Exercise 3

TKT4150 - Biomechanics

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Contents

§1) Assignment 3

§1.1) Exercise 1: Principal and equivalent stress in femur bone of running human

The running human from the previous exercise is investigated further. The mechanics of the femur (the thigh bone)are in thes potlight this time. Fracture stress and density of bone are both given in Figure1. Experiments reveal that point P has the largest stress and that the yield strength of the femur is $\sigma_{\text{max}} = 130$ MPa. Preliminary calculations and experiments suggest the following stress in point P:

$$T = \begin{bmatrix} -50 & 35 & 0 \\ 35 & -70 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

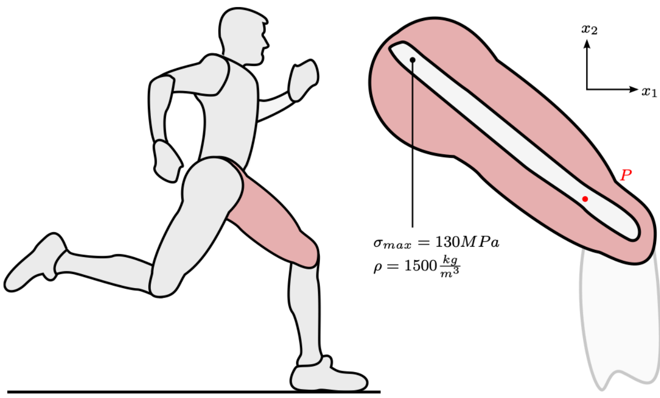

Figure 1: Running human.

§1.1.a) Principal stresses

Given the stress tensor at point P:

$$T = \begin{bmatrix} -50 & 35 & 0 \\ 35 & -70 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa} \tag{2}$$

This is a 2D plane stress problem since $\sigma_{33} = 0$. We extract the 2D stress matrix:

$$T_{2D} = \begin{bmatrix} -50 & 35 \\ 35 & -70 \end{bmatrix} \text{ MPa} \tag{3}$$

Principal Stress Calculation:

Using eigenvalue analysis, the characteristic equation is:

$$\det(T_{2D} - \sigma I) = 0 \tag{4}$$

$$\det \begin{bmatrix} -50 - \sigma & 35 \\ 35 & -70 - \sigma \end{bmatrix} = 0 \tag{5}$$

$$(50 + \sigma)(70 + \sigma) - 35^2 = 0 \tag{6}$$

$$\sigma^2 + 120\sigma + 2275 = 0 \tag{7}$$

Solving this quadratic equation:

$$\sigma_{1,2} = \frac{-120 \pm \sqrt{120^2 - 4 \cdot 2275}}{2} = (-120 \pm \frac{\sqrt{5300\}}}{2} \tag{8}$$

Principal Stresses:

- $\sigma_1 = -23.60$ MPa (least compressive)
- $\sigma_2 = -96.40$ MPa (most compressive)
- $\sigma_3 = 0.00$ MPa (plane stress condition)

Principal Directions:

The principal directions make the following angles with the x_1-axis:

- Direction 1 (for σ_1): 37.03°
- Direction 2 (for σ_2): 127.03° (or $37.03° + 90°$)
- Direction 3 (for σ_3): along x_3-axis

Maximum Shear Stress:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{-23.60 - (-96.40)}{2} = 36.40 \text{ MPa} \tag{9}$$

The analysis shows that the femur bone at point P experiences:

- Maximum compressive stress of 96.4 MPa
- The stress state is entirely compressive in the plane (both principal stresses are negative)
- Maximum shear stress of 36.4 MPa occurs at 45° to the principal directions

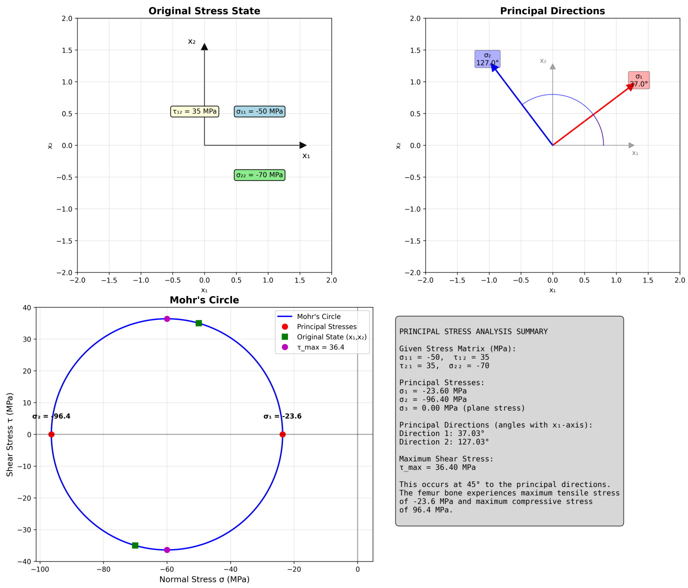

Figure 2: Principal stress analysis for femur bone at point P. Top left: original stress state, top right: principal directions with angles, bottom left: Mohr's circle, bottom right: summary.

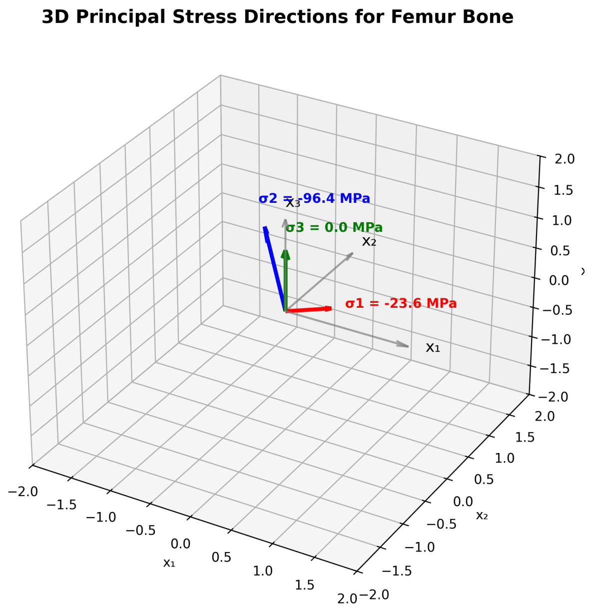

Figure 3: 3D visualization of principal stress directions relative to the coordinate system.

The computational analysis was performed using Python (`ex3.1.1a.py`) and verified using both eigenvalue decomposition and analytical formulas for 2D principal stress analysis.

§1.1.b) Stress lead to fracture?

The von Mises equivalent stress is calculated using:

$$\sigma_{\text{eq}} = \frac{\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}}{\sqrt{2}} \tag{10}$$

Substituting our principal stresses ($\sigma_1 = -23.60$ MPa, $\sigma_2 = -96.40$ MPa, $\sigma_3 = 0$):

$$\sigma_{\text{eq}} = \frac{\sqrt{(-23.60 - (-96.40))^2 + (-96.40 - 0)^2 + (0 - (-23.60))^2}}{\sqrt{2}} \tag{11}$$

$$\sigma_{\text{eq}} = \frac{\sqrt{72.80^2 + 96.40^2 + 23.60^2}}{\sqrt{2}} = \frac{\sqrt{15149.76}}{\sqrt{2}} = 87.15 \text{ MPa} \tag{12}$$

Result: $\sigma_{\text{eq}} = 87.15$ MPa < 130 MPa The given stress matrix does not lead to fracture in the feamur.

§1.2) Exercise 2: Deformation measures

§1.2.a) Homogeneous deformation analysis

Consider the homogeneous deformation:

$$x_1 = X_1 + aX_2 \tag{13}$$

$$x_2 = (1 + a)X_2 \tag{14}$$

where $a = 0.1$.

Part 1: Visualization of square deformation

The square with corners A = (0,0), B = (1,0), C = (1,1), D = (0,1) deforms as follows:

Original square corners:
- A = (0, 0) → A' = (0.0, 0.0)
- B = (1, 0) → B' = (1.0, 0.0)
- C = (1, 1) → C' = (1.1, 1.1)
- D = (0, 1) → D' = (0.1, 1.1)

The square becomes a parallelogram with shear deformation.

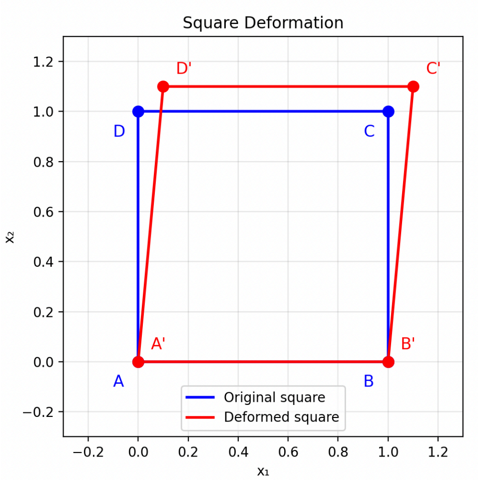

Figure 4: Deformation of square ABCD to A'B'C'D' under the given mapping with a=0.1.

Part 2: Stretch calculations from geometry

Stretch along AC (diagonal):
- Original |AC| = √(1² + 1²) = √2 = 1.414
- Deformed |A'C'| = √(1.1² + 1.1²) = √2.42 = 1.556
- Stretch ratio: λ_AC = 1.556/1.414 = 1.100

Stretch along BD (diagonal):
- Original |BD| = √(1² + 1²) = √2 = 1.414
- Deformed |B'D'|: from (1,0) to (0.1, 1.1)

- |B'D'| = √((0.1-1)² + (1.1-0)²) = √(0.81 + 1.21) = √2.02 = 1.421
- Stretch ratio: λ_BD = 1.421/1.414 = 1.005

Part 3: Shear strain calculation

The shear strain γ between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$:

- Original angle between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$: π/2
- After deformation, $\boldsymbol{e}_1$ remains (1,0), $\boldsymbol{e}_2$ becomes (0.1, 1.1)
- New angle α: $\cos\alpha = \frac{e_1 \cdot e_{2'}}{|e_1\|e_{2'}|} = \frac{0.1}{1\times\sqrt{0.1^2+1.1^2}} = \frac{0.1}{\sqrt{1.22}} = 0.0905$
- $\alpha = \arccos(0.0905) \approx 1.480$ rad
- Shear strain: $\gamma = \frac{\pi}{2} - \alpha \approx 1.571 - 1.480 = 0.091$ rad $= 5.2°$

Part 4: Deformation gradient F and Green strain E

Deformation gradient tensor F: From the deformation mapping:

$$F = \begin{bmatrix} \partial \frac{x_1}{\partial} X_1 & \partial \frac{x_1}{\partial} X_2 \\ \partial \frac{x_2}{\partial} X_1 & \partial \frac{x_2}{\partial} X_2 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1+a \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1.1 \end{bmatrix} \quad (15)$$

Green-Lagrange strain tensor E:

$$E = \frac{1}{2}(F^T F - I) \quad (16)$$

$$F^T = \begin{bmatrix} 1 & 0 \\ 0.1 & 1.1 \end{bmatrix} \quad (17)$$

$$F^T F = \begin{bmatrix} 1 & 0 \\ 0.1 & 1.1 \end{bmatrix}\begin{bmatrix} 1 & 0.1 \\ 0 & 1.1 \end{bmatrix} = \begin{bmatrix} 1 & 0.11 \\ 0.1 & 1.21 \end{bmatrix} \quad (18)$$

$$E = \frac{1}{2}\left(\begin{bmatrix} 1 & 0.11 \\ 0.1 & 1.21 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0.055 \\ 0.05 & 0.105 \end{bmatrix} \quad (19)$$

Part 5: Stretch ratio for arbitrary direction

For direction $n_0 = \cos\theta e_1 + \sin\theta e_2$:

- Deformed vector: $n = F n_0 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1.1 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta + 0.1\sin\theta \\ 1.1\sin\theta \end{bmatrix}$
- $|n_0| = 1$
- $|n| = \sqrt{(\cos\theta + 0.1\sin\theta)^2 + (1.1\sin\theta)^2}$
- $\lambda(\theta) = \sqrt{\cos^2\theta + 0.2\cos\theta\sin\theta + 0.01\sin^2\theta + 1.21\sin^2\theta}$
- $\lambda(\theta) = \sqrt{\cos^2\theta + 0.2\cos\theta\sin\theta + 1.22\sin^2\theta}$

Longitudinal strain: $\varepsilon = \lambda - 1$

For comparison with Green strain tensor: $\varepsilon = n_0^T E n_0 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}\begin{bmatrix} 0 & 0.055 \\ 0.05 & 0.11 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $\varepsilon = 0.055\cos\theta\sin\theta + 0.05\cos\theta\sin\theta + 0.11\sin^2\theta = 0.105\cos\theta\sin\theta + 0.11\sin^2\theta$

Part 6: Maximum and minimum stretch directions

To find extrema, differentiate λ²(ϑ):

$$(\lambda^2)' = -2\cos\theta\sin\theta + 0.2(\cos^2\theta - \sin^2\theta) + 2.44\sin\theta\cos\theta = 0 \quad (20)$$

$$2.42\cos\theta\sin\theta + 0.2\cos 2\theta = 0 \quad (21)$$

$$1.21\sin 2\theta + 0.2\cos 2\theta = 0 \quad (22)$$

$$\tan 2\theta = -\frac{0.2}{1.21} = -0.165 \tag{23}$$

Solutions:
- $2\theta_1 = \arctan(-0.165) \approx -0.164 \text{ rad} \to \theta_1 \approx -0.082 \text{ rad}$
- $2\theta_2 = \pi + \arctan(-0.165) \approx 2.978 \text{ rad} \to \theta_2 \approx 1.489 \text{ rad}$

Maximum stretch: $\lambda_{\max} \approx 1.122$ (at $\theta \approx 68.8°$ or 1.20 rad) **Minimum stretch:** $\lambda_{\min} \approx 0.980$ (at $\theta \approx 158.9°$ or 2.77 rad)

Principal strains from eigenvalues:
- $E_1 = -0.01933$ (minimum principal strain)
- $E_2 = 0.12933$ (maximum principal strain)

Part 7: Principal directions and verification

Principal strains (eigenvalues of E):
- $E_1 = -0.01933$ (minimum principal strain, compression)
- $E_2 = 0.12933$ (maximum principal strain, extension)

Relationship to principal stretches:
- $\lambda_1 = \sqrt{2E_1 + 1} = \sqrt{-0.03866 + 1} \approx 0.980$
- $\lambda_2 = \sqrt{2E_2 + 1} = \sqrt{0.25866 + 1} \approx 1.122$

Verification: These match our computed maximum/minimum stretches, confirming the principal strain analysis.

§1.2.b) Longitudinal strain and stretch relationships

Task: Show that the longitudinal strain and the stretch in the line element aligned with the direction vector $\boldsymbol{e}$ are, respectively:

$$\varepsilon = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{24}$$

(3)

$$\lambda = \varepsilon + 1 \tag{25}$$

(4)

where longitudinal strain is defined as:

$$\varepsilon_l = \frac{ds - ds_0}{d} s_0 = d\frac{s}{d} s_0 - 1 \tag{26}$$

(5)

and the stretch ratio λ is defined as:

$$\lambda = d\frac{s}{d} s_0 \tag{27}$$

(6)

where ds is the deformed length and ds_0 is the reference length of the material element.

Derivation from continuum mechanics

Step 1: Consider a line element in the reference configuration

Let $d\boldsymbol{X}$ be an infinitesimal line element in the reference configuration with length:

$$ds_0 = |d\boldsymbol{X}| = \sqrt{dX_i dX_i} \tag{28}$$

If this element is aligned with unit direction vector $\boldsymbol{e}$, then:

$$d\boldsymbol{X} = ds_0\boldsymbol{e} = ds_0 e_i \boldsymbol{E}_i \tag{29}$$

Step 2: Deformed configuration

In the deformed configuration, this element becomes $\boldsymbol{dx}$ with length:

$$ds = |d\boldsymbol{x}| = \sqrt{dx_i dx_i} \tag{30}$$

The relationship between deformed and reference configurations is:

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j = F_{ij} dX_j \tag{31}$$

Step 3: Length in deformed configuration

For our line element aligned with $\boldsymbol{e}$:

$$dx_i = F_{ij} ds_0 e_j \tag{32}$$

The deformed length squared is:

$$(ds)^2 = dx_i dx_i = F_{ij} ds_0 e_j F_{ik} ds_0 e_k = (ds_0)^2 F_{ij} F_{ik} e_j e_k \tag{33}$$

$$(ds)^2 = (ds_0)^2 \left(\boldsymbol{F}^T\boldsymbol{F}\right)_{jk} e_j e_k \tag{34}$$

Step 4: Green strain tensor relationship

The Green-Lagrange strain tensor is defined as:

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) \tag{35}$$

Therefore:

$$\boldsymbol{F}^T\boldsymbol{F} = 2\boldsymbol{E} + \boldsymbol{I} \tag{36}$$

Substituting:

$$(ds)^2 = (ds_0)^2 \left(2E_{jk} + \delta_{jk}\right) e_j e_k \tag{37}$$

$$(ds)^2 = (ds_0)^2 \left(2e_j E_{jk} e_k + e_j e_j\right) \tag{38}$$

Since $\boldsymbol{e}$ is a unit vector: $e_j e_j = 1$

$$(ds)^2 = (ds_0)^2 \left(1 + 2e_j E_{jk} e_k\right) \tag{39}$$

Step 5: Stretch ratio

The stretch ratio is:

$$\lambda = d\frac{s}{d}s_0 = \sqrt{1 + 2e_i E_{ij} e_j} \tag{40}$$

Step 6: Longitudinal strain

The longitudinal strain is:

$$\varepsilon = \frac{ds - ds_0}{d} s_0 = d\frac{s}{d} s_0 - 1 = \lambda - 1 \tag{41}$$

Therefore:

$$\varepsilon = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{42}$$

Conclusion: We have proven both relationships:

- $$\lambda = \sqrt{1 + 2e_i E_{ij} e_j} \tag{43}$$

  (stretch ratio)
- $$\varepsilon = \lambda - 1 = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{44}$$

  (longitudinal strain)

These equations show how the Green strain tensor $\boldsymbol{E}$ directly relates to measurable quantities (stretch and strain) in any direction $\boldsymbol{e}$.

§1.2.c) Longitudinal strain for coordinate axes

Task: Use, $\varepsilon = \sqrt{1 + 2e_i E_{ij} e_j} - 1$, to determine the general expression for the longitudinal strain and stretch ratio in a line element aligned with the x_k-axis of the coordinate system where k may be 1, 2, or 3.

Step 1: Direction vector for x₁-axis

For a line element aligned with the x_1-axis, the unit direction vector is:

$$\boldsymbol{e} = \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{45}$$

So: $e_1 = 1$, $e_2 = 0$, $e_3 = 0$

Step 2: Apply equation (3) for x₁-direction

Using $\varepsilon = \sqrt{1 + 2e_i E_{ij} e_j} - 1$:

$$\varepsilon_1 = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{46}$$

Expanding the sum:

$$e_i E_{ij} e_j = e_1 E_{1j} e_j + e_2 E_{2j} e_j + e_3 E_{3j} e_j \tag{47}$$

Since $e_1 = 1$ and $e_2 = e_3 = 0$:

$$e_i E_{ij} e_j = 1 \cdot E_{1j} e_j = E_{11} e_1 + E_{12} e_2 + E_{13} e_3 \tag{48}$$

$$e_i E_{ij} e_j = E_{11} \cdot 1 + E_{12} \cdot 0 + E_{13} \cdot 0 = E_{11} \tag{49}$$

Therefore:

$$\varepsilon_1 = \sqrt{1 + 2E_{11}} - 1 \tag{50}$$

$$\lambda_1 = \sqrt{1 + 2E_{11}} \tag{51}$$

Step 3: Generalization for arbitrary k

Following the same procedure for any coordinate axis x_k:

For $\boldsymbol{e} = \boldsymbol{e}_k$, the unit vector has:
- $e_k = 1$ (component in the k-direction)
- $e_i = 0$ for all $i \neq k$

Using the general formula:

$$e_i E_{ij} e_j = e_k E_{kj} e_j = E_{kk} e_k = E_{kk} \tag{52}$$

Step 4: General expressions

For a line element aligned with the x_k-axis (where $k = 1, 2, 3$):

Longitudinal strain:

$$\varepsilon_k = \sqrt{1 + 2E_{kk}} - 1 \tag{53}$$

Stretch ratio:

$$\lambda_k = \sqrt{1 + 2E_{kk}} \tag{54}$$

Step 5: Physical interpretation

These results show that:
- The longitudinal strain in the k-direction depends only on the diagonal component E_{kk} of the Green strain tensor
- Off-diagonal terms (shear components) do not contribute to the longitudinal strain along coordinate axes
- For small strains: $E_{kk} \ll 1$, so $\varepsilon_k \approx E_{kk}$ (using $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for small x)

Examples:
- x_1-direction: $\varepsilon_1 = \sqrt{1 + 2E_{11}} - 1$, $\lambda_1 = \sqrt{1 + 2E_{11}}$
- x_2-direction: $\varepsilon_2 = \sqrt{1 + 2E_{22}} - 1$, $\lambda_2 = \sqrt{1 + 2E_{22}}$
- x_3-direction: $\varepsilon_3 = \sqrt{1 + 2E_{33}} - 1$, $\lambda_3 = \sqrt{1 + 2E_{33}}$

§1.3) Exercise 3: Laplace's Law for Membranes

Laplace's law states:

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p}{t} \tag{55}$$

where σ_i is the stress along x_i, r_i the radius of the shell in x_i-direction, p the internal pressure and t the thickness of the membrane. For a thin-walled sphere, the following equation holds:

$$\sigma = \sigma_\theta = \sigma_\varphi = \frac{r}{2t} p \tag{56}$$

For a thin-walled cylinder with capped ends, we have the following equations:

$$\sigma_z = \frac{r}{2t} p, \ \ \sigma_\theta = \frac{r}{t} p \tag{57}$$

§1.3.a) Derive spherical membrane stress

Task: Use Laplace's law, given in Equation 55, to derive the formula for the membrane stress in Equation 56, for a spherical membrane.

Solution:

For a sphere, the geometry is symmetric in all directions, meaning:

- $r_1 = r_2 = r$ (same radius of curvature in both principal directions)
- $\sigma_1 = \sigma_2 = \sigma$ (same stress in both principal directions due to symmetry)

Substituting into Laplace's law Equation 55:

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p}{t} \tag{58}$$

$$\frac{\sigma}{r} + \frac{\sigma}{r} = \frac{p}{t} \tag{59}$$

$$\frac{2\sigma}{r} = \frac{p}{t} \tag{60}$$

Solving for σ:

$$\sigma = \frac{rp}{2t} \tag{61}$$

This matches Equation 56: $\sigma = \sigma_\theta = \sigma_\varphi = \frac{r}{2t}p$

Physical interpretation: The stress in a spherical pressure vessel is half that of a cylindrical vessel with the same radius and pressure, due to the biaxial nature of the stress distribution.

§1.3.b) Derive cylindrical membrane stress

Task: Use Laplace's law to derive the membrane stress σ_θ in Equation 57, for a cylindrical membrane.

Solution:

For a cylinder, we consider the hoop (circumferential) stress σ_θ. The geometry has:

- One principal radius: $r_1 = r$ (circumferential direction)
- Other principal radius: $r_2 = \infty$ (axial direction is straight, no curvature)

For the hoop stress analysis, we consider a circumferential section where:

- $\sigma_1 = \sigma_\theta$ (hoop stress)
- $\sigma_2 = 0$ (no stress contribution from axial curvature since $r_2 = \infty$)

Applying Laplace's law:

$$\frac{\sigma_\theta}{r} + \frac{0}{\infty} = \frac{p}{t} \tag{62}$$

$$\frac{\sigma_\theta}{r} = \frac{p}{t} \tag{63}$$

Solving for σ_θ:

$$\sigma_\theta = \frac{rp}{t} \tag{64}$$

This matches the hoop stress in Equation 57: $\sigma_\theta = \frac{r}{t}p$

Note: The axial stress $\sigma_z = \frac{r}{2t}p$ comes from equilibrium of the end caps, which we'll derive in part c.

§1.3.c) Free body diagram and axial stress derivation

Task: A thin-walled cylindrical container is subjected to an internal pressure p_i. The stress σ_z on a plane perpendicular to the axis of the cylinder is given in Equation 57. Sketch a suitable free-body-diagram of the container, and derive the formula for σ_z by requiring equilibrium of the free body.

Solution:

Free Body Diagram: Consider a cylindrical section cut by a plane perpendicular to the cylinder axis. The free body consists of:

- Cylindrical wall with internal radius r and wall thickness t
- Internal pressure p_i acting on the circular cross-section
- Axial stress σ_z acting on the cylindrical wall cross-section

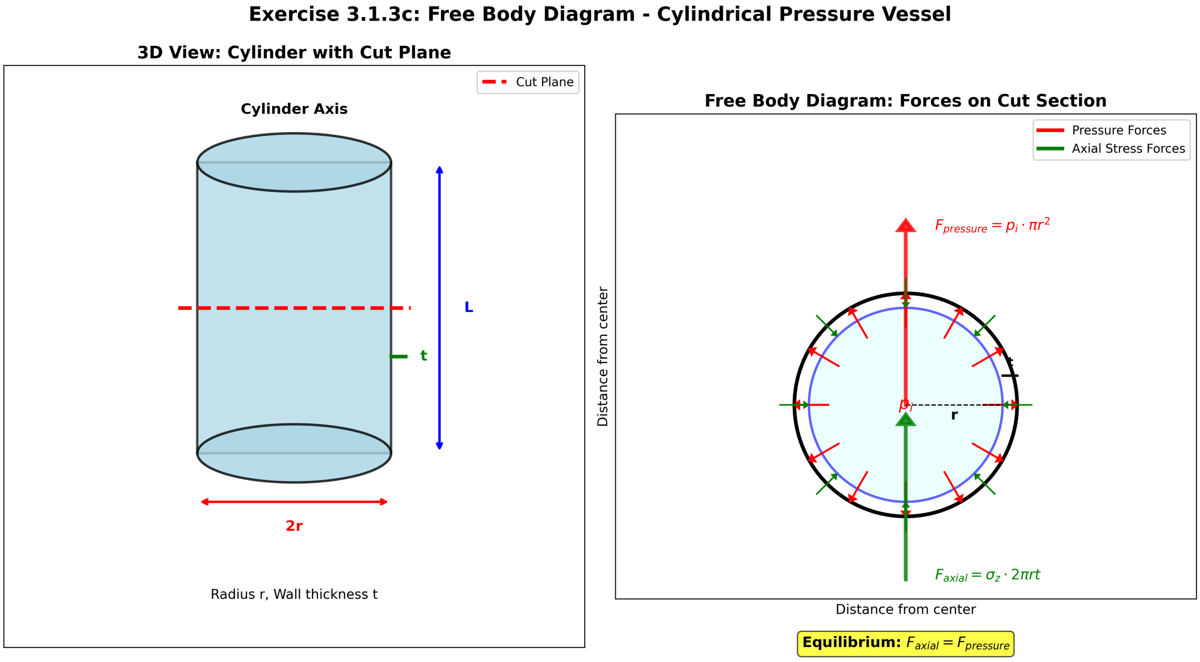

Figure 5: Free body diagram for cylindrical pressure vessel showing pressure forces (red) and axial stress forces (green).

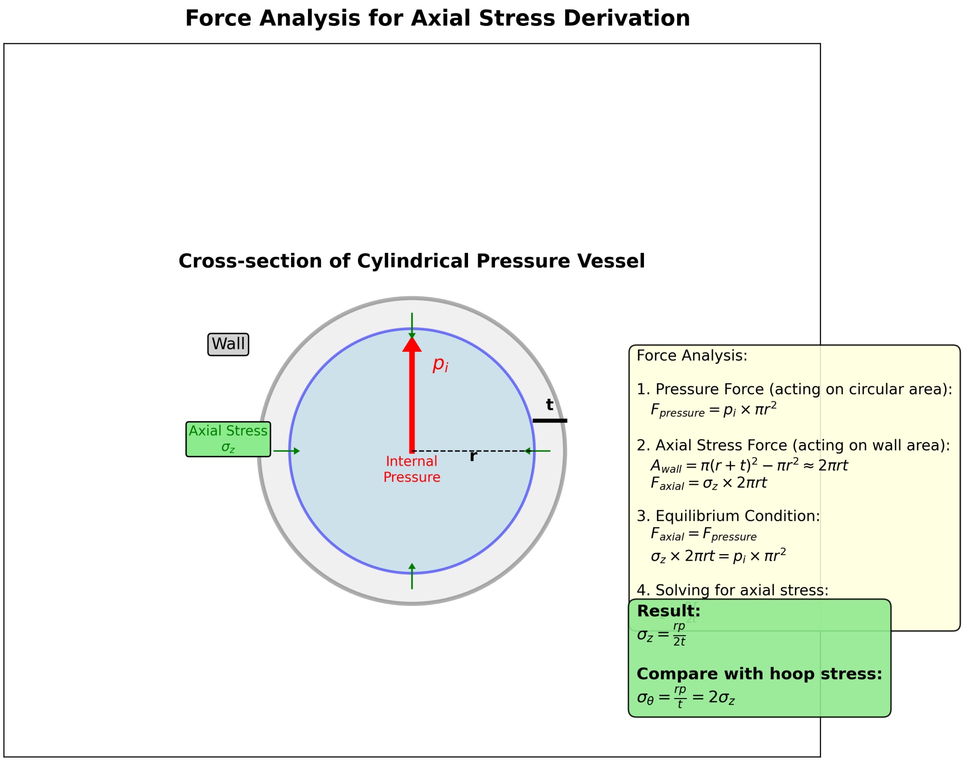

Figure 6: Detailed force analysis and mathematical derivation for axial stress.

Force Analysis:

Forces due to internal pressure:

- Acts on circular area: $A_{\text{pressure}} = \pi r^2$
- Total pressure force: $F_{\text{pressure}} = p_i \times \pi r^2$ (acting axially outward)

Forces due to axial stress:

- Acts on wall cross-sectional area: $A_{\text{wall}} = \pi(r+t)^2 - \pi r^2 \approx 2\pi rt$ (for thin wall: $t \ll r$)
- Total axial stress force: $F_{\text{axial}} = \sigma_z \times 2\pi rt$ (acting axially inward)

Equilibrium condition: For static equilibrium, forces must balance:

$$F_{\text{axial}} = F_{\text{pressure}} \tag{65}$$

$$\sigma_z \times 2\pi rt = p_i \times \pi r^2 \tag{66}$$

Solving for σ_z:

$$\sigma_z = \frac{p_i \times \pi r^2}{2\pi rt} = \frac{p_i r}{2t} \tag{67}$$

This gives us: $\sigma_z = \frac{r}{2t}p$, confirming Equation 57.

Physical interpretation:

- Hoop stress $\sigma_\theta = r\frac{p}{t}$ is twice the axial stress $\sigma_z = r\frac{p}{2t}$
- This is because the hoop stress resists circumferential expansion, while axial stress only needs to balance the pressure on the end caps
- The factor of 2 difference is fundamental to cylindrical pressure vessel design