Programming Languages and Compiler Design

Optimization Using Data-flow Analysis

Yliès Falcone, Jean-Claude Fernandez

Master of Sciences in Informatics at Grenoble (MoSIG)
Master 1 info

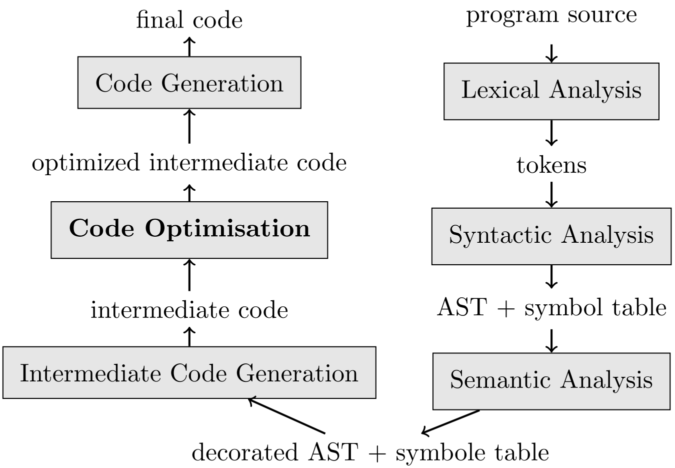
Univ. Grenoble Alpes
(Université Joseph Fourier, Grenoble INP)

Academic Year 2015 - 2016

Outline - Optimization Using Data-flow Analysis

Optimization

Where are we in the compiler steps?



Outline - Optimization Using Data-flow Analysis

Outline - Optimization Using Data-flow Analysis

Objectives of this chapter

Give some hints on general optimization techniques:

- **data-flow analysis**,
- register allocation,
- software pipelining,
- etc.

Describe the main data-structures used:

- Control Flow Graph (CFG),
- intermediate code (e.g., 3-address code),
- Static Single Assignment form (SSA),
- etc.

See some concrete examples.

But not a complete panorama of the whole optimization process.

Objective of the optimization phase

Improve the **efficiency** of the target code, while preserving the source semantics.

Several (antagonist) criteria:

- execution time,
- code size,
- used memory,
- energy consumption,
- etc.

$\Rightarrow$ no optimal solution, no general algorithm

A bunch of optimization techniques:

- dependent from each other,
- sometimes based on heuristics.

Outline - Optimization Using Data-flow Analysis

Intra-procedural 3-address code (TAC)

"High-level" assembly code:

- binary logic and arithmetic operators,
- use of temporary memory location `ti`,
- assignments to variables, temporary locations,
- a label can be assigned to an instruction,
- conditional jumps `goto`.

Example (3-address code)

- `l:  x := y op x`
- `l:  x := op y`
- `l42:  x := y`
- `l9:  goto l'`
- `l':  if x oprel y goto l''`

Basic block (BB)

Definition and how to compute them

Definition (Basic Block)

A maximal instruction sequence $S = i_1. \cdots i_n$ such that:

- S execution is never "broken" by a jump
  $\Rightarrow$ no `goto` instruction in $i_1 \cdots i_{n-1}$
- S execution cannot start somewhere in the middle
  $\Rightarrow$ no `label` in $i_2 \cdots i_n$

$\Rightarrow$ execution of a BB is "atomic".

Partitioning a 3-address code into BBs

1. computation of BB heads:
   1st inst., inst. target of a jump, inst. following a jump
2. computation of BB tails:
   last inst., inst. before a BB head

$\Rightarrow$ a single traversal of the TAC.

Control-Flow Graph (CFG)

A representation of how the execution may progress inside the TAC.

Definition (Control-Flow Graph)

A graph (V, E) such that:

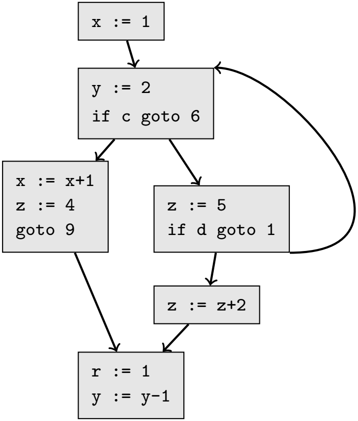
$$
\begin{aligned}
V &= \{B_i \mid B_i \text{ is a basic block}\} \\
E &= \{(B_i, B_j) \mid \\
&\quad \text{“tail of } B_i \text{ is a jump to head of } B_j\text{”} \\
&\quad \text{or} \\
&\quad \text{“head of } B_j \text{ follows the tail of } B_i \text{ in the TAC”}\}
\end{aligned}
$$

Basic Block and Control-Flow Graph: example

Example/Exercise

Give the Basic Blocks and CFG associated to the following TAC sequence:

```
0. x := 1          6. z := 5
1. y := 2          7. if d goto 1
2. if c goto 6     8. z := z+2
3. x := x+1        9. r := 1
4. z := 4          10 y := y-1
5. goto 9
```



- Heads of blocks: 0, 1, 3, 6, 8, 9.
- Tails of blocks: 2, 5, 7, 10.

Outline - Optimization Using Data-flow Analysis

Two kinds of optimization techniques

Optimization independent from the target machine

- Objective: optimize the performance of the program.
- "source level" or "assembly level" pgm transformations.

Example (Optimization independent from the target machine)

- constant propagation, constant folding
- common sub-expressions elimination
- dead code elimination
- code motion

Optimization dependent from the target machine

- Objective: optimize the use of hardware resources.

Example (Optimization dependent from the target machine)

- machine instruction,
- memory hierarchy (registers, cache, pipeline, etc.).

Main principles of optimisation techniques

Input: initial intermediate code

Output: optimized intermediate code

Several steps:

1. generation of a control flow graph (CFG)
2. analysis of the CFG
3. transformation of the CFG
4. generation of the output code

Analysis and transformations

Analysis	**Transformation**
Available expressions common sub-expressions	Elimination of redundant computation
Live Variables	Elimination of useless code
Constant propagation	Replacing variables by their constant value
Induction Variable	Strength reduction
Loop Invariant	Moving the invariant code outside the loop
Dead-code elimination	Suppress useless instructions (which do not influence the execution)
Constant folding	Performing operations between constants
Copy propagation	Suppress useless variables (i.e., equal to another one or to a constant)
Algebraic simplification Strength reduction	Replace costly computations by less expensive ones

Optimization techniques performed on the CFG

Two levels: local and global.

Local optimization techniques

- Computed inside each BB.
- BBs are transformed independently from each other.

Global optimizations techniques

- Computed on the CFG.
- Transformation of the CFG:
  - code motion between BBs,
  - transformation of BBs,
  - modification of the CFG edges.

Outline - Optimization Using Data-flow Analysis

Examples of local optimization techs

Initial code:

```
a := x ** 2
b := 3
c := x
d := c * c
e := b * 2
f := a + d
g := e * f
```

Examples of local optimization techs

Algebraic simplification:

```
a := x ** 2        a := x * x
b := 3             b := 3
c := x             c := x
d := c * c         d := c * c
e := b * 2         e := b << 1
f := a + d         f := a + d
g := e * f         g := e * f
```

Examples of local optimization techs

Copy propagation:

`a := x * x`	`a := x * x`
`b := 3`	`b := 3`
`c := x`	`c := x`
`d := c * c`	`d := x * x`
`e := b << 1`	`e := 3 << 1`
`f := a + d`	`f := a + d`
`g := e * f`	`g := e * f`

Examples of local optimization techs

Constant folding:

```
a := x * x        a := x * x
b := 3            b := 3
c := x            c := x
d := x * x        d := x * x
e := 3 << 1       e := 6
f := a + d        f := a + d
g := e * f        g := e * f
```

Examples of local optimization techs

Elimination of common sub-expressions:

`a := x * x`	`a := x * x`
`b := 3`	`b := 3`
`c := x`	`c := x`
`d := x * x`	`d := a`
`e := 6`	`e := 6`
`f := a + d`	`f := a + d`
`g := e * f`	`g := e * f`

Examples of local optimization techs

Copy propagation:

`a := x * x`	`a := x * x`
`b := 3`	`b := 3`
`c := x`	`c := x`
`d := a`	`d := a`
`e := 6`	`e := 6`
`f := a + d`	`f := a + a`
`g := e * f`	`g := 6 * f`

Example of local optimizations

Dead code elimination (+ strength reduction):

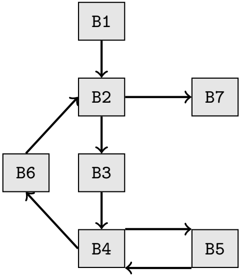
```
a := x * x          a := x * x          a:= x * x
b := 3
c := x
d := a
e := 6
f := a + a          f := a + a          f := a << 1
g := 6 * f          g := 6 * f          g := 6 * f
```

Local optimization: a more concrete example

Initial source program: addition of matrices

```
for (i=0 ; i < 10 ; i ++)
     for (j=0 ; j < 10 ; j++)
        S[i,j] := A[i,j] + B[i,j]
```

B1: $i := 0$
B2: if $i > 10$ goto B7
B3: $j := 0$
B4: if $j > 10$ goto B6
$\boxed{\text{B5}}$
B6: $i := i + 1$
goto B2
B7: end



Initial Block B5

```
B5:  t1 := 4 * i
     t2 := 40 * j
     t3 := t1 + t2
     t4 := A[t3]
     t5 := 4 * i
     t6 := 40 * j
     t7 := t5 + t6
     t8 := B[t7]
     t9 := t4 + t8
     t10:= 4 * i
     t11:= 40 * j
     t12:= t10 + t11
     S[t12] := t9
     j := j + 1
     goto B4
```

Optimization of B5 (1/4)

```
B5:  t1 := 4 * i
     t2 := 40 * j
     t3 := t1 + t2
     t4 := A[t3]
     t5 := 4 * i
     t6 := 40 * j
     t7 := t5 + t6
     t8 := B[t7]
     t9 := t4 + t8
     t10:= 4 * i
     t11:= 40 * j
     t12:= t10 + t11
     S[t12] := t9
     j := j + 1
     goto B4
```

The same value is assigned to temporary locations t1, t5, t10.

Optimization of B5 (2/4)

```
B5:  t1 := 4 * i            t8 := B[t7]
     t2 := 40 * j           t9 := t4 + t8
     t3 := t1 + t2          t11:= 40 * j
     t4 := A[t3]            t12:= t1 + t11
     t6 := 40 * j           S[t12] := t9
     t7 := t1 + t6          j := j + 1
                            goto B4
```

A same value is assigned to temporary locations t2, t6, t11.

Optimization of B5 (3/4)

```
B5:  t1 := 4 * i             t8 := B[t7]
     t2 := 40 * j            t9 := t4 + t8
     t3 := t1 + t2           t12:= t1 + t2
     t4 := A[t3]             S[t12] := t9
     t7 := t1 + t2           j := j + 1
                             goto B4
```

A same value is assigned to temporary locations t3, t7, t12.

Optimization of B5 (4/4): the final code

```
B5:   t1 := 4 * i
      t2 := 40 * j
      t3 := t1 + t2
      t4 := A[t3]
      t8 := B[t3]
      t9 := t4 + t8
      S[t3] := t9
      j := j + 1
      goto B4
```

Outline - Optimization Using Data-flow Analysis

Global optimization techniques

Example (Global optimization techniques)

- constant propagation trough several basic blocks
- elimination of global redundancies
- code motion: move invariant computations outside loops
- dead code elimination

How to extend local optimization to the whole CFG?

1. Associate (local) properties to entry/exit points of BBs (e.g., set of live variables, set of available expressions, etc.)
2. Propagate them along CFG paths
   $\rightarrow$ enforce consistency w.r.t. the CFG structure
3. Update each BB (and CFG edges) according to these global properties.

$\Rightarrow$ a possible technique: data-flow analysis

Outline - Optimization Using Data-flow Analysis

Data-flow analysis

Static computation of data-related properties of programs.

Data-flow problem

- (local) Properties φ_i associated to some pgm locations i
- Set of data-flow equations:
  $\rightarrow$ how the φ_i's are transformed along pgm executions.
- Regarding propagation:
  - forward vs backward propagation (depending on φ_i)
  - the property can depend on its previous value on either all paths or at least one path
  - cycles inside the control flow $\Rightarrow$ fix-point equations!

Solving the equation system

- A solution of this equation system assigns "globaly consistent" values to each φ_i.
- Such a solution may not exist...
- Decidability may require abstractions and/or approximations.

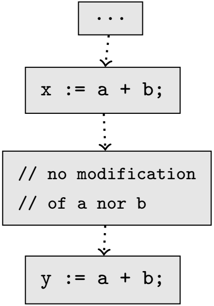
Outline - Optimization Using Data-flow Analysis

Computing available expressions

Getting intuition on examples

Let us consider expression `a + b`.

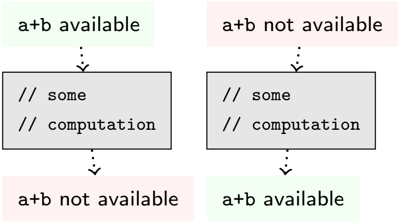
How to determine whether `a + b` is available, i.e., whether its value has been previously computed and does not need to be computed again.



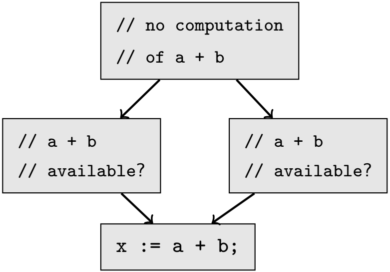
- Which computation of `a+b` is not needed?
- How does the information *"being previously computed"* propagate?

Computing available expressions

Getting intuition on examples



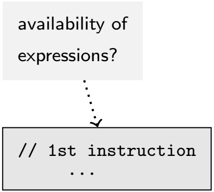
- What can make a+b not available?
- What can make a+b available?



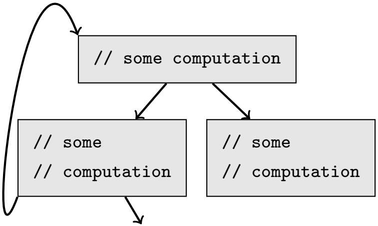
- What is needed for a+b to be available in the last block?
- How does the notion of availability depend on previous paths?

Computing available expressions

Getting intuition on examples



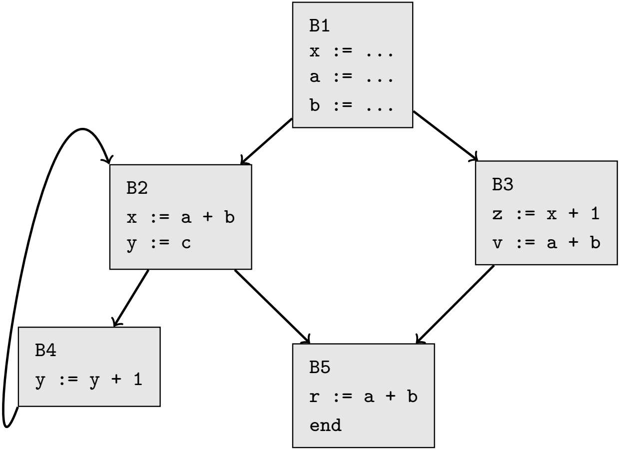
- What is available when the program starts?



- The CFG structure gives us relations between the availabilities at the entrances and exits of blocks.
- Availability at the entrance is obtained from the availability at the exits.
- How to deal with cycles?

Elimination of redundant computation with available expressions

Running example



Available expressions and redundant computations

Definition

We consider the set of expressions appearing in the program.

Definition (Available expression)

An expression e is available at location i iff

- it is computed on every path going to location i, and
- on each of the paths leading to i operands of e are not modified between the last computation of e and location i

Definition (Redundant computation (of an expression))

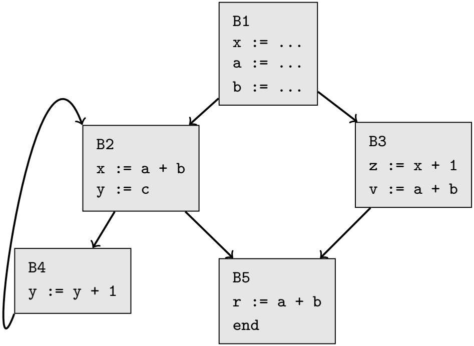
The computation of an expression e is redundant at location i iff it is available at location i, **and** it is computed at location i,

Remark We consider syntactic equality. □

Running example

Available expressions and redundant computations

Available and redundant computation



- local availability
- global availability

- `y + 1` is not available anywhere.
- `x + 1` is only available after being computed and at the exit of B3.
- `a + b` is available at the exits of B2, B3, and at the entrance of B5.
- the computation of `a + b` in B5 is redundant.

Data-flow equations for available expressions (1/3)

For a basic block B, we note:

- $\mathit{Kill}(B)$: expressions made non available by B (because an operand of e is modified by B).
- $\mathit{Gen}(B)$: expressions made available by B (computed in B, operands not modified afterwards).
- $\mathit{In}(B)$: available expressions when entering B.
- $\mathit{Out}(B)$: available expressions when exiting B

$$\mathit{Out}(B) = (\mathit{In}(B) \setminus \mathit{Kill}(B)) \cup \mathit{Gen}(B) = F_b(\mathit{In}(B)),$$

where: F_B: transfer function of block B

Data-flow equations for available expressions (2/3)

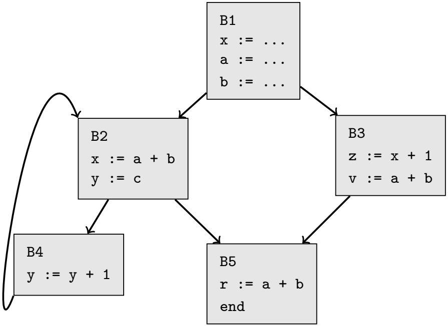
To define *Gen* and *Kill*, for a block B, we introduce local functions Gen_l and $Kill_l$:

$$
\begin{array}{|lcl|}
\hline
Gen(B) & = & Gen_l(B, \emptyset) \\
Kill(B) & = & Kill_l(B, \emptyset) \\
\hline
Gen_l(\texttt{x} := \texttt{a}; B, X) & = & Gen_l(B, X \setminus \{e' \mid x \in Used(e')\} \\
 & & \qquad \cup \{a \mid x \notin Used(a)\}) \\
Gen_l(\texttt{if b goto l}, X) & = & X \cup \{b\} \\
Gen_l(\texttt{goto l}, X) & = & X \\
Gen_l(\epsilon, X) & = & X \\
\hline
Kill_l(\texttt{x} := \texttt{a}; B, X) & = & Kill_l(B, X \cup \{e' \mid x \in Used(e')\} \\
Kill_l(\texttt{if b goto l}, X) & = & X \\
Kill_l(\texttt{goto l}, X) & = & X \\
Kill_l(\epsilon, X) & = & X \\
\hline
\end{array}
$$

Running example

Computing *Gen* and *Kill*

Gen and *Kill*



- $Gen(B1) = \emptyset$, $Kill(B1) = \{\mathtt{a+b}, \mathtt{x+1}\}$
- $Gen(B2) = \{\mathtt{x+1}, \mathtt{a+b}\}$, $Kill(B2) = \{\mathtt{y+1}\}$
- $Gen(B3) = \{\mathtt{a+b}, \mathtt{x+1}\}$, $Kill(B3) = \emptyset$
- $Gen(B4) = \emptyset$, $Kill(B4) = \{\mathtt{y+1}\}$
- $Gen(B5) = \{\mathtt{a+b}\}$, $Kill(B5) = \emptyset$

Data-flow equations for available expressions (3/3)

How to compute $In(b)$?

- if b is the initial block:

$$In(b) = \emptyset$$

- if b is not the initial block: An expression e is available at its entry point iff it is available at the exit point of all predecessor of b in the CFG.

$$In(b) = \bigcap_{b' \in Pre(b)} Out(b')$$

$\Rightarrow$ forward data-flow analysis along the CFG paths.

How to deal with cycles inside the CFG? fix-points computation!

We want to as much available expressions possible: greatest fix-point.

Using data-flow equations to compute available expressions

Initialisation

It is a forward analysis $\Rightarrow$ initialisation concerns the $In(b)$ sets.

- Initialise $In(B_{\mathrm{init}})$ to $\emptyset$: there is no available expression at the beginning of the program.
- Initialise $In(b)$, for $b \neq B_{\mathrm{init}}$ to the maximal element, i.e., to the set of all expressions.

Iteration until stabilisation

Iterate the following steps until stabilisation, i.e., when the $In(b)$ sets are the same as in the previous step. At stabilisation, one has found the (greatest) fix-point.

1. Compute $Out(b)$ sets using $Out(b) = (In(b) \setminus Kill(b)) \cup Gen(b)$.
2. Compute the (new) $In(b)$ sets using $In(b) = \bigcap_{b' \in Pre(b)} Out(b')$.

Suppressing redundant computations

We look at all computed available expressions in each block ($In(b)$ sets).

Is an available expression redundant?

Let e be an available expression at the entry of a basic block b.
If the two following conditions are met:

- e appears in b, and
- none of the operand of e is assigned from the beginning of the block until the use of e

$\Rightarrow$ the computation of e is *redundant*.

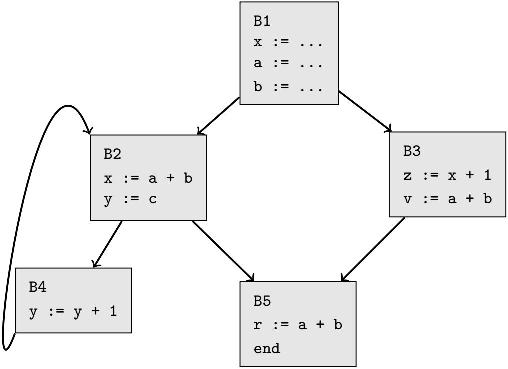
Suppressing redundant computation

Let e be an expression which computation is redundant in a block b.

- Introduce a new variable, say u.
- Using a backward analysis from b, locate each occurence of $\mathtt{x} := \mathtt{e}$, and replace it with $\left\{ \begin{array}{l} \mathtt{u} := \mathtt{e}; \\ \mathtt{x} := \mathtt{u}; \end{array} \right.$
- Replace the occurence of e where its computation is redundant.

Running example

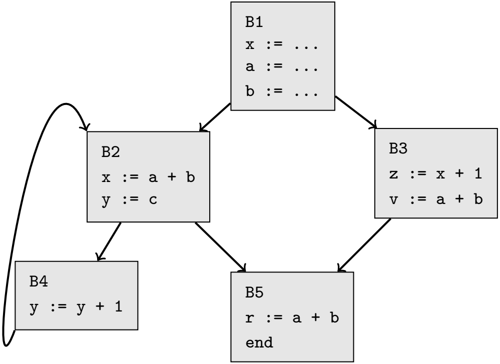
Computing available expressions



$$\Sigma = \{a+b, x+1, y+1\}$$

Running example

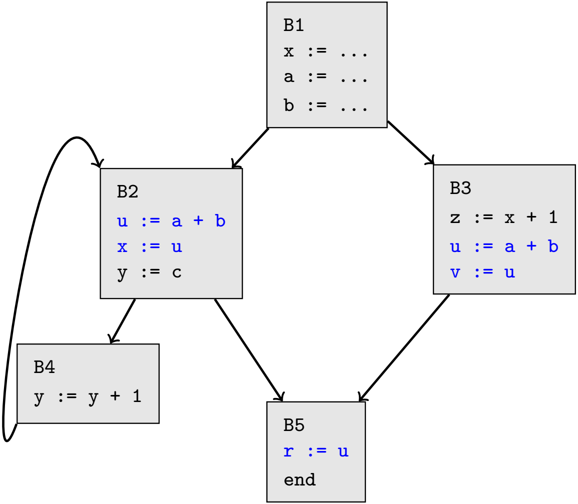
Determining redundant computation



- $In(B1) = In(B2) = In(B3) = \emptyset$.
- $In(B4) = \{a + b\}$, but there is no computation of $a + b$ in B4.
- $In(B5) = \{a + b\}$, $a + b$ is computed in B5, and there is no modification of its operands from the beginning of the block to its computation. Hence, the computation of $a + b$ in B5 is *redundant*.

Running example

Final CFG



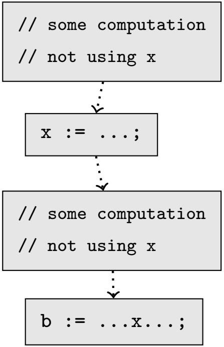
Outline - Optimization Using Data-flow Analysis

Computing live variables

Getting intuition on examples

Let us consider a variable `x` appearing in a program.

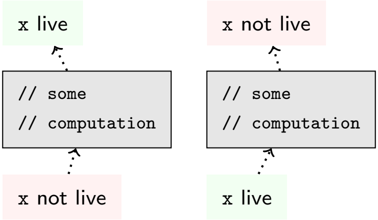
How to determine whether `x` is *live*, i.e., whether its value is needed for computation.



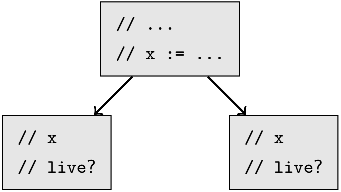
- Where is the value of `x` needed?
- How does the information *"being needed"* propagate?

Computing live variables

Getting intuition on examples



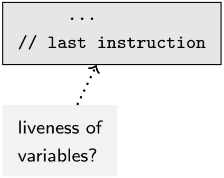
- What can make x live?
- What can make x not live?



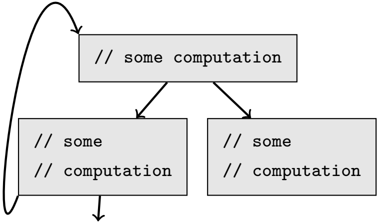
- What is needed for x to be live in the first block?
- How does the notion of liveness depend on previous paths? (note, previous refers to a successor block)

Computing live variables

Getting intuition on examples



- What is live when the program terminates?



- The CFG structure gives us relations between liveness at the exits and entrances of blocks.
- Liveness at exits is obtained from the liveness at the entrance.
- How to deal with cycles?

Live Variables

Objective: remove useless instructions.

Definition (Live Variable)

A variable `x` is live at location `i` if it is *used* in at least one CFG-path going from `i` to `j`, where `j` is:

- either a final instruction, or
- an assignment to `x`.

Definition (Useless instructions)

An instruction `x := e` at location `i` is useless if `x` is dead at location `i`.

Remark Used means "in the right-hand side of an assignment or in a branch condition". □

Data-flow analysis for dead variables

We compute the set of live variables...

Local analysis

Gen(b) is the set of variables x s.t. x is used in block b, and, in this block, any assignment to x happens after the (first) use of x.

Kill(i) is the set of variables x assigned in block b.

Global analysis

Backward analysis, $\exists$ a CFG-path (least solution).

- $In(b) = (Out(b) \setminus Kill(b)) \cup Gen(b)$.
- $$\texttt{Out}(b) = \begin{cases} \emptyset & \text{if } b \text{ is final,} \\ \bigcup_{b' \in Succ(b)} In(b') & \text{otherwise.} \end{cases}$$

Computation of functions *Gen* and *Kill*

Recursively defined on the syntax of a basic block B:

$$\mathtt{B} ::= \varepsilon \mid \mathtt{B}\ ;\ \mathtt{x} := \mathtt{a} \mid \mathtt{B}\ ;\ \mathtt{if\ b\ goto\ l} \mid \mathtt{B}\ ;\ \mathtt{goto\ l}$$

$$\begin{array}{|lcl|}
\hline
Gen(B) & = & Gen_l(\mathtt{B}, \emptyset) \\
Kill(B) & = & Kill_l(\mathtt{B}, \emptyset) \\
\hline
Gen_l(\mathtt{B}\ ;\ \mathtt{x} := \mathtt{a}, X) & = & Gen_l(\mathtt{B}, X \setminus \{\mathtt{x}\} \cup Used(\mathtt{a})) \\
Gen_l(\mathtt{B}\ ;\ \mathtt{if\ b\ goto\ l}, X) & = & Gen_l(\mathtt{B}, X \cup Used(\mathtt{b})) \\
Gen_l(\mathtt{B}\ ;\ \mathtt{goto\ l}, X) & = & Gen_l(\mathtt{B}, X) \\
Gen_l(\varepsilon, X) & = & X \\
\hline
Kill_l(\mathtt{B}\ ;\ \mathtt{x} := \mathtt{a}, X) & = & Kill_l(\mathtt{B}, X \cup \{\mathtt{x}\}) \\
Kill_l(\mathtt{B}\ ;\ \mathtt{if\ b\ goto\ l}, X) & = & Kill_l(\mathtt{B}, X) \\
Kill_l(\mathtt{B}\ ;\ \mathtt{goto\ l}, X) & = & Kill_l(\mathtt{B}, X) \\
Kill_l(\varepsilon, X) & = & X \\
\hline
\end{array}$$

$Used(\mathtt{e})$: set of variables appearing in expression e.

Removal of useless instructions

1. Compute the sets $In(B)$ and $Out(B)$ of live variables at entry and exit points of each block.
2. Let $F : Code \times 2^{Var} \rightarrow Code$
   $F(B, X)$ is the code obtained when removing useless assignments inside B, assuming that variables of X are live at the end of B execution.

$$
\begin{array}{rcl}
F(\texttt{B ; x } := \texttt{a}, X) & = & \begin{cases} F(B, X) & \text{if } x \notin X \\ F(B, (X \setminus \{x\}) \cup Used(a)); x := a & \text{if } x \in X \end{cases} \\
F(\texttt{B ; if b goto l}, X) & = & F(B, X \cup Used(b)); \texttt{if b goto l} \\
F(\texttt{B ; goto l}, X) & = & F(B, X); \texttt{goto l} \\
F(\epsilon, X) & = & \epsilon
\end{array}
$$

3. Replace each block B by $F(B, Out(B))$.

Remark This transformation may produce new dead variables... □

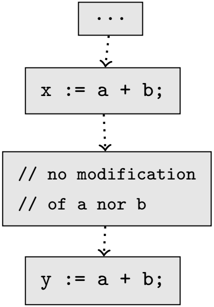
Outline - Optimization Using Data-flow Analysis

Computing anticipaple expressions

Getting intuition on examples

Let us consider expression `a + b`.

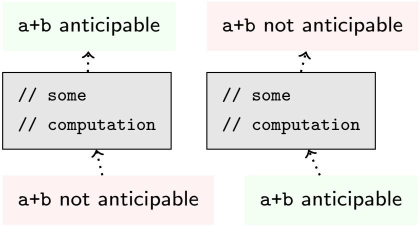
How to determine whether `a + b` is anticipaple, i.e., whether its value is computed and does not need to be computed again (because its operands remain with the same value).



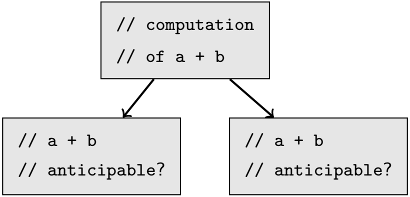
- Which computation of `a+b` is not needed?
- How does the information *"being previously computed"* propagate?

Computing anticipable expressions

Getting intuition on examples



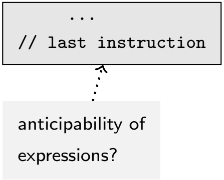
- What can make a+b anticipable?
- What can make a+b not anticipable?



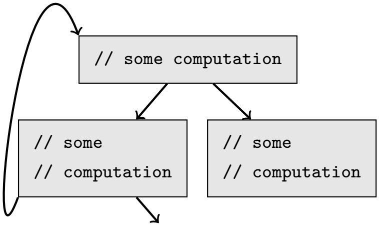
- What is needed for a+b to be anticipable in the first block?
- How does the notion of anticipable depend on paths?

Computing anticipable expressions

Getting intuition on examples



- What is anticipable when the program terminates?



- The CFG structure gives us relations between the availabilities at the entrances and exits of blocks.
- Anticipabillity at the exits is obtained from the anticipable at the entrances.
- How to deal with cycles?

Anticipable expressions

Definition

Definition (Anticipable expression)

An expression e is anticipable at program point p if on every path from p to the end node,

- e is computed, and
- e's operands are not defined before the first computation of e.

Data-flow equations for anticipable expressions (1/2)

For a basic block b, we note:

- $In(b)$: anticipable expressions when exiting b
- $Kill(b)$: expressions made non anticipable by b
  (because an operand of e is modified by b)
- $Gen(b)$: expressions made anticipable by block b
  (computed in b, operands not modified before)
- $Out(b)$: anticipable expressions when entering b

$$In(b) = (Out(b) \setminus Kill(b)) \cup Gen(b) = F_b(Out(b))$$

F_b: transfer function of block b

Data-flow equations for anticipable expressions (2/2)

How to compute $Out(b)$?

- if b is the final block:

$$Out(b) = \emptyset$$

- if b is not the final block: An expression e is anticipable at its exit point iff it is anticipable at the entry point of each successor of b in the CFG

$$Out(b) = \bigcap_{b' \in Suc(b)} In(b')$$

$\Rightarrow$ backward data-flow analysis along the CFG paths

Outline - Optimization Using Data-flow Analysis

Reaching definition

Definition

Definition (Reaching definition)

A definition at location d of a variable `x` reaches a program location i if there is a control-flow path from d to `i` that does not contain a definition of variable x.

Data-flow equations for reaching definitions (1/2)

Each instruction receives a unique label; this label represents the program counter (instructions are numbered from 1 to the number of instructions). Thus, `x := e` becomes `I: x := e` (normally, instructions are numbered in the 3-address code by an integer modeling the control point).

$$\begin{array}{rl} \mathit{Gen}(b) & \text{contains the downward exposed definitions in } b, \\ \mathit{Kill}(b) & \text{contains all definition of all variables modified in } b. \end{array}$$

Consider function `TheDef`(x) which, to a variable `x` associates all definitions of `x`, of the form `I: x := e`.

$$\boxed{\begin{array}{rcl} \mathit{Gen}_l(I : x := e; b, X) & = & \mathit{Gen}_l(b, X \setminus \mathtt{TheDef}(x) \cup \{I\}) \\ \mathit{Kill}_l(I : x := e; b, X) & = & \mathtt{TheDef}(x) \setminus \{I\} \end{array}}$$

Data-flow equations for reaching definitions (2/2)

Forward analysis, $\exists$ a CFG-path (least solution)

- $$Out(b) = (In(b) \setminus Kill(b)) \cup Gen(b)$$
- $$\mathtt{In}(b) = \begin{cases} \emptyset & \text{if } b \text{ is Start} \\ \displaystyle\bigcup_{b' \in Pred(b)} aOut(b') & \text{otherwise.} \end{cases}$$

Applications of reaching definitions

We restrict the definition to copy and a definition `x:=y` is contained in

- *Gen*(b) if it is downward exposed in b in the sense of not being followed by a definition of `x` and `y` and in
- *Kill*(b) if b contains a definition of `x` or `y`

Loop-invariant code motion

Consider a loop containing an instruction i. If all of the reaching definitions of the variables used in i are outside the loop, then i can be moved out of the loop.

Outline - Optimization Using Data-flow Analysis

Recall data-flow analysis

Static computation of data-related properties of programs:

Data-flow problem

- (local) Properties φ_i associated to some pgm locations i
- Set of data-flow equations:
  $\rightarrow$ how φ_i are transformed along pgm execution
- Regarding propagation:
  - forward vs backward propagation (depending on φ_i)
  - cycles inside the control flow $\Rightarrow$ fix-point equations!

Solving the equation system

- A solution of this equation system assigns "globaly consistent" values to each φ_i.
- Such a solution may not exist...
- Decidability may require abstractions and/or approximations

Generalization

Data-flow properties are expressed as finite sets associated to entry/exit points of basic blocks: *In*(b), *Out*(b).

Forward analysis

- property is "false" ($\bot$) at entry of initial block
- $Out(b) = F_b(In(b))$
- *In*(b) depends on *Out*(b'), where $b' \in Pred(b)$
  ($\sqcap$ for "$\forall$ paths", $\sqcup$ for "$\exists$ path")

Backward analysis

- property is "false" ($\bot$) at exit of final block
- *In*(b) $= F_b$ (*Out*(b))
- *Out*(b) depends on *In*(b'), where $b' \in Succ(b)$
  ($\sqcap$ for "$\forall$ paths", $\sqcup$ for "$\exists$ path")

Data-flow equations: forward analysis

Forward analysis, least fix-point	$In(b) = \begin{cases} \bot & \text{if } b \text{ is initial} \\ \bigsqcup_{b' \in Pre(b)} Out(b') & \text{otherwise.} \end{cases}$ $Out(b) = F_b(In(b))$
Forward analysis, greatest fix-point	$In(b) = \begin{cases} \bot & \text{if } b \text{ is initial} \\ \bigsqcap_{b' \in Pre(b)} Out(b') & \text{otherwise.} \end{cases}$ $Out(b) = F_b(In(b))$

Data-flow equations: backward analysis

Backward analysis, least fix-point	$Out(b) = \begin{cases} \bot & \text{if } b \text{ is final} \\ \bigsqcup_{b' \in Succ(b')} In(b') & \text{otherwise.} \end{cases}$ $In(b) = F_b(Out(b))$
Backward analysis, greatest fix-point	$Out(b) = \begin{cases} \bot & \text{if } b \text{ is final} \\ \bigsqcap_{b' \in Succ(b)} In(b') & \text{otherwise.} \end{cases}$ $In(b) = F_b(Out(b))$

Solving the data-flow equations (1/2)

Let $(E, \leq)$ a partial order.

- For $X \subseteq E, a \in E$:
  - a is an upper bound of X if $\forall x \in X : x \leq a$,
  - a is a lower bound of X if $\forall x \in X : a \leq x$.
- The least upper bound (*lub*, $\sqcup$) is the smallest upper bound.
- The greatest lower bound (*glb*, $\sqcap$) is the largest lower bound.
- $(E, \leq)$ is a lattice if any two elements of E admit a *lub* and a *glb*.
- A function $f : 2^E \rightarrow 2^E$ is increasing if:

$$\forall X, Y \subseteq E \quad X \leq Y \; \Rightarrow \; f(X) \leq f(Y)$$

- $X = \{x_0, x_1, \ldots x_n, \ldots\} \subseteq E$ is an (increasing) chain if $x_0 \leq x_1 \leq \ldots x_n \leq \ldots$
- A function $f : 2^E \rightarrow 2^E$ is ($\sqcup$-)continuous if $\forall$ increasing chain X, $f(\sqcup X) = \sqcup f(X)$

Solving the data-flow equations (2/2)

Fix-point equation: solution?

- properties are finite sets of expressions $\mathcal{E}$
- $(2^{\mathcal{E}}, \subseteq)$ is a complete lattice
  $\perp$: least element (aka minimum), $\top$: greatest element (aka maximum)
  $\sqcap$: greatest lower bound (aka supremum), $\sqcup$: least upper bound (aka infimum)
- data-flow equations are defined on monotonic and continuous operators ($\cup$, $\cap$) on $(2^{\mathcal{E}}, \subseteq)$.
- Kleene and Tarski theorems:
  - the set of solutions is a complete *lattice*
  - the greatest (resp. least) solution can be obtained by successive iterations w.r.t. the greatest (resp. least) element of $2^{\mathcal{E}}$

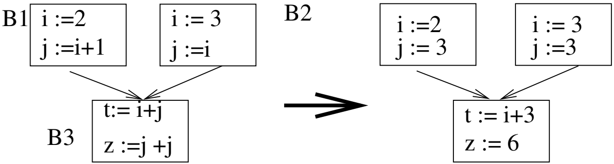
$$\mathrm{lfp}(f) = \sqcup\{f^i(\perp) \mid i \in \mathbb{N}\} \qquad \mathrm{gfp}(f) = \sqcap\{f^i(\top) \mid i \in \mathbb{N}\}$$

Outline - Optimization Using Data-flow Analysis

Constant propagation

A variable is constant at location `l` if its value at this location can be computed at compilation time.

Example (Constant propagation principle)



- At exit point of B1 and B2, `i` and `j` are constants.
- At entry point of B3, `i` is not constant, `j` is constant.

Constant propagation: the lattice

Intuitively, the property is an "abstraction" of the memory:

- Each variable takes its value in $D = \mathbb{N} \cup \{\top, \bot\}$, where:
  - $\top$ means "non constant value"
  - $\bot$ means "no information"

Defining the lattice:

- Partial order relation $\leq$:

  if $v \in D$ then $\bot \leq v$ and $v \leq \top$.

- The least upper bound $\sqcup$:

  for $x \in D$ and $v_1, v_2 \in \mathbb{N}$

$x \sqcup \top = \top$	$x \sqcup \bot = x$	$v_1 \sqcup v_2 = \top$ if $v_1 \neq v_2$	$v_1 \sqcup v_1 = v_1$

Remark Relation $\leq$ is extended to functions $Var \to D$

$f1 \leq f2$ iff $\forall x : f1(x) \leq f2(x)$. □

Constant propagation: data-flow equations

A basic block = sequence of assignments

$$\texttt{b} \quad ::= \quad \epsilon \mid \texttt{x:=e ; b}$$

- Property at location `l` is a function $Var \rightarrow D$.
- Forward analysis:

$$\begin{array}{lcl} In(b) & = & \begin{cases} \lambda x.\bot & \text{if } b \text{ is initial,} \\ \sqcup_{b' \in Pred(b)} Out(b') & \text{otherwise} \end{cases} \\ Out(b) & = & F_b(In(b)) \end{array}$$

Transfer function F_b by syntactic induction

$$\begin{array}{rcl} F_{\texttt{x:=e ; b}}(f) & = & F_{\texttt{b}}(f[x \mapsto f(e)]) \quad \text{(assuming variable initialization)} \\ F_\epsilon(f) & = & f \end{array}$$

Program transformation

$\forall$ block b, $f \in In(b), f(\text{e}) = v \Rightarrow$ `x:=e` replaced by `x:=v`

Remark We assume that variables are properly initialized. □

Outline - Optimization Using Data-flow Analysis

Partially available expressions

1. Compute
   - Local anticipable expressions at entry of each block, $\texttt{AntGen}(b)$, which is the local property $\texttt{Gen}(b)$ of anticipability analysis,
   - Partial avalaible expressions at entry of each bloc, $\texttt{Pavin}(b)$,
   - Avalaible expressions at exit of each bloc, $\texttt{Avout}(b)$.
   - $\texttt{Kill}(b)$ is the same for all analyses involving expressions : Available expressions analysis, partially available expressions analysis, anticipable expressions analysis.
2. Compute Possible Placement at entry and exit of each bloc :

$$
\begin{aligned}
PPIN(b) \quad &= \quad \texttt{Pavin}(b) \cap (\texttt{AntGen}(b) \cup \\
&\quad\;\; (\texttt{PPOUT}(b) \setminus \texttt{Kill}(b))) \\
&\quad\;\; \cap \bigcap_{b' \in \texttt{Pre}(b)} (\texttt{PPOUT}(b') \cup \texttt{Avout}(b'))
\end{aligned}
$$

$$
\texttt{PPOUT}(b) \quad = \quad \begin{cases} \emptyset \text{ if } b \text{ is the end block,} \\ \displaystyle\bigcap_{b' \in \texttt{Succ}(b)} \texttt{PPIN}(b') \end{cases}
$$

$$\texttt{Insert}(b) = \texttt{PPOUT}(b) \cap (\overline{PPIN(b)} \cup kill(b))$$

$$\texttt{Suppress}(b) = \texttt{AntGen}(b) \cap PPIN(b)$$

Def-Use, Use-Def chains

Def-Use chains A Def-Use chain associates with each definition a list of statements that are reached by the definition and contains a use of variable being defined,

Use-Def chains A Use-Def chains associates with each use of a variable a list of statements containing a definition of the variable that reach the use.

Def-Use, Use-Def chains: data flow equations

Data flow equations for Def-Use chain. Similar to Live variable analysis, we give only $gen(b)$ and $kill(b)$:

- $gen(b) =$
  $\{(I, x) \mid I$ is an instruction of b, I uses x and x is not defined before $I\}$
- $kill(b) =$
  $\{(I, x) \mid I$ is an instruction not in b, I uses x and x is defined in $b\}$

Data flow equations for Use-Def chain. Similar to Reaching Definitions, we give only $gen(b)$ and $kill(b)$:

- $gen(b) =$
  $\{(I, x) \mid I$ is an instruction of b, I defines x and x is not defined after $I\}$
- $kill(b) =$
  $\{(I, x) \mid I$ is an instruction not in b, I defines x and x is defined in $b\}$

Applications of UD chains

Lopp invariant Given a loop,

1. Mark invariant all the instructions such that operands are either a constant or have reaching definitions outside the loop,
2. repeat next step until stabilization :
3. Mark invariant all the instructions such that operands are either a constant, have reaching definitions outside the loop or have exactly on reaching definition inside the loop.

Exercices

- Dead Code Elimination,
- Constant Propagation
- Induction variables

Static Single Assignment (SSA)

About SSA

- Program representation
- Variables are split into instances : every new assignment (or definition) of a variable results in a new instances
- Variables instances are numbered
- In a program in SSA form, there is exactly one definition reaching each use of a variable.

Translating a CFG into SSA form

1. Special assignment statements, called ϕ-*functions* are inserted at certain points in the program
2. Each variable x numbered $x_1, \cdots x_n$ one x_i for exactly one definition of x.

Outline - Optimization Using Data-flow Analysis

Summary

Code Optimization using data-flow analysis

- Objective: produce a semantically equivalent version of 3-address code that is *optimized*.
- Focused on optimizations independent from the target machine:
  - available expressions for the suppression of redundant computations,
  - live variables for useless assignments
  - constant propagation for replacing variables with their (fixed) values.