TKT 4150 Assignement 3

2022

1 Assignment 3

1.1 Exercise 1: Principal and equivalent stress in femur bone of running human

The running human from the previous exercise is investigated further. The mechanics of the femur (the thigh bone) are in the spotlight this time. Fracture stress and density of bone are both given in Figure 1. Experiments reveal that point P has the largest stress and that the yield strength of the femur is $\sigma_{\text{max}} = 130\text{MPa}$. Preliminary calculations and experiments suggest the following stress in point P:

$$T = \begin{bmatrix} -50 & 35 & 0 \\ 35 & -70 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa} \tag{1}$$

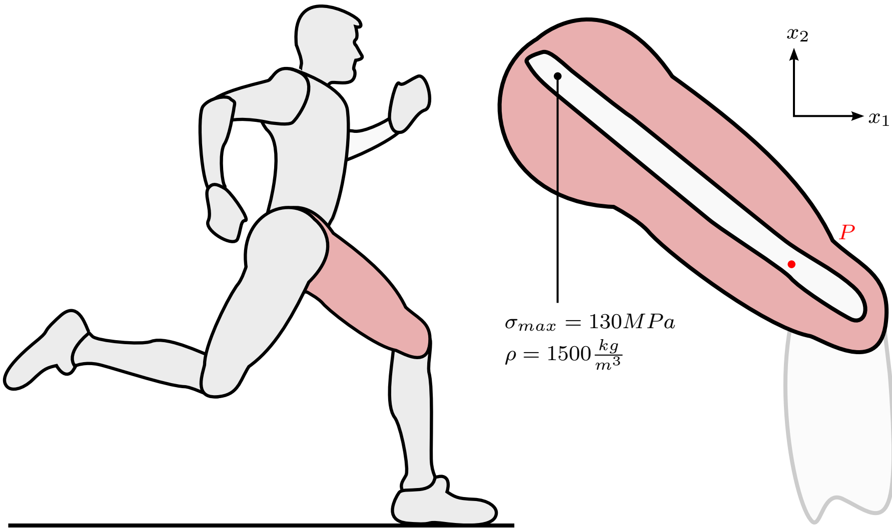

Figure 1: Running human.

a) Calculate the principal stresses in this point, and sketch them together with their corresponding angles relative to the x_1, x_2, and x_3 coordinate system.

b) One way of evaluating whether a material is stressed beyond its strength is to compare the equivalent, also called von Mises, stress to the yield strength ($\sigma_{\text{max}} = 130\text{MPa}$). The equivalent (von Mises) stress is given by

$$\sigma_{eq} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]} \tag{2}$$

where σ_1, σ_2, and σ_3 are the **ordered** principal stresses.

Calculate the equivalent stress (von Mises). Does the given stress matrix lead to fracture in the femur?

1.2 Exercise 2: Deformation measures

a) Consider the homogenous deformation

$$x_1 = X_1 + aX_2$$
$$x_2 = (1 + a)X_2$$

where $a = 0.1$.

Draw a figure to show how the square with corners $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, $D = (0,1)$ deforms.

Attempt to calculate the following from geometrical understanding, i.e. don't use the formulas from the compendium.

The stretch λ of a line element is defined as the ratio of its stretched length to unstretched length. What is the stretch for a line element along AC? What about along BD?

The shear strain γ is defined as the change in angle between two line elements which are perpendicular in the reference configuration, i.e. $\gamma = \frac{\pi}{2} - \alpha$ where α is the angle bewteen the deformed line elements. What is the shear between $\mathbf{e}_1$ and $\mathbf{e}_2$?

Compute $\mathbf{F}$ and $\mathbf{E}$ for the given deformation.

Note that any direction may be denoted $\mathbf{n}_0 = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2$. Use $\mathbf{F}$ to determine the deformed vector $\mathbf{n}$. Calculate the stretch ratio $\lambda = \frac{\|\mathbf{n}\|}{\|\mathbf{n}_0\|}$. Note that longitudinal strain $\epsilon = \lambda - 1$, use this to compare the previous results to the longitudinal strain calculated using the Green strain tensor $\mathbf{E}$.

Use calculus to determine which directions have the largest and smallest stretch ratio λ.

Calculate $\mathbf{En}$ for these directions. Do you notice anything?

b) Show that the longitudinal strain and the stretch in the line element aligned with the direction vector e are, respectively:

$$\varepsilon = \sqrt{1 + 2e_i E_{ij} e_j} - 1 \tag{3}$$
$$\lambda = \varepsilon + 1 \tag{4}$$

where longitudinal strain is defined as:

$$\epsilon^l = \frac{ds - ds_0}{ds_0} = \frac{ds}{ds_0} - 1 \tag{5}$$

and the stretch ratio λ is defined as

$$\lambda = \frac{ds}{ds_0} \tag{6}$$

where ds is the deformed length and ds_0 is the reference length of the material element.

Hint. Reference the section in the compendium titled "The Green strain tensor".

c) Use (3) determine the general expression for the longitudinal strain and the stretch ratio in a line element aligined with the x_k-axis of the coordinate system where k may be 1, 2, or 3.

Hint. Try to do it for the direction $\mathbf{e} = \mathbf{e}_1 = \begin{pmatrix}1\&0 & 0\end{pmatrix}^\intercal$ first, then generalize the formula for arbitrary k.

1.3 Exercise 3: Laplace's law for membranes

Laplace's law states

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p}{t} \tag{7}$$

where σ_i is the stress along x_i, r_i the radius of the shell in x_i-direction, p the internal pressure and t the thickness of the membrane. For a thin-walled sphere, the following equation holds:

$$\sigma = \sigma_\theta = \sigma_\phi = \frac{r}{2t}p \tag{8}$$

For a thin-walled cylinder with capped ends, we have the following equations:

$$\sigma_z = \frac{r}{2t}p, \quad \sigma_\theta = \frac{r}{t}p \tag{9}$$

a) Use Laplace's law, given in Equation (7), to derive the formula for the membrane stress in Equation (8), for a spherical membrane.

b) Use Laplace's law to derive the membrane stress σ_θ in Equation (9), for a cylindrical membrane.

c) A thin-walled cylindrical container is subjected to an internal pressure p_i. The stress σ_z, on a plane perpendicular to the axis of the cylinder is given in Equation (9). Sketch a suitable free-body-diagram of the container, and derive the formula for σ_z, by requiring equilibrium of the free body.