Exercise 4

TKT4150 - Biomechanics

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Contents

§1) Assignment 4

§1.1) Exercise 1: Cauchy equations

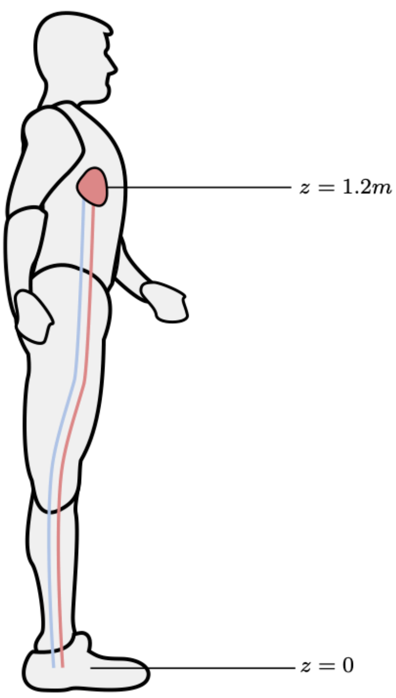

Figure 1: Arteries and veins

§1.1.a) Formula for pressure in a still body of fluid

Starting with the Cauchy equation:

$$T_{ij,j} = \frac{\partial}{\partial x_j}(T_{ij}) = \frac{\partial}{\partial x_j}(-p\delta_{ij}) = -\frac{\partial p}{\partial x_j}\delta_{ij} \tag{1}$$

$$= -\frac{\partial p}{\partial x_1}\delta_{i1} - \frac{\partial p}{\partial x_2}\delta_{i2} - \frac{\partial p}{\partial x_3}\delta_{i3} \tag{2}$$

The pressure only depends on the vertical distance, so we can then reduce this to:

$$T_{ij,j} = -\frac{\partial p}{\partial x_3} \quad \text{for } i = 3 \tag{3}$$

$$= 0 \quad \text{otherwise} \tag{4}$$

For equilibrium with no external acceleration, the Cauchy equation becomes:

$$-\frac{\partial p}{\partial z} + \rho g = 0 \tag{5}$$

which gives:

$$\nabla p = \frac{\partial p}{\partial z} = \rho g \tag{6}$$

Integrating:

$$p(z) = \int \rho g dz = \rho g z + C \tag{7}$$

With boundary condition $p(z = 0) = p_0$, we get $C = p_0$, thus:

Result: $p(z) = \rho g z + p_0$

§1.1.b) Blood pressure in foot

Using the formula from the task above:

$$p(z) = \rho g z + p_0 \tag{8}$$

Veins:

$$\begin{aligned} p_v(1.2m) &= p_{vh} + \rho g h \\ &= 10kPa + 1000\frac{kg}{m^3} \cdot 9.81\left(\frac{m}{s^2}\right) \cdot 1.2(m) \\ &= 10kPa + 11.8kPa = 21.8kPa \end{aligned} \tag{9}$$

Arteries:

$$\begin{aligned} p_a(1.2m) &= p_{ah} + \rho g h \\ &= 16kPa + 1000\frac{kg}{m^3} \cdot 9.81\left(\frac{m}{s^2}\right) \cdot 1.2(m) \\ &= 16kPa + 11.8kPa = 27.8kPa \end{aligned} \tag{10}$$

§1.2) Exercise 2: Hooke's law

§1.2.a) Superposition

Apply superposition of three uni-axial cases:

- σ_1 only: $\varepsilon_1 = \frac{\sigma_1}{\eta}$, $\varepsilon_2 = \varepsilon_3 = -\nu\frac{\sigma_1}{\eta}$
- σ_2 only: $\varepsilon_2 = \frac{\sigma_2}{\eta}$, $\varepsilon_1 = \varepsilon_3 = -\nu\frac{\sigma_2}{\eta}$
- σ_3 only: $\varepsilon_3 = \frac{\sigma_3}{\eta}$, $\varepsilon_1 = \varepsilon_2 = -\nu\frac{\sigma_3}{\eta}$

Total strain in direction 1:

$$\varepsilon_1 = \frac{\sigma_1}{\eta} - \nu\frac{\sigma_2}{\eta} - \nu\frac{\sigma_3}{\eta} = \frac{\sigma_1}{\eta} + \nu\frac{\sigma_1}{\eta} - \frac{\nu}{\eta(\sigma_1 + \sigma_2 + \sigma_3)} \tag{11}$$

$$\varepsilon_1 = \frac{1+\nu}{\eta}\sigma_1 - \frac{\nu}{\eta}\text{ tr }\boldsymbol{T} \tag{12}$$

Similarly for ε_2 and ε_3, giving the general result.

§1.2.b) Index notation tensor form

The following shear stress-strain relation is assumed:

$$E_{ij} = \frac{1+\nu}{\eta}T_{ij}\ i \neq j \tag{13}$$

From this, establish the index notation tensor version of Hooke's law, with strain on left hand side, that incorporates both normal and shear components.

From part a), we use Equation 12.

From Equation 13, strains: $E_{ij} = \frac{1+\nu}{\eta}T_{ij}$ for $i \neq j$

We can unify these using the Kronecker delta δ_{ij}:

$$E_{ij} = \frac{1+\nu}{\eta}T_{ij} - \frac{\nu}{\eta}\delta_{ij}\text{ tr }\boldsymbol{T} \tag{14}$$

Verification:
- When $i = j$: $E_{ii} = \frac{1+\nu}{\eta}T_{ii} - \frac{\nu}{\eta}\text{ tr }\boldsymbol{T}$
- When $i \neq j$: $E_{ij} = \frac{1+\nu}{\eta}T_{ij} - 0 = \frac{1+\nu}{\eta}T_{ij}$

Complete tensor form of Hooke's law:

$$\boldsymbol{E} = \frac{1+\nu}{\eta}\boldsymbol{T} - \frac{\nu}{\eta}(\text{tr }\boldsymbol{T})\boldsymbol{I} \tag{15}$$

§1.2.c) No task, only an equaion?

§1.2.d) Plane stress conditions

Given: Inverse relation:

$$T_{ij} = \frac{\eta}{1+\nu}\left(E_{ij} + \frac{\nu}{1-2\nu}E_{kk}\delta_{ij}\right) \tag{16}$$

Task: Under plane stress conditions $T_{i3} = 0$ for $i = 1, 2, 3$, show that Equation 16 becomes:

$$T_{\alpha\beta} = \frac{\eta}{1+\nu}\left(E_{\alpha\beta} + \frac{\nu}{1-\nu}E_{\rho\rho}\delta_{\alpha\beta}\right)\ \alpha = 1, 2 \tag{17}$$

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Solution:

From $T_{33} = 0$ in equation (4):

$$E_{33} + \frac{\nu}{1-2\nu} E_{kk} = 0 \tag{18}$$

Solving: $E_{33} = -\frac{\nu}{1-2\nu} E_{kk}$

Substituting back: $E_{kk} = E_{11} + E_{22} + E_{33} = E_{\rho\rho} - \frac{\nu}{1-2\nu} E_{kk}$

Therefore: $E_{kk}(1 + \frac{\nu}{1-2\nu}) = E_{\rho\rho}$, giving $E_{kk} = \frac{1-2\nu}{1-\nu} E_{\rho\rho}$

Substituting into equation (4): $\frac{\nu}{1-2\nu} E_{kk} = \frac{\nu}{1-2\nu} \cdot \frac{1-2\nu}{1-\nu} E_{\rho\rho} = \frac{\nu}{1-\nu} E_{\rho\rho}$

Result: $T_{\alpha\beta} = \frac{\eta}{1+\nu}(E_{\alpha\beta} + \frac{\nu}{1-\nu} E_{\rho\rho} \delta_{\alpha\beta})$

§1.3) Exercise 3: Navier's equations

The Cauchy equations read out:

$$T_{ij,j} + \rho b_i = \rho \ddot{u}_i \tag{19}$$

Compatibility requires:

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{20}$$

By introducing Hooke's law and Green's strain tensor, the Cauchy equations are simplified. The result is called Navier's equations.

§1.3.a) Derive Navier's equations

From Hooke's law:

$$T_{ij} = \frac{\eta}{1+\nu}\left(E_{ij} + \frac{\nu}{1-2\nu} E_{kk} \delta_{ij}\right) \tag{21}$$

From compatibility:

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{22}$$

Therefore:

$$E_{kk} = E_{11} + E_{22} + E_{33} = \frac{1}{2}(2u_{1,1} + 2u_{2,2} + 2u_{3,3}) = u_{k,k} \tag{23}$$

Substituting into Hooke's law:

$$T_{ij} = \frac{\eta}{1+\nu}\left(\frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{\nu}{1-2\nu} u_{k,k} \delta_{ij}\right) \tag{24}$$

Taking divergence for Cauchy equation:

$$T_{ij,j} = \frac{\eta}{1+\nu}\left(\frac{1}{2}(u_{i,jj} + u_{j,ij}) + \frac{\nu}{1-2\nu} u_{k,kj} \delta_{ij}\right) \tag{25}$$

Since $u_{j,ij} = u_{j,ji} = u_{k,ki}$ and $\delta_{ij,j} = 0$:

$$T_{ij,j} = \frac{\eta}{1+\nu}\left(\frac{1}{2} u_{i,jj} + \frac{1}{2} u_{k,ki} + \frac{\nu}{1-2\nu} u_{k,ki}\right) \tag{26}$$

$$= \frac{\eta}{1+\nu}\left(\frac{1}{2} u_{i,jj} + \left(\frac{1}{2} + \frac{\nu}{1-2\nu}\right) u_{k,ki}\right) \tag{27}$$

$$= \frac{\eta}{1+\nu}\left(\frac{1}{2}u_{i,jj} + \frac{1-2\nu+2\nu}{2(1-2\nu)}u_{k,ki}\right) \tag{28}$$

$$= \frac{\eta}{1+\nu}\left(\frac{1}{2}u_{i,jj} + \frac{1}{2(1-2\nu)}u_{k,ki}\right) \tag{29}$$

Substituting into Cauchy equation (7):

$$\frac{\eta}{1+\nu}\left(\frac{1}{2}u_{i,jj} + \frac{1}{2(1-2\nu)}u_{k,ki}\right) + \rho b_i = \rho \ddot{\mathrm{u}}_i \tag{30}$$

Navier's equation:

$$\frac{\eta}{2(1+\nu)}u_{i,jj} + \frac{\eta}{2(1+\nu)(1-2\nu)}u_{k,ki} + \rho b_i = \rho \ddot{\mathrm{u}}_i \tag{31}$$

§1.4) Exercise 4: Hyperelasticity

A material is hyperelastic if there exists a potential function $\varphi = \varphi(E_{ij})$ that satisfies $T_{ij} = \frac{\partial\varphi}{\partial E_{ij}}$, i.e. the stress is defined by a potential function of the strain. An equivalent definition of hyperelasticity requires that the stress power ω may be derived from a scalar valued potential $\varphi(E_{ij})$:

$$\omega = \dot{\varphi} = \frac{\partial\varphi}{\partial E_{ij}}\dot{E}_{ij} \tag{32}$$

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§1.4.a) Stress power

Stress power ω is the rate of energy dissipation or storage per unit volume in a deforming material. It represents:

- Power = Energy/Time per unit volume
- $\omega = T_{ij}\dot{E}_{ij}$ (stress × strain rate)
- Physical meaning: Rate of work done by internal stresses during deformation

§1.4.b) Equivalence of definitions

Definition 1:

$$T_{ij} = \frac{\partial\varphi}{\partial E_{ij}} \tag{33}$$

(stress from potential) **Definition 2:**

$$\omega = \dot{\varphi} = \frac{\partial\varphi}{\partial E_{ij}}\dot{E}_{ij} \tag{34}$$

(stress power from potential rate)

From Definition 1, substitute into stress power:

$$\omega = T_{ij}\dot{E}_{ij} = \frac{\partial\varphi}{\partial E_{ij}}\dot{E}_{ij} \tag{35}$$

Using chain rule: $\dot{\varphi} = \frac{\partial\varphi}{\partial E_{ij}}\dot{E}_{ij}$

Therefore: $\omega = \dot{\varphi}$, which is Definition 2.

Conversely, if $\omega = \dot{\varphi} = \frac{\partial \varphi}{\partial E_{ij}} \dot{E}_{ij}$ and $\omega = T_{ij} \dot{E}_{ij}$, then:

$$T_{ij} \dot{E}_{ij} = \frac{\partial \varphi}{\partial E_{ij}} \dot{E}_{ij} \tag{36}$$

Since this holds for arbitrary $\dot{E}_{ij}$: $T_{ij} = \frac{\partial \varphi}{\partial E_{ij}}$

The definitions are equivalent.

§1.5) Exercise 5: Piola-Kirchhoff stress

§1.5.a) Tensor form relations

Cauchy stress (true stress): $\boldsymbol{\sigma}$ - stress per unit deformed area

First Piola-Kirchhoff stress: $\boldsymbol{P}$ - force per unit reference area

$$\boldsymbol{P} = J \boldsymbol{\sigma} \boldsymbol{F}^{-T} \tag{37}$$

Second Piola-Kirchhoff stress: $\boldsymbol{S}$ - reference configuration stress

$$\boldsymbol{S} = \boldsymbol{F}^{-1} \boldsymbol{P} = J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T} \tag{38}$$

Relations:
- $\boldsymbol{P} = J \boldsymbol{\sigma} \boldsymbol{F}^{-T}$ (Cauchy → First PK)
- $\boldsymbol{S} = \boldsymbol{F}^{-1} \boldsymbol{P}$ (First PK → Second PK)
- $\boldsymbol{S} = J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T}$ (Cauchy → Second PK)

where $J = \det(\boldsymbol{F})$ is the Jacobian of deformation.

§1.5.b) Symmetry of second Piola-Kirchhoff stress

From angular momentum balance, the Cauchy stress is symmetric: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$

The second Piola-Kirchhoff stress is:

$$\boldsymbol{S} = J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T} \tag{39}$$

Taking the transpose:

$$\boldsymbol{S}^T = \left(J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T}\right)^T \tag{40}$$

$$= J \left(\boldsymbol{F}^{-T}\right)^T \boldsymbol{\sigma}^T \left(\boldsymbol{F}^{-1}\right)^T \tag{41}$$

$$= J \boldsymbol{F}^{-1} \boldsymbol{\sigma}^T \boldsymbol{F}^{-T} \tag{42}$$

Since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ (Cauchy stress is symmetric):

$$\boldsymbol{S}^T = J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T} = \boldsymbol{S} \tag{43}$$

Therefore, the second Piola-Kirchhoff stress tensor is symmetric: $\boldsymbol{S} = \boldsymbol{S}^T$