STAT 2141A

Applied Probability and Statistics for Engineers

6. Conditional Probability and Independence

(Textbook Section 2.4 & 2.5)

Conditional Probability

Conditional Probability

Let A and B be two events from the sample space ($\mathcal{S}$) such that $P(B) > 0$.

The conditional probability of A given that B has occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Multiplication Rule

Rearranging the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

yields the multiplication rule

$$P(A \cap B) = P(A|B)P(B).$$

Example 2.4: Norma, Norma, Norma…

Norma is a very forgetful person. Let

- A be the event that she remembers her keys
- B be the event that she remembers her wallet
- C be the event that she remembers her lunch

We know that

- $P(A) = .50$
- $P(A \cap B) = .25$
- $P(B|A') = .60$
- $P(C|A) = .80$
- $P(A \cup C) = .90$

Example 2.4: Norma, Norma, Norma…

a) What is the probability that Norma remembers her wallet if she remembers her keys? Provide an interpretation for this quantity.

b) What is the probability that Norma remembers her lunch?

P(B|A)=P(A&B)/P(A)   =0.25/0.50=0.50

P(A U C)= P(A)+P(C)-P(A &C)
P(C)=P(AUC)-P(A)+P(A&C)
P(A&C)=P(C|a)P(A)
=.80(.50)
=.40
P(C)=.90-.50+.40
=.80

Law of Total Probability

Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events:

- $A_i \cap A_j = \emptyset$ for any pair of events
- $A_1 \cup \cdots \cup A_k = \mathcal{S}$.

Let B be any other event.

Then

$$P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i).$$

Bayes' Theorem

Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events:

- $A_i \cap A_j = \emptyset$ for any pair of events
- $A_1 \cup \cdots \cup A_k = \mathcal{S}$.

Let B be any other event such that $P(B) > 0$.

Then

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}.$$

* Your text calls $P(A_j)$ a prior probability. This is irrelevant.

Example 2.4: Norma, Norma, Norma…

c) What is the probability that Norma remembers her wallet?

d) Is it more likely that Norma remembers her keys when she does or does not remember her wallet?

Independence

Independence

Two events, A and B, are independent if

$$P(A|B) = P(A).$$

Two events that are not independent are dependent.

Multiplication Rule for Independent Events

Rearranging the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

yields the multiplication rule

$$P(A \cap B) = P(A|B)P(B).$$

If A and B are independent them

$$P(A \cap B) = P(A)P(B).$$

Example 2.4: Norma, Norma, Norma…

e) Are A and B independent events? Explain what this means in words.

f) Are A and C independent events? Explain what this means in words.

g) What would the value of $P(B \cup C)$ be if B and C are independent?

Independent vs Mutually Exclusive

Note that if A and B are mutually exclusive then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = 0.$$

Hence, A and B can only be independent if $P(A) = 0$ as well.

Any pair of mutually exclusive events *with positive probabilities* must be dependent.

Independence of >2 Events

Events $A_1, \ldots, A_n$ are mutually independent if the probability of the intersection of any subset of the events is equal the product of their probabilities.

I.e., for any indices $i_1, \ldots, i_k$

$$P\left(A_{i_1} \cap \cdots \cap A_{i_k}\right) = \prod_{j=1}^{k} P(A_{i_j})$$

for all $k = 2, \ldots, n$.

Example 2.4: Norma, Norma, Norma…

h) Are A, B, and C mutually independent?