LogisticRegresssionBinary

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Logistic Regression

$$\sigma(w^T x) = \frac{1}{1+e^{-w^T x}}$$

$$\frac{d}{dx}(x^T a) = \frac{d(a^T x)}{dx} = a^T$$

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

$$\sigma'(x) = \sigma(x)(1-\sigma(x))$$

$$\left.\begin{aligned} p(y=1 \mid x) &= \sigma(w^T x) = \frac{1}{1+e^{-w^T x}} \\ p(y=0 \mid x) &= 1 - p(y=1 \mid x) = \frac{e^{-w^T x}}{(1+e^{-w^T x})} \end{aligned}\right\} \text{2class}$$

$$\boxed{p(y_n \mid x_n) = p(y_n = 1 \mid x)^{y_n} \cdot p(y_n = 0 \mid x_n)^{1-y_n}}$$

$$p(y_1, \ldots, y_n \mid x_1, \ldots, x_n) = \prod_{n=1}^{N} p(y_n \mid x_n)$$

$$y_n \in \binom{0}{1}$$

$$\mathcal{L}(w) = \prod_{n=1}^{N} \log p(y_n \mid x_n)$$

$$= \prod_{n=1}^{N} \log \left[p(y_n = 1 \mid x)^{y_n} \cdot p(y_n = 0 \mid x_n)^{1-y_n} \right]$$

$$= \sum_{n=1}^{N} \log p(y=1 \mid x)^{y_n} + \log p(y=0 \mid x_n)^{1-y_n}$$

$$= \sum_{n=1}^{N} y_n \log p(y=1 \mid x) + \sum_{n=1}^{N} (1-y_n) \log p(y=0 \mid x_n)$$

$$\mathcal{L}(w) = \sum_{n=1}^{N} y_n \log \sigma(w^T x) + \sum_{n=1}^{N} (1-y_n) \log(1 - \sigma(w^T x))$$

$$\boxed{\mathcal{L}(w) = -\frac{1}{N} \sum_{n=1}^{N} y_n \log \sigma(w^T x) + \sum_{n=1}^{N} (1-y_n) \log(1 - \sigma(w^T x))}$$

min this and $w^{(t+1)} = w^{(t)} - \underbrace{\eta}_{\text{learning rate}} \cdot \nabla \mathcal{L}(w^{(t)})$

$$\frac{\partial \mathcal{L}(w)}{\partial w} = -\frac{1}{N} \sum_{n=1}^{N} y_n \left(\frac{1}{\cancel{\sigma(w^T x)}}\right)\left(\cancel{\sigma(w^T x)}(1-\sigma(w^T x))\right) x^T + \sum_{n=1}^{N} \frac{(1-y_n)}{\cancel{1-\sigma(w^T x)}} (-\sigma(w^T x)) \left(\cancel{1-\sigma(w^T x)}\right) x^T$$

$$= -\frac{1}{N} \sum_{n=1}^{N} (y_n - \cancel{y_n \sigma(w^T x)}) x^T + \sum_{n=1}^{N} \left[(-\sigma(w^T x)) + \cancel{y_n \sigma(w^T x)}\right] x^T$$

$$= -\frac{1}{N} \sum_{n=1}^{N} \left[y_n + \sum_{n=1}^{N} -\sigma(w^T x) \right] x^T$$

$$= -\frac{1}{N} \sum_{n=1}^{N} (y_n - \sigma(w^T x)) x^T$$

$$= \frac{1}{N} \sum_{n=1}^{N} (\sigma(w^T x) - y_n) x^T \quad (\text{no. of weights} \times 1)$$

$$\partial \mathcal{L}(w) = -\frac{1}{N} \sum_{n=1}^{N} y_n \log \sigma(w^T x + b) + \sum_{n=1}^{N} (1-y_n) \log(1 - \sigma(w^T x + b))$$ (if bias is considered)

$$\frac{\partial \mathcal{L}(w)}{\partial b} = -\frac{1}{N} \sum_{n=1}^{N} y_n \frac{\cancel{\sigma(w^T x + b)}(1 - \sigma(w^T x + b))}{\cancel{\sigma(w^T x + b)}} + \sum_{n=1}^{N} (1-y_n) \frac{-\sigma(w^T x + b)\cancel{(1 - \sigma(w^T x + b))}}{\cancel{(1 - \sigma(w^T x + b))}}$$

$$= -\frac{1}{N} \sum_{n=1}^{N} y_n - \cancel{y_n \sigma(w^T x + b)} + \sum_{n=1}^{N} -\sigma(w^T x + b) + \cancel{y_n \sigma(w^T x + b)}$$

$$= -\frac{1}{N} \sum_{n=1}^{N} y_n - \sigma(w^T x + b)$$

$$= \frac{1}{N} \sum_{n=1}^{N} \sigma(w^T x + b) - y_n$$ (be careful there's only one element here)