Exercise 0

TKT4150 - Biomechanics

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Contents

§0.1) Preliminaries

This section was only lots of recap of earlier courses, there did not seem to be any exercised to be completed here.

§0.2) Linear Algebra and Matrix analysis

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\boldsymbol{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$, and $\boldsymbol{b} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}^T$

§0.2.a) Calculate trA, detA, the Frobenius norm $\|A\|$, $A\boldsymbol{a}$, $\boldsymbol{a}^T\boldsymbol{b}$ and $\boldsymbol{b}^T\boldsymbol{a}$.

trA is given formualicly as trA$= \sum_{i=1}^{n} a_{ii}$ We can simply state this as the sum of the diagonal elements of the matrix A. Which gives us: trA $= 1 + 5 + 9 = \mathbf{15}$

detA is simply calculated as:

$$\begin{aligned} \det(A) &= 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 - 2 \cdot 4 \cdot 9 - 1 \cdot 6 \cdot 8 \\ &= 45 + 84 + 96 - 21 - 32 - 48 = \mathbf{0} \end{aligned}$$

The Forbenius norm is the square root of the sum of all the elements squared. It is defined and caluclated as follows:

$$\begin{aligned} \|A\|_F &= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} \\ &= \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2} \\ &= \sqrt{285} \approx \mathbf{16.88} \end{aligned}$$

Calculating $A\boldsymbol{a}$ we get:

$$A\boldsymbol{a} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot1+2\cdot2+3\cdot3 \\ 4\cdot1+5\cdot2+6\cdot3 \\ 7\cdot1+8\cdot2+9\cdot3 \end{bmatrix} = \begin{bmatrix} \mathbf{14} \\ \mathbf{32} \\ \mathbf{50} \end{bmatrix}$$

Calculating $\boldsymbol{a}^T\boldsymbol{b}$ we get:

$$\boldsymbol{a}^T\boldsymbol{b} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = \mathbf{32}$$

Calculating $\boldsymbol{b}^T\boldsymbol{a}$ we get:

$$\boldsymbol{b}^T\boldsymbol{a} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 = \mathbf{32}$$

§0.2.b) A and B are 3x3 matrices. a and b are 3x1 matrices (i.e. vectors). Prove the following implications:

- If $\boldsymbol{a}^T A\boldsymbol{b} = 0$ for all $\boldsymbol{a}$ and $\boldsymbol{b}$, then $A = 0$.

Assuming that $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors, we can prove this by contradiction.
Assume that A is not the zero matrix, meaning that at least one element of A is non-zero.
Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ where at least one $a_{ij} \neq 0$.
We can choose specific vectors for $\boldsymbol{a}$ and $\boldsymbol{b}$ to make $\boldsymbol{a}^T A\boldsymbol{b} \neq 0$.
For example, if we choose $\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then:

$$\boldsymbol{a}^T A\boldsymbol{b} = [1 \;\; 0 \;\; 0] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a_{11}$$

If $a_{11} \neq 0$, then $\boldsymbol{a}^T A\boldsymbol{b} \neq 0$, which contradicts our assumption that $\boldsymbol{a}^T A\boldsymbol{b} = 0$ for all $\boldsymbol{a}$ and $\boldsymbol{b}$.
Therefore, our assumption that A is not the zero matrix must be false, and we conclude that $A = 0$.

- $A^T = -A \Leftrightarrow \boldsymbol{a}^T A\boldsymbol{a} = 0$ for all $\boldsymbol{a}$

  Assume that $A^T = -A$. Then, for any vector $\boldsymbol{a}$, we have:

$$\boldsymbol{a}^T A\boldsymbol{a} = \boldsymbol{a}^T(-A^T)\boldsymbol{a} = -(\boldsymbol{a}^T A^T \boldsymbol{a}) = -(\boldsymbol{a}^T A\boldsymbol{a})$$

  This implies that

$$\boldsymbol{a}^T A\boldsymbol{a} = -(\boldsymbol{a}^T A\boldsymbol{a})$$

  which can only be true if

$$\boldsymbol{a}^T A\boldsymbol{a} = 0$$

  Therefore, if

$$A^T = -A$$

  then

$$\boldsymbol{a}^T A\boldsymbol{a} = 0$$

  for all $\boldsymbol{a}$.

- If $\boldsymbol{a}^T A\boldsymbol{a} = a^T B\boldsymbol{a}$ for all $\boldsymbol{a}$, then $A + A^T = B + B^T$.

  We start with the given condition:

$$\boldsymbol{a}^T A\boldsymbol{a} = \boldsymbol{a}^T B\boldsymbol{a}$$

  for all $\boldsymbol{a}$
  This can be rewritten as:

$$\boldsymbol{a}^T(A - B)\boldsymbol{a} = 0$$

for all $\boldsymbol{a}$
Let $C = A - B$. Then we have:

$$\boldsymbol{a}^T C \boldsymbol{a} = 0$$

for all $\boldsymbol{a}$
This implies that C is a skew-symmetric matrix, meaning that $C^T = -C$. Therefore, we have:

$$(A - B)^T = -(A - B)$$

Expanding this gives us:

$$A^T - B^T = -A + B$$

Rearranging terms, we get:

$$A + A^T = B + B^T$$

Thus, if

$$\boldsymbol{a}^T A \boldsymbol{a} = \boldsymbol{a}^T B \boldsymbol{a}$$

for all $\boldsymbol{a}$, then it follows that

$$A + A^T = B + B^T$$