end{pmatrix}, B = \begin{pmatrix} X + i\ c_0 Y & (c_0 - 1) e^{\frac{i 2\pi}{3}} Y \\ (1 + c_0) Y e^{-\frac{i 2\pi}{3}} & X - i\ c_0 Y \end{pmatrix},$$

$$C = \begin{pmatrix} X + i\ c_0 Y & (c_0 - 1) e^{\frac{i 4\pi}{3}} Y \\ (1 + c_0) Y e^{-\frac{i 4\pi}{3}} & X - i\ c_0 Y \end{pmatrix}$$

here $X = \cos\frac{\alpha}{2}, Y = \sin\frac{\alpha}{2}, c_0 \in \mathbb{C}$. Isometries A, B and C, represent the rotation by angle α around each singular component of the cone-manifold $6_1^3(\alpha)$. The relations of fundamental group (20) have the following holonomy map φ images

$$L_A = C^{-1}AB^{-1}, L_B = A^{-1}BC^{-1}, L_C = B^{-1}CA^{-1}, \tag{21}$$

and so they are matrices from $\mathbb{PSL}(2, \mathbb{C})$.

Let us introduce a non-convex 48-faced polyhedron F with a set of vertices $\{N_i\}$, $\{M_i\}$, $\{L_i\}$, $i = 2, \dots, 8$ and pole vertises n and s [Fig. 4]. The axes a, b and c of polyhedron F lay on the axes of isometries A, B and C, respectively, with orientations as follows

$$L_6 \to N_4, N_6 \to M_4 \text{ and } M_6 \to L_4.$$

Moreover, vertices a_2, b_2 and c_2 are the middle points of the lines a, b and c respectively.

Also, we put under the consideration another isometries of $\mathbb{H}^3_u$, which make sense to use due to the symmetry of manifold $6_1^3(\alpha)$, they are

$$\phi = \begin{pmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{pmatrix}$$

is a rotation by angle $\pi/3$ about applicate axis, and

$$\tau = \begin{pmatrix} 0 & \frac{e^{-i2\pi/3}\sqrt{c_0-1}}{\sqrt{1+c_0}} \\ -\frac{e^{i2\pi/3}\sqrt{c_0+1}}{\sqrt{c_0-1}} & 0 \end{pmatrix}, \ c_0 \in \mathbb{C}$$

is a rotation of order two about axis which pass through the center of the polyhedron F and its vertex N_5.

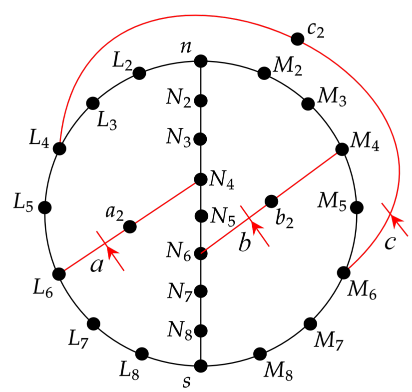

Fig. 4: *Abstract construction of polyhedron F.*

For instance, we place the pole vertex n on the applicate axis, then it has the following form

$$n = \begin{pmatrix} 0 & -t \\ \frac{1}{t} & 0 \end{pmatrix}, \ t \in \mathbb{R}, \ t > 0.$$

From relation $C^{-1}BCB^{-1}(n) = s$ we get that the pole vertice s depends on parameters t and α, c_0. It is not hard to understand that

$$N_7 = A^{-1}(s)A^*, N_5 = B(N_7)(B^{-1})^*.$$

Since vertex N_5 is fixed under the action of τ, then the solution of equation

$$\tau N_5 \left(\tau^{-1}\right)^* = N_5$$

gives the exact value of t. Thus, the rest of vertices with odd indexes could be found with use of the obvious relations

$$\begin{aligned}
&N_3 = A(N_5)(A^{-1})^*, M_7 = \phi(N_7)(\phi^{-1})^*,\\
&L_7 = \phi(M_7)(\phi^{-1})^*, M_5 = \phi(N_5)(\phi^{-1})^*,\\
&L_5 = \phi(M_5)(\phi^{-1})^*, M_3 = \phi(N_3)(\phi^{-1})^*,\\
&L_3 = \phi(M_3)(\phi^{-1})^*
\end{aligned}$$

and all of them depend only on parameters α and c_0. Now, let's move to the vertices with even indexes. Suppose that vertex N_4 has the form:

$$N_4 = \begin{pmatrix} \frac{x+iy}{\gamma} & -\frac{x^2+y^2+\gamma^2}{\gamma} \\ \frac{1}{\gamma} & \frac{-x+iy}{\gamma} \end{pmatrix}, \ x, y, \gamma \in \mathbb{R}, \ \gamma > 0.$$

Then, all remaining even vertices inherit the parameters of the point N_4:

$$\begin{aligned}
&N_8 = B^{-1}(N_4)(B)^*, L_2 = A(N_8)(A^{-1})^*,\\
&N_2 = \phi(L_2)(\phi^{-1})^*, M_2 = \phi(N_2)(\phi^{-1})^*,\\
&M_4 = \phi(N_4)(\phi^{-1})^*, L_4 = \phi(M_4)(\phi^{-1})^*,\\
&L_6 = C^{-1}(L_2)(C)^*, N_6 = \phi(L_6)(\phi^{-1})^*,\\
&M_6 = \phi(N_6)(\phi^{-1})^*, M_8 = \phi(N_8)(\phi^{-1})^*,\\
&L_8 = \phi(M_8)(\phi^{-1})^*.
\end{aligned}$$

By taking into account that N_4 is a fixed point of isometry A, one could find the following expressions for y, γ expressed in x and c_0

$$y = \frac{c_0 - \overline{c_0} + ix\left(\overline{c_0} + c_0 + 2\right)}{c_0 - \overline{c_0}},$$

$$\gamma = 2\sqrt{\frac{x^2(1+c_0)(1+\overline{c_0}) - ix(c_0 - \overline{c_0})}{(c_0 - \overline{c_0})^2}}.$$

At last, to find the value of the parameter x in words of α and c_0, one can use following relation

$$\tau(L_4)(\tau^{-1})^* = M_6.$$

In conclusion, the geometrical construction of a non-convex 48-faced polyhedron F was achieved and all its geometric components depend on parameters α and c_0.

The next step to achive is to use the same fact as in the proof of Theorem 1, that the cone-manifold $6_1^3(\alpha)$ admits hyperbolic structure if its fundamental polyhedron exists in $\mathbb{H}_u^3$. And the main goal for now is to find the conditions of existance of fundamental polyhedron F.

By taking into account the conditions of the problem under consideration, from [[26], p. 47] follows the two motions A and L_A, same for (B and L_B, or C and L_C) of $\mathbb{H}_u^3$ commute, then respectively, their axes a and l_A, or (b and l_B, or c and l_C) coincide (but their orientation need not agree [[26], p. 66]). This fact help us to prove next

Theorem 3. *Let $\alpha = \frac{2\pi}{3}$ or $\alpha = \frac{2\pi}{4}$ and parameter c_0 satisfies the following cubic equation:*

$$3Y^2c_0^3 - \sqrt{3}XYc_0^2 + c_0(2X^2 - Y^2) + \sqrt{3}XY = 0,$$

where $X = \cos\frac{\alpha}{2}, Y = \sin\frac{\alpha}{2}$ and $c_0 \in \mathbb{C}$ with imaginary part $\Im[c_0] > 0$. Then, the cone-manifold $6_1^3(\alpha)$ admits a hyperbolic structure.

Proof. Since A and L_A, same for (B and L_B, or C and L_C) have their axes coinside, then the common normal $AB - BA$ of axes of A and B is the common normal of the axis of L_A. Thus, the following condition must be satisfied

$$\mathrm{Tr}(L_A(AB - BA)) = 0.$$

By analogy we get

$$\mathrm{Tr}(L_B(BC - CB)) = 0,$$

$$\mathrm{Tr}(L_C(CA - AC)) = 0.$$

The system of equations is reducing to solution of one cubic equation

$$3Y^2c_0^3 - \sqrt{3}XYc_0^2 + c_0(2X^2 - Y^2) + \sqrt{3}XY = 0,$$

with $c_0 \in \mathbb{C}$. With sutable replacement, the equation takes the following form

$$G^2 + G\left(\frac{c_0^2 - 1}{2c_0}\right) + \frac{1}{6}(3c_0^2 - 1) = 0, \tag{22}$$

where $G \in \mathbb{R}, G = -\frac{X}{\sqrt{3}Y}$. From all possible cases, the equation (22) has suitable set of solutions G, when real part $\Re[c_0] = x$ of c_0 satisfies

$$\frac{\sqrt{-13\ (2^{11/3})q^{2/3} + (2^{4/3})q^{4/3} - 5q}}{3\sqrt{7}q^{1/2}} < x < \frac{1}{24}\left(\sqrt{33} + 9\right), \tag{23}$$

where $q = 571{+}189\sqrt{17}$, and the square Y of imaginary part $\Im[c_0] = y$, $\Im[c_0] > 0$ of c_0 satisfies the cubic equation

$$-3Y^3 + \left(15x^2 - 7\right)Y^2 + \left(39x^4 - 2x^2 - 5\right)Y + 21x^6 + 5x^4 + 7x^2 - 1 = 0. \tag{24}$$

Then, from (22), (23) and (24), one can receive $\alpha = \frac{2\pi}{3}$ or $\alpha = \frac{2\pi}{4}$, and conditions of existance of polyhedron F in $\mathbb{H}^3_u$ were found.

To prove that the polyhedron F is the fundamental polyhedron of cone-manifold $6^3_1(\alpha)$, one can use definition 1, but modified for this specific case. The condition *(1)* is easy to check, the condition *(2)* and *(3)* are equal, due to symmetry of the cone-manifold $6^3_1(\alpha)$. So, one can simply show that planes intersecting in the line, for example a, and one of these planes passes through the point L_5 (is P_{L_5}), and the other through the point N_3 (is P_{N_3}), form an angle $\phi(P_{L_5}, P_{N_3})$ from the positive normal of plane P_{L_5} to the positive normal of plane P_{N_3}. And in conclusion angle $\phi(P_{L_5}, P_{N_3})$ is a dihedral angle equal to cone angle α. □

Using one-to-one projection, (3) one can get a visual construction of the fundamental polyhedron of cone-manifold $6^3_1(\alpha)$ [Fig. 5]

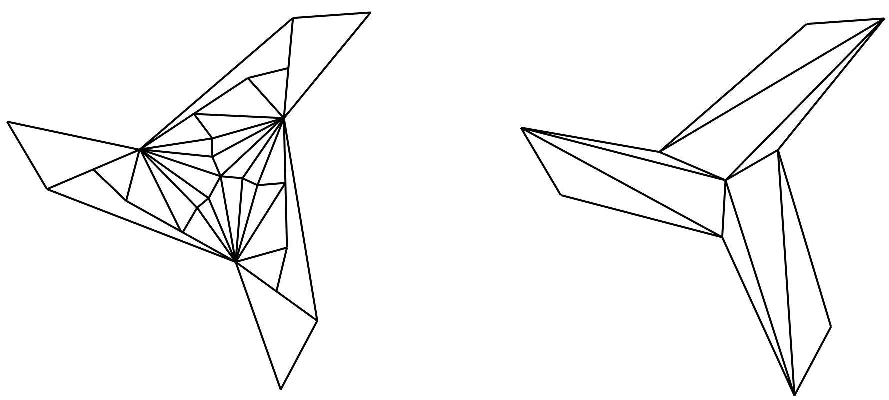

Fig. 5: *Polyhedron F* [*left: the superior view and right: the inferior view*].

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