Homework #2
APM115: Mathematical Modeling

Individual component

For problems 1-6 (for 10 points each), you may approach them analytically first and then use Monte Carlo simulations to confirm the results. You could also explore the answer first with Monte Carlo simulations and then try to understand the results analytically.

1. In a Monopoly game, players roll two (fair) dice each with the six faces marked from 1 to 6 to determine how many steps to move forward. What's the probability of the total being 2, 3, 4,…, 11, 12, respectively?

2. What is the probability of at least one of the dice shows a "one"? And by symmetry, this should be the case for "two", "three", etc., as well.

3. What is the probability of one die showing a "one" and the other showing a "six"?

4. What is the probability of both dice showing "one"?

5. Verify the law of total probability: Label the two dice A and B. The probability of die A showing a "one" is equal to sum of the joint probabilities of the two dice with die A showing a "one" and all possible outcomes for die B.

6. After someone rolled the two dice in secret, it is revealed that one of the dice (but not necessarily only one) shows a "six". What is the chance that the other die shows a "one"?

For problems 1-6, turn in the code, results, and your explanations.

7. (20 points) **Optimal pool size for COVID-19 surveillance, Part 2.** For some background, review Part 1 on the Canvas Site (go to "Modules" then "Course Overview Slides"). We want to choose the pool size to minimize the total number of tests needed to identify the people who have the virus. We will use the two-stage pooling strategy: if a pooled sample tests positive then all people in the group will be given an individual test, if a pooled sample tests negative, then all people in the group are considered virus free. We will assume the test is sensitive enough such that if one person in the pool is infected, the pooled sample will test positive. Let N be the total population, p be the probability of a randomly selected person having the virus (i.e. the prevalence of the virus) already determined through random testing, and s be the pool size.

a) Express the probability of a pooled sample with pool size s testing positive in terms of p and s, assuming the probabilities of individuals in the pool carrying the virus are independent of each other.
b) Express the expected value of the total number of tests needed T in terms of N, p, and s.
c) Vary p between 0.001 and 0.4, and for each value of p, find the value of s that minimizes the expected value of T, E(T). To find the optimal s, you can either use

an optimization function in Matlab or Python or set the derivative of E(T) with respect to s to zero and use a root finding function in Matlab or Python. You could also try a symbolic solver, but it is not required.

Group component (To be completed in groups of 3 or 4. Members of the same group can turn in the same code and write-up. Please list the names of your group members.)

8. (20 points) **Climate.** Complete the climate exercise. Turn in the code and a brief description of your findings. Note that when the parameters are changed slowly, the system has enough time to adjust and can stay close to a fixed point corresponding to the parameters at the time.

9. (20 points) **Stochastic population.** Modify the code shown in class (stochastic_simulation_logistic.m) to study the evolution of the probability distribution with time. Note that the distribution that we derived analytically in the notes is the equilibrium probability distribution, toward which the probability distribution will evolve.

To do so, make a large number (say 5000) of realizations of the same system. While the different realizations have identical parameters, because we are dealing with stochastic systems (and we are using a random number generator to simulate it), the different realizations will evolve differently.

Now run the 5000 realizations forward in time for 10000 time-steps. After every 100 time-steps, record the distribution of population among the 5000 realizations and plot it. You could do this for every time step but doing this after every 100 time-steps reduces the number of plots that you make.

Observe the evolution of the probability distributions.

Turn in your code and a brief report with plots of some representative probability distributions and a brief discussion of what are shown.

10. (10 points) **Valentine's day.** Valentine's day is around the corner. Suppose you would like to set up a flower stand. How many roses would you order? If you order too many, roses are perishable and those that are not sold by Valentine's day's end will be lost. If you order too few, you will lose valuable sales opportunities. What factors would you consider, what information would you gather, what approaches (deterministic or probabilistic) would you take towards building a mathematical model to help the decision making? You are only asked to sketch out some ideas; no equations are necessary at this stage. Happy Valentine's Day, everyone!