19 Lie groups and Lie algebras

19.1 Lie groups

In Section 10.8 we defined a topological group as a group G that is also a topological space such that the map $\psi : (g, h) \mapsto gh^{-1}$ is continuous. If G has the structure of a differentiable manifold and ψ is a smooth map it is said to be a **Lie group**. The arguments given in Section 10.8 to show that the maps $\phi : (g, h) \mapsto gh$ and $\tau : g \mapsto g^{-1}$ are both continuous are easily extended to show that both are differentiable in the case of a Lie group. The map τ is clearly a diffeomorphism of G since $\tau^{-1} = \tau$. Details of proofs in Lie group theory can sometimes become rather technical. We will often resort to outline proofs, when the full proof is not overly instructive. Details can be found in a number of texts, such as [1–7]. Warner [6] is particularly useful in this respect.

Example 19.1 The additive group $\mathbb{R}^n$, described in Example 10.20, is an n-dimensional abelian Lie group, as is the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

Example 19.2 The set $M_n(\mathbb{R})$ of $n \times n$ real matrices is a differentiable manifold, diffeomorphic to $\mathbb{R}^{n^2}$, with global coordinates $x^i{}_j(\mathsf{A}) = A^i{}_j$, where A is the matrix $[A^i{}_j]$. The general linear group $GL(n, \mathbb{R})$ is an n^2-dimensional Lie group, which is an open submanifold of $M_n(\mathbb{R})$ and a Lie group since the function ψ is given by

$$x^i{}_j(\psi(\mathsf{A}, \mathsf{B})) = x^i{}_k(\mathsf{A})x^k{}_j(\mathsf{B}^{-1}),$$

which is differentiable since $x^k{}_j(\mathsf{B}^{-1})$ are rational polynomial functions of the components $B^i{}_j$ with non-vanishing denominator on $\det^{-1}(\dot{\mathbb{R}})$. In a similar way, $GL(n, \mathbb{C})$ is a Lie group, since it is an open submanifold of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$.

Left-invariant vector fields

If G is a Lie group, the operation of **left translation** $L_g : G \to G$ defined by $L_g h \equiv L_g(h) = gh$ is a diffeomorphism of G onto itself. Similarly, the operation of **right translation** $R_g : G \to G$ defined by $R_g h = h_g$ is a diffeomorphism.

A left translation L_g induces a map on the module of vector fields, $L_{g*} : \mathcal{T}(M) \to \mathcal{T}(M)$ by setting

$$(L_{g*}X)_a = L_{g*}X_{g^{-1}a} \tag{19.1}$$

where X is any smooth vector field. On the right-hand side of Eq. (19.1) L_{g*} is the tangent map at the point $g^{-1}a$. A vector field X on G is said to be **left-invariant** if $L_{g*}X = X$ for all $g \in G$.

Given a tangent vector A at the identity, $A \in T_e(G)$, define the vector field X on G by $X_g = L_{g*}A$. This vector field is left-invariant for, by Eq. (15.17),

$$L_{g*}X_h = L_{g*} \circ L_{h*}A = (L_g \circ L_h)_*A = L_{gh*}A = X_{gh}.$$

It is clearly the unique left-invariant vector field on G such that $X_e = A$. We must show, however, that X is a differentiable vector field. In a local coordinate chart $(U, \varphi; x^i)$ at the identity e let the composition law be represented by n differentiable functions $\psi^i : \varphi(U) \times \varphi(U) \to \varphi(U)$,

$$x^i(gh^{-1}) = \psi^i(x^1(g), \dots, x^n(g), x^1(h), \dots, x^n(h)) \quad \text{where} \quad \psi^i = x^i \circ \psi.$$

For any smooth function $f : G \to \mathbb{R}$

$$\begin{aligned} Xf = (L_{g*}A)f &= A(f \circ L_g) \\ &= A^i \frac{\partial f \circ L_g}{\partial x^i}\bigg|_{x^i(e)} \\ &= A^i \frac{\partial}{\partial y^i} \hat{f}\big(\psi^1(x^1(g), \dots, x^n(g), \mathbf{y}), \dots, \psi^n(x^1(g), \dots, x^n(g), \mathbf{y})\big)\bigg|_{y^i = x^i(e)} \end{aligned}$$

where $\hat{f} = f \circ \varphi^{-1}$. Hence Xf is differentiable at e since it is differentiable on the neighbourhood U. If g is an arbitrary point of G then gU is an open neighbourhood of g and every point $h \in gU$ can be written $h = gg'$ where $g' \in U$, so that

$$\begin{aligned} (Xf)(h) &= A(f \circ L_h) \\ &= A(f \circ L_g \circ L_{g'}) \\ &= X(f \circ L_g)(g') \\ &= X(f \circ L_g) \circ L_{g^{-1}}(h). \end{aligned}$$

Thus $Xf = X(f \circ L_g) \circ L_{g^{-1}}$ on gU, and it follows that Xf is differentiable at g.

Hence X is the unique differentiable left-invariant vector field everywhere on G such that $X_e = A$. Left-invariant vector fields on G are therefore in one-to-one correspondence with tangent vectors at e, and form a vector space of dimension n, denoted $\mathcal{G}$.

Lie algebra of a Lie group

Given a smooth map $\varphi : M \to N$ between manifolds M and M', we will say vector fields X on M and X' on M' are φ**-related** if $\varphi_* X_p = X'_{\varphi(p)}$ for every $p \in M$. In general there does not exist a vector field on M' that is φ-related to a given vector field X on M unless φ is a diffeomorphism (see Section 15.4).

Lemma 19.1 *If $\varphi : M \to M'$ is a smooth map and X and Y are two vector fields on M, φ-related respectively to X' and Y' on M', then their Lie brackets $[X, Y]$ and $[X', Y']$ are φ-related.*

Proof: If $f : M' \to \mathbb{R}$ is a smooth map on M' then for any $p \in M$

$$(X' f) \circ \varphi = X(f \circ \varphi)$$

since $X'_{\varphi(p)} f = (\varphi_* X_p) f = X_p(f \circ \varphi)$. Hence

$$\begin{aligned}[X', Y']_{\varphi(p)} f &= X'_{\varphi(p)}(Y' f) - Y'_{\varphi(p)}(X' f) \\ &= X_p\big((Y' f) \circ \varphi\big) - Y_p\big((X' f) \circ \varphi\big) \\ &= X_p\big(Y(f \circ \varphi)\big) - Y_p\big(X(f \circ \varphi)\big) \\ &= [X, Y]_p(f \circ \varphi) \\ &= \varphi_*[X, Y]_p f,\end{aligned}$$

as required. ■

Exercise: Show that if X is a left-invariant vector field then $(Xf) \circ L_g = X(f \circ L_g)$.

If X and Y are left-invariant vector fields on a Lie group G it follows from Lemma 19.1 that

$$L_{g*}[X, Y] = [X, Y]. \tag{19.2}$$

The vector space $\mathcal{G}$ therefore forms an n-dimensional Lie algebra called the **Lie algebra of the Lie group** G. Because of the one-to-one correspondence between $\mathcal{G}$ and $T_e(G)$ it is meaningful to write $[A, B]$ for any pair $A, B \in T_e(G)$, and the Lie algebra structure can be thought of as being imposed on the tangent space at the identity e.

Let $A_1, \dots, A_n$ be a basis of the tangent space at the identity $T_e(G)$, and $X_1, \dots, X_n$ the associated set of left-invariant vector fields forming a basis of $\mathcal{G}$. As in Section 6.5 define the structure constants $C^k_{ij} = -C^k_{ji}$ by

$$[X_i,\ X_j] = C^k_{ij}\, X_k.$$

Exercise: Show that the Jacobi identities (15.24) are equivalent to

$$C^k_{mi} C^m_{jl} + C^k_{mj} C^m_{li} + C^k_{ml} C^m_{ij} = 0. \tag{19.3}$$

Example 19.3 Let $\mathbb{R}^n$ be the additive abelian Lie group of Example 19.1. The vector field X generated by a tangent vector $A = A^i \big(\partial_{x^i}\big)_e$ has components

$$X^i(g) = (Xx^i)(g) = (L_{g*}A)x^i = A(x^i \circ L_g).$$

Now $x^i \circ L_g = g^i + x^i$ where $g^i = x^i(g)$, whence

$$X^i(g) = A^j \frac{\partial}{\partial x^j}(g^i + x^i)\Big|_{x^i=0} = A^j \delta^i_j = A^i.$$

If $X = A^i \partial_{x^i}$ and $Y = B^j \partial_{x^j}$ are left-invariant vector fields, then for any function f

$$[X, Y]f = A^i \frac{\partial}{\partial x^i}\Big(B^j \frac{\partial f}{\partial x^j}\Big) - B^j \frac{\partial}{\partial x^j}\Big(A^i \frac{\partial f}{\partial x^i}\Big)$$
$$= A^i B^j (f_{,ji} - f_{,ij}) = 0.$$

Hence $[X, Y] = 0$ for all left-invariant vector fields. The Lie algebra of the abelian Lie group $\mathbb{R}^n$ is commutative.

Example 19.4 Let A be a tangent vector at the identity element of $GL(n, \mathbb{R})$,

$$A = A^i{}_j \Big(\frac{\partial}{\partial x^i{}_j}\Big)_{\mathsf{X}=\mathsf{I}}.$$

The tangent space at $e = \mathsf{I}$ is thus isomorphic with the vector space of $n \times n$ real matrices $M_n(\mathbb{R})$. The left-invariant vector field X generated by this tangent vector is

$$X = X^i{}_j \frac{\partial}{\partial x^i{}_j}$$

with components

$$\begin{aligned} X^i{}_j(\mathsf{G}) &= (L_{\mathsf{G}*}A)x^i{}_j \\ &= A\big(x^i{}_j \circ L_{\mathsf{G}}\big) \\ &= A^p_q \frac{\partial}{\partial x^p_q}\big(G^i{}_k x^k{}_j\big)\Big|_{\mathsf{X}=\mathsf{I}} \quad \text{where } G^i{}_j = x^i{}_j(\mathsf{G}) \\ &= A^p_q G^i{}_k \delta^k{}_p \delta^q{}_j \\ &= x^i{}_k(\mathsf{G}) A^k{}_j. \end{aligned}$$

Hence

$$X = x^i{}_k A^k{}_j \frac{\partial}{\partial x^i{}_j}.$$

If X and Y are left-invariant vector fields such that $X_e = A$ and $Y_e = B$, then their Lie bracket has components

$$[X, Y]^i{}_j = [X, Y]x^i{}_j = x^p{}_m A^m{}_q \frac{\partial}{\partial x^p_q}\big(x^i{}_k B^k{}_j\big) - x^p{}_m B^m{}_q \frac{\partial}{\partial x^p_q}\big(x^i{}_k A^k{}_j\big)$$
$$= x^i{}_m\big(A^m_k B^k_j - B^m_k A^k{}_j\big).$$

At the identity element $e = \mathsf{I}$ the components of $[X, Y]$ are therefore formed by taking the matrix commutator product $\mathsf{AB} - \mathsf{BA}$ where $\mathsf{A} = [A^i{}_j]$ and $\mathsf{B} = [B^i{}_j]$, and the Lie algebra of $GL(n, \mathbb{R})$ is isomorphic to the Lie algebra formed by taking commutators of $n \times n$ matrices from $M_n(\mathbb{R})$, known as $\mathcal{GL}(n, \mathbb{R})$.

Maurer–Cartan relations

We say that a differential form α is **left-invariant** if $L_g^*\alpha = \alpha$ for all $g \in G$. Its exterior derivative $\mathrm{d}\alpha$ is also left-invariant, for $L_g^*\,\mathrm{d}\alpha = \mathrm{d}L_g^*\alpha = \mathrm{d}\alpha$. If ω is a left-invariant 1-form and X a left-invariant vector field then $\langle \omega, X\rangle$ is constant over G, for

$$\langle \omega_g, X_g\rangle = \langle \omega_g, L_{g*}X_e\rangle = \langle L_g^*\omega_g, X_e\rangle = \langle \omega_e, X_e\rangle.$$

By the Cartan identity, Eq. (16.14), we therefore have

$$\begin{aligned} \mathrm{d}\omega(X, Y) &= \tfrac{1}{2}\big[X(\langle Y, \omega\rangle) - Y(\langle X, \omega\rangle) - \langle \omega, [X, Y]\rangle\big] \\ &= -\tfrac{1}{2}\langle \omega, [X, Y]\rangle. \end{aligned} \tag{19.4}$$

Let $E_1, \dots, E_n$ be a left-invariant set of vector fields, forming a basis of the Lie algebra $\mathcal{G}$ and $\varepsilon^1, \dots, \varepsilon^n$ the dual basis of differential 1-forms such that

$$\langle \varepsilon^i, E_i\rangle = \delta^i_{\ j}.$$

These 1-forms are left-invariant, for $L_g^*\varepsilon^i = \varepsilon^i$ for $i = 1, \dots, n$ as

$$\langle L_g^*\varepsilon^i, E_j\rangle = \langle \varepsilon^i, L_{g*}E_j\rangle = \langle \varepsilon^i, E_j\rangle = \delta^i_{\ j}.$$

Hence, by (19.4),

$$\mathrm{d}\varepsilon^i(E_j, E_k) = -\tfrac{1}{2}\langle \varepsilon^i, [E_j, E_k]\rangle = -\tfrac{1}{2}\langle \varepsilon^i, C^l_{jk}E_l\rangle = -\tfrac{1}{2}C^i_{jk},$$

from which we can deduce the **Maurer–Cartan relations**

$$\mathrm{d}\varepsilon^i = -\tfrac{1}{2}C^i_{jk}\varepsilon^j \wedge \varepsilon^k. \tag{19.5}$$

Exercise: Show that the Jacobi identities (19.3) follow by taking the exterior derivative of the Maurer–Cartan relations.

Theorem 19.2 *A Lie group G has vanishing structure constants if and only if it is isomorphic to the abelian group $\mathbb{R}^n$ in some neighbourhood of the identity.*

Proof: If there exists a coordinate neighbourhood of the identity such that $(gh)^i = g^i + h^i$ then from Example 19.3 $[X, Y] = 0$ throughout this neighbourhood. Thus $C^i_{jk} = 0$, since the Lie algebra structure is only required in a neighbourhood of e.

Conversely, if all structure constants vanish then the Maurer–Cartan relations imply $\mathrm{d}\varepsilon^i = 0$. By the Poincaré lemma 17.5, there exist functions y^i in a neighbourhood of the identity such that $\varepsilon^i = \mathrm{d}y^i$. Using these as local coordinates at e, we may assume that the identity is at the origin of these coordinates, $y^i(e) = 0$. For any a, g in the domain of these coordinates

$$L_g^*(\varepsilon^i)_{L_g a} = (\varepsilon^i)_a = \mathrm{d}a^i \quad \text{where } a^i = y^i(a),$$

and

$$L_g^*(\varepsilon^i)_{L_g a} = L_g^*(\mathrm{d}y^i)_{ga} = \big(\mathrm{d}(y^i \circ L_g)\big)_{ga}.$$

Writing $\phi(g, h) = gh$, we have for a fixed g

$$\mathrm{d}a^i = \mathrm{d}\big(y^i \circ \phi(g, a)\big) = \frac{\partial \phi^i(g^1, \dots, g^n, a^1, \dots, a^n)}{\partial a^j}\mathrm{d}a^j$$

where $\phi^i = y^i \circ \phi$. Thus

$$\frac{\partial \phi^i}{\partial a^j} = \delta^i{}_j,$$

equations that are easily integrated to give

$$\phi^i(g^1, \dots, g^n, a^1, \dots, a^n) = f^i(g^1, \dots, g^n) + a^i.$$

If $a = e$, so that $a^i = 0$, we have $\phi^i(g^1, \dots, g^n, 0, \dots, 0) = f^i(g^1, \dots, g^n) = g^i$. The required local isomorphism with $\mathbb{R}^n$ follows immediately,

$$\phi^i(g^1, \dots, g^n, a^1, \dots, a^n) = g^i + a^i.$$

■

Problems

Problem 19.1 Let $E^i{}_j$ be the matrix whose (i, j)th component is 1 and all other components vanish. Show that these matrices form a basis of $\mathcal{GL}(n, \mathbb{R})$, and have the commutator relations

$$[E^i{}_j, E^k{}_l] = \delta^i{}_l E^k{}_j - \delta^k{}_j E^i{}_l.$$

Write out the structure constants with respect to this algebra in this basis.

Problem 19.2 Let $E^i{}_j$ be the matrix defined as in the previous problem, and $F^i{}_j = iE^i{}_j$ where $i = \sqrt{-1}$. Show that these matrices form a basis of $\mathcal{GL}(n, \mathbb{C})$, and write all the commutator relations between these generators of $\mathcal{GL}(n, \mathbb{C})$.

Problem 19.3 Define the $T_e(G)$-valued 1-form θ on a Lie group G, by setting

$$\theta_g(X_g) = L_{g^{-1}*} X_g$$

for any vector field X on G (not necessarily left-invariant). Show that θ is left-invariant, $L_a^* \theta_g = \theta_{a^{-1}g}$ for all $a, g \in G$.

With respect to a basis E_i of left-invariant vector fields and its dual basis ε^i, show that

$$\theta = \sum_{i=1}^{n} \big(E_i\big)_e \varepsilon^i.$$

19.2 The exponential map

A **Lie group homomorphism** between two Lie groups G and H is a differentiable map $\varphi : G \to H$ that is a group homomorphism, $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in G$. In the case where it is a diffeomorphism φ is said to be a **Lie group isomorphism**.

Theorem 19.3 *A Lie group homomorphism $\varphi : G \to H$ induces a Lie algebra homomorphism $\varphi_* : \mathcal{G} \to \mathcal{H}$. If φ is an isomorphism, then φ_* is a Lie algebra isomorphism.*

Proof: The tangent map at the origin, $\varphi_* : T_e(G) \to T_e(H)$, defines a map between Lie algebras $\mathcal{G}$ and $\mathcal{H}$. If X is a left-invariant vector field on G then the left-invariant vector field $\varphi_* X$ on H is defined by

$$(\varphi_* X)_h = L_{h*}\varphi_* X_e.$$

Since φ is a homomorphism

$$\varphi \circ L_a(g) = \varphi(ag) = \varphi(a)\varphi(g) = L_{\varphi(a)} \circ \varphi(g)$$

for arbitrary $g \in G$. Thus the vector fields X and $\varphi_* X$ are φ-related,

$$\begin{aligned}\varphi_* X_a &= \varphi_* \circ L_{a*} X_e \\ &= L_{\varphi(a)*} \circ \varphi_* X_e \\ &= (\varphi_* X)_{\varphi(a)}.\end{aligned}$$

It follows by Theorem 19.1 that the Lie brackets $[X, Y]$ and $[\varphi_* X, \varphi_* Y]$ are φ-related,

$$[\varphi_* X, \varphi_* Y] = \varphi_*[X, Y],$$

so that φ_* is a Lie algebra homomorphism. In the case where φ is an isomorphism, φ_* is one-to-one and onto at G_e. ■

A **one-parameter subgroup of** G is a homomorphism $\gamma : \mathbb{R} \to G$,

$$\gamma(t + s) = \gamma(t)\gamma(s).$$

Of necessity, $\gamma(0) = e$. The tangent vector at the origin generates a unique left-invariant vector field X such that $X_e = \dot{\gamma}(0)$, where we use the self-explanatory notation $\dot{\gamma}(t)$ for $\dot{\gamma}_{\gamma(t)}$. The vector field X is everywhere tangent to the curve, $X_{\gamma(t)} = \dot{\gamma}(t)$, for if $f : G \to \mathbb{R}$ is any differentiable function then

$$\begin{aligned}X_{\gamma(t)} f &= \big(L_{\gamma(t)*}\dot{\gamma}(0)\big) f \\ &= \dot{\gamma}(0)(f \circ L_{\gamma(t)}) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} f\big(\gamma(t)\gamma(u)\big)\Big|_{u=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}u} f\big(\gamma(t+u)\big)\Big|_{u=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f\big(\gamma(t)\big) = \dot{\gamma}(t) f.\end{aligned}$$

Conversely, given any left-invariant vector field X, there is a unique one-parameter group $\gamma : \mathbb{R} \to G$ that generates it. The proof uses the result of Theorem 15.2 that there exists a local one-parameter group of transformations σ_t on a neighbourhood U of the identity e such that $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for $0 \le |t|, |s|, |t + s| < \epsilon$, and for all $h \in U$ and smooth functions f on U

$$X_h f = \frac{\mathrm{d} f\big(\sigma_t(h)\big)}{\mathrm{d}t}\Big|_{t=0}.$$

For all $h, g \in U$ such that $g^{-1}h \in U$ we have

$$\begin{aligned}
X_h f &= X_{L_g(g^{-1}h)} f \\
&= \big(L_{g*} X_{g^{-1}h}\big) f \\
&= X_{g^{-1}h}\big(f \circ L_g\big) \\
&= \frac{\mathrm{d} f \circ L_g \circ \sigma_t(g^{-1}h)}{\mathrm{d}t}\bigg|_{t=0} \\
&= \frac{\mathrm{d} f\big(\sigma'_t(h)\big)}{\mathrm{d}t}\bigg|_{t=0}
\end{aligned}$$

where

$$\sigma'_t = L_g \circ \sigma_t \circ L_{g^{-1}}.$$

The maps σ'_t form a local one-parameter group of transformations, since

$$\sigma'_t \circ \sigma'_s = L_g \circ \sigma_t \circ \sigma_s \circ L_{g^{-1}} = L_g \circ \sigma_{t+s} \circ L_{g^{-1}} = \sigma'_{t+s},$$

and generate the same vector field X as generated by σ_t. Hence $\sigma'_t = \sigma_t$ on a neighbourhood U of e, so that

$$\sigma_t \circ L_g = L_g \circ \sigma_t. \tag{19.6}$$

Setting $\gamma(t) = \sigma_t(e)$, we have

$$\sigma_t(g) = g\sigma_t(e) = g\gamma(t)$$

for all $g \in U$, and the one-parameter group property follows for γ for $0 \le |t|, |s|, |t + s| < \epsilon$,

$$\begin{aligned}
\gamma(t+s) = \sigma_{t+s}(e) = \sigma_t \circ \sigma_s(e) &= \sigma_t\big(\gamma(s)\big) \\
&= \gamma(s)\sigma_t(e) = \gamma(s)\gamma(t) = \gamma(t)\gamma(s).
\end{aligned}$$

The local one-parameter group may be extended to all values of t and s by setting

$$\gamma(t) = \big(\gamma(t/n)\big)^n$$

for n a positive integer chosen such that $|t/n| < \epsilon$. The group property follows for all t, s from

$$\gamma(t+s) = \big(\gamma((t+s)/n)\big)^n = \big(\gamma(t/n)\gamma(s/n)\big)^n = \big(\gamma(t/n)\big)^n\big(\gamma(s/n)\big)^n = \gamma(t)\gamma(s).$$

It is straightforward to verify that this one-parameter group is tangent to X for all values of t.

Exponential map

The **exponential map** $\exp : \mathcal{G} \to G$ is defined by

$$\exp X \equiv \exp(X) = \gamma(1)$$

where $\gamma : \mathbb{R} \to G$, the one-parameter subgroup generated by the left-invariant vector field X. Then

$$\gamma(s) = \exp sX. \tag{19.7}$$

For, let α be the one-parameter subgroup defined by $\alpha(t) = \gamma(st)$,

$$\alpha(t + t') = \gamma(s(t + t')) = \gamma(st)\gamma(st') = \alpha(t)\alpha(t').$$

If f is any smooth function on G, then

$$\dot{\alpha}(0)f = \frac{\mathrm{d}}{\mathrm{d}t} f\big(\gamma(st)\big)\Big|_{t=0} = s\frac{\mathrm{d}}{\mathrm{d}u} f\big(\gamma(u)\big)\Big|_{u=0} = sX_e f.$$

Thus α is the one-parameter subgroup generated by the left-invariant vector field sX, and $\exp sX = \alpha(1) = \gamma(s)$.

We further have that

$$(Xf)(\exp tX) = X_{\gamma(t)} f = \dot{\gamma}(t) f = \frac{\mathrm{d}}{\mathrm{d}t} f\big(\gamma(t)\big)$$

so that

$$(Xf)(\exp tX) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp tX). \tag{19.8}$$

The motivation for the name ‘exponential map’ lies in the identity

$$\exp(sX)\exp(tX) = \gamma(s)\gamma(t) = \gamma(s + t) = \exp(s + t)X. \tag{19.9}$$

Example 19.5 Let

$$X = x^i{}_k A^k{}_j \frac{\partial}{\partial x^i{}_j}$$

be a left-invariant vector field on the general linear group $GL(n, \mathbb{R})$. Setting $f = x^i{}_j$ we have

$$Xf = Xx^i{}_j = x^i{}_k A^k{}_j$$

and substitution in Eq. (19.8) gives

$$A^k{}_j x^i{}_k(\exp tX) = \frac{\mathrm{d}}{\mathrm{d}t} x^i{}_j(\exp tX).$$

If $y^i_k = x^i{}_k(\exp tX)$ are the components of the element $\exp(tX)$, and the matrix $\mathsf{Y} = [y^i{}_k]$ satisfies the linear differential equation

$$\frac{\mathrm{d}\mathsf{Y}}{\mathrm{d}t} = \mathsf{YA}$$

with the initial condition $\mathsf{Y}(0) = \mathsf{I}$, having unique solution

$$\mathsf{Y}(t) = \mathrm{e}^{t\mathsf{A}} = \mathsf{I} + t\mathsf{A} + \frac{t^2\mathsf{A}^2}{2!} + \cdots,$$

then if X_e is identified with the matrix $\mathsf{A} \in \mathcal{G}$, the matrix of components of the element $\exp(tX_e) \in GL(n, \mathbb{R})$ is $\mathrm{e}^{t\mathsf{A}}$.

For any left-invariant vector field X we have, by Eq. (19.8),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f\big(L_g(\exp tX)\big) &= \big(X(f \circ L_g)\big)(\exp tX) \\ &= (L_{g*}X\,f)(\exp tX) \\ &= (Xf)(g \exp tX)\end{aligned}$$

since $L_{g*}X = X$. The curve $t \mapsto L_g \circ \exp tX$ is therefore an integral curve of X through g at $t = 0$. Since g is an arbitrary point of G, the maps $\sigma_t : g \to g \exp tX = R_{\exp tX}g$ form a one-parameter group of transformations that generate X, and we conclude that every left-invariant vector field is complete.

The exponential map is a diffeomorphism of a neighbourhood of the zero vector $0 \in \mathcal{G}$ onto a neighbourhood of the identity $e \in G$. The proof may be found in [6]. If $\varphi : H \to G$ is a Lie group homomorphism then

$$\varphi \circ \exp = \exp \circ \varphi_* \tag{19.10}$$

where $\varphi_* : \mathcal{G} \to \mathcal{H}$ is the induced Lie group homomorphism of Theorem 19.3. For, let $\gamma : \mathbb{R} \to G$ be the curve defined by $\gamma(t) = \varphi(\exp tX)$. Since φ is a homomorphism, this curve is a one-parameter subgroup of G

$$\gamma(t)\gamma(s) = \varphi(\exp tX \exp sX) = \varphi(\exp(t+s)X) = \gamma(t+s).$$

Its tangent vector at $t = 0$ is $\varphi_* X_e$ since

$$\dot\gamma(0) f = \frac{\mathrm{d} f \circ \varphi(\exp tX)}{\mathrm{d}t}\Big|_{t=0} = X_e(f \circ \varphi) = (\varphi_* X_e) f,$$

and γ is the one-parameter subgroup generated by the left-invariant vector field $\varphi_* X$. Hence $\varphi(\exp tX) = \exp t\varphi_* X$ and Eq. (19.10) follows on setting $t = 1$.

Problems

Problem 19.4 A function $f : G \to \mathbb{R}$ is said to be an analytic function on G if it can be expanded as a Taylor series at any point $g \in G$. Show that if X is a left-invariant vector field and f is an analytic function on G then

$$f(g \exp tX) = \big(\mathrm{e}^{tX} f\big)(g)$$

where, for any vector field Y, we define

$$\mathrm{e}^Y f = f + Yf + \frac{1}{2!}Y^2 f + \frac{1}{3!}Y^3 f + \dots = \sum_{i=0}^{\infty} \frac{Y^n}{n!} f.$$

The operator Y^n is defined inductively by $Y^n f = Y(Y^{n-1} f)$.

Problem 19.5 Show that $\exp tX \exp tY = \exp t(X + Y) + O(t^2)$.

19.3 Lie subgroups

A **Lie subgroup** H of a Lie group G is a subgroup that is a Lie group, and such that the natural injection map $i : H \to G$ defined by $i(g) = g$ makes it into an embedded submanifold of G. It is called a **closed subgroup** if in addition H is a closed subset of G. In this case the embedding is regular and its topology is that induced by the topology of G (see Example 15.12). The injection i induces a Lie algebra homomorphism $i_* : \mathcal{H} \to (G)$, which is clearly an isomorphism of $\mathcal{H}$ with a Lie subalgebra of $\mathcal{G}$. We may therefore regard the Lie algebra of the Lie subgroup as being a Lie subalgebra of $\mathcal{G}$.

Example 19.6 Let $T^2 = S^1 \times S^1$ be the 2-torus, where S^1 is the one-dimensional Lie group where composition is addition modulo 1. This is evidently a Lie group whose elements can be written as pairs of complex numbers $(\mathrm{e}^{i\theta}, \mathrm{e}^{i\phi})$, where

$$(\mathrm{e}^{i\theta}, \mathrm{e}^{i\phi})(\mathrm{e}^{i\theta'}, \mathrm{e}^{i\phi'}) = (\mathrm{e}^{i(\theta+\theta')}, \mathrm{e}^{i(\phi+\phi')}).$$

The subset

$$H = \{(\mathrm{e}^{iat}, \mathrm{e}^{ibt}) \mid -\infty < t < \infty\}$$

is a Lie subgroup for arbitrary values of a and b. If a/b is rational it is isomorphic with S^1, the embedding is regular and it is a closed submanifold. If a/b is irrational then the subgroup winds around the torus an infinite number of times and is arbitrarily close to itself everywhere. In this case the embedding is not regular and the induced topology does not correspond to the submanifold topology. It is still referred to as a Lie subgroup. This is done so that all Lie subalgebras correspond to Lie subgroups.

The following theorem shows that there is a one-to-one correspondence between Lie subgroups of G and Lie subalgebras of $\mathcal{G}$. The details of the proof are a little technical and the interested reader is referred to the cited literature for a complete proof.

Theorem 19.4 *Let G be a Lie group with Lie algebra $\mathcal{G}$. For every Lie subalgebra $\mathcal{H}$ of $\mathcal{G}$, there exists a unique connected Lie subgroup $H \subseteq G$ with Lie algebra $\mathcal{H}$.*

Outline proof: The Lie subalgebra $\mathcal{H}$ defines a distribution D^k on G, by

$$D^k(g) = \{X_g \mid X \in \mathcal{H}\}.$$

Let the left-invariant vector fields $E_1, \dots, E_k$ be a basis of the Lie algebra $\mathcal{H}$, so that a vector field X belongs to D^k if and only if it has the form $X = X^i E_i$ where X^i are real-valued functions on G. The distribution D^k is involutive, for if X and Y belong to D^k, then so does their Lie bracket $[X, Y]$:

$$[X, Y]_g = X^i(g)Y^j(g)C^k_{ij}E_k + X^i(g)E_i Y^j(g)E_j - Y^j(g)E_j X^i(g)E_i.$$

By the Frobenius theorem 15.4, every point $g \in G$ has an open neighbourhood U such that every $h \in U$ lies in an embedded submanifold N_h of G whose tangent space spans $\mathcal{H}$ at all points $h' \in N_h$. More specifically, it can be proved that through any point of G there exists a unique maximal connected integral submanifold – see [2, p. 92] or [6, p. 48]. Let H be the maximal connected integral submanifold through the identity $e \in G$. Since D^k

is invariant under left translations $g^{-1}H = L_{g^{-1}}H$ is also an integral submanifold of D^k. By maximality, we must have $g^{-1} \subseteq H$. Hence, if $g, h \in H$ then $g^{-1}h \in H$, so that H is a subgroup of G. It remains to show that $(g, h) \mapsto g^{-1}h$ is a smooth function with respect to the differentiable structure on H, and that H is the unique subgroup having $\mathcal{H}$ as its Lie algebra. Further details may be found in [6, p. 94]. ■

If the Lie subalgebra is set to be $\mathcal{H} = \mathcal{G}$, namely the Lie algebra of G itself, then the unique Lie subgroup corresponding to $\mathcal{G}$ is the connected component of the identity, often denoted G_0.

Matrix Lie groups

All the groups discussed in Examples 2.10–2.15 of Section 2.3 are instances of **matrix Lie groups**; that is, they are all Lie subgroups of the general linear group $GL(n, \mathbb{R})$. Their Lie algebras were discussed heuristically in Section 6.5.

Example 19.7 As seen in Example 19.4 the Lie algebra of $GL(n, \mathbb{R})$ is isomorphic to the Lie algebra of all $n \times n$ matrices with respect to commutator products $[\mathsf{A}, \mathsf{B}] = \mathsf{AB} - \mathsf{BA}$. The set of all trace-free matrices $\mathcal{H} = \{\mathsf{A} \in M_n(\mathbb{R}) \mid \operatorname{tr}\mathsf{A} = 0\}$ is a Lie subalgebra of $\mathcal{GL}(n, \mathbb{R})$ since it is clearly a vector subspace of $M_n(\mathbb{R})$,

$$\operatorname{tr}\mathsf{A} = 0, \operatorname{tr}\mathsf{B} = 0 \implies \operatorname{tr}(\mathsf{A} + a\mathsf{B}) = 0,$$

and is closed with respect to taking commutators,

$$\operatorname{tr}[\mathsf{A}, \mathsf{B}] = \operatorname{tr}(\mathsf{AB}) - \operatorname{tr}(\mathsf{BA}) = 0.$$

It therefore generates a unique connected Lie subalgebra $H \subset GL(n, \mathbb{R})$ (see Theorem 19.4). To show that this Lie subgroup is the unimodular group $SL(n, \mathbb{R})$, we use the well-known identity

$$\det \mathrm{e}^{\mathsf{A}} = \mathrm{e}^{\operatorname{tr}\mathsf{A}}. \tag{19.11}$$

Thus, for all $\mathsf{A} \in \mathcal{H}$

$$\det \mathrm{e}^{t\mathsf{A}} = \mathrm{e}^{t \operatorname{tr}\mathsf{A}} = \mathrm{e}^0 = 1,$$

and by Example 19.5 the entire one-parameter subgroup $\exp(t\mathsf{A})$ lies in the unimodular group $SL(n, \mathbb{R})$.

Since the map exp is a diffeomorphism from an open neighbourhood U of 0 in $\mathcal{GL}(n, \mathbb{R})$ onto $\exp(U)$, every non-singular matrix X in a connected neighbourhood of I is uniquely expressible as an exponential, $\mathsf{X} = \mathrm{e}^{\mathsf{A}}$. Note the importance of connectedness here: the set of non-singular matrices has two connected components, being the inverse images of the two components of $\dot{\mathbb{R}}$ under the continuous map $\det : GL(n, \mathbb{R}) \to \mathbb{R}$. The matrices of negative determinant clearly cannot be connected to the identity matrix by a smooth curve in $GL(n, \mathbb{R})$ since the determinant would need to vanish somewhere along such a curve. In particular the subgroup $SL(n, \mathbb{R})$ is connected since it is the inverse image of the connected set $\{1\}$ under the determinant map. Every $\mathsf{X} \in SL(n, \mathbb{R})$ has $\det \mathsf{X} = 1$ and is therefore of the form $\mathsf{X} = \mathrm{e}^{\mathsf{A}}$ where, by Eq. (19.11), $\mathsf{A} \in \mathcal{H}$. Let H be the unique connected

Lie subgroup H whose Lie algebra is $\mathcal{H}$, according to Theorem 19.4. In a neighbourhood of the identity every $\mathsf{X} = e^{\mathsf{A}}$ for some $\mathsf{A} \in \mathcal{H}$, and every matrix of the form e^{A} belongs to H. Hence $SL(n, \mathbb{R}) = H$ is the connected Lie subgroup of $GL(n, \mathbb{R})$ with Lie algebra $\mathcal{SL}(n, \mathbb{R}) = \mathcal{H}$. Since a Lie group and its Lie algebra are of equal dimension, the Lie group $\dim SL(n, \mathbb{R}) = \dim \mathcal{H} = n^2 - 1$.

The Lie group $GL(n, \mathbb{C})$ has Lie algebra isomorphic with $M_n(\mathbb{C})$, with bracket $[\mathsf{A}, \mathsf{B}]$ again the commutator of the complex matrices A and B. As discussed in Chapter 6 this complex Lie algebra must be regarded as the complexification of a real Lie algebra by restricting the field of scalars to the real numbers. In this way any complex Lie algebra of dimension n can be considered as being a real Lie algebra of dimension $2n$. As a *real* Lie group $GL(n, \mathbb{C})$ has dimension $2n^2$, as does its Lie algebra $\mathcal{GL}(n, \mathbb{C})$. A similar discussion to that above can be used to show that the unimodular group $SL(n, \mathbb{C})$ of complex matrices of determinant 1 has Lie algebra consisting of trace-free complex $n \times n$ matrices. Both $SL(n, \mathbb{C})$ and $\mathcal{SL}(n, \mathbb{C})$ have (real) dimension $2n^2 - 2$.

Example 19.8 The orthogonal group $O(n)$ consists of real $n \times n$ matrices R such that

$$\mathsf{R}\mathsf{R}^T = \mathsf{I}.$$

A one-parameter group of orthogonal transformations has the form $\mathsf{R}(t) = \exp(t\mathsf{A}) = e^{t\mathsf{A}}$, whence

$$e^{t\mathsf{A}}\big(e^{t\mathsf{A}}\big)^T = e^{t\mathsf{A}}e^{t\mathsf{A}^T} = \mathsf{I}.$$

Performing the derivative with respect to t of this matrix equation results in

$$\mathsf{A}e^{t\mathsf{A}}e^{t\mathsf{A}^T} + e^{t\mathsf{A}}e^{t\mathsf{A}^T}\mathsf{A}^T = \mathsf{A} + \mathsf{A}^T = \mathsf{O},$$

so that A is a skew-symmetric matrix.

The set of skew-symmetric $n \times n$ matrices $\mathcal{O}(n)$ forms a Lie algebra since it is a vector subspace of $M_n(\mathbb{R})$ and is closed with respect to commutator products,

$$\begin{aligned}[\mathsf{A}, \mathsf{B}]^T &= (\mathsf{A}\mathsf{B} - \mathsf{B}\mathsf{A})^T \\ &= \mathsf{B}^T\mathsf{A}^T - \mathsf{A}^T\mathsf{B}^T \\ &= -[\mathsf{A}^T, \mathsf{B}^T] = -[\mathsf{A}, \mathsf{B}].\end{aligned}$$

Since every matrix e^{A} is orthogonal for a skew-symmetric matrix A,

$$\mathsf{A} = -\mathsf{A}^T \implies e^{\mathsf{A}}\big(e^{\mathsf{A}}\big)^T = e^{\mathsf{A}}e^{\mathsf{A}^T} = e^{\mathsf{A}}e^{-\mathsf{A}} = \mathsf{I},$$

$\mathcal{O}(n)$ is the Lie algebra corresponding to the connected Lie subgroup $SO(n) = O(n) \cap SL(n, \mathbb{R})$. The dimensions of this Lie group and Lie algebra are clearly $\frac{1}{2}n(n-1)$.

Similar arguments show that the unitary group $U(n)$ of complex matrices such that

$$\mathsf{U}\mathsf{U}^\dagger \equiv \mathsf{U}\overline{\mathsf{U}}^T = \mathsf{I}$$

is a Lie group with Lie algebra $\mathcal{U}(n)$ consisting of skew-hermitian matrices, $\mathsf{A} = -\mathsf{A}^\dagger$. The dimensions of $U(n)$ and $\mathcal{U}(n)$ are both n^2. The group $SU(n) = U(n) \cap SL(n, \mathbb{C})$ has Lie algebra consisting of trace-free skew-hermitian matrices and has dimension $n^2 - 1$.

Problems

Problem 19.6 For any $n \times n$ matrix A, show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \mathrm{e}^{t\mathsf{A}}\Big|_{t=0} = \operatorname{tr} \mathsf{A}.$$

Problem 19.7 Prove Eq. (19.11). One method is to find a matrix S that transforms A to upper-triangular Jordan form by a similarity transformation as in Section 4.2, and use the fact that both determinant and trace are invariant under such transformations.

Problem 19.8 Show that $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ are connected Lie groups. Is $U(n)$ a connected group?

Problem 19.9 Show that the groups $SL(n, \mathbb{R})$ and $SO(n)$ are closed subgroups of $GL(N, \mathbb{R})$, and that $U(n)$ and $SU(n)$ are closed subgroups of $GL(n, \mathbb{C})$. Show furthermore that $SO(n)$ and $U(n)$ are compact Lie subgroups.

Problem 19.10 As in Example 2.13 let the symplectic group $Sp(n)$ consist of $2n \times 2n$ matrices S such that

$$\mathsf{S}^T \mathsf{J} \mathsf{S} = \mathsf{J}, \qquad \mathsf{J} = \begin{pmatrix} \mathsf{O} & \mathsf{I} \\ -\mathsf{I} & \mathsf{O} \end{pmatrix},$$

where O is the $n \times n$ zero matrix and I is the $n \times n$ unit matrix. Show that the Lie algebra $\mathcal{S}p(n)$ consists of matrices A satisfying

$$\mathsf{A}^T \mathsf{J} + \mathsf{J}\mathsf{A} = \mathsf{O}.$$

Verify that these matrices form a Lie algebra and generate the symplectic group. What is the dimension of the symplectic group? Is it a closed subgroup of $GL(2n, \mathbb{R})$? Is it compact?

19.4 Lie groups of transformations

Let M be a differentiable manifold, and G a Lie group. By an **action** of G on M we mean a differentiable map $\phi : G \times M \to M$, often denoted $\phi(g, x) = gx$ such that

(i) $ex = x$ for all $x \in X$, where e is the identity element in G,
(ii) $(gh)x = g(hx)$.

This agrees with the conventions of a left action as defined in Section 2.6 and we refer to G as a **Lie group of transformations of** M. We may, of course, also have right actions $(g, x) \mapsto xg$ defined in the natural way.

Exercise: For any fixed $g \in G$ show that the map $\phi_g : M \to M$ defined by $\phi_g(x) = \phi(g, x) = gx$ is a diffeomorphism of M.

The action of G on M is said to be **effective** if e leaves every point $x \in M$ fixed,

$$gx = x \quad \text{for all } x \in M \implies g = e.$$

As in Section 2.6 the **orbit** Gx of a point $x \in M$ is the set $Gx = \{gx \mid g \in G\}$, and the action of G on M is said to be **transitive** if the whole of M is the orbit of some point in

M. In this case $M = Gy$ for all $y \in M$ and it is commonly said that M is a **homogeneous manifold** of G.

Example 19.9 Any Lie group G acts on itself by left translation $L_g : G \to G$, in which case the map $\phi : G \times G \to G$ is defined by $\phi(g, h) = gh = L_g h$. The action is both effective and transitive. Similarly G acts on itself to the right with right translations $R_g : G \to G$, where $R_g h = hg$.

Let H be a closed subgroup of a Lie group G, and $\pi : G \to G/H$ be the natural map sending each element of g to the left coset to which it belongs, $\pi(g) = gH$. As in Section 10.8 the factor space G/H is given the natural topology induced by π. Furthermore, G/H has a unique manifold structure such that π is C^∞ and G is a transitive Lie transformation group of G/H under the action

$$\phi(g, hH) = g(hH) = ghH.$$

A proof of this non-trivial theorem may be found in [6, p. 120]. The key result is the existence everywhere on G of *local sections*; every coset $gH \in G/H$ has a neighbourhood W and a smooth map $\alpha : W \to G$ with respect to the differentiable structure on G/H such that $\pi \circ \alpha = \mathrm{id}$ on $\alpha(W)$.

Every homogeneous manifold can be cast in the form of a left action on a space of cosets. Let G act transitively to the left on the manifold M and for any point $x \in M$ define the map $\phi_x : G \to M$ by $\phi_x(g) = gx$. This map is smooth, as it is the composition of two smooth maps $\phi_x = \phi \circ i_x$ where $i_x : G \to G \times M$ is the injection defined by $i_x(g) = (g, x)$. The isotropy group G_x of x, defined in Section 2.6 as $G_x = \{g \mid gx = x\}$, is therefore a closed subgroup of G since it is the inverse image of a closed singleton set, $G_x = \phi_x^{-1}(\{x\})$. Let the map $\rho : G/G_x \to M$ be defined by $\rho(gG_x) = gx$. This map is one-to-one, for

$$\begin{aligned} \rho(gGx) = \rho(g'G_x) &\Longrightarrow gx = g'x \\ &\Longrightarrow g^{-1}g' \in G_x \\ &\Longrightarrow g' \in gG_x \\ &\Longrightarrow g'G_x = gG_x. \end{aligned}$$

Furthermore, with respect to the differentiable structure induced on G/G_x, the map ρ is C^∞ since for any local section $\alpha : W \to G$ on a neighbourhood W of a given coset gG_x we can write $\rho = \pi \circ \alpha$, which is a composition of smooth functions. Since $\rho(gK) = g(\rho(K))$ for all cosets $K = hG_x \in G/G_x$, the group G has the 'same action' on G/G_x as it does on M.

Exercise: Show that ρ is a continuous map with respect to the factor space topology induced on G/G_x.

Example 19.10 The orthogonal group $O(n + 1)$ acts transitively on the unit n-sphere

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid (x^1)^2 + (x^2)^2 + \cdots + (x^{n+1})^2 = 1\} \subset \mathbb{R}^{n+1},$$

since for any $\mathbf{x} \in S^n$ there exists an orthogonal transformation A such that $\mathbf{x} = \mathsf{A}\mathbf{e}$ where $\mathbf{e} = (0, 0, \dots, 0, 1)$. In fact any orthogonal matrix with the last column having the same components as $\mathbf{x}$, i.e. $A^i_{n+1} = x^i$, will do. Such an orthogonal matrix exists by a Schmidt orthonormalization in which $\mathbf{x}$ is transformed to the $(n+1)$th unit basis vector.

Let H be the isotropy group of the point $\mathbf{e}$, consisting of all matrices of the form

$$\begin{pmatrix} & & & 0 \\ [B^i_j] & & & \vdots \\ & & & 0 \\ \dots & & 0 & 1 \end{pmatrix} \quad \text{where} \quad [B^i_j] \text{ is } n \times n \text{ orthogonal.}$$

Hence $H \cong O(n)$. The map $O(n+1)/H \to S^n$ defined by $\mathsf{A}H \mapsto \mathsf{A}\mathbf{e}$ is clearly one-to-one and continuous with respect to the induced topology on $O(n+1)/H$. Furthermore, $O(n+1)/H$ is compact since it is obtained by identification from an equivalence relation on the compact space $O(n+1)$ (see Example 10.16). Hence the map $O(n+1)/H \to S^n$ is a homeomorphism since it is a continuous map from a compact Hausdorff space onto a compact space (see Problem 10.17). Smoothness follows from the general results outlined above. Thus $O(n+1)/O(n)$ is diffeomorphic to S^n, and similarly it can be shown that $SO(n+1)/SO(n) \cong S^n$.

Example 19.11 The group of matrix transformations $\mathbf{x}' = \mathsf{L}\mathbf{x}$ leaving invariant the inner product $\mathbf{x} \cdot \mathbf{y} = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4$ of Minkowski space is the Lorentz group $O(3, 1)$. The group of all transformations leaving this form invariant including translations,

$$x'^i = L^i_j x^j + b^j,$$

is the Poincaré group $P(4)$ (see Example 2.30 and Chapter 9). The isotropy group of the origin is clearly $O(3, 1)$. The factor space $P_4/O(3, 1)$ is diffeomorphic to $\mathbb{R}^4$, for two Poincaré transformations P and P' belong to the same coset if and only if their translation parts are identical,

$$\begin{aligned} P^{-1}P' \in O(3,1) &\iff \mathsf{L}^{-1}(\mathsf{L}'\mathbf{x} + \mathbf{b}') - \mathsf{L}^{-1}\mathbf{b} = \mathsf{K}\mathbf{x} \quad \text{for } \mathsf{K} \in O(3,1) \\ &\iff \mathbf{b}' = \mathbf{b}. \end{aligned}$$

Normal subgroups

Theorem 19.5 *Let H be a closed normal subgroup of a Lie group G. Then the factor group G/H is a Lie group.*

Proof: The map $\Psi : (aH, bH) \mapsto ab^{-1}H$ from $G/H \times G/H \to G/H$ is C^∞ with respect to the natural differentiable structure on G/H. For, if $(K_a, \alpha_a : K_a \to G)$ and $(K_b, \alpha_b : K_b \to G)$ are any pair of local sections at a and b then on $K_a \times K_b$

$$\Psi = \pi \circ \psi \circ (\alpha_a \times \alpha_b)$$

where $\psi : G \times G \to G$ is the map $\psi(a, b) = ab^{-1}$. Hence Ψ is everywhere locally a composition of smooth maps, and is therefore C^∞. ■

Now suppose $\varphi : G \to H$ is any Lie group homomorphism, and let $N = \varphi^{-1}(\{e'\})$ be the kernel of the homomorphism, where e' is the identity of the Lie group H. It is clearly a closed subgroup of G. The tangent map $\varphi_* : \mathcal{G} \to \mathcal{H}$ induced by the map φ is a Lie algebra homomorphism. Its kernel $\mathcal{N} = (\varphi_*)^{-1}(0)$ is an ideal of $\mathcal{G}$ and is the Lie algebra corresponding to the Lie subgroup N, since

$$\begin{aligned} Z \in \mathcal{N} &\iff \varphi_* Z = 0 \\ &\iff \exp t\varphi_* Z = e' \\ &\iff \varphi(\exp tZ) = e' \quad \text{by Eq. (19.10)} \\ &\iff \exp tZ \in N. \end{aligned}$$

Thus, if N is any closed normal subgroup of G then it is the kernel of the homomorphism $\pi : G \to H = G/N$, and its Lie algebra is the kernel of the Lie algebra homomorphism $\pi_* : \mathcal{G} \to \mathcal{H}$. That is, $\mathcal{H} \cong \mathcal{G}/\mathcal{N}$.

Example 19.12 Let G be the additive abelian group $G = \mathbb{R}^n$, and H the discrete subgroup consisting of all points with integral coordinates. Evidently H is a closed normal subgroup of G, and its factor group

$$T^n = \mathbb{R}^n / H$$

is the n-dimensional torus (see Example 10.14). In the torus group two vectors are identified if they differ by integral coordinates, $[\mathbf{x}] = [\mathbf{y}]$ in T^n if and only if $\mathbf{x} - \mathbf{y} = (k_1, k_2, \dots, k_n)$ where k_i are integers. The one-dimensional torus $T = T^1$ is diffeomorphic to the unit circle in $\mathbb{R}^2$, and the n-dimensional torus group is the product of one-dimensional groups $T^n = T \times T \times \dots \times T$. It is a compact group.

Induced vector fields

Let G be a Lie group of transformations of a manifold M with action to the right defined by a map $\rho : G \times M \to M$. We set $\rho(p, g) = pg$, with the stipulations $pe = p$ and $p(gh) = (pg)h$. Every left-invariant vector field X on G induces a vector field $\tilde{X}$ on M by setting

$$\tilde{X}_p f = \frac{\mathrm{d}}{\mathrm{d}t} f(p \exp tX)\Big|_{t=0} \quad (p \in M) \tag{19.12}$$

for any smooth function $f : M \to \mathbb{R}$. This is called the vector field **induced** by the left-invariant vector field X.

Exercise: Show that $\tilde{X}$ is a vector field on M by verifying linearity and the Leibnitz rule at each point $p \in M$, Eqs. (15.3) and (15.4).

Theorem 19.6 *Lie brackets of a pair of induced vector fields correspond to the Lie products of the corresponding left-invariant vector fields,*

$$\widetilde{[X, Y]} = [\tilde{X}, \tilde{Y}]. \tag{19.13}$$

Proof: Before proceeding with the main part of the proof, we need an expression for $[X, Y]_e$. Let σ_y be a local one-parameter group of transformations on G generated by the vector field X. By Eq. (19.6),

$$\sigma_t(g) = \sigma_t(ge) = g\sigma_t(e) = g \exp tX = R_{\exp tX}(g)$$

where the operation R_h is right translation by h. Hence

$$[X, Y]_e = \lim_{t\to 0} \frac{1}{t}\Big[Y_e - \big(\sigma_{t*}Y\big)_e\Big] = \lim_{t\to 0} \frac{1}{t}\Big[Y_e - \big(R_{\exp(tX)*}Y\big)_e\Big]. \tag{19.14}$$

Define the maps $\rho_p : G \to M$ and $\rho_g : M \to M$ for any $p \in M$, $g \in G$ by

$$\rho_p(g) = \rho_g(p) = \rho(p, g) = pg.$$

Then

$$\tilde{X}_p = \rho_{p*} X_e \tag{19.15}$$

for if f is any smooth function on M then, on making use of (19.8), we have

$$\begin{aligned}\rho_{p*}X_e f = X_e(f \circ \rho_p) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(f \circ \rho_p(\exp tX)\big)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(f(p \exp tX)\big)\Big|_{t=0} = \tilde{X}_p f.\end{aligned}$$

The maps $\tilde{\sigma}_t : M \to M$ defined by

$$\tilde{\sigma}_t(p) = p \exp tX = \rho_{\exp tX}(p)$$

form a one-parameter group of transformations of M since, using Eq. (19.9),

$$\tilde{\sigma}_{t+s}(p) = \tilde{\sigma}_t \circ \tilde{\sigma}_t(p)$$

for all t and s. By (19.12) they induce the vector field $\tilde{X}$, whence

$$[\tilde{X}, \tilde{Y}]_p = \lim_{t\to 0} \frac{1}{t}\Big[\tilde{Y}_p - \big(\tilde{\sigma}_{t*}\tilde{Y}\big)_p\Big]. \tag{19.16}$$

Applying the definition of $\tilde{\sigma}_t$ we have

$$\begin{aligned}\big(\tilde{\sigma}_{t*}\tilde{Y}\big)_p &= \big(\rho_{\exp(tX)*}\tilde{Y}\big)_p \\ &= \rho_{\exp(tX)*}\rho_{p\exp(-tX)*}Y_e \\ &= \big(\rho_{\exp tX} \circ \rho_{p\exp(-tX)}\big)_* Y_e.\end{aligned}$$

The map in the brackets can be written

$$\rho_{\exp tX} \circ \rho_{p\exp(-tX)} = \rho_p \circ R_{\exp tX} \circ L_{\exp(-tX)}$$

since

$$\rho_{\exp tX} \circ \rho_{p\exp(-tX)}(g) = p\exp(-tX)g\exp tX = \rho_p \circ R_{\exp tX} \circ L_{\exp(-tX)}(g).$$

Hence, since Y is left-invariant

$$\begin{aligned}\big(\tilde{\sigma}_{t*}\tilde{Y}\big)_p &= \rho_{p*} \circ R_{\exp(tX)*} \circ L_{\exp(-tX)*}Y_e \\ &= \rho_{p*} \circ R_{\exp(tX)*}\big(Y_{\exp(-tX)}\big) \\ &= \rho_{p*}\big(R_{\exp(tX)*}Y_{\exp(-tX)}\big)_e.\end{aligned}$$

Since $\tilde{Y}_p = \rho_{p*} Y_e$ by Eq. (19.15), substitution in Eq. (19.16) and using Eq. (19.14) gives

$$[\tilde{X}, \tilde{Y}]_p = \rho_{p*}\Big(\lim_{t\to 0} \frac{1}{t}\Big[Y_e - \big(R_{\exp(tX)*}Y\big)_e\Big]\Big)$$
$$= \rho_{p*}[X, Y]_e = \widetilde{[X, Y]}_p,$$

which proves Eq. (19.13). ■

Problems

Problem 19.11 Show that a group G acts effectively on G/H if and only if H contains no normal subgroup of G. [*Hint*: The set of elements leaving all points of G/H fixed is $\bigcap_{a\in G} aHa^{-1}$.]

Problem 19.12 Show that the special orthogonal group $SO(n)$, the pseudo-orthogonal groups $O(p, q)$ and the symplectic group $Sp(n)$ are all closed subgroups of $GL(n, \mathbb{R})$.

(a) Show that the complex groups $SL(n, \mathbb{C})$, $O(n, \mathbb{C})$, $U(n)$, $SU(n)$ are closed subgroups of $GL(n, \mathbb{C})$.
(b) Show that the unitary groups $U(n)$ and $SU(n)$ are compact groups.

Problem 19.13 Show that the centre Z of a Lie group G, consisting of all elements that commute with every element $g \in G$, is a closed normal subgroup of G.

Show that the general complex linear group $GL(n + 1, \mathbb{C})$ acts transitively but not effectively on complex projective n-space CP^n defined in Problem 15.4. Show that the centre of $GL(n + 1, \mathbb{C})$ is isomorphic to $GL(1, \mathbb{C})$ and $GL(n + 1, \mathbb{C})/GL(1, \mathbb{C})$ is a Lie group that acts effectively and transitively on CP^n.

Problem 19.14 Show that $SU(n + 1)$ acts transitively on CP^n and the isotropy group of a typical point, taken for convenience to be the point whose equivalence class contains $(0, 0, \dots, 0, 1)$, is $U(n)$. Hence show that the factor space $SU(n + 1)/U(n)$ is homeomorphic to CP^n. Show similarly, that

(a) $SO(n + 1)/O(n)$ is homeomorphic to real projective space P^n.
(b) $U(n + 1)/U(n) \cong SU(n + 1)/SU(n)$ is homeomorphic to S^{2n+1}.

Problem 19.15 As in Problem 9.2 every Lorentz transformation $\mathsf{L} = [L^i{}_j]$ has $\det \mathsf{L} = \pm 1$ and either $L^4{}_4 \geq 1$ or $L^4{}_4 \leq -1$. Hence show that the Lorentz group $G = O(3, 1)$ has four connected components,

$$G_0 = G^{++} : \ \det \mathsf{L} = 1,\ L^4{}_4 \geq 1 \qquad G^{+-} : \ \det \mathsf{L} = 1,\ L^4{}_4 \leq -1$$
$$G^{-+} : \ \det \mathsf{L} = -1,\ L^4{}_4 \geq 1 \qquad G^{--} : \ \det \mathsf{L} = -1,\ L^4{}_4 \leq -1.$$

Show that the group of components G/G_0 is isomorphic with the discrete abelian group $Z_2 \times Z_2$.

Problem 19.16 Show that the component of the identity G_0 of a locally connected group G is generated by any connected neighbourhood of the identity e: that is, every element of G_0 can be written as a product of elements from such a neighbourhood.

Hence show that every discrete normal subgroup N of a connected group G is contained in the centre Z of G.

Find an example of a discrete normal subgroup of the disconnected group $O(3)$ that is not in the centre of $O(3)$.

Problem 19.17 Let $\mathcal{A}$ be a Lie algebra, and X any element of $\mathcal{A}$.

(a) Show that the linear operator $\mathrm{ad}_X : \mathcal{A} \to \mathcal{A}$ defined by $\mathrm{ad}_X(Y) = [X, Y]$ is a Lie algebra homomorphism of $\mathcal{A}$ into $\mathcal{GL}(\mathcal{A})$ (called the **adjoint representation**).

(b) For any Lie group G show that each inner automorphism $C_g : G \to G$ defined by $C_g(a) = gag^{-1}$ (see Section 2.4) is a Lie group automorphism, and the map $\mathrm{Ad} : G \to GL(\mathcal{G})$ defined by $\mathrm{Ad}(g) = C_{g*}$ is a Lie group homomorphism.

(c) Show that $\mathrm{Ad}_* = \mathrm{ad}$.

Problem 19.18 (a) Show that the group of all Lie algebra automorphisms of a Lie algebra $\mathcal{A}$ form a Lie subgroup of $\mathrm{Aut}(\mathcal{A}) \subseteq GL(\mathcal{A})$.
(b) A linear operator $D : \mathcal{A} \to \mathcal{A}$ is called a *derivation* on $\mathcal{A}$ if $D[X, Y] = [DX, Y] + [X, DY]$. Prove that the set of all derivations of $\mathcal{A}$ form a Lie algebra, $\partial(\mathcal{A})$, which is the Lie algebra of $\mathrm{Aut}(\mathcal{A})$.

19.5 Groups of isometries

Let (M, g) be a pseudo-Riemannian manifold. An **isometry** of M is a transformation $\varphi : M \to M$ such that $\widetilde{\varphi} g = g$, where $\widetilde{\varphi}$ is the map induced on tensor fields as defined in Section 15.5. This condition amounts to requiring

$$g_{\varphi(p)}\big(\varphi_* X_p, \varphi_* Y_p\big) = g_p\big(X_p, Y_p\big)$$

for all $X_p, Y_p \in T_p(M)$.

Let G be a Lie group of isometries of (M, g), and $\mathcal{G}$ its Lie algebra of left-invariant vector fields. If $A \in \mathcal{G}$ is a left-invariant vector field then the induced vector field $X = \widetilde{A}$ is called a **Killing vector** on M. If σ_t is the one-parameter group of isometries generated by A then, by Eq. (15.33), we have

$$\mathcal{L}_X g = \lim_{t \to 0} \frac{1}{t}\big(g - \widetilde{\sigma}_t g\big) = 0.$$

In any coordinate chart $(U; x^i)$ let $X = \xi^i \partial_{x^i}$ and, by Eq. (15.39), this equation becomes

$$\mathcal{L}_X g_{ij} = g_{ij,k}\xi^k + \xi^k{}_{,i} g_{kj} + \xi^k{}_{,j} g_{ik} = 0, \tag{19.17}$$

known as **Killing's equations**. In a local chart such that $\xi^i = (1, 0, \dots, 0)$ (see Theorem 15.3), Eq. (19.17) reads

$$g_{ij,1} = \frac{\partial g_{ij}}{\partial x^1} = 0$$

and the components of g are independent of the coordinate x^1, $g_{ij} = g_{ij}(x^2, \dots, x^n)$. By direct computation from the Christoffel symbols or by considering the equation in geodesic coordinates, ordinary derivatives may be replaced by covariant derivatives in Eq. (19.17)

$$g_{ij;k}\xi^k + \xi^k{}_{;i} g_{kj} + \xi^k{}_{;j} g_{ik} = 0,$$

and since $g_{ij;k} = 0$ Killing's equations may be written in the *covariant form*:

$$\xi_{i;j} + \xi_{j;i} = 0. \tag{19.18}$$

By Theorem 19.6, if $X = \widetilde{A}$ and $Y = \widetilde{B}$ then $[X, Y] = \widetilde{[A, B]}$. We also conclude from Problem 15.18 that if X and Y satisfy Killing's equations then so does $[X, Y]$. In fact, there can be at most a finite number of linearly independent Killing vectors. For, from (19.18) and the Ricci identities (18.29), $\xi_{k;ij} - \xi_{k;ji} = \xi_a R^a{}_{kij}$ (no torsion), we have

$$\xi_{k;ij} + \xi_{j;ki} = \xi_a R^a{}_{kij}.$$

From the cyclic first Bianchi identity (18.26), $R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0$, we have $\xi_{i;jk} + \xi_{j;ki} + \xi_{k;ij} = 0$, whence

$$\xi_{i;jk} = -\xi_{j;ki} - \xi_{k;ij} = -\xi_a R^a{}_{kij} = \xi_a R^a{}_{kji}. \tag{19.19}$$

Thus if we know the components ξ_i and $\xi_{i;j}$ in a given pseudo-Riemannian space, all covariant derivatives of second order of ξ_i may be calculated from Eq. (19.19). All higher orders may then be found by successively forming higher order covariant derivatives of this equation. Assuming that ξ_i can be expanded in a power series in a neighbourhood of any point of M (this is not actually an additional assumption as it turns out), we only need to know ξ_i and $\xi_{i;j} = -\xi_{j;i}$ at a specified point p to define the entire Killing vector field in a neighbourhood of p. As there are $n + \binom{n}{2} = n(n+1)/2$ linearly independent initial values at p, the maximum number of linearly independent Killing vectors in any neighbourhood of M is $n(n+1)/2$. In general of course there are fewer than these, say r, and the general Killing vector is expressible as a linear combination of r Killing vectors $X_1, \dots, X_r$,

$$X = \sum_{i=1}^{r} a^i X_i, \quad (a^i = \text{const.})$$

generating a Lie algebra of dimension r with structure constants $C^k_{ij} = -C^k_{ji}$,

$$[X_i, X_j] = C^k_{ij} X_k.$$

Maximal symmetries and cosmology

A pseudo-Riemannian space is said to have **maximal symmetry** if it has the maximum number $n(n+1)/2$ of Killing vectors. Taking a covariant derivative of Eq. (19.19),

$$\xi_{i;jkl} = \xi_{a;l} R^a{}_{kji} + \xi_a R^a{}_{kji;l},$$

and using the generalized Ricci identities given in Problem 18.9,

$$\begin{aligned}\xi_{i;jkl} - \xi_{i:jlk} &= \xi_{a;j} R^a{}_{ikl} + \xi_{i;a} R^a{}_{jkl} \\ &= \xi_{a;j} R^a{}_{ikl} - \xi_{a;i} R^a{}_{jkl},\end{aligned}$$

we have

$$\xi_a \big(R^a{}_{kji;l} - R^a{}_{lji;k} \big) = \xi_{a;b} \big(R^a{}_{ikl} \delta^b_j - R^a{}_{jkl} \delta^b_i - R^a{}_{kji} \delta^b_l + R^a{}_{lji} \delta^b_k \big).$$

Since for maximal symmetry ξ_a and $\xi_{a;b} = -\xi_{b;a}$ are arbitrary at any point, the antisymmetric part with respect to a and b of the term in parentheses on the right-hand side vanishes,

$$R^a{}_{ikl} \delta^b_j - R^a{}_{jkl} \delta^b_i - R^a{}_{kji} \delta^b_l + R^a{}_{lji} \delta^b_k = R^b{}_{ikl} \delta^a_j - R^b{}_{jkl} \delta^a_i - R^b{}_{kji} \delta^a_l + R^b{}_{lji} \delta^a_k.$$

Contracting this equation with respect to indices b and l, we find on using the cyclic symmetry (18.26),

$$(n-1)R^a{}_{kji} = R_{ik}\delta^a_j - R_{jk}\delta^a_i.$$

Another contraction with respect to k and i gives $nR^a_j = R\delta^a_j$ and substituting back in the expression for $R^a{}_{kji}$, we find on lowering the index and making a simple permutation of index symbols

$$R_{ijkl} = \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{19.20}$$

The contracted Bianchi identity (18.60) implies that the Ricci scalar is constant for $n > 2$ since

$$R^a{}_{j;a} = \tfrac{1}{2}R_{,j} \implies nR_{,j} = \tfrac{1}{2}R_{,j} \implies R_{,j} = 0.$$

Spaces whose Riemann tensor has this form are known as spaces of **constant curvature**. Example 18.4 provides another motivation for this nomenclature and shows that the 3-sphere of radius a is a space of constant curvature, with $R = 6/a^2$. The converse is in fact true – every space of constant curvature has maximal symmetry. We give a few instances of this statement in the following examples.

Example 19.13 Euclidean 3-space $\mathrm{d}s^2 = \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ of constant curvature zero. To find its Killing vectors, we must find all solutions of Killing's equations (19.18),

$$\xi_{i,j} + \xi_{j,i} = 0.$$

Since this implies $\xi_{1,1} = \xi_{2,2} = \xi_{3,3} = 0$ we have

$$\xi_{i,11} = -\xi_{1,1i} = 0, \qquad \xi_{i,22} = \xi_{i,33} = 0,$$

whence there exist constants $a_{ij} = -a_{ji}$ and b_i such that

$$\xi_i = a_{ij}x^j + b_i.$$

Setting $a_{ij} = -\epsilon_{ijk}a^k$ and $b^k = b_k$ we can express the general Killing vector in the form

$$X = a^1X_1 + a^2X_2 + a^3X_3 + b^1Y_1 + b^2Y_2 + b^3Y_3$$

where $X_1 = x^2\partial_3 - x^2\partial_3$, $X_2 = x^3\partial_1 - x^1\partial_3$, $X_3 = x^1\partial_2 - x^2\partial_1$ and $Y_1 = \partial_1$, $Y_2 = \partial_2$, $Y_3 = \partial_3$. As these are six independent Killing vectors, the space has maximal symmetry. Their Lie algebra commutators are

$$[X_1, X_2] = -X_3, \qquad [X_2, X_3] = -X_1, \qquad [X_3, X_1] = -X_2,$$
$$[Y_1, Y_2] = [Y_1, Y_3] = [Y_2, Y_3] = 0,$$

$$\begin{array}{lll} [X_1, Y_1] = 0, & [X_2, Y_1] = Y_3, & [X_3, Y_1] = -Y_2, \\ [X_1, Y_2] = -Y_3, & [X_2, Y_2] = 0, & [X_3, Y_2] = Y_1, \\ [X_1, Y_3] = Y_2, & [X_2, Y_3] = -Y_1, & [X_3, Y_3] = 0. \end{array}$$

This is known as the Lie algebra of the **Euclidean group**.

Example 19.14 The 3-sphere of Example 18.4,

$$ds^2 = a^2(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)),$$

has Killing's equations

$$\xi_{1,1} = 0, \tag{19.21}$$
$$\xi_{1,2} + \xi_{2,1} - 2\cot\chi\,\xi_2 = 0, \tag{19.22}$$
$$\xi_{1,3} + \xi_{3,1} - 2\cot\chi\,\xi_3 = 0, \tag{19.23}$$
$$\xi_{2,2} + \sin\chi\cos\chi\,\xi_1 = 0, \tag{19.24}$$
$$\xi_{2,3} + \xi_{3,2} - 2\cot\chi\,\xi_3 = 0, \tag{19.25}$$
$$\xi_{3,3} + \sin\chi\cos\chi\sin^2\theta\,\xi_1 + \sin\theta\cos\theta\,\xi_2 = 0. \tag{19.26}$$

From (19.21) we have $\xi_1 = F(\theta, \phi)$ and differentiating (19.22) with respect to $x^1 = \chi$ we have a differential equation for ξ_2,

$$\xi_{2,11} - 2\cot\chi\,\xi_{2,1} + 2\,\mathrm{cosec}^2\chi\,\xi_2 = 0.$$

The general solution of this linear differential equation is not hard to find:

$$\xi_2 = -\sin\chi\cos\chi\, f(\theta, \phi) + \sin^2\chi\, G(\theta, \phi).$$

Substituting back into (19.22) we find $f = -F_{,2}$ where $x^2 = \theta$. Similarly,

$$\xi_3 = F_{,3}\sin\chi\cos\chi + H(\theta, \phi)\sin^2\chi.$$

Substituting these expressions in the remaining equations results in the following general solution of Killing's equations dependent on six arbitrary constants a^1, a^2, a^3, b^1, b^2, b^3:

$$X = \xi^i\partial_i = a^i X_i + b^j Y_j$$

where $\xi^1 = \xi_1$, $\xi^2 = \xi_2/\sin^2\chi$, $\xi^3 = \xi_3/\sin^2\chi\sin^2\theta$, and

$$\begin{aligned}
X_1 &= \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi,\\
X_2 &= \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi,\\
X_3 &= \partial_\phi,\\
Y_1 &= \sin\theta\sin\phi\,\partial_\chi + \cot\chi\cos\theta\sin\phi\,\partial_\theta + \frac{\cot\chi}{\sin\theta}\cos\phi\,\partial_\phi,\\
Y_2 &= -\sin\theta\cos\phi\,\partial_\chi - \cot\chi\cos\theta\cos\phi\,\partial_\theta + \frac{\cot\chi}{\sin\theta}\sin\phi\,\partial_\phi,\\
Y_3 &= \cos\theta\,\partial_\chi - \sin\theta\cot\chi\,\partial_\theta.
\end{aligned}$$

The Lie algebra brackets are tedious to calculate compared with those in the previous example, but the results have similarities to those of the Euclidean group:

$$\begin{array}{lll} [X_1,\ X_2] = -X_3, & [X_2,\ X_3] = -X_1, & [X_3,\ X_1] = -X_2, \\ [Y_1,\ Y_2] = -X_3, & [Y_2,\ Y_3] = -X_1, & [Y_3,\ Y_1] = -X_2, \\ [X_1,\ Y_1] = 0, & [X_1,\ Y_2] = -Y_3, & [X_1,\ Y_3] = Y_2, \\ [X_2,\ Y_1] = Y_3, & [X_2,\ Y_2] = 0, & [X_2,\ Y_3] = -Y_1, \\ [X_3,\ Y_1] = -Y_2, & [X_3,\ Y_2] = Y_1, & [X_3,\ Y_3] = 0. \end{array}$$

Not surprisingly this Lie algebra is isomorphic to the Lie algebra of the four-dimensional rotation group, $SO(4)$.

The Robertson–Walker cosmologies of Section 18.9 all have maximally symmetric spatial sections. The sections $t =$ const. of the open model (18.105) are 3-spaces of constant negative curvature, called *pseudo-spheres*. These models are called homogeneous and isotropic. It is not hard to see that these space-times have the same number of independent Killing vectors as their spatial sections. In general they have six Killing vectors, but some special cases may have more. Of particular interest is the **de Sitter universe**, which is a maximally symmetric space-time, having 10 independent Killing vectors:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \cosh^2(t/a)\big[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\big].$$

This is a space-time of constant curvature, which may be thought of as a hyperboloid embedded in five-dimensional space,

$$x^2 + y^2 + z^2 + w^2 - v^2 = a^2.$$

Since it is a space of constant curvature, $R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}$, the Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{4}Rg_{\mu\nu} = \frac{3}{a^2}g_{\mu\nu}.$$

This can be thought of in two ways. It can be interpreted as a solution of Einstein's field equations $G_{\mu\nu} = \kappa T_{\mu\nu}$ with a perfect fluid $T_{\mu\nu} \propto g_{\mu\nu}$ having negative pressure $P = -\rho c^2$. However it is more common to interpret it as a *vacuum* solution $T_{\mu\nu} = 0$ of the modified Einstein field equations with **cosmological constant** Λ,

$$\mathbb{G}_{\mu\nu} = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \quad (\Lambda = 3a^{-2}).$$

This model is currently popular with advocates of the *inflationary cosmology*. Interesting aspects of its geometry are described in [8, 9].

Sometimes cosmologists focus on cosmologies having fewer symmetries. A common technique is to look for homogeneous models that are not necessarily isotropic, equivalent to relaxing the Lie algebra of Killing vectors from six to three, and assuming the orbits are three-dimensional subspaces of space-time. All three-dimensional Lie algebras may be categorized into one of nine **Bianchi types**, usually labelled by Roman numerals. A detailed discussion may be found in [10]. The Robertson–Walker models all fall into this classification, the flat model being of Bianchi type I, the closed model of type IX, and the open model of type V. To see how such a relaxation of symmetry gives rise to more general models, consider type I, which is the commutative Lie algebra

$$[X_1,\ X_2] = [X_2,\ X_3] = [X_1,\ X_3] = 0.$$

It is not hard to show locally that a metric having these symmetries must have the form

$$ds^2 = e^{2\alpha_1(t)}(dx^1)^2 + e^{2\alpha_2(t)}(dx^2)^2 + e^{2\alpha_3(t)}(dx^3)^2 - c^2 dt^2.$$

The vacuum solutions of this metric are (see [11])

$$\alpha_i(t) = a_i \ln t \quad (a_1 + a_2 + a_3 = a_1^2 + a_2^2 + a_3^2 = 1),$$

called **Kasner solutions**. The pressure-free dust cosmologies of this type are called **Heckmann–Schücking solutions** (see the article by E. Heckmann and O. Schücking in [12]) and have the form

$$\alpha_i = a_i \ln(t - t_1) + b_i \ln(t - t_2) \quad \big(b_i = \tfrac{2}{3} - a_i,\ \textstyle\sum_{i=1}^3 a_i = \sum_{i=1}^3 (a_i)^2 = 1\big).$$

It is not hard to show that $\sum_{i=1}^3 b_i = \sum_{i=1}^3 (b_i)^2 = 1$. The density in these solutions evolves as

$$\rho = \frac{1}{6\pi G(t - t_1)(t - t_2)}.$$

The flat Friedmann model arises as the limit $t_1 = t_2$ of this model.

Spherical symmetry

A space-time is said to be **spherically symmetric** if it has three spacelike Killing vectors X_1, X_2, X_3 such that they span a Lie algebra isomorphic with $\mathcal{SO}(3)$,

$$[X_i,\ X_j] = -\epsilon_{ijk} X_k \quad (i, j, k \in \{1, 2, 3\})$$

and such that the orbits of all points are two-dimensional surfaces, or possibly isolated points. The idea is that the orbits generated by the group of transformations are in general 2-spheres that could be represented as $r = \text{const.}$ in appropriate coordinates. There should therefore be coordinates $x = r,\ x^2 = \theta,\ x^3 = \phi$ such that the X_i are spanned by ∂_θ and ∂_ϕ, and using Theorem 15.3 it should be locally possible to choose these coordinates such that

$$X_3 = \partial_\phi, \qquad X_1 = \xi^1 \partial_\theta + \xi^2 \partial_\phi, \qquad X_2 = \eta^1 \partial_\theta + \eta^2 \partial_\phi.$$

We then have

$$\begin{aligned}
[X_3,\ X_1] = -X_2 &\implies \eta^1 = -\xi^1_{,\phi},\ \ \eta^2 = -\xi^2_{,\phi} \\
[X_3,\ X_2] = X_1 \quad &\implies \xi^1 = \eta^1_{,\phi},\quad \xi^2 = \eta^2_{,\phi}
\end{aligned}$$

whence $\xi^i_{,\phi\phi} = -\xi^i$ $(i = 1, 2)$, so that

$$\begin{aligned}
\xi^1 &= f \sin\phi + g \cos\phi, & \xi^2 &= h \sin\phi + k \cos\phi \\
\eta^1 &= -f \cos\phi + g \sin\phi, & \eta^2 &= -h \cos\phi + k \sin\phi
\end{aligned}$$

where the functions f, g, h, k are arbitrary functions of θ, r and t. The remaining commutation relation $[X_1, X_2] = -X_3$ implies, after some simplification,

$$fg_\theta - gf_\theta + gk + fh = 0 \tag{19.27}$$

$$fk_\theta - gh_\theta + h^2 + k^2 = -1 \tag{19.28}$$

where $g_\theta \equiv \partial g/\partial\theta$, etc. A coordinate transformation $\phi' = \phi + F(\theta, r, t), \theta = G(\theta, r, t)$ has the effect

$$\partial_\phi = \partial_{\phi'}, \qquad \partial_\theta = F_\theta\, \partial_{\phi'} + G_\theta\, \partial_{\theta'}$$

and therefore

$$X_1 = \xi^1 G_\theta\, \partial_{\theta'} + \big(\xi^1 F_\theta + \xi^2\big)\, \partial_{\phi'}.$$

Hence, using addition of angle identities for the functions sin and cos,

$$\begin{aligned}(\xi^1)' = \xi^1 G_\theta &= \big(f \sin(\phi' - F) + g \cos(\phi' - F)\big)\, G_\theta \\ &= (f \cos F + g \sin F)\, G_\theta \sin\phi' + (-f \sin F + g \cos F)\, G_\theta \cos\phi'.\end{aligned}$$

Choosing

$$\tan F = -\frac{f}{g} \quad \text{and} \quad G_\theta = \frac{1}{g \cos F - f \sin F}$$

we have $(\xi^1)' = \cos\phi'$. We have thus arrived at the possibility of selecting coordinates θ and ϕ such that $f = 0,\ g = 1$. Substituting in Eqs. (19.27) and (19.28) gives $k = 0$ and $h = -\cot(\theta - \theta_0(r, t))$. Making a final coordinate transformation $\theta \to \theta - \theta_0(r, t)$, which has no effect on ξ^1, we have

$$X_1 = \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\phi, \qquad X_2 = \sin\phi\, \partial_\theta + \cot\theta \cos\phi\, \partial_\phi, \qquad X_3 = \partial_\phi.$$

From Killing's equations (19.17) with $X = X_3$ we have $g_{\mu\nu} = g_{\mu\nu}(r, \theta, t)$ and for $X = X_2$, X_3, we find that these equations have the form

$$\xi^2 \partial_\theta g_{\mu\nu} + \xi^2_{,\mu} g_{2\nu} + \xi^3_{,\mu} g_{3\nu} + \xi^2_{,\nu} g_{2\mu} + \xi^3_{,\nu} g_{3\mu} = 0$$

and successively setting $\mu\nu = 11,\ 12, \dots$ we obtain

$$\begin{gathered} g_{11} = g_{11}(r, t), \qquad g_{14} = g_{14}(r, t), \qquad g_{44} = g_{44}(r, t), \\ g_{12} = g_{13} = g_{42} = g_{43} = g_{23} = 0, \\ g_{22} = f(r, t), \qquad g_{33} = f(r, t) \sin^2\theta. \end{gathered}$$

As there is still an arbitrary coordinate freedom in the radial and time coordinate,

$$r' = F(r, t), \qquad t' = G(r, t)$$

it is possible to choose the new radial coordinate to be such that $f = r'^2$ and the time coordinate may then be found so that $g'_{14} = 0$. The resulting form of the metric is that postulated in Eq. (18.90),

$$\mathrm{d}s^2 = g_{11}(r,t)\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) - |g_{44}(r,t)|c^2\,\mathrm{d}t^2.$$

If g_{11} and g_{44} are independent of the time coordinate then the vector $X = \partial_t$ is a Killing vector. Any space-time having a timelike Killing vector is called **stationary**. For the case considered here the Killing vector has the special property that it is orthogonal to the 3-surfaces $t =$ const., and is called a **static** space-time. The condition for a space-time to be static is that the covariant version of the Killing vector be proportional to a gradient $\xi_\mu = g_{\mu\nu}\xi^\nu = \lambda f_{,\mu}$ for some functions λ and f. Equivalently, if ξ is the 1-form $\xi = \xi_\mu\,\mathrm{d}x^\mu$, then $\xi = \lambda\,\mathrm{d}f$ which, by the Frobenius theorem 16.4, can hold if and only if $\mathrm{d}\xi \wedge \xi = 0$. For the spherically symmetric metric above, $\xi = g_{44}c\,\mathrm{d}t$ and $\mathrm{d}\xi \wedge \xi = \mathrm{d}g_{44} \wedge c\,\mathrm{d}t \wedge g_{44}c\,\mathrm{d}t = 0$, as required. An important example of a metric that is stationary but not static is the Kerr solution, representing a rotating body in general relativity. More details can be found in [8, 13].

Problem

Problem 19.19 Show that the non-translational Killing vectors of pseudo-Euclidean space with metric tensor $g_{ij} = \eta_{ij}$ are of the form

$$X = A_j^{\ k} x^j \partial_{x^k} \quad \text{where} \quad A_{kl} = A_k^{\ j}\eta_{jl} = -A_{lk}.$$

Hence, with reference to Example 19.3, show that the Lie algebra of $SO(p,q)$ is generated by matrices I_{ij} with $i < j$, having matrix elements $(I_{ij})_a^{\ b} = \eta_{ia}\delta_j^{\ b} - \delta_i^{\ b}\eta_{ja}$. Show that the commutators of these generators can be written (setting $I_{ij} = -I_{ji}$ if $i > j$)

$$[I_{ij}, I_{kl}] = I_{il}\eta_{jk} + I_{jk}\eta_{il} - I_{ik}\eta_{jl} - I_{jl}\eta_{ik}.$$

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