Homework 4

FE621 Computational Methods in Finance
due 23:55ET, Sunday April 23, 2017

Specifications. For all the problems in this assignment you need to design and use a computer program, output and present the results in nicely formatted tables and figures. The computer program may be written in any programming language you want. Please submit an archive containing a written report (pdf), where you detail your results and copy your code into an Appendix. The archive should also contain the code with comments. Any part of the problems that asks for implementation should contain a reference to the relevant code submitted.

1 Parameter estimation from historical data

Please refer to the attached file "sample_data.csv". In this data set, you will observe 5 columns, all representing stock prices. Each column is data generated by one or more of the models below:

- $dS_t = \theta_1 S_t dt + \theta 2 S_t^{\theta_3} dW_t$
- $dS_t = (\theta_1 + \theta_2 S_t)dt + \theta_3 S_t^{\theta_4} dW_t$
- $dS_t = (\theta_1 + \theta_2 S_t)dt + \theta_3 \sqrt{S_t} dW_t$
- $dS_t = \theta_1 dt + \theta_2 S_t^{\theta 3} dWt$
- $dS_t = \theta_1 S_t dt + (\theta_2 + \theta_3 S_t^{\theta_4} dWt)$

where the time between two observations is $\Delta t = 1/365$. Specifically, each column is a realization for a model above. It is not known which of the models was used for each column. We also do not know if one model was used for all 5 columns or if each column corresponds to a different model.

Given this information please answer the following questions:

1. Decide which of the models could be the source of each of the appropriate columns in the dataset. Use the AIC criterion and read the document posted in the course site.
2. Implement Euler method, Ozaki method, Shoji-Ozaki method and Kessler method to estimate parameters for the model that you chose as the best

model. Report the model and your parameter estimates in a nice table. You will be judged on your reasoning (most points) but also on how far your estimates are from the known numbers.

3. In your opinion which method gives you the best estimates?

2 SABR model parameter estimation

From Fabrice Douglas Rouah's paper [5], the author shows how to estimate parameter for a SABR model. In this question, you are required to use swaption data from the file "2017_2_15_mid.xlsx" to estimate SABR parameters.

Here is the detailed information for data structure in that file:

Each column represents the maturity of a swaption contract. More specifically, $1Yr, 2Yr, \ldots, 30Yr$ represent the maturity T in equation (3) from Rouah paper. In this data set, all swaption are European Call option with the underlying a swap.

Each row, which is marked with 1Mo, 2Mo, ... , 30Yr represent the maturity of the swap you have the option to enter at the exercising time of the swaption.

"Vol" is at-the-money volatility for each swaption contract and "strike" is the strike price K for each of the contracts. NOTE all the contracts are AT-THE-MONEY.

Pick one maturity (1Yr, 2Yr, ..., 30Yr) you are interested in (one column) and answer the following questions.

1. Assume $\beta = 0.5$, implement equation (5) to estimate parameter α, ρ and ν. To solve this equation, you can use any standard optimization method such as Newton-Raphson method. For this question, you could use any available package to solve it.

2. Set $\beta = 0.7$ and 0.4, and repeat part 1. That is, estimate the corresponding α, ρ and ν values.

3. Compare the parameters you obtained based on different β values. What do you observe? Write a paragraph to explain your observations.

4. Using the maximum (minimum) of the function you optimized, tell us which model would give you best estimation.

5. Using the best parameter values do the following: Select another contract you have not used in the previous question (a different column) and treat it as a benchmark. That is, calculate at-the-money volatility using the parameters you obtained and compare the numbers you obtain to the benchmark values. What do you observe? Please write a paragraph to comment.

3 Option valuation using the Fast Fourier Transform.

In the paper [1], the authors show how the Fast Fourier Transform (FFT) may be used to value options when the characteristic function is known analytically. As it turns out, this method is highly efficient from the numerical point of view. Please read the paper [1], and pay special attention to sections 2, 3.1, and 4. Please solve the following tasks.

(a) Let $C_T(k)$ be the desired value of a T–maturity call option with strike e^k, where we denoted $k := \ln(K)$. The authors proved that (see equation (5)):

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} \psi_T(v) \mathrm{d}v, \tag{1}$$

where i is the imaginary unit, and $\psi_T(v)$ is given in equation (6). Note that $\psi_T(v)$ is given in terms of the characteristic function of the log price, i.e., $\phi_T(u) = \mathbb{E}[e^{ius_T}]$, $s_T := \ln(S_T)$. To implement the formula in (1), one needs to compute the integral. But this can be done by using the FFT method, please see section 4 and equation (24). Consider now the Black–Scholes model, for which the log price is normally distributed. Implement the FFT transform method for the Black–Scholes model, and price an European call option with synthetic parameters. Compare your result with the Black–Scholes formula. You can use any package or library to do this.

BONUS (10pts) We note that the method described here is model independent. Consider now the Heston stochastic volatility model. In order to implement the FFT method in this setting, we need the characteristic function of the log price process. For this, please see the presentation by George Hong here, where the author obtains an explicit formula for $\phi_T(u)$. Price an European call option under this assumptions. Compare this result with the prices obtained in Homework 1, Problem 3 via the analytical formula. Consider the same numerical values as in [3].

4 BONUS (30pts). A quadratic volatility process

Consider a drift-less, time-homogeneous stochastic process X_t, $t \geq 0$, i.e.,

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t, \quad \text{with } X_0 = x_0,$$

and assume that

$$\sigma(x) = \alpha x^2 + \beta x + \gamma.$$

In this case, the transition probability density for X_t, denoted with $p(t, x|x_0)$, satisfies the PDE:

$$\frac{\partial}{\partial t} p = \frac{1}{2}\sigma^2(x) \frac{\partial^2}{\partial x^2} p, \tag{2}$$

with the initial condition

$$p(t=0, s|s_0) = \delta(s - s_0).$$

Here, $\delta(x)$ is the delta function which is equal to 1 at $x = 0$ and zero elsewhere. It can be shown that, by using the following transformation in the space domain, the transformed PDE becomes analytically solvable, see [2]:

$$s(x) := \int_{x_0}^{x} \frac{1}{\sigma(u)} \mathrm{d}u.$$

In the transformed space $s(x)$, we obtain

$$\frac{\partial}{\partial t} P(t, s|s_0) = \frac{\partial^2}{\partial s^2} P(t, s|s_0) + QP(t, s|s_0),$$

where $Q = \frac{\alpha\gamma}{2} - \frac{\beta^2}{8}$. It turns out that $P(t, s|s_0)$ is explicitly invertible. The inverted solution is:

$$p(t, x|x_0) = \frac{1}{\sigma(x)\sqrt{2\pi t}} \frac{\sigma(x_0)}{\sigma(x)} \exp\left(-\frac{1}{2t} \left(\int_{x_0}^{x} \frac{\mathrm{d}u}{\sigma(u)} \right)^2 + Qt \right).$$

Furthermore, the price of an European call option with zero interest rate may be obtained in explicit form using this transition density:

$$\begin{aligned} C(\tau, K|x_0) &= \max(x_0 - K, 0) + \frac{\sigma(K)\sigma(x_0)}{2\sqrt{-2Q}} \times \\ & \times \left(e^{s\sqrt{-Q}} \Phi\left(-\frac{s}{\sqrt{2\tau}} - \sqrt{-2Q\tau} \right) - e^{-s\sqrt{-Q}} \Phi\left(-\frac{s}{\sqrt{2\tau}} + \sqrt{-2Q\tau} \right) \right), \end{aligned}$$

where $s = \left| \int_{x_0}^{K} \frac{1}{\sigma(u)} \mathrm{d}u \right|$, and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal. Please answer the following questions.

(a) Let $t = t_1$ be fixed. Compute and plot in 3 dimensions the surface $p(t, x|x_0)$, for some parameters of your choice. To show the dynamics of this surface, plot the same surface for a different time t_2. Does this surface seem appropriate? Please comment.

(b) Check numerically, by using finite difference approximations, that the transition probability density satisfies the initial PDE (2).

(c) Pick a strike and a maturity value. Compute the price of an European call option assuming that the prices dynamics is described by a quadratic volatility process (use the formula above). Compare your results with the Black–Scholes price of the same option.

5 BONUS (30 points). Calibration of Heston's model to market data.

We consider the paper [3] (which we have seen before), and we focus on section 4 in the paper. We know that the price of an European call option is given by $C(S_0, K, V_0, t, T)$, please see equation (1.4). To calibrate this model, we will look here at two approaches: a simpler one, and the more complex procedure used in [3].

(a) Implement a function that returns the price $C_H(S_0, K, V_0, t, T)$, with input parameters S_0, K, V_0, t, T and $\kappa, \theta, \sigma, \rho, V_0, \lambda$.

(b) Download from the Bloomberg terminal option prices for different maturities and strike prices. To calibrate our model, we must minimize the following square error:

$$f(\Theta) = \sum_i \sum_j \left(C_{MP}(K_i, T_j) - C_H(S_0, K_i, \sigma_{ij}, t, T, \Theta)\right)^2,$$

where $\Theta = \{\kappa, \theta, \sigma, \rho, V_0, \lambda\}$, σ_{ij} is the implied volatility for the pair (K_i, T_j), and C_{MP} is the market price. Solve the optimization problem $\min f(\Theta)$ subject to $2\kappa\theta - \sigma^2 > 0$ and obtain the parameters Θ. To this end, take a close look into the methods described here [4].

(c) In section 4 of [3], a more advanced optimization procedure is proposed. Please read this section and try to calibrate the Heston model using the market data that you have downloaded.

References

[1] Carr, Peter and Madan, Dilip. *Option valuation using the fast Fourier transform.* Journal of computational finance 2.4 (1999): 61-73.

[2] Zuhlsdorff, Christian. *The pricing of derivatives on assets with quadratic volatility.* Applied Mathematical Finance 8.4 (2001): 235-262.

[3] Mikhailov, Sergei and Nögel, Ulrich. *Heston's stochastic volatility model: Implementation, calibration and some extensions* 2004, John Wiley and Sons.

[4] Nimalin Moodley. *The Heston Model: A Practical Approach with Matlab Code.* Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa, 2005.

[5] Rouah, Fabrice Douglas. "The SABR Model." ht tp://www. volopta. com.