Initial value problems and the normal form

An ordinary initial value problem is of form

$$\begin{cases} y'(t) &= f(y(t),t) \\ y(0) &= y_0 \end{cases},$$

where the unknown $y(t)$ is a vector-valued function. Function $f(y(t),t)$ and the initial value y_0 are known. A more concrete example:

$$\begin{cases} y'(t) &= \begin{bmatrix} 3 & 5 & 0 \\ -1 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} y(t) + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ y(0) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

Matlab can solve problems which are given in this form. If the problem is of higher order, it can be expressed as this kind of problem using the normal form.

The normal form

Consider an initial value problem

$$\begin{cases} \alpha_n y^{(n)}(t) + \alpha_{n-1} y^{(n-1)}(t) + \cdots + \alpha_1 y'(t) + \alpha_0 y(t) &= f(y(t),t) \\ y^{(n-1)}(0) = a_{n-1}, \quad y^{(n-2)}(0) = a_{n-2}, \quad \ldots, \quad y(0) = a_0 \end{cases}, \tag{1}$$

where $\alpha_n \neq 0$. Define functions $x_i(t)$ such that

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \quad \ldots \quad x_{n-1}(t) = y^{(n-2)}(t), \quad x_n(t) = y^{(n-1)}(t).$$

Using these definitions and equation (1) we obtain the following set of equations:

$$\begin{cases} x_1'(t) &= y'(t) = x_2(t) \\ x_2'(t) &= y^{(2)}(t) = x_3(t) \\ &\vdots \\ x_{n-1}'(t) &= y^{(n-1)}(t) = x_n(t) \\ x_n'(t) &= \frac{1}{\alpha_n}\left(-\alpha_{n-1} y^{(n-1)}(t) - \cdots - \alpha_1 y'(t) - \alpha_0 y(t) + f(y(t),t)\right) \\ &= \frac{1}{\alpha_n}\left(-\alpha_{n-1} x_n(t) - \cdots - \alpha_1 x_2(t) - \alpha_0 x_1(t) + f(x_1(t),t)\right) \end{cases},$$

which can be expressed in matrix form

$$x'(t) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & -\frac{\alpha_0}{\alpha_n} & -\frac{\alpha_1}{\alpha_n} & \cdots & -\frac{\alpha_{n-1}}{\alpha_n} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x_1(t),t) \end{bmatrix}$$

We get the initial value of this vector-valued problem from equation (1):

$$x(0) = \begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

A more concrete example

Let $a \neq 0$ and

$$\begin{cases} ay''(t) + by'(t) + cy(t) &= 0 \\ y(0) = z_0, \quad y'(0) = z_1 \end{cases},$$

Define functions

$$x_1(t) = y(t), \quad x_2(t) = y'(t).$$

Now

$$\begin{cases} x_1'(t) &= y'(t) = x_2(t) \\ x_2'(t) &= -\frac{1}{a}\left(by'(t) + cy(t)\right) = -\frac{1}{a}\left(bx_2(t) + cx_1(t)\right) \\ x_1(0) &= y(0) = z_0 \\ x_2(0) &= y'(0) = z_1 \end{cases},$$

and thus

$$\begin{cases} x'(t) &= \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} x(t) \\ \\ x(0) &= \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \end{cases}$$