The 12-Balls Problem as an Illustration of the Application of Information Theory
Author(s): Robert H. Thouless
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Examples

(*a*) $n = 9$, $r = 2$. Here $j = 2$, $k = 3 = r + 1$, $m = 2$, so

$$9_2 = 3 + 33 + 0 + 21 = 57.$$

(*b*) $n = 8$, $r = 4$. Here $j = 1$, $k = 5 = r + 1$, $m = 0$, so

$$8_4 = 3 + 12 + 1 + 0 = 16.$$

(*c*) $n = 7$, $r = 2$. Here $j = 1$, $k = 4 = r + 2$, $m = 1$, so

$$7_2 = 7 + 15 + 0 + 9 = 31.$$

Department of Mathematics,
The Polytechnic
Kingston-upon-Thames.

F. GERRISH

THE 12-BALLS PROBLEM AS AN ILLUSTRATION OF THE APPLICATION OF INFORMATION THEORY

BY ROBERT H. THOULESS

One is sometimes asked for an elementary introduction to the principles of information theory. I suggest that this need may be met by showing its application to the well-known problem of twelve balls. The problem is as follows:

You have 12 *balls of the same size.* 11 *of these balls are of the same weight, one is different; you do not know whether this ball is heavier or lighter than the others. You have also a balance on which any group of the balls can be weighed against any other group. How can you discover by means of three weighings which is the different ball and whether it is heavier or lighter than the other balls?*

The following two basic principals of information theory may be assumed to be known

(1) That the unit of information is the "bit," which is the information required to discriminate between two equally probable alternatives.

(2) That the number of bits of information required to discriminate between N equally probable possibilities is $\log_2 N$.

These principles can be applied to the 12-balls problem by considering the amount of information given by three weighings and that required for the solution of the problem. It is a necessary condition for the solubility of the problem that the first of these quantities should be not less than the second. This, however, is not also a sufficient condition of solubility; it is also necessary that there should be a possible strategy of solving such that at no point in the path towards the solution is the information still required greater than that which can be supplied by the remaining weighings.

Since there are three possible positions of the balance in weighing, it follows that the information which can be given by each weighing is $\log_2 3$, and that given by r weighings is $\log_2 3^r$. So if there are three weighings the maximum amount of information that can be extracted is $\log_2 27$. One must enquire whether this is sufficient information to solve the problem of the balls.

There are two things that have to be found out; which of the balls is the different one and whether it is heavier or lighter than the others. If n is the total number of balls, the first of these is a choice between n alternatives requiring $\log_2 n$ bits of information, while the second is a choice between two alternatives requiring one bit of information. So the total information required is $(\log_2 n + 1)$ i.e. $\log_2 2n$. If n is twelve, the amount of information required is $\log_2 24$. The problem should therefore be soluble with three weighings if a strategy can be found which, at all stages, leaves the information required less than or equal to that which can be given by the remaining weighings.

It is clearly necessary that the balls should be divided into two sets. One of these sets (necessarily an even number) will be divided into two equal parts which will be weighed against each other; the remaining balls will not be weighed at this time. The three possible outcomes of this weighing are that the balance remains level or that either one side or the other side of the balance goes down.

The first case (of the balance remaining level) indicates that the different ball is amongst those not yet weighed. So we are left with a problem parallel to the original one: for how many balls can one discover in two weighings which is the different one and whether it is heavier or lighter? The amount of information we can get from two weighings is $\log_2 9$, while the amount we need for solving the problem with n balls is $\log_2 2n$. It follows that n must not be more than 4.

The other possibilities are that one or the other side of the balance goes down. This indicates that the different ball is somewhere in the weighed set. We do not know on which side of the balance it is because we do not know whether it is heavier or lighter than the others. We do have, however, a useful additional item of information beyond the fact that the different ball is in this set; we also know of each ball whether, if it is the different one, it differs by being lighter or by being heavier than the others. The information required from the next two weighings is, therefore, only as to which of these balls is the different one; we no longer need to discover in what direction it differs from the others.

The amount of information required from the next two weighings is, in this case, therefore, $\log_2 n$ if n is the number of balls in this group. The amount of information that can be obtained from two weighings is $\log_2 9$, so the number of balls in this lot cannot be more than nine. Since they had to be divided into two equal parts for the weighing, the number cannot in fact be more than eight, but this is enough for the solution of the problem since there were four balls in the other lot.

It appears, therefore, when we look at the problem from the point of view of information theory, that the best first step is to divide the twelve balls into a group of four to be left unweighed at the first weighing and another group of eight to be weighed four against four. This will lead to a solution if the remaining steps can be carried out without at any stage introducing a situation where more information is required than can be provided by the remaining weighings.

In the first case, that the different ball was amongst the four unweighed in the first weighing, the next step is determined by the fact that the information equation shows that only one ball must be left for the final weighing, since if there were two balls left the information required would be $\log_2 4$ which is more than the $\log_2 3$ bits of information provided by a single weighing. This leaves three balls to be weighed on the second weighing which can be done only if the number is made up to four by adding one ball that is known to be of the standard weight. This can easily be done since it has already been discovered that all eight balls of the first weighing are of standard weight. If, on the second weighing, one side of the balance goes down, the problem is fully solved by weighing against each other the two balls that are on the side of the balance that does not contain the standard ball. If the balance remains even on the second weighing we know that the hitherto unweighed ball is the odd one and we can find out whether it is lighter or heavier by weighing it against a ball which has already been discovered to be of standard weight. This last step provides only one bit of information but at this stage it is all that is required for the complete solution of the problem.

The other case, with one side of balance going down on the first weighing, also leads to a solution. We now have four balls of which we know that if the different one occurs amongst them it differs from the standard balls by being lighter, and four other balls of which we know that if the different one occurs amongst them it must be heavier than the other balls. We may conveniently refer to these two kinds of balls as the L balls and the H balls respectively. All that remains to be determined is which of these eight balls is the different one. For 8 balls, the amount of information required is $\log_2 8$ and the amount of information given by two weighings is greater than this, i.e. $\log_2 9$. For the final weighing there must be left not more than three balls since the information given by a single weighing is $\log_2 3$. This end can be secured by making the second weighing a weighing of three against three. The three positions of the balance will then indicate in which of the groups 3:3:2 the different ball is to be found; the final weighing will be of 1 against 1 in the group indicated. There is, however, one limitation; the second weighing must be arranged with the same number of Hs and Ls on both sides of the balance. It must be HHL/HHL or HLL/HLL; otherwise some outcome will be found to lead to a situation in which more information is required than can be obtained in the final weighing.

It is apparent, therefore, that a solution can be reached for the problem of 12 balls. It is also suggested, however, by the consideration of information requirements, that the problem might be soluble for 13 balls. The information which can be obtained from three weighings is $\log_2 27$, whereas only $\log_2 26$ bits are required for answering the problem with 13 balls. The 13 balls problem would indeed be soluble if we had one extra ball known to be of standard weight so that the first weighing was of 5 balls against 5 balls of which one ball was known, leaving 9 balls to be discriminated in the next two weighings if the different ball was found to be in the weighed set. It may be noted that an extra ball is also needed for the second weighing if four unknown balls are left by the first weighing, and for the third weighing if 1 unknown ball is left from the first two weighings. This, however, creates no difficulty since previous weighings will, in these cases, have shown the balls already weighed are of the standard weight.

In order that the initial number of the balls may be 13 and still provide a soluble problem, it is necessary to restate the problem in such a way that a ball of known standard weight is provided. This may be done in the following way:

You have 1 *red ball and* 13 *white balls all of the same size.* 12 *of the white balls are of the same weight as the red ball but one differs from it in weight; you do not know whether this ball is heavier or lighter than the red one. You have also a balance on which any group of the balls can be weighed against any other group. How can you discover by means of three weighings which is the different ball and whether it is heavier or lighter than the other balls?*

One can, of course, also generalize the problem by asking for how many balls it can be solved with any number of weighings greater than three. If one is allowed r weighings, the amount of information given is $\log_2 (3^r)$ and, if the problem is stated in my revised form (with an additional ball known to be of standard weight), the number of balls between which the required discrimination can be made is given by the relation $2n \ngtr 3^r$.

Corpus Christi College,
Cambridge.

ROBERT H. THOULESS