18 Connections and curvature

18.1 Linear connections and geodesics

There is no natural way of comparing tangent vectors Y_p and Y_q at p and q, for if they had identical components in one coordinate system this will not generally be true in a different coordinate chart covering the two points. In a slightly different light, consider the partial derivatives $Y^i_{,j} = \partial Y^i/\partial x^i$ of a vector field Y in a coordinate chart $(U; x^i)$. On performing a transformation to coordinates $(U'; x'^{i'})$, we have from Eq. (15.13)

$$Y'^{i'}_{,j'} = \frac{\partial Y'^{i'}}{\partial x'^{j'}} = \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial}{\partial x^j}\Big(Y^i \frac{\partial x'^{i'}}{\partial x^i}\Big),$$

whence

$$Y'^{i'}_{,j'} = \frac{\partial x'^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x'^{j'}} Y^i_{,j} + Y^i \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial^2 x'^{i'}}{\partial x^i \partial x^j}. \tag{18.1}$$

The first term on the right-hand side has the form of a tensor transformation term, as in Eq. (15.15), but the second term is definitely not tensorial in character. Thus, if Y^i has constant components in the chart $(U; x^i)$ (so that $Y^i_{,j} = 0$) this will not be true in the chart $(U'; x'^{i'})$ unless the coordinate transformation functions are linear,

$$x'^{i'} = A^{i'}_{j} x^j \iff \frac{\partial^2 x'^{i'}}{\partial x^i \partial x^j} = 0. \tag{18.2}$$

Suppose we had a well-defined notion of 'directional derivative' of a vector field Y with respect to a tangent vector X_p at p, rather like the concept of directional derivative of a function f with respect to X_p. It would then be possible to define a 'constant' vector field $Y(t)$ along a parametrized curve γ connecting p and q by requiring that the directional derivative with respect to the tangent vector $\dot\gamma$ to the curve be zero for all t in the parameter range. The resulting tangent vector Y_q at q may, however, be dependent on the choice of connecting curve γ. If we write the action of a vector field X on a scalar field f as $D_X f \equiv Xf$, then for any real-valued function g

$$D_{gX} f = gXf = gD_X f. \tag{18.3}$$

This property is essential for D_X to be a *local action*, as the action of D_{gX} at p only depends on the value $g(p)$, not on the behaviour of the function g in an entire neighbourhood of p,

$$\big(D_{gX} f\big)(p) = g(p) D_X f(p).$$

We may therefore write $D_{X_p} f = (D_X f)(p)$ without ambiguity, for if $X_p = Y_p$ then $(D_X f)(p) = (D_Y f)(p)$.

Exercise: Show that the last assertion follows from Eq. (18.3).

Extending this idea to vector fields Y, we seek a derivative $D_X Y$ having the property

$$D_{gX} Y = g D_X Y$$

for any function $g : M \to \mathbb{R}$. We will show that the derivative of Y relative to a tangent vector X_p at p can then be defined by setting $D_{X_p} Y = (D_X Y)_p$ – the result will be independent of the choice of vector field X reducing to X_p at p.

The Lie derivative $\mathcal{L}_X Y = [X, Y]$ (see Section 15.5) is not a derivative in the sense required here since, for a general function $g \in \mathcal{F}(M)$,

$$\mathcal{L}_{gX} Y = [gX, Y] = g[X, Y] - Y(g)X \neq g\mathcal{L}_X Y.$$

To calculate the Lie derivative $[X, Y]_p$ at a point p it is not sufficient to know the tangent vector X_p at p – we must know the behaviour of the vector field X in an entire neighbourhood of a point p.

A **connection**, also called a **linear** or **affine connection**, on a differentiable manifold M is a map $D : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$, where $\mathcal{T}(M)$ is the module of differentiable vector fields on M, such that the map $D_X : \mathcal{T}(M) \to \mathcal{T}(M)$ defined by $D_X Y = D(X, Y)$ satisfies the following conditions for arbitrary vector fields X, Y, Z and scalar fields f, g:

(Con1) $D_{X+Y} Z = D_X Z + D_Y Z$,
(Con2) $D_{gX} Y = g D_X Y$,
(Con3) $D_X(Y + Z) = D_X Y + D_X Z$,
(Con4) $D_X(fY) = (Xf)Y + f D_X Y = (D_X f)Y + f D_X Y$.

A linear connection is not inherent in the original manifold structure – it must be imposed as an extra structure on the manifold. Given a linear connection D on a manifold M, for every vector field Y there exists a tensor field DY of type (1, 1) defined by

$$DY(\omega, X) = \langle \omega, D_X Y \rangle \tag{18.4}$$

for every 1-form ω and vector field X. The tensor nature of DY follows from linearity in both arguments. Linearity in ω is trivial, while linearity in X follows immediately from (Con1) and (Con2). The tensor field DY is called the **covariant derivative** of the vector field Y. The theory of connections as described here is called a *Koszul connection* [1–6], while the 'old-fashioned' coordinate version that will be deduced below appears in texts such as [7, 8].

A connection can be restricted to any open submanifold $U \subset M$ in a natural way. For example, if $(U; x^i)$ is a coordinate chart on M and ∂_{x^i} the associated local basis of vector fields, we may set $D_k = D_{\partial_{x^k}}$. Expanding the vector fields $D_k \partial_{x^j}$ in terms of the local basis,

$$D_k \partial_{x^j} = \Gamma^i_{jk} \partial_{x^i} \tag{18.5}$$

where Γ^i_{jk} are real-valued functions on U, known as the **components of the connection** D with respect to the coordinates $\{x^i\}$. Using (Con3) and (Con4) we can compute the covariant derivative of any vector field $Y = Y^i \partial_{x^i}$ on U:

$$\begin{aligned} D_k Y = D_k\big(Y^j \partial_{x^j}\big) &= \big(\partial_{x^k} Y^j\big)\partial_{x^j} + Y^j \Gamma^i_{jk}\partial_{x^i} \\ &= Y^i{}_{;k}\partial_{x^i} \end{aligned}$$

where

$$Y^i{}_{;k} = Y^i{}_{,k} + \Gamma^i_{jk} Y^j. \tag{18.6}$$

The coefficients $Y^i{}_{;k}$ are the components of the covariant derivative with respect to these coordinates since, by Eq. (18.4),

$$\begin{aligned} (DY)^i{}_k = DY\big(\mathrm{d}x^i, \partial_{x^k}\big) &= \langle \mathrm{d}x^i, D_k Y\rangle \\ &= \langle \mathrm{d}x^i, Y^j{}_{;k}\rangle \partial_{x^j} \\ &= Y^j{}_{;k}\delta^i{}_j = Y^i{}_{;k}. \end{aligned}$$

Thus

$$DY = Y^i{}_{;k}\, \partial_{x^i} \otimes \mathrm{d}x^k$$

and the components of $D_X Y$ with respect to the coordinates x^i are

$$\big(D_X Y\big)^i = DY(\mathrm{d}x^i, X) = Y^i{}_{;k} X^k. \tag{18.7}$$

As anticipated above, it is possible to define the covariant derivative of a vector field Y with respect to a tangent vector X_p at a point p as $D_{X_p} Y = \big(D_X Y\big)_p \in T_p(M)$, where X is any vector field that 'reduces' to X_p at p. For this definition to make sense, we must show that it is independent of the choice of vector field X. Suppose X' is a second vector field such that $X'_p = X_p$. The vector field $Z = X - X'$ vanishes at p, and we have

$$\begin{aligned} \big(D_X Y\big)_p - \big(D_{X'} Y\big)_p &= \big(D_{X-X'} Y\big)_p \\ &= \big(D_Z Y\big)_p \\ &= Y^i_{;k}(p)(Z_p)^k\, \partial_{x^i} = 0 \quad \text{by Eq. (18.7).} \end{aligned}$$

The **covariant derivative of a vector field** Y **along a curve** $\gamma : \mathbb{R} \to M$ is defined to be

$$\frac{DY}{\mathrm{d}t} = D_{\dot\gamma} Y$$

where $X = \dot\gamma$ is the tangent vector to the curve. By (18.7), the components are

$$\begin{aligned} \frac{DY^i}{\mathrm{d}t} \equiv \Big(\frac{DY}{\mathrm{d}t}\Big)^i &= \frac{\mathrm{d}x^k}{\mathrm{d}t}\Big(\frac{\partial Y^i}{\partial x^k} + \Gamma^i_{jk} Y^j\Big) \\ &= \frac{\mathrm{d}Y^i(t)}{\mathrm{d}t} + \Gamma^i_{jk} Y^j \frac{\mathrm{d}x^k}{\mathrm{d}t}. \end{aligned} \tag{18.8}$$

We will say the vector field Y is **parallel along the curve** γ if $DY(t)/\mathrm{d}t = 0$ for all t in the curve's parameter range. A curve will be called a **geodesic** if its tangent vector is everywhere parallel along the curve, $D_{\dot\gamma}\dot\gamma = 0$ – note that the expression on the right-hand side of

Eq. (18.8) depends only on the values of the components $Y^i(t) \equiv Y^i\big(\gamma(t)\big)$ along the curve. By (18.8) a geodesic can be written locally as a set of differential equations

$$\frac{D}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0. \tag{18.9}$$

The above discussion can also be reversed. Let p be any point of M and $\gamma : [a, b] \to M$ a curve such that $p = \gamma(a)$. In local coordinates $(U; x^i)$ the equations for a vector field to be parallel along the curve are a linear set of differential equations

$$\frac{\mathrm{d}Y^i(t)}{\mathrm{d}t} + \Gamma^i_{jk}Y^j(t)\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0. \tag{18.10}$$

By the existence and uniqueness theorem of differential equations, for any tangent vector Y_p at p there exists a unique vector field $Y(t)$ parallel along $\gamma \cap U$ such that $Y(a) = Y_p$. The curve segment is a compact set and can be covered by a finite family of charts, so that existence and uniqueness extends over the entire curve $a \le t \le b$. Furthermore, as the differential equations are linear the map $P_t : T_p(M) \to T_{\gamma(t)}(M)$ such that $P_t(Y_p) = Y(t)$ is a linear map, called **parallel transport** along γ from $p = \gamma(a)$ to $\gamma(t)$. Since the parallel transport map can be reversed by changing the parameter to $t' = -t$, the map P_t is one-to-one and must be a linear isomorphism.

The uniqueness of a maximal solution to a set of differential equations also shows that if p is any point of M there exists a unique maximal geodesic $\sigma : [0, a)$ where $a \le \infty$ starting with any specified tangent vector $\dot\sigma(0) = X_p$ at $p = \sigma(0)$. The parameter t such that a geodesic satisfies Eq. (18.9) is called an **affine parameter**. Under a parameter transformation $t' = f(t)$ the tangent vector becomes

$$\dot\sigma' = \frac{\mathrm{d}x^i}{\mathrm{d}t'}\partial_{x^i} = \frac{1}{f'(t)}\frac{\mathrm{d}x^i}{\mathrm{d}t}\partial_{x^i}$$

where $f'(t) = \mathrm{d}f/\mathrm{d}t$ and, using Eq. (18.9), we have

$$\begin{aligned}\frac{\mathrm{d}^2x^i}{\mathrm{d}t'^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t'}\frac{\mathrm{d}x^k}{\mathrm{d}t'} &= \frac{1}{f'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{f'(t)}\Big)\frac{\mathrm{d}x^i}{\mathrm{d}t} \\ &= -\frac{f''(t)}{(f'(t))^2}\frac{\mathrm{d}x^i}{\mathrm{d}t'}.\end{aligned}$$

The new parameter t' is an affine parameter if and only if $f''(t) = 0$ – that is, an affine transformation, $t' = at + b$. Herein lies the reason behind the term *affine parameter*.

Coordinate transformations

Consider a coordinate transformation from a chart $(U; x^i)$ to a chart $(U'; x'^{i'})$. In the overlap $U \cap U'$ we have, using the transformations between coordinate bases given by

Eq. (15.12),

$$\begin{aligned} D_{\partial_{x'^{k'}}}\partial_{x'^{j'}} &= \frac{\partial x^k}{\partial x'^{k'}} D_k\Big(\frac{\partial x^j}{\partial x'^{j'}}\partial_{x^j}\Big) \\ &= \frac{\partial x^k}{\partial x'^{k'}}\frac{\partial x^j}{\partial x'^{j'}}\Gamma^i_{jk}\frac{\partial x'^{i'}}{\partial x^i}\partial_{x'^{i'}} + \frac{\partial^2 x^j}{\partial x'^{k'}\partial x'^{j'}}\partial_{x^j} \\ &= \Gamma'^{i'}_{j'k'}\partial_{x'^{i'}} \end{aligned}$$

where

$$\Gamma'^{i'}_{j'k'} = \frac{\partial x^k}{\partial x'^{k'}}\frac{\partial x^j}{\partial x'^{j'}}\frac{\partial x'^{i'}}{\partial x^i}\Gamma^i_{jk} + \frac{\partial^2 x^i}{\partial x'^{k'}\partial x'^{j'}}\frac{\partial x'^{i'}}{\partial x^i}. \tag{18.11}$$

This is the law of transformation of components of a connection.

The first term on the right-hand side of (18.11) is tensorial in nature, but the second term adds a complication that only vanishes for linear transformations. It is precisely the expression needed to counteract the non-tensorial part of the transformation of the derivative of a vector field given in Eq. (18.1) – see Problem 18.1.

Problems

Problem 18.1 Show directly from the transformation laws (18.1) and (18.11) that the components of the covariant derivative (18.6) of a vector field transform as a tensor of type (1, 1).

Problem 18.2 Show that the transformation law (18.11) can be written in the form

$$\Gamma'^{i'}_{j'k'} = \frac{\partial x^k}{\partial x'^{k'}}\frac{\partial x^j}{\partial x'^{j'}}\frac{\partial x'^{i'}}{\partial x^i}\Gamma^i_{jk} - \frac{\partial^2 x'^{k'}}{\partial x^i\partial x^j}\frac{\partial x^i}{\partial x'^{i'}}\frac{\partial x^j}{\partial x'^{j'}}.$$

18.2 Covariant derivative of tensor fields

For every vector field X we define a map $D_X : \mathcal{T}^{(r,s)}(M) \to \mathcal{T}^{(r,s)}(M)$ that extends the covariant derivative to general tensor fields by requiring:

(Cov1) For scalar fields, $f \in \mathcal{F}(M) = \mathcal{T}^{(0,0)}(M)$, we set $D_X f = Xf$.

(Cov2) For 1-forms $\omega \in \mathcal{T}^{(0,1)}(M)$ assume a Leibnitz rule for $\langle\, ,\rangle$,

$$D_X\langle\omega, Y\rangle = \langle D_X\omega, Y\rangle + \langle\omega, D_XY\rangle.$$

(Cov3) $D_X(T+S) = D_XT + D_XS$ for any pair of tensor fields $T, S \in \mathcal{T}^{(r,s)}(M)$.

(Cov4) The Leibnitz rule holds with respect to tensor products

$$D_X(T\otimes S) = (D_XT)\otimes S + T\otimes D_XS.$$

These requirements define a unique tensor field D_XT for any smooth tensor field T. Firstly, let $\omega = w_i\mathrm{d}x^i$ be any 1-form defined on a coordinate chart $(U; x^i)$ covering p. Setting $Y = \partial_{x^i}$, condition (Cov1) gives $D_X\langle\omega, \partial_{x^i}\rangle = D_X(w_i) = X^k w_{i,k}$, while (Cov2) implies

$$D_X\langle\omega, \partial_{x^i}\rangle = \langle D_X\omega, \partial_{x^i}\rangle + \langle\omega, X^kD_k\partial_{x^i}\rangle.$$

Hence, using (18.5) and $\big(D_X\omega\big)_i = \langle D_X\omega, \partial_{x^i}\rangle$, we find

$$\big(D_X\omega\big)_i = w_{i;k}X^k \tag{18.12}$$

where

$$w_{i;k} = w_{i,k} - \Gamma^j_{ik}w_j. \tag{18.13}$$

Exercise: Verify that Eq. (18.13) implies the coordinate expression for condition (Cov2),

$$\big(w_iY^i\big)_{,k}X^k = w_{i;k}X^kY^i + w_iY^i{}_{;k}X^k.$$

For a general tensor T, expand in terms of the basis consisting of tensor products of the ∂_{x^i} and $\mathrm{d}x^j$ and use (Cov3) and (Cov4). For example, if

$$T = T^{ij\dots}{}_{kl\dots}\partial_{x^i}\otimes\partial_{x^j}\otimes\dots\otimes\mathrm{d}x^k\otimes\mathrm{d}x^l\otimes\dots$$

a straightforward calculation results in

$$D_XT = T^{ij\dots}{}_{kl\dots;p}X^p\partial_{x^i}\otimes\partial_{x^j}\otimes\dots\otimes\mathrm{d}x^k\otimes\mathrm{d}x^l\otimes\dots$$

where

$$\begin{aligned} T^{ij\dots}{}_{kl\dots;p} = {} & T^{ij\dots}{}_{kl\dots,p} + \Gamma^i_{ap}T^{aj\dots}{}_{kl\dots} + \Gamma^j_{ap}T^{ia\dots}{}_{kl\dots} + \dots \\ & -\Gamma^a_{kp}T^{ij\dots}{}_{al\dots} - \Gamma^a_{lp}T^{ij\dots}{}_{ka\dots} - \dots \end{aligned} \tag{18.14}$$

This demonstrates that (Cov1)–(Cov4) can be used to compute the components of D_XT at any point $p \in M$ with respect to a coordinate chart $(U; x^i)$ covering p, and thus uniquely define the tensor field DXT throughout M.

For every tensor field T of type (r, s) its **covariant derivative** DT is the tensor field of type $(r, s+1)$ defined by

$$DT(\omega^1, \omega^2, \dots, \omega^r, Y_1, Y_2, \dots, Y_s, X) = D_XT(\omega^1, \omega^2, \dots, \omega^r, Y_1, Y_2, \dots, Y_s).$$

Exercise: Show that, with respect to any local coordinates, the tensor field DT has components $T^{ij\dots}{}_{kl\dots;p}$ defined by Eq. (18.14).

Exercise: Show that

$$\delta^i{}_{j;k} = 0. \tag{18.15}$$

The covariant derivative commutes with all contractions on a tensor field

$$D\big(C^k_lT\big) = C^k_lDT. \tag{18.16}$$

This relation is most easily shown in local coordinates, where it reads

$$\big(T^{i_1\dots a\dots i_r}{}_{j_1\dots a\dots j_s}\big)_{;k} = T^{i_1\dots a\dots i_r}{}_{j_1\dots a\dots j_s;k}, \tag{18.17}$$

the upper index a being in the kth position, the lower in the lth. If we expand the right-hand side according to Eq. (18.14) the terms corresponding to these indices are

$$\Gamma^a_{bk}T^{i_1\dots b\dots i_r}{}_{j_1\dots a\dots j_s} - \Gamma^b_{ak}T^{i_1\dots a\dots i_r}{}_{j_1\dots b\dots j_s} = 0$$

and what remains reduces to the expression formed by expanding the left-hand side of (18.17).

A useful corollary of this property is the following relation:

$$\begin{aligned} X(T(\omega^1, \dots, \omega^r, Y_1, \dots, Y_s)) &= D_X T(\omega^1, \dots, \omega^r, Y_1, \dots, Y_s) \\ &+ T(D_X\omega^1, \dots, \omega^r, Y_1, \dots, Y_s) + \dots + T(\omega^1, \dots, D_X\omega^r, Y_1, \dots, Y_s) \\ &+ T(\omega^1, \dots, \omega^r, D_X Y_1, \dots, Y_s) + \dots + T(\omega^1, \dots, \omega^r, Y_1, \dots, D_X Y_s). \end{aligned} \tag{18.18}$$

Problems

Problem 18.3 Show directly from (Cov1)–(Cov4) that $D_{fX}T = f D_X T$ for all vector fields X, tensor fields T and scalar functions $f : M \to \mathbb{R}$.

Problem 18.4 Verify from the coordinate transformation rule (18.11) for Γ^i_{jk} that the components of the covariant derivative of an arbitrary tensor field, defined in Eq. (18.14), transform as components of a tensor field.

Problem 18.5 Show that the identity (18.18) follows from Eq. (18.16).

18.3 Curvature and torsion

Torsion tensor

In the transformation law of the components Γ^i_{jk}, Eq. (18.11), the term involving second derivatives of the transformation functions is symmetric in the indices jk. It follows that the antisymmetrized quantity $T^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}$ does transform as a tensor, since the non-tensorial parts of the transformation law (18.11) cancel out.

To express this idea in an invariant non-coordinate way, observe that for any vector field Y the tensor field DY defined in Eq. (18.4) satisfies the identities

$$DY(f\omega, X) = f DY(\omega, X), \qquad DY(\omega, fX) = f DY(\omega, X)$$

for all functions $f \in \mathcal{F}(M)$. These are called $\mathcal{F}$**-linearity** in the respective arguments. On the other hand, the map $D' : \mathcal{T}^*(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{F}(M)$ defined by

$$D'(\omega, X, Y) = \langle \omega, D_X Y \rangle$$

is not a tensor field of type $(1, 2)$, since $\mathcal{F}$-linearity fails for the third argument by (Con4),

$$\begin{aligned} D'(\omega, X, fY) &= \langle \omega, D_X(fY) \rangle = \langle \omega, (Xf)Y + f D_X Y \rangle \\ &= (Xf)\langle \omega, Y \rangle + f D'(\omega, X, Y) \neq f D'(\omega, X, Y). \end{aligned}$$

Exercise: Show that the ‘components’ of D' are $D'(\mathrm{d}x^i, \partial_{x^j}, \partial_{x^k}) = \Gamma^i_{jk}$.

Now let the *torsion map* $\tau : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ be defined by

$$\tau(X, Y) = D_X Y - D_Y X - [X, Y] = -\tau(Y, X). \tag{18.19}$$

This map is $\mathcal{F}$-linear in the first argument,

$$\begin{aligned}\tau(fX, Y) &= D_{fX}Y - D_Y(fX) - [fX, Y] \\ &= fD_XY - (Yf)X - fD_YX - f[X, Y] + (Yf)X \\ &= f\tau(X, Y),\end{aligned}$$

and by antisymmetry it is also $\mathcal{F}$-linear in the second argument Y. Hence τ gives rise to a tensor field T of type (1, 2) by

$$T(\omega, X, Y) = \langle \omega, \tau(X, Y)\rangle, \tag{18.20}$$

known as the **torsion tensor** of the connection D. In a local coordinate chart $(U; x^i)$ its components are precisely the antisymmetrized connection components:

$$T^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}. \tag{18.21}$$

The proof follows from setting $T^i_{jk} = \langle \mathrm{d}x^i, \tau\big(\partial_{x^j}, \partial_{x^k}\big)\rangle$ and sustituting Eq. (18.19). We call a connection **torsion-free** or **symmetric** if its torsion tensor vanishes, $T = 0$; equivalently, its components are symmetric with respect to all coordinates, $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Exercise: Prove Eq. (18.21).

Curvature tensor

A similar problem occurs when commuting repeated covariant derivatives on a vector or tensor field. If X, Y and Z are any vector fields on M then $D_X D_Y Z - D_Y D_X Z$ is obviously a vector field, but the map $P : \mathcal{T}^*(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{F}(M)$ defined by $P(\omega, X, Y, Z) = \langle \omega, D_X D_Y Z - D_Y D_X Z\rangle$ fails to be a tensor field of type (1, 3) as it is not $\mathcal{F}$-linear in the three vector field arguments. The remedy is similar to that for creating the torsion tensor.

For any pair of vector fields X and Y define the operator $\rho_{X,Y} : \mathcal{T}(M) \to \mathcal{T}(M)$ by

$$\rho_{X,Y}Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z = -\rho_{Y,X}Z. \tag{18.22}$$

This operator is $\mathcal{F}$-linear with respect to X, and therefore Y, since

$$\begin{aligned}\rho_{fX,Y}Z &= D_{fX}D_Y Z - D_Y D_{fX}Z - D_{[fX,Y]}Z \\ &= fD_X D_Y Z - (Yf)D_X Z - fD_Y D_X Z - fD_{[X,Y]}Z + (Yf)D_X Z \\ &= f\rho_{X,Y}Z.\end{aligned}$$

$\mathcal{F}$-linearity with respect to Z follows from

$$\begin{aligned}\rho_{X,Y}fZ &= D_X D_Y(fZ) - D_Y D_X(fZ) - D_{[X,Y]}(fZ) \\ &= D_X\big((Yf)Z + fD_Y Z\big) - D_Y\big((Xf)Z + fD_X Z\big) - ([X, Y]f)Z - fD_{[X,Y]}Z \\ &= (Yf)D_X Z + X(Yf)Z + (Xf)D_Y Z + fD_X D_Y Z - Y(Xf)Z - (Xf)D_Y Z \\ &\quad - (Yf)D_X Z - fD_Y D_X Z - ([X, Y]f)Z - fD_{[X,Y]}Z \\ &= f\rho_{X,Y}Z.\end{aligned}$$

We can therefore define a tensor field R of type $(1, 3)$, by setting

$$R(\omega, Z, X, Y) = \langle \omega, \rho_{X,Y} Z \rangle = -R(\omega, Z, Y, X), \tag{18.23}$$

called the **curvature tensor** of the connection D.

For a torsion-free connection, $D_X Y - D_Y X = [X, Y]$, there is a cyclic identity:

$$\begin{aligned}
\rho_{X,Y} Z + \rho_{Y,Z} X + \rho_{Z,X} Y &= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \\
&\quad + D_Y D_Z X - D_Z D_Y X - D_{[Y,Z]} X + D_Z D_X Y \\
&\quad - D_X D_Z Y - D_{[Z,X]} Y \\
&= D_X [Y, Z] + D_Y [Z, X] + D_Z [X, Y] - D_{[Y,Z]} X \\
&\quad - D_{[Z,X]} Y - D_{[X,Y]} Z \\
&= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0
\end{aligned}$$

using $\tau(X, [Y, Z]) = 0$, etc. and the Jacobi identity (15.24). For $T = 0$ we thus have the so-called **first Bianchi identity**,

$$R(\omega, Z, X, Y) + R(\omega, X, Y, Z) + R(\omega, Y, Z, X) = 0. \tag{18.24}$$

In a coordinate system $(U; x^i)$, using Eq. (18.5) and $[\partial_{x^k}, \partial_{x^l}] = 0$ for all k, l, the components of the curvature tensor are

$$\begin{aligned}
R^i{}_{jkl} &= R(\mathrm{d}x^i, \partial_{x^j}, \partial_{x^k}, \partial_{x^l}) \\
&= \langle \mathrm{d}x^i, D_k D_l \partial_{x^j} - D_l D_k \partial_{x^j} - D_{[\partial_{x^k}, \partial_{x^l}]} \partial_{x^j} \rangle \\
&= \langle \mathrm{d}x^i, D_k \big(\Gamma^m_{jl} \partial_{x^m} \big) - D_l \big(\Gamma^m_{jk} \partial_{x^m} \big) \rangle \\
&= \langle \mathrm{d}x^i, \Gamma^m_{jl,k} \partial_{x^m} + \Gamma^m_{jl} \Gamma^p_{mk} \partial_{x^p} - \Gamma^m_{jk,l} \partial_{x^m} - \Gamma^m_{jk} \Gamma^p_{ml} \partial_{x^p} \rangle
\end{aligned}$$

where $\Gamma^m_{jk,l} = \partial \Gamma^k_{jk} / \partial x^l$. Hence

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl} \Gamma^i_{mk} - \Gamma^m_{jk} \Gamma^i_{ml} = -R^i{}_{jlk}. \tag{18.25}$$

Setting $\omega = \mathrm{d}x^i,\ X = \partial_{x^k}$, etc. in the first Bianchi identity (18.24) gives

$$R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0. \tag{18.26}$$

Another class of identities, known as **Ricci identities**, are sometimes used to define the torsion and curvature tensors in a coordinate region $(U; x^i)$. For any smooth function f on U, set $f_{;ij} \equiv (f_{;i})_{;j} = (f_{,i})_{;j}$. Then, by Eq. (18.13),

$$f_{;ij} - f_{;ji} = f_{,ij} - \Gamma^k_{ij} f_{,k} - f_{,ji} + \Gamma^k_{ji} f_{,k} = T^k_{ij} f_{,k}. \tag{18.27}$$

Similarly, for a smooth vector field $X = X^k \partial_{x^k}$, Eq. (18.25) gives rise to

$$X^k{}_{;ij} - X^k{}_{;ji} = X^a R^k{}_{aji} + T^a_{ij} X^k{}_{;a}, \tag{18.28}$$

and for a 1-form $\omega = w_i\, \mathrm{d}x^i$,

$$w_{k;ij} - w_{k;ji} = w_a R^a{}_{kij} + T^a_{ij} w_{k;a}. \tag{18.29}$$

Problems

Problem 18.6 Let f be a smooth function, $X = X^i \partial_{x^i}$ a smooth vector field and $\omega = w_i \, \mathrm{d}x^i$ a differential 1-form. Show that

$$\begin{aligned}(D_j D_i - D_i D_j) f &= 0,\\ (D_j D_i - D_i D_j) X &= X^a R^k{}_{aji} \partial_{x^k},\\ (D_j D_i - D_i D_j)\omega &= w_a R^a{}_{kji} \, \mathrm{d}x^k.\end{aligned}$$

Why does the torsion tensor not appear in these formulae, in contrast with the Ricci identities (18.27)–(18.29)?

Problem 18.7 Show that the coordinate expression for the Lie derivative of a vector field may be written

$$(\mathcal{L}_X Y)^i = [X, Y]^i = Y^i{}_{;j} X^j - X^i{}_{;j} Y^j + T^i_{jk} X^k Y^j. \tag{18.30}$$

For a torsion-free connection show that the Lie derivative (15.39) of a general tensor field S of type (r, s) may be expressed by

$$\begin{aligned}\big(\mathcal{L}_X S\big)^{ij\dots}{}_{kl\dots} = S^{ij\dots}{}_{kl\dots;m} X^m - S^{mj\dots}{}_{kl\dots} X^i{}_{;m} - S^{im\dots}{}_{kl\dots} X^j{}_{;m} - \dots\\ + S^{ij\dots}{}_{ml\dots} X^m{}_{;k} + S^{ij\dots}{}_{km\dots} X^m{}_{;l} + \dots\end{aligned} \tag{18.31}$$

Write down the full version of this equation for a general connection with torsion.

Problem 18.8 Prove the Ricci identities (18.28) and (18.29).

Problem 18.9 For a torsion-free connection prove the generalized Ricci identities

$$\begin{aligned}S^{kl\dots}{}_{mn\dots;ij} - S^{kl\dots}{}_{mn\dots;ji} = S^{al\dots}{}_{mn\dots} R^k{}_{aji} + S^{ka\dots}{}_{mn\dots} R^l{}_{aji} + \dots\\ + S^{kl\dots}{}_{an\dots} R^a{}_{mij} + S^{kl\dots}{}_{ma\dots} R^a{}_{nij} + \dots\end{aligned}$$

How is this equation modified in the case of torsion?

Problem 18.10 For arbitrary vector fields Y, Z and W show that the operator $\Sigma_{Y,Z,W} : \mathcal{T}(M) \to \mathcal{T}(M)$ defined by

$$\Sigma_{Y,Z,W} X = D_W\big(\rho_{Y,Z} X\big) - \rho_{Z,[Y,W]} X - \rho_{Y,Z}\big(D_W X\big)$$

has the cyclic symmetry

$$\Sigma_{Y,Z,W} X + \Sigma_{Z,W,Y} X + \Sigma_{W,Y,Z} X = 0.$$

Express this equation in components with respect to a local coordinate chart and show that it is equivalent to the **(second) Bianchi identity**

$$R^i{}_{jkl;m} + R^i{}_{jlm;k} + R^i{}_{jmk;l} = R^i{}_{jpk} T^p_{ml} + R^i{}_{jpl} T^p_{km} + R^i{}_{jpm} T^p_{lk}. \tag{18.32}$$

Problem 18.11 Let $Y^i(t)$ be a vector that is parallel propagated along a curve having coordinate representation $x^j = \overset{0}{x}{}^j + A^j t$. Show that for $t \ll 1$

$$Y^i(t) = \overset{0}{Y}{}^i - \overset{0}{\Gamma}{}^i_{ja} \overset{0}{Y}{}^j A^a t + \frac{t^2}{2}\big(\overset{0}{\Gamma}{}^i_{ka} \overset{0}{\Gamma}{}^k_{ja} - \overset{0}{\Gamma}{}^j_{ja,b}\big) A^a A^b \overset{0}{Y}{}^j + O(t^3)$$

where $\overset{0}{\Gamma}{}^i_{jk} = \Gamma^i_{jk}(\overset{0}{x}{}^a)$ and $\overset{0}{Y}{}^i = Y^i(0)$. From the point P, having coordinates $\overset{0}{x}{}^i$, parallel transport the tangent vector $\overset{0}{Y}{}^i$ around a coordinate rectangle $PQRSP$ whose sides are each of parameter length t and are along the a- and b-axes successively through these points. For example, the a-axis through P is the curve $x^j = \overset{0}{x}{}^j + \delta^j_a t$. Show that to order t^2, the final vector at P has components

$$Y^i = \overset{0}{Y}{}^i + t^2 \overset{0}{R}{}^i{}_{jba} \overset{0}{Y}{}^j$$

where $\overset{0}{R}{}^i{}_{jba}$ are the curvature tensor components at P.

18.4 Pseudo-Riemannian manifolds

A tensor field g of type $(0, 2)$ on a manifold M is said to be **non-singular** if $g_p \in T^{(0,2)}$ is a non-singular tensor at every point $p \in M$. A **pseudo-Riemannian manifold** (M, g) consists of a differentiable manifold M together with a symmetric non-singular tensor field g of type $(0, 2)$, called a **metric tensor**. This is equivalent to defining an inner product $X_p \cdot Y_p = g_p(X_p, Y_p)$ on the tangent space $T_p(M)$ at every point $p \in M$ (see Chapters 5 and 7). We will assume g is a differentiable tensor field, so that for every pair of smooth vector fields $X, Y \in \mathcal{T}(M)$ the inner product is a differentiable function,

$$g(X, Y) = g(Y, X) \in \mathcal{F}(M).$$

In any coordinate chart $(U; x^i)$ we can write

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \quad \text{where} \quad g_{ij} = g_{ji} = g\big(\partial_{x^i}, \partial_{x^j}\big),$$

and $\mathsf{G} = [g_{ij}]$ is a non-singular matrix at each point $p \in M$. As in Example 7.7 there exists a smooth **inverse metric tensor** g^{-1} on M, a symmetric tensor field of type (2, 0), such that in any coordinate chart $(U; x^i)$

$$g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \quad \text{where} \quad g^{ik} g_{kj} = \delta^i_{\ j}.$$

It is always possible to find a set of orthonormal vector fields $e_1, \dots, e_n$ on a neighbourhood of any given point p, spanning the tangent space at each point of the neighbourhood, such that

$$g(e_i, e_j) = \eta_{ij} = \begin{cases} \eta_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where $\eta_i = \pm 1$. At p one can set up coordinates such that $e_i(p) = \big(\partial_{x^i}\big)_p$, so that

$$g_p = \eta_{ij} (\mathrm{d}x^i)_p \otimes (\mathrm{d}x^j)_p,$$

but in general it is not possible to achieve that $e_i = \partial_{x^i}$ over an entire coordinate chart unless all Lie brackets of the orthonormal fields vanish, $[e_i, e_j] = 0$. We say (M, g) is a **Riemannian manifold** if the metric tensor is everywhere positive definite,

$$g_p(X_p, X_p) > 0 \qquad \text{for all } X_p \neq 0 \in T_p(M),$$

or equivalently, $g(X, X) \geq 0$ for all vector fields $X \in \mathcal{T}(M)$. In this case all $\eta_i = 1$ in the above expansion. The word *Riemannian* is also applied to the negative definite case, all $\eta_i = -1$. If the inner product defined by g_p on every tangent space is Minkowskian, as defined in Section 5.1, we say (M, g) is a **Minkowskian** or **hyperbolic** manifold. In this case there exists a local orthonormal set of vector fields e_i such that the associated coefficients are $\eta_1 = \epsilon, \eta_2 = \cdots = \eta_n = -\epsilon$ where $\epsilon = \pm 1$.

If $\gamma : [a, b] \to M$ is a parametrized curve on a Riemannian manifold, its **length** between t_0 and t is defined to be

$$s = \int_{t_0}^{t} \sqrt{g(\dot\gamma(u), \dot\gamma(u))}\, \mathrm{d}u. \tag{18.33}$$

If the curve is contained in a coordinate chart $(U; x^i)$ and is written $x^i = x^i(t)$ we have

$$s = \int_{t_0}^{t} \sqrt{g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}u} \frac{\mathrm{d}x^j}{u}}\, \mathrm{d}u.$$

Exercise: Verify that the length of the curve is independent of parametrization; i.e., s is unaltered under a change of parameter $u' = f(u)$ in the integral on the right-hand side of (18.33).

Let the value of t_0 in (18.33) be fixed. Then $\mathrm{d}s/\mathrm{d}t = \sqrt{g(\dot\gamma(t), \dot\gamma(t))}$, and

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t}. \tag{18.34}$$

If the parameter along the curve is set to be the distance parameter s, the tangent vector is a unit vector along the curve,

$$g(\dot\gamma(s), \dot\gamma(s)) = g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}s} \frac{\mathrm{d}x^j}{\mathrm{d}s} = 1. \tag{18.35}$$

Sometimes Eq. (18.34) is written symbolically in the form

$$\mathrm{d}s^2 = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j, \tag{18.36}$$

commonly called the *metric* of the space. It is to be thought of as a symbolic expression for displaying the components of the metric tensor and replaces the more correct $g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$. This may be done even in the case of an indefinite metric where, strictly speaking, we can have $\mathrm{d}s^2 < 0$.

The Riemannian space R^n with metric

$$\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \cdots + \left(\mathrm{d}x^n\right)^2$$

is called **Euclidean space** and is denoted by the symbol $\mathbb{E}^n$. Of course other coordinates such as polar coordinates may be used, but when we use the symbol $\mathbb{E}^n$ we shall usually assume that the rectilinear system is being adopted unless otherwise specified.

Example 18.1 If $(M, \varphi : M \to \mathbb{E}^n)$ is any submanifold of $\mathbb{E}^n$, it has a naturally induced metric tensor

$$g = \varphi^*\big(\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \cdots + \mathrm{d}x^n \otimes \mathrm{d}x^n\big).$$

Let M be the 2-sphere of radius a, $x^2 + y^2 + z^2 = a^2$, and adopt polar coordinates

$$x = a \sin\theta \cos\phi, \qquad y = a \sin\theta \sin\phi, \qquad z = a \cos\theta.$$

It is straightforward to evaluate the induced metric tensor on S^2,

$$
\begin{aligned}
g &= \varphi^*(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z) \\
&= (\cos\theta \cos\phi \,\mathrm{d}\theta - \sin\theta \sin\phi \,\mathrm{d}\phi) \otimes (\cos\theta \cos\phi \,\mathrm{d}\theta - \sin\theta \sin\phi \,\mathrm{d}\phi) + \cdots \\
&= a^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2 \mathrm{d}\phi \otimes \mathrm{d}\phi).
\end{aligned}
$$

Alternatively, let $\theta = \theta(t),\ \phi = \phi(t)$ be any curve lying in M. The components of its tangent vector in $\mathbb{E}^3$ are

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = \cos\theta(t)\cos\phi(t)\frac{\mathrm{d}\theta}{\mathrm{d}t} - \sin\theta(t)\sin\phi(t)\frac{\mathrm{d}\phi}{\mathrm{d}t}, \text{ etc.}
$$

and the length of the curve in $\mathbb{E}^3$ is

$$
\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^2 = \Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big)^2 + \Big(\frac{\mathrm{d}y}{\mathrm{d}t}\Big)^2 + \Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)^2 = a^2\bigg(\Big(\frac{\mathrm{d}\theta}{\mathrm{d}t}\Big)^2 + \sin^2\theta\Big(\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big)^2\bigg).
$$

The metric induced on M from $\mathbb{E}^3$ may thus be written

$$
\mathrm{d}s^2 = a^2\,\mathrm{d}\theta^2 + a^2\cos^2\theta\,\mathrm{d}\phi^2.
$$

Riemannian connection

A pseudo-Riemannian manifold (M, g) has a natural connection D defined on it that is subject to the following two requirements:

(i) D is torsion-free.
(ii) The covariant derivative of the metric tensor field vanishes, $Dg = 0$.

This connection is called the **Riemannian connection** defined by the metric tensor g. An interesting example of a physical theory that does not impose condition (i) is the *Einstein–Cartan theory* where torsion represents spin [9]. Condition (ii) has the following consequence. Let γ be a curve with tangent vector $X(t) = \dot{\gamma}(t)$, and let Y and Z be vector fields parallel transported along γ, so that $D_X Y = D_X Z = 0$. By Eq. (18.18) it follows that their inner product $g(Y, Z)$ is constant along the curve:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}g(Y,Z) = \frac{D}{\mathrm{d}t}g(Y,Z) &= D_X(g(Y,Z)) \\
&= (D_X g)(Y,Z) + g(D_X Y, Z) + g(Y, D_X Z) \\
&= Dg(Y, Z, X) = 0.
\end{aligned}
$$

In particular every vector field Y parallel transported along γ has constant magnitude $g(Y, Y)$ along the curve, a condition that is in fact necessary and sufficient for condition (ii) to hold.

Exercise: Prove the last statement.

Conditions (i) and (ii) define a unique connection, for let $(U; x^i)$ be any local coordinate chart, and $\Gamma^i_{jk} = \Gamma^i_{kj}$ the components of the connection with respect to this chart. Condition (ii) can be written using Eq. (18.14)

$$
g_{ij;k} = g_{ij,k} - \Gamma^m_{ik} g_{mj} - \Gamma^m_{jk} g_{im} = 0. \tag{18.37}
$$

Interchanging pairs of indices i, k and j, k results in

$$g_{kj;i} = g_{kj,i} - \Gamma^m_{ki} g_{mj} - \Gamma^m_{ji} g_{km} = 0, \tag{18.38}$$

$$g_{ik;j} = g_{ik,j} - \Gamma^m_{ij} g_{mk} - \Gamma^m_{kj} g_{im} = 0. \tag{18.39}$$

The combination (18.37) + (18.38) – (18.39) gives, on using the symmetry of g_{ij} and Γ^m_{ij},

$$g_{ij,k} + g_{kj,i} - g_{ik,j} = 2g_{mj}\Gamma^m_{ik}.$$

Multiply through by g^{jl} and, after a change of indices, we have

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{im}\big(g_{mj,k} + g_{mk,j} - g_{jk,m}\big). \tag{18.40}$$

These expressions are called **Christoffel symbols**; they are the explicit expression for the components of the Riemannian connection in any coordinate system.

Exercise: Show that

$$g^{ij}{}_{;k} = 0. \tag{18.41}$$

Let $\gamma : \mathbb{R} \to M$ be a geodesic with affine parameter t. As the tangent vector $X(t) = \dot\gamma$ is parallel propagated along the curve,

$$\frac{DX}{\mathrm{d}t} = D_X X = 0,$$

it has constant magnitude,

$$g(X, X) = g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = \text{const.}$$

A scaling transformation can be applied to the affine parameter such that

$$g(X, X) = g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = \pm 1 \text{ or } 0.$$

In Minkowskian manifolds, the latter case is called a **null geodesic**. If (M, g) is a Riemannian space and $p = \gamma(0)$ then $g(X, X) = 1$ and the affine parameter t is identical with the distance parameter along the geodesic.

Exercise: Show directly from the geodesic equation (18.9) and the Christoffel symbols (18.40) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\Big) = 0.$$

Example 18.2 In a pseudo-Riemannian manifold the geodesic equations may be derived from a variation principle (see Section 16.5). Geodesics can be thought of as curves of *stationary length*,

$$\delta s = \delta \int_{t_1}^{t_2} \sqrt{|g(\dot\gamma(t), \dot\gamma(t))|}\,\mathrm{d}t = 0.$$

Let $\gamma : [t_1, t_2] \times [-a, a] \to M$ be a variation of the given curve $\gamma : [t_1, t_2] \to M$, such that $\gamma(t, 0) = \gamma(t)$ and the end points of all members of the variation are fixed, $\gamma(t_1, \lambda) =$

$\gamma(t_1), \gamma(t_1, \lambda) = \gamma(t_1)$ for all $\lambda \in [-a, a]$. Set the Lagrangian $L : TM \to \mathbb{R}$ to be $L(X_p) = \sqrt{|g(X_p, X_p)|}$, and we follow the argument leading to the Euler–Lagrange equations (16.25):

$$
\begin{aligned}
0 = \delta s &= \int_{t_1}^{t_2} \delta L(\dot\gamma(t))\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}\lambda}\sqrt{|g(\dot\gamma(t,\lambda), \dot\gamma(t,\lambda))|}\Big|_{\lambda=0}\mathrm{d}t \\
&= \pm \int_{t_1}^{t_2} \frac{1}{2L}\big(\delta g_{ij}\dot x^i \dot x^j + 2g_{ij}\dot x^i \delta \dot x^j\big)\,\mathrm{d}t \\
&= \pm \int_{t_1}^{t_2} \Big\{\frac{1}{2L}\delta g_{ij,k}\dot x^i \dot x^j \delta x^k - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{g_{ik}\dot x^i}{L}\Big)\delta x^k\Big\}\mathrm{d}t + \Big[\frac{g_{ij}\dot x^i}{L}\delta \dot x^j\Big]_{t_1}^{t_2} \\
&= \pm \int_{t_1}^{t_2} \Big\{\frac{1}{2L}\delta g_{ij,k}\dot x^i \dot x^j - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{g_{ik}\dot x^i}{L}\Big)\Big\}\delta x^k \mathrm{d}t
\end{aligned}
$$

since $\delta x^k = 0$ at the end points $t = t_1$ and $t = t_2$. Since δx^k is arbitrary,

$$\frac{1}{2L}\delta g_{ij,k}\dot x^i \dot x^j - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{g_{ik}\dot x^i}{L}\Big) = 0$$

and expanding the second term on the left and multiplying the resulting equation by Lg^{km}, we find

$$\frac{\mathrm{d}^2 x^m}{\mathrm{d}t^2} + \Gamma^m_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\mathrm{d}x^m}{\mathrm{d}t}, \tag{18.42}$$

where Γ^m_{ij} are the Christoffel symbols given by Eq. (18.40). If we set t to be the distance parameter $t = s$, then $L = 1$ so that $\mathrm{d}L/\mathrm{d}s = 0$ and Eq. (18.42) reduces to the standard geodesic equation with affine parameter (18.9).

While we might think of this as telling us that geodesics are curves of 'shortest distance' connecting any pair of points, this is by no means true in general. More usually there is a critical point along any geodesic emanating from a given point, past which the geodesic is 'point of inflection' with respect to distance along neighbouring curves. In pseudo-Riemannian manifolds some geodesics may even be curves of 'longest length'. For timelike geodesics in Minkowski space this is essentially the time dilatation effect – a clock carried on an arbitrary path between two events will indicate less elapsed time than an inertial clock between the two events.

Geodesic coordinates

In cartesian coordinates for Euclidean space we have $g_{ij} = \delta_{ij}$ and by Eq. (18.40) all components of the Riemannian connection vanish, $\Gamma^i_{jk} = 0$. It therefore follows from Eq. (18.25) that all components of the curvature tensor R vanish. Conversely, if all components of the connection vanish in a coordinate chart $(U; x^i)$, we have $g_{ij,k} = 0$ by Eq. (18.37) and the metric tensor components g_{ij} are constant through the coordinate region U.

In Section 18.7 we will show that a necessary and suffic̄ent condition for $\Gamma^i_{jk} = 0$ in a coordinate chart $(V; y^i)$ is that the curvature tensor vanish throughout an open region of the manifold. However, as long as the torsion tensor vanishes, it is *always* possible to find coordinates such that $\Gamma^i_{jk}(p) = 0$ at any given point $p \in M$. For simplicity assume that p has coordinates $x^i(p) = 0$. We attempt a local coordinate transformation of the form

$$x^i = B^i_{i'} y^{i'} + A^i_{j'k'} y^{j'} y^{k'}$$

where $B = [B^i_{i'}]$ and $A^i_{j'k'} = A^i_{k'j'}$ are constant coefficients. Since

$$\frac{\partial x^i}{\partial y^{i'}}\bigg|_p = B^i_{i'}$$

the transformation is invertible in a neighbourhood of p only if $B = [B^i_{i'}]$ is a non-singular matrix. The new coordinates of p are again zero, $y^{i'}(p) = 0$, and using the transformation formula (18.11), we have

$$\Gamma'^{i'}_{j'k'}(p) = B^j_{j'} B^k_{k'} (B^{-1})^{i'}_i \Gamma^i_{jk}(p) + 2A^i_{j'k'}(B^{-1})^{i'}_i = 0$$

if we set

$$A^i_{j'k'} = -\tfrac{1}{2} B^j_{j'} B^k_{k'} \Gamma^i_{jk}(p).$$

Any such coordinates $(V; y^{j'})$ are called **geodesic coordinates**, or **normal coordinates**, at p. Their effect is to make geodesics appear locally 'straight' in a vanishingly small neighbourhood of p.

Exercise: Why does this procedure fail if the connection is not torsion free?

In the case of a pseudo-Riemannian manifold all derivatives of the metric tensor vanish in geodesic coordinates at p, $g_{ij,k}(p) = 0$. The constant coefficients $B^j_{j'}$ in the above may be chosen to send the metric tensor into standard diagonal form at p, such that $g'_{i'j'}(p) = g_{ij}(p) B^i_{i'} B^j_{j'} = \eta_{i'j'}$ has values ± 1 along the diagonal. Higher than first derivatives of g_{ij} will not in general vanish at p. For example, in normal coordinates at p the components of the curvature tensor can be expressed, using (18.40) and (18.25), in terms of the second derivatives $g_{ij,kl}(p)$:

$$\begin{aligned} R^i{}_{jkl}(p) &= \Gamma^i_{jl,k}(p) - \Gamma^i_{jk,l}(p) \\ &= \tfrac{1}{2}\eta^{im}\big(g_{ml,jk} + g_{jk,ml} - g_{mk,jl} - g_{jl,mk}\big)\big|_p. \end{aligned} \tag{18.43}$$

Problems

Problem 18.12 (a) Show that in a pseudo-Riemannian space the action principle

$$\delta \int_{t_1}^{t_2} L \, \mathrm{d}t = 0$$

where $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ gives rise to geodesic equations with affine parameter t.
(b) For the sphere of radius a in polar coordinates,

$$\mathrm{d}s^2 = a^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2),$$

use this variation principle to write out the equations of geodesics, and read off from them the Christoffel symbols $\Gamma^\mu_{\nu\rho}$.

(c) Verify by direct substitution in the geodesic equations that $L = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2$ is a constant along the geodesics and use this to show that the general solution of the geodesic equations is given by

$$b\cot\theta = -\cos(\phi - \phi_0) \quad \text{where} \quad b, \phi_0 = \text{const.}$$

(d) Show that these curves are great circles on the sphere.

Problem 18.13 Show directly from the tensor transformation laws of g_{ij} and g^{ij} that the Christoffel symbols

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{ia}(g_{aj,k} + g_{ak,j} - g_{jk,a})$$

transform as components of an affine connection.

18.5 Equation of geodesic deviation

We now give a geometrical interpretation of the curvature tensor, which will subsequently be used in the measurement of the gravitational field (see Section 18.8). Let $\gamma : I \times J \to M$, where $I = [t_1, t_2]$ and $J = [\lambda_1, \lambda_2]$ are closed intervals of the real line, be a one-parameter family of curves on M. We will assume that the restriction of the map γ to $I' \times J'$, where I' and J' are the open intervals (t_1, t_2) and (λ_1, λ_2) respectively, is an embedded two-dimensional submanifold of M. We think of each map $\gamma_\lambda : I \to M$ defined by $\gamma_\lambda(t) = \gamma(t, \lambda)$ as being the curve represented by $\lambda = \text{const.}$ and t as the parameter along the curve. The one-parameter family of curves γ will be said to be **from** p **to** q if

$$p = \gamma(t_1, \lambda), \qquad q = \gamma(t_2, \lambda)$$

for all $\lambda_1 \le \lambda \le \lambda_2$.

The tangent vectors to the curves of a one-parameter family constitute a vector field X on the two-dimensional submanifold $\gamma(I' \times J')$. If the curves are all covered by a single coordinate chart $(U; x^i)$, then

$$X = \frac{\partial \gamma^i(t,\lambda)}{\partial t}\frac{\partial}{\partial x^i} \quad \text{where} \quad \gamma^i(t,\lambda) = x^i\big(\gamma(t,\lambda)\big).$$

The **connection vector field** Y defined by

$$Y = \frac{\partial \gamma^i(t,\lambda)}{\partial \lambda}\frac{\partial}{\partial x^i}$$

is the tangent vector field to the curves connecting points having the same parameter value, $t = \text{const.}$ (see Fig. 18.1). The covariant derivative of the vector field Y along the curves γ_λ is given by

$$\begin{aligned}
\frac{DY^i}{\partial t} &\equiv \frac{\partial Y^i}{\partial t} + \Gamma^i_{jk}Y^jX^k \\
&= \frac{\partial^2\gamma^i}{\partial t\partial\lambda} + \Gamma^i_{jk}\frac{\partial\gamma^j}{\partial\lambda}\frac{\partial\gamma^k}{\partial t} \\
&= \frac{\partial^2\gamma^i}{\partial\lambda\partial t} + \Gamma^i_{jk}\frac{\partial\gamma^k}{\partial t}\frac{\partial\gamma^j}{\partial\lambda}.
\end{aligned}$$

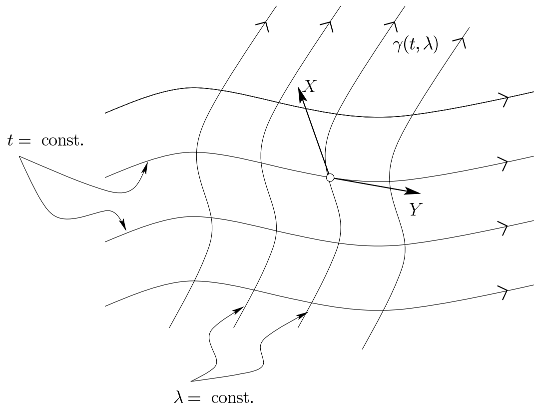

Figure 18.1 Tangent and connection vectors of a one-parameter family of geodesics

Hence

$$\frac{DY^i}{\partial t} = \frac{DX^i}{\partial \lambda}. \tag{18.44}$$

Alternatively, we can write

$$D_X Y = D_Y X. \tag{18.45}$$

If $A = A^i \partial_{x^i}$ is any vector field on U then

$$\begin{aligned}\Big(\frac{D}{\partial t}\frac{D}{\partial \lambda} - \frac{D}{\partial \lambda}\frac{D}{\partial t}\Big)A^i &= \frac{D}{\partial t}\big(A^i{}_{;j}Y^j\big) - \frac{D}{\partial \lambda}\big(A^i{}_{;j}X^j\big)\\ &= A^i{}_{;jk}Y^j X^k + A^i{}_{;j}\frac{DY^j}{\partial t} - A^i{}_{;jk}X^j Y^k - A^i{}_{;j}\frac{DX^j}{\partial \lambda}.\end{aligned}$$

From Eq. (18.44) and the Ricci identity (18.28)

$$\Big(\frac{D}{\partial t}\frac{D}{\partial \lambda} - \frac{D}{\partial \lambda}\frac{D}{\partial t}\Big)A^i = R^i{}_{ajk}A^a X^j Y^k. \tag{18.46}$$

Let M be a pseudo-Riemannian manifold and $\gamma(\lambda, t)$ a one-parameter family of geodesics, such that the geodesics $\lambda =$ const. all have t as an affine parameter,

$$D_X X = \frac{DX^i}{\partial t} = 0,$$

the parametrization chosen to have the same normalization on all geodesics, $g(X, X) = \pm 1$ or 0. It then follows that $g(X, Y)$ is constant along each geodesic, since

$$\begin{aligned}\frac{\partial}{\partial t}\big(g(X, Y)\big) = D_X\big(g(X, Y)\big) &= (D_X g)(X, Y) + g(D_X X, Y) + g(X, D_X Y)\\ &= g(X, D_Y X) \quad \text{by (18.45)}\\ &= \frac{1}{2}D_Y\big(g(X, X)\big) = \frac{1}{2}\frac{\partial}{\partial \lambda}g(X, X) = 0.\end{aligned}$$

Thus, if the tangent and connection vector are initially orthogonal on a geodesic of the one-parameter family, $g(X_p, Y_p) = 0$, then they are orthogonal all along the geodesic.

In Eq. (18.46) set $A^i = X^i$ – this is possible since it is only necessary to have A^i defined in terms of t and λ (see Problem 18.14). With the help of Eq. (18.44) we have

$$\frac{D}{\partial t}\frac{DY^i}{\partial t} = R^i{}_{ajk} X^a X^j Y^k, \tag{18.47}$$

known as the **equation of geodesic deviation**. For two geodesics, labelled by constants λ and $\lambda + \delta\lambda$, let δx^i be the tangent vector

$$\delta x^i \frac{\partial \gamma^i}{\partial \lambda}\delta\lambda = Y^i \delta\lambda.$$

For vanishingly small $\Delta\lambda$ it is usual to think of δx^i as an 'infinitesimal separation vector'. Since $\delta\lambda$ is constant along the geodesic we have

$$\delta\ddot{x}^i = R^i{}_{ajk} X^a X^j \delta x^k \tag{18.48}$$

where $\cdot \equiv D/\partial t$. Thus $R^i{}_{jkl}$ measures the relative 'acceleration' between geodesics.

Problem

Problem 18.14 Equation (18.46) has strictly only been proved for a vector field A. Show that it holds equally for a vector field whose components $A^i(t, \lambda)$ are only defined on the one-parameter family of curves γ.

18.6 The Riemann tensor and its symmetries

In a pseudo-Riemannian manifold (M, g) it is possible to lower the contravariant index of the curvature tensor to form a tensor $\overline{R}$ of type $(0, 4)$,

$$\overline{R}(W, Z, X, Y) = R(\omega, Z, X, Y) \quad \text{where} \quad g(W, A) = \langle \omega, A \rangle.$$

This tensor will be referred to as the **Riemann curvature tensor** or simply the **Riemann tensor**. Setting $W = \partial_{x^i}$, $Z = \partial_{x^j}$, etc. then $\omega = g_{ia}\,\mathrm{d}x^a$, whence

$$\overline{R}_{ijkl} = g_{ia} R^a{}_{jkl}.$$

In line with the standard index lowering convention, we denote the components of $\overline{R}$ by R_{ijkl}.

The following symmetries apply to the Riemann tensor:

$$R_{ijkl} = -R_{ijlk}, \tag{18.49}$$

$$R_{ijkl} = -R_{jikl}, \tag{18.50}$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0, \tag{18.51}$$

$$R_{ijkl} = R_{klij}. \tag{18.52}$$

Proof: Antisymmetry in the second pair of indices, Eq. (18.49), follows immediately from the definition of the curvature tensor (18.23) – it is not changed by the act of lowering the first index. Similarly, (18.51) follows immediately from Eq. (18.26). The remaining symmetries (18.50) and (18.52) may be proved by adopting geodesic coordinates at any given point p and using the expression (18.43) for the components of $R^i{}_{jkl}$:

$$R_{ijkl}(p) = \tfrac{1}{2}\big(g_{ml,jk} + g_{jk,ml} - g_{mk,jl} - g_{jl,mk}\big)\big|_p. \tag{18.53}$$

A more 'invariant' proof of (18.50) is to apply the generalized Ricci identities, Problem 18.9, to the metric tensor g,

$$0 = g_{ij;kl} - g_{ij;lk} = g_{aj} R^a{}_{ikl} + g_{ia} R^a{}_{jkl} = R_{jikl} + R_{ijkl}.$$

The symmetry (18.52) is actually a consequence of the first three symmetries, as may be shown by performing cyclic permutations on all four indices of Eq. (18.51):

$$R_{jkli} + R_{jlik} + R_{jikl} = 0, \tag{18.51a}$$

$$R_{klij} + R_{kijl} + R_{kjli} = 0, \tag{18.51b}$$

$$R_{lijk} + R_{ljki} + R_{lkij} = 0. \tag{18.51c}$$

The combination (18.51) – (18.51a) – (18.51b) + (18.51c) gives, after several cancellations using the symmetries (18.49) and (18.50),

$$2R_{ijkl} - 2R_{klij} = 0.$$

This is obviously equivalent to Eq. (18.52). ■

Exercise: Prove from these symmetries that the cyclic symmetry also holds for any three indices; for example

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$

These symmetries permit us to count the number of independent components of the Riemann tensor. Since a skew symmetric tensor of type (0, 2) on an n-dimensional vector space has $\frac{1}{2}n(n-1)$ independent components, a tensor of type (0, 4) subject to symmetries (18.49) and (18.50) will have $\frac{1}{4}n^2(n-1)^2$ independent components. For fixed $i = 1, 2, \dots, n$ only unequal triples $j \neq k \neq l$ need be considered in the symmetry (18.52), for if some pair are equal nothing new is added, by the cyclic identity: for example, if $k = l = 1$ then $R_{ij11} + R_{i11j} + R_{i1j1} = 0$ merely reiterates the skew symmetry on the second pair of indices, $R_{i11j} = -R_{i1j1}$. For every triple $j \neq k \neq l$ the total number of relations generated by (18.52) that are independent of (18.49) and (18.50) is therefore the number of such triples of numbers $j \neq k \neq l$ in the range $1, \dots, n$, namely $n\binom{n}{3} = n^2(n-1)(n-2)/6$. By the above proof we need not consider the symmetry (18.52), and the total number of independent components of the Riemann tensor is

$$N = \frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6}$$
$$= \frac{n^2(n^2-1)}{12}. \tag{18.54}$$

For low dimensions the number of independent components of the Riemann tensor is

$$\begin{aligned}
n &= 1: \quad N = 0,\\
n &= 2: \quad N = 1,\\
n &= 3: \quad N = 6,\\
n &= 4: \quad N = 20.
\end{aligned}$$

A tensor of great interest in general relativity is the **Ricci tensor**, defined by

$$\mathrm{Ric} = C^1_2 R.$$

It is common to write the components of $\mathrm{Ric}(\partial_{x^i}, \partial_{x^j})$ in any chart $(U; x^i)$ as R_{ij}:

$$R_{ij} = R^a{}_{iaj} = g^{ab} R_{aibj}. \tag{18.55}$$

This tensor is symmetric since, by symmetry (18.52),

$$R_{ij} = g^{ab} R_{aibj} = g^{ab} R_{bjai} = R^a{}_{jai} = R_{ji}.$$

Contracting again gives the quantity known as the **Ricci scalar**,

$$R = R^i{}_i = g^{ij} R_{ij}. \tag{18.56}$$

Bianchi identities

For a torsion-free connection we have, on setting $T^p_{km} = 0$ in Eq. (18.32) of Problem 18.10, the **second Bianchi identity**

$$R^i{}_{jkl;m} + R^i{}_{jlm;k} + R^i{}_{jmk;l} = 0. \tag{18.57}$$

These are often referred to simply as the **Bianchi identities**. An alternative demonstration is to use normal coordinates at any point $p \in M$, such that $\Gamma^i{}_{jk}(p) = 0$. Making use of Eqs. (18.14) and (18.25) we have

$$\begin{aligned}
R^i{}_{jkl;m}(p) &= R^i{}_{jkl,m}(p) + \big(\Gamma^i_{am} R^a{}_{jkl} - \Gamma^a_{jm} R^i{}_{akl} - \Gamma^a_{km} R^i{}_{jal} - \Gamma^a_{lm} R^i{}_{jka}\big)(p)\\
&= R^i{}_{jkl,m}(p)\\
&= \Gamma^i_{jl,km}(p) - \Gamma^i_{jk,lm}(p) + \Gamma^a_{jl,m}\Gamma^i_{ak}(p) + \Gamma^a_{jl}(p)\Gamma^i_{ak,m} - \Gamma^a_{jk,m}\Gamma^i_{al}(p)\\
&\quad - \Gamma^a_{jk}(p)\Gamma^i_{al,m}\\
&= \Gamma^i_{jl,km}(p) - \Gamma^i_{jk,lm}(p).
\end{aligned}$$

If we substitute this expression in the left-hand side of (18.57) and use $\Gamma^i_{jk} = \Gamma^i_{kj}$, all terms cancel out.

Contracting Eq. (18.57) over i and m gives

$$R^i{}_{jkl;i} - R_{jl;k} + R_{jk;l} = 0. \tag{18.58}$$

Contracting Eq. (18.58) again by multiplying through by g^{jl} and using Eq. (18.41) we find

$$R^i{}_{k;i} - R_{;k} + R^j{}_{k;j} = 0,$$

or equivalently, the **contracted Bianchi identities**

$$R^j_{k;j} - \tfrac{1}{2}R_{,k} = 0. \tag{18.59}$$

A useful way of writing (18.59) is

$$G^j{}_{k;j} = 0, \tag{18.60}$$

where $G^i{}_j$ is the **Einstein tensor**,

$$G^i{}_j = R^i{}_j - \tfrac{1}{2}R\delta^i_j. \tag{18.61}$$

This tensor is symmetric when its indices are lowered,

$$G_{ij} = g_{ia}G^a{}_j = R_{ij} - \tfrac{1}{2}Rg_{ij} = G_{ji}. \tag{18.62}$$

18.7 Cartan formalism

Cartan's approach to curvature is expressed entirely in terms of differential forms. Let e_i $(i = 1, \dots, n)$ be a local basis of vector fields, spanning $\mathcal{T}(U)$ over an open set $U \subseteq M$ and $\{\varepsilon^i\}$ the dual basis of $\mathcal{T}^*(U)$, such that $\langle e_i, \varepsilon^j\rangle = \delta^j_i$. For example, in a coordinate chart $(U; x^i)$ we may set $e_i = \partial_{x^i}$ and $\varepsilon^j = \mathrm{d}x^j$, but such a coordinate system will exist for an arbitrary basis $\{e_i\}$ if and only if $[e_i, e_j] = 0$ for all $i, j = 1, \dots, n$. We define the **connection 1-forms** $\omega^i{}_j : \mathcal{T}(U) \to \mathcal{F}(U)$ by

$$D_X e_j = \omega^i{}_j(X)e_i. \tag{18.63}$$

These maps are differential 1-forms on U since they are clearly $\mathcal{F}$-linear,

$$D_{X+fY}e_j = D_X e_j + f D_Y e_j \implies \omega^i{}_j(X + fY) = \omega^i{}_j(X) + f\omega^i{}_j(Y).$$

If $\tau : \mathcal{T}(U) \times \mathcal{T}(U) \to \mathcal{T}(U)$ is the torsion operator defined in Eq. (18.19), set

$$\tau(X, Y) = \tau^i(X, Y)e_i.$$

The maps $\tau^i : \mathcal{T}(U) \times \mathcal{T}(U) \to \mathcal{F}(U)$ are $\mathcal{F}$-linear in both arguments by the $\mathcal{F}$-linearity of τ, and are antisymmetric $\tau^i(X, Y) = -\tau^i(Y, X)$. They are therefore differential 2-forms on U, known as the **torsion 2-forms**.

Exercise: Show that $\tau^i = T^i_{jk}\varepsilon^j \wedge \varepsilon^k$ where $T^i_{jk} = \langle \tau(e_j, e_k), \varepsilon^i\rangle$.

From the identity $Z = \langle \varepsilon^i, Z\rangle e_i$ for any vector field Z on U, we have

$$\begin{aligned}
\tau(X, Y) &= D_X(\langle \varepsilon^i, Y\rangle e_i) - D_Y(\langle \varepsilon^i, X\rangle e_i) - \langle \varepsilon^i, [X, Y]\rangle e_i \\
&= X(\langle \varepsilon^i, Y\rangle)e_i + \langle \varepsilon^i, Y\rangle \omega^k_i(X)e_k - Y(\langle \varepsilon^i, X\rangle)e_i - \langle \varepsilon^i, X\rangle \omega^k_i(X)e_k \\
&\quad - \langle \varepsilon^i, [X, Y]\rangle e_i \\
&= 2\,\mathrm{d}\varepsilon^i(X, Y)e_i + 2\omega^k_i \wedge \varepsilon^i(X, Y)e_k,
\end{aligned}$$

using the Cartan identity (16.14). Thus

$$\tau^i(X, Y)e_i = 2\big(\mathrm{d}\varepsilon^i(X, Y) + 2\omega^i{}_k \wedge \varepsilon^k(X, Y)\big)e_i,$$

and by $\mathcal{F}$-linear independence of the vector fields e_i we have **Cartan's first structural equation**

$$\mathrm{d}\varepsilon^i = -\omega^i{}_k \wedge \varepsilon^k + \tfrac{1}{2}\tau^i \,. \tag{18.64}$$

Define the **curvature 2-forms** $\rho^i{}_j$ by

$$\rho_{X,Y} e_j = \rho^i{}_j(X, Y) e_i, \tag{18.65}$$

where $\rho_{X,Y} : \mathcal{T}(M) \to \mathcal{T}(M)$ is the curvature operator in Eq. (18.22).

Exercise: Show that the $\rho^i{}_j$ are differential 2-forms on U; namely, they are $\mathcal{F}$-linear with respect to X and Y and $\rho^i{}_j(X, Y) = -\rho^i{}_j(Y, X)$.

Changing the dummy suffix on the right-hand side of Eq. (18.65) from i to k and applying $\langle \varepsilon^i, .\rangle$ to both sides of the equation we have, with the help of Eq. (18.23),

$$\begin{aligned}
\rho^i{}_j(X, Y) &= \langle \varepsilon^i, \rho_{X,Y} e_j \rangle \\
&= R(\varepsilon^i, e_j, X, Y) \\
&= R^i{}_{jkl} X^k Y^l \quad \text{where } R^i{}_{jkl} = R(\varepsilon^i, e_j, e_k, e_l) \\
&= \tfrac{1}{2} R^i{}_{jkl} (\varepsilon^k \otimes \varepsilon^l - \varepsilon^l \otimes \varepsilon^k)(X, Y).
\end{aligned}$$

Hence

$$\rho^i{}_j = R^i{}_{jkl} \varepsilon^k \wedge \varepsilon^l. \tag{18.66}$$

A similar analysis to that for the torsion operator results in

$$\begin{aligned}
\rho_{X,Y} e_j &= D_X D_Y e_j - D_Y D_X e_j - D_{[X,Y]} e_j \\
&= D_X\big(\langle \omega^i{}_j, Y\rangle e_i\big) - D_Y\big(\langle \omega^i{}_j, X\rangle e_i\big) - \langle \omega^i{}_j, [X, Y]\rangle e_i \\
&= 2\big[\mathrm{d}\omega^i{}_j(X, Y) + \omega^i{}_k \wedge \omega^k{}_j(X, Y)\big] e_i,
\end{aligned}$$

and **Cartan's second structural equation**

$$\mathrm{d}\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j + \tfrac{1}{2}\rho^i{}_j. \tag{18.67}$$

Example 18.3 With respect to a coordinate basis $e_i = \partial_{x^i}$, the Cartan structural equations reduce to formulae found earlier in this chapter. For any vector field $X = X^k \partial_{x^k}$,

$$D_X \partial_{x^j} = X^k D_k \partial_{x^j} = X^k \Gamma^i_{jk} \partial_{x^i}.$$

Hence, by Eq. (18.63), we have $\omega^i{}_j(X) = X^k \Gamma^i{}_{jk}$, so that

$$\omega^i{}_j = \Gamma^i_{jk}\, \mathrm{d}x^k. \tag{18.68}$$

Thus the components of the connection 1-forms with respect to a coordinate basis are precisely the components of the connection.

Setting $\varepsilon^i = \mathrm{d}x^i$ in Cartan's first structural equation (18.64), we have

$$\mathrm{d}\varepsilon^i = \mathrm{d}^2 x^i = 0 = -\omega^i{}_j \wedge \mathrm{d}x^j + \tfrac{1}{2}\tau^i.$$

Hence $\tau^i = 2\Gamma^i_{jk}\,\mathrm{d}x^k \wedge \mathrm{d}x^j$, and it follows that the components of the torsion 2-forms are identical with those of the torsion tensor T in Eq. (18.21),

$$\tau^i = T^i_{jk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k \quad \text{where} \quad T^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}. \tag{18.69}$$

Finally, Cartan's second structural equation (18.67) reduces in a coordinate basis to

$$\mathrm{d}\Gamma^i_{jk} \wedge \mathrm{d}x^k = -\Gamma^i_{kl}\,\mathrm{d}x^l \wedge \Gamma^k_{jm}\,\mathrm{d}x^m + \tfrac{1}{2}\rho^i{}_j,$$

whence, on using the decomposition (18.66),

$$R^i{}_{jkl}\,\mathrm{d}x^k \wedge \mathrm{d}x^l = 2\Gamma^i_{jk,l}\,\mathrm{d}x^l \wedge \mathrm{d}x^k + 2\Gamma^i_{ml}\Gamma^m_{jk}\,\mathrm{d}x^l \wedge \mathrm{d}x^k.$$

We thus find, in agreement with Eq. (18.25),

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml} = -R^i{}_{jlk}.$$

The big advantage of Cartan's structural equations over these various coordinate expressions is that they give expressions for torsion and curvature for arbitrary vector field bases.

Bianchi identities

Taking the exterior derivative of (18.64) gives, with the help of (18.67),

$$\begin{aligned}
\mathrm{d}^2\varepsilon^i = 0 &= -\mathrm{d}\omega^i{}_k \wedge \varepsilon^k + \omega^i{}_k \wedge \mathrm{d}\varepsilon^k + \tfrac{1}{2}\mathrm{d}\tau^i \\
&= \omega^i{}_j \wedge \omega^j{}_k \wedge \varepsilon^k - \tfrac{1}{2}\rho^i{}_k \wedge \varepsilon^k - \omega^i{}_k \wedge \omega^k{}_j \wedge \varepsilon^j + \tfrac{1}{2}\omega^i{}_k \wedge \tau^k + \tfrac{1}{2}\mathrm{d}\tau^i \\
&= \tfrac{1}{2}\big(-\rho^i{}_k \wedge \varepsilon^k + \omega^i{}_k \wedge \tau^k + \mathrm{d}\tau^i\big).
\end{aligned}$$

Hence, we obtain the *first Bianchi identity*

$$\mathrm{d}\tau^i = \rho^i{}_k \wedge \varepsilon^k - \omega^i{}_k \wedge \tau^k. \tag{18.70}$$

Its relation to the earlier identity (18.24) is left as an exercise (see Problem 18.15).

Similarly

$$\begin{aligned}
\mathrm{d}^2\omega^i{}_j = 0 &= -\mathrm{d}\omega^i{}_k \wedge \omega^k{}_j + \omega^i{}_k \wedge \mathrm{d}\omega^k{}_j + \tfrac{1}{2}\mathrm{d}\rho^i{}_j \\
&= \tfrac{1}{2}\big(-\rho^i{}_k \wedge \omega^k{}_j + \omega^i{}_k \wedge \rho^k{}_j + \mathrm{d}\rho^i{}_j\big),
\end{aligned}$$

resulting in the *second Bianchi identity*

$$\mathrm{d}\rho^i{}_j = \rho^i{}_k \wedge \omega^k{}_j - \omega^i{}_k \wedge \rho^k{}_j. \tag{18.71}$$

Pseudo-Riemannian spaces in Cartan formalism

In a pseudo-Riemannian manifold with metric tensor g, set $g_{ij} = g(e_i, e_j)$. Since g_{ij} is a scalar field for each $i, j = 1, \dots, n$ and $D_X g = 0$, we have

$$\begin{aligned}
\langle X, \mathrm{d}g_{ij}\rangle = X(g_i j) &= D_X\big(g(e_i, e_j)\big) \\
&= g(D_X e_i, e_j) + g(e_i, D_X e_j) \\
&= g\big(\omega^k_i(X)e_k, ej\big) + g\big(e_i, \omega^k_j(X)e_k\big) \\
&= g_{kj}\omega^k_i(X) + g_{ik}\omega^k_j(X) \\
&= \langle X, \omega_{ji}\rangle + \langle X, \omega_{ij}\rangle
\end{aligned}$$

where

$$\omega_{ij} = g_{ki}\omega^k{}_j.$$

As X is an arbitrary vector field,

$$\mathrm{d}g_{ij} = \omega_{ij} + \omega_{ji}. \tag{18.72}$$

For an orthonormal basis e_i, such that $g_{ij} = \eta_{ij}$, we have $\mathrm{d}g_{ij} = 0$ and

$$\omega_{ij} = -\omega_{ji}. \tag{18.73}$$

In particular, all diagonals vanish, $g_{ii} = 0$ for $i = 1, \dots, n$.

Lowering the first index on the curvature 2-forms $\rho_{ij} = g_{ik}\rho^k{}_j$, we have from the second Cartan structural equation (18.67),

$$\begin{aligned}\rho_{ij} &= 2\big(\mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega^k{}_j\big)\\ &= 2\big(-\mathrm{d}\omega_{ji} + \omega^k{}_j \wedge \omega_{ki}\big)\\ &= 2\big(-\mathrm{d}\omega_{ji} - \omega_{jk} \wedge \omega^k{}_i\big),\end{aligned}$$

whence

$$\rho_{ij} = -\rho_{ji}. \tag{18.74}$$

Exercise: Show that (18.74) is equivalent to the symmetry $R_{ijkl} = -R_{jikl}$.

Example 18.4 The 3-sphere of radius a is the submanifold of $\mathbb{R}^4$,

$$S^3(a) = \{(x, y, z, w) \,|\, x^2 + y^2 + z^2 + w^2 = a^2\} \subset \mathbb{R}^4.$$

Spherical polar coordinates $\chi,\ \theta,\ \phi$ are defined by

$$\begin{aligned}x &= a \sin\chi \sin\theta \cos\phi\\ y &= a \sin\chi \sin\theta \sin\phi\\ z &= a \sin\chi \cos\theta\\ w &= a \cos\chi\end{aligned}$$

where $0 < \chi, \theta < \pi$ and $0 < \phi < 2\pi$. These coordinates cover all of $S^3(a)$ apart from the points $y = 0,\ x \geq 0$. The Euclidean metric on $\mathbb{R}^4$

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2$$

induces a metric on $S^3(a)$ as for the 2-sphere in Example 18.1:

$$\mathrm{d}s^2 = a^2\big[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\big].$$

An orthonormal frame is

$$\begin{aligned}e_1 &= \frac{1}{a}\frac{\partial}{\partial\chi}, & e_2 &= \frac{1}{a\sin\chi}\frac{\partial}{\partial\theta}, & e_3 &= \frac{1}{a\sin\chi\sin\theta}\frac{\partial}{\partial\phi}\\ \varepsilon^1 &= a\,\mathrm{d}\chi, & \varepsilon^2 &= a\sin\chi\,\mathrm{d}\theta, & \varepsilon^3 &= a\sin\chi\sin\theta\,\mathrm{d}\phi\end{aligned}$$

where

$$g_{ij} = g(e_i, e_j) = \delta_{ij}, \qquad g = \varepsilon^1 \otimes \varepsilon^1 + \varepsilon^2 \otimes \varepsilon^2 + \varepsilon^3 \otimes \varepsilon^3.$$

Since the metric connection is torsion-free, $\tau^i = 0$, the first structural equation reads

$$\mathrm{d}\varepsilon^i = -\omega^i_{\ k} \wedge \varepsilon^k = -\omega_{ik} \wedge \varepsilon^k$$

setting

$$\omega_{ij} = A_{ijk}\varepsilon^k$$

where $A_{ijk} = -A_{jik}$. By interchanging dummy suffixes j and k we may also write

$$\mathrm{d}\varepsilon^i = -A_{ikj}\varepsilon^j \wedge \varepsilon^k = A_{ikj}\varepsilon^k \wedge \varepsilon^j = A_{ijk}\varepsilon^j \wedge \varepsilon^k.$$

For $i = 1$, using $A_{11k} = 0$,

$$\mathrm{d}\varepsilon^1 = a\mathrm{d}^2\chi = 0 = A_{121}\varepsilon^2 \wedge \varepsilon^1 + A_{131}\varepsilon^3 \wedge \varepsilon^1 + (A_{123} - A_{132})\varepsilon^2 \wedge \varepsilon^3,$$

whence

$$A_{121} = A_{131} = 0, \qquad A_{123} = A_{132}.$$

For $i = 2$,

$$\begin{aligned}\mathrm{d}\varepsilon^2 &= a \cos\chi \,\mathrm{d}\chi \wedge \mathrm{d}\theta = a^{-1} \cot\chi\, \varepsilon^1 \wedge \varepsilon^2 \\ &= A_{212}\varepsilon^1 \wedge \varepsilon^2 + A_{232}\varepsilon^3 \wedge \varepsilon^2 + (A_{213} - A_{231})\varepsilon^1 \wedge \varepsilon^3,\end{aligned}$$

which implies

$$A_{212} = a^{-1}\cot\chi, \quad A_{213} = A_{231}, \quad A_{232} = 0.$$

Similarly the $i = 3$ equation gives

$$A_{312} = A_{321}, \qquad A_{313} = a^{-1}\cot\chi, \qquad A_{323} = a^{-1}\frac{\cot\theta}{\sin\chi}.$$

All coefficients having all three indices different, such as A_{123}, must vanish since

$$A_{123} = A_{132} = -A_{312} = -A_{321} = A_{231} = A_{213} = -A_{123} \implies A_{123} = 0.$$

There is enough information now to write out the connection 1-forms:

$$\begin{aligned}\omega_{12} &= -\omega_{21} = -a^{-1}\cot\chi\,\varepsilon^2, \\ \omega_{13} &= -\omega_{31} = -a^{-1}\cot\chi\,\varepsilon^3, \\ \omega_{23} &= -\omega_{32} = -a^{-1}\frac{\cot\theta}{\sin\chi}\varepsilon^3.\end{aligned}$$

The second structural relations (18.67) can now be used to calculate the curvature 2-forms:

$$\begin{aligned}\rho_{12} &= 2\big(\mathrm{d}\omega_{12} + \omega_{1k} \wedge \omega^k_{\ 2}\big) \\ &= 2\big(a^{-1}\mathrm{cosec}^2\,\chi\,\mathrm{d}\chi \wedge \varepsilon^2 - a^{-1}\cot\chi\,\mathrm{d}\varepsilon^2 + \omega_{13} \wedge \omega_{32}\big) \\ &= 2a^{-2}\big(\mathrm{cosec}^2\,\chi\,\varepsilon^1 \wedge \varepsilon^2 - \cot^2\chi\,\varepsilon^1 \wedge \varepsilon^2\big) \\ &= 2a^{-2}\varepsilon^1 \wedge \varepsilon^2,\end{aligned}$$

and similarly

$$\rho_{13} = 2\big(\mathrm{d}\omega_{13} + \omega_{12} \wedge \omega_{23}\big) = 2a^{-2}\varepsilon^1 \wedge \varepsilon^3,$$
$$\rho_{23} = 2\big(\mathrm{d}\omega_{23} + \omega_{21} \wedge \omega_{13}\big) = 2a^{-2}\varepsilon^2 \wedge \varepsilon^3.$$

The components of the Riemann curvature tensor can be read off using Eq. (18.66):

$$R_{1212} = R_{1313} = R_{2323} = a^{-2},$$

and all other components of the Riemann tensor are simply related to these components by symmetries; for example, $R_{2121} = -R_{1221} = a^{-2}$, $R_{1223} = 0$, etc. It is straightforward to verify the relation

$$R_{ijkl} = \frac{1}{a^2}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

For any Riemannian space (M, g) the **sectional curvature** of the vector 2-space spanned by a pair of tangent vectors X and Y at any point p is defined to be

$$K(X, Y) = \frac{R(X, Y, X, Y)}{A(X, Y)}$$

where $A(X, Y)$ is the 'area' of the parallelogram spanned by X and Y,

$$A(X, Y) = g(X, Y)g(X, Y) - g(X, X)g(Y, Y).$$

For the 3-sphere

$$K(X, Y) = \frac{1}{a^2}\frac{X_iY^iX_kY^k - X_iX^iY_kY^k}{X_iY^iX_kY^k - X_iX^iY_kY^k} = \frac{1}{a^2}$$

independent of the point $p \in S^3(a)$ and the choice of tangent vectors X, Y. For this reason the 3-sphere is said to be a **space of constant curvature**.

Locally flat spaces

A manifold M with affine connection is said to be **locally flat** if for every point $p \in M$ there is a chart $(U; x^i)$ such that all components of the connection vanish throughout U. This implies of course that both torsion tensor and curvature tensor vanish throughout U, but more interesting is that these conditions are both necessary and sufficient. This result is most easily proved in Cartan's formalism, and requires the transformation of the connection 1-forms under a change of basis

$$\varepsilon'^i = A^i{}_j\varepsilon^j \quad \text{where} \quad A^i{}_j = A^i{}_j(x^1, \dots, x^n).$$

Evaluating $\mathrm{d}\varepsilon'^i$, using Eq. (18.64), gives

$$\mathrm{d}\varepsilon'^i = \mathrm{d}A^i{}_j \wedge \varepsilon^j + A^i{}_j\,\mathrm{d}\varepsilon^j = -\omega'^i{}_k \wedge \varepsilon'^k + \tfrac{1}{2}\tau'^i$$

where $\tau'^i = A^i{}_j\tau^j$ and

$$A^k{}_j\omega'^i{}_k = \mathrm{d}A^i{}_j - A^i{}_k\omega^k{}_j. \tag{18.75}$$

Exercise: Show from this equation and Eq. (18.67) that if a transformation exists such that $\omega'^i{}_k = 0$ then the curvature 2-forms vanish, $\rho^k{}_j = 0$.

Theorem 18.1 *A manifold with symmetric connection is locally flat everywhere if and only if the curvature 2-forms vanish, $\rho^k{}_j = 0$.*

Proof: The *only if* part follows from the above comments. For the converse, we suppose $\tau^i = \rho^i{}_j = 0$ everywhere. If $(U; x^i)$ is any chart on M, let $N = U \times \mathbb{R}^{n^2}$ and denote coordinates on $\mathbb{R}^{n^2}$ by $z^j{}_k$. Using the second structural formula (18.67) with $\rho^i{}_j = 0$, the 1-forms $\alpha^i{}_j = \mathrm{d}z^i{}_j - z^i{}_k \omega^k{}_j$ on N satisfy

$$\begin{aligned}
\mathrm{d}\alpha^i{}_j &= -\mathrm{d}z^i{}_k \wedge \omega^k{}_j - z^i{}_k\, \mathrm{d}\omega^k{}_j \\
&= -\big(\alpha^i{}_k + z^i{}_m\, \mathrm{d}\omega^m{}_k\big) \wedge \omega^k{}_j + z^i{}_k \omega^k{}_m \wedge \omega^m{}_j \\
&= \omega^k{}_j \wedge \alpha^i{}_k.
\end{aligned} \tag{18.76}$$

By the Frobenius theorem 16.4, $\mathrm{d}\alpha^i{}_j = 0$ is an integrable system on N, and has a local integral submanifold through any point x^i_0, A^j_{0k} where $\det[A^j_{0k}] \neq 0$, that may be assumed to be of the form

$$z^j_i = A^j_i(x^1, \dots, x^n) \quad \text{where} \quad A^j_i(x^1_0, \dots, x^n_0) = A^j_{0i}.$$

We may assume that $\det[A^j_k]$ is non-singular in a neighbourhood of $\mathbf{x}_0$. Hence

$$\alpha^i{}_j = 0 \implies \mathrm{d}A^i{}_j - A^i{}_k \omega^k{}_j = 0$$

and substituting in Eq. (18.75) results in

$$\omega'^i{}_k = 0 \quad \text{if} \quad \varepsilon'^i = A^i{}_j \varepsilon^j.$$

Finally, the structural equation (18.64) gives $\mathrm{d}\varepsilon'^i = 0$, and the Poincaré lemma 17.5 implies that there exist local coordinates y^i such that $\varepsilon'^i = \mathrm{d}y^i$. ■

In the case of a pseudo-Riemannian space, locally flat coordinates such that $\Gamma^i_{jk} = 0$ imply that $g_{ij,k} = 0$ by Eq. (18.37). Hence $g_{ij} = \text{const.}$ throughout the coordinate region, and a linear transformation can be used to diagonalize the metric into standard diagonal form $g_{ij} = \eta_{ij}$ with ± 1 along the diagonal.

Problems

Problem 18.15 Let $e_i = \partial_{x^i}$ be a coordinate basis.

(a) Show that the first Bianchi identity reads

$$R^i{}_{[jkl]} = T^i_{[jk;l]} - T^a_{[jk} T^i_{l]a},$$

and reduces to the cyclic identity (18.26) in the case of a torsion-free connection.

(b) Show that the second Bianchi identity becomes

$$R^i{}_{j[kl;m]} = R^i{}_{ja[k} T^a_{ml]},$$

which is identical with Eq. (18.32) of Problem 18.10.

Problem 18.16 In a Riemannian manifold (M, g) show that the sectional curvature $K(X, Y)$ at a point p, defined in Example 18.4, is independent of the choice of basis of the 2-space; i.e., $K(X', Y') = K(X, Y)$ if $X' = aX + bY$, $Y' = cX + \mathrm{d}Y$ where $ad - bc \neq 0$.

The space is said to be *isotropic* at $p \in M$ if $K(X, Y)$ is independent of the choice of tangent vectors X and Y at p. If the space is isotropic at each point p show that

$$R_{ijkl} = f(g_{ik}g_{jl} - g_{il}g_{jk})$$

where f is a scalar field on M. If the dimension of the manifold is greater than 2, show *Schur's theorem*: a Riemannian manifold that is everywhere isotropic is a space of constant curvature, $f = \text{const.}$ [*Hint*: Use the contracted Bianchi identity (18.59).]

Problem 18.17 Show that a space is locally flat if and only if there exists a local basis of vector fields $\{e_i\}$ that are *absolutely parallel*, $De_i = 0$.

Problem 18.18 Let (M, φ) be a *surface of revolution* defined as a submanifold of $\mathbb{E}^3$ of the form

$$x = g(u)\cos\theta, \qquad y = g(u)\sin\theta, \qquad z = h(u).$$

Show that the induced metric (see Example 18.1) is

$$\mathrm{d}s^2 = \big(g'(u)^2 + h'(u)^2\big)\,\mathrm{d}u^2 + g^2(u)\,\mathrm{d}\theta^2.$$

Picking the parameter u such that $g'(u)^2 + h'(u)^2 = 1$ (interpret this choice!), and setting the basis 1-forms to be $\varepsilon^1 = \mathrm{d}u$, $\varepsilon^2 = g\,\mathrm{d}\theta$, calculate the connection 1-forms $\omega^i{}_j$, the curvature 1-forms $\rho^i{}_j$, and the curvature tensor component R_{1212}.

Problem 18.19 For the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

show that the sectional curvature is given by

$$K = \Big(\frac{x^2bc}{a^3} + \frac{y^2ac}{b^3} + \frac{z^2ab}{c^3}\Big)^{-2}.$$

18.8 General relativity

The principle of equivalence

The Newtonian gravitational force on a particle of mass m is $\mathbf{F} = -m\nabla\phi$ where the scalar potential ϕ satisfies *Poisson's equation*

$$\nabla^2\phi = 4\pi G\rho. \tag{18.77}$$

Here $G = 6.672 \times 10^{-8}\ \mathrm{g}^{-1}\,\mathrm{cm}^3\,\mathrm{s}^{-2}$ is Newton's gravitational constant and ρ is the density of matter present. While it is in principle possible to generalize this theory by postulating a relativistically invariant equation such as

$$\Box\phi = \nabla^2\phi - \frac{\partial^2}{\partial t^2}\phi = 4\pi G\rho,$$

there are a number of problems with this theory, not least of which is that it does not accord with observations.

Key to the formulation of a correct theory is the **principle of equivalence** that, in its simplest version, states that all particles fall with equal acceleration in a gravitational field, a fact first attributed to Galileo in an improbable tale concerning the leaning tower of Pisa. While the derivation is by simple cancellation of m from both sides of the Newtonian equation of motion

$$m\ddot{x} = mg,$$

the masses appearing on the two sides of the equation could conceivably be different. On the left we really have *inertial mass*, m_i, which measures the particle's resistance to *any* force, while on the right the mass should be identified as *gravitational mass*, measuring the particle's response to gravitational fields – its 'gravitational charge' so to speak. The principle of equivalence can be expressed as saying that the ratio of inertial to gravitational mass will be the same for all bodies irrespective of the material from which they are composed. This was tested to one part in 10^8 for a vast variety of materials in 1890 by Eötvös using a torsion balance, and repeated in the 1960s to one part in 10^{11} by Dicke using a solar balancing mechanism (see C. M. Will's article in [10]).

The principle of equivalence essentially says that it is impossible to distinguish inertial forces such as centrifugal or Coriolis forces from gravitational ones. A nice example of the equivalence of such forces is the *Einstein elevator*. An observer in an elevator at rest sees objects fall to the ground with acceleration g. However if the elevator is set in free fall, objects around the observer will no longer appear to be subject to forces, much as if he were in an inertial frame in outer space. It has in effect been possible to *transform away* the gravitational field by going to a freely falling laboratory. Conversely as all bodies 'fall' to the floor in an accelerated rocket with the same acceleration, an observer will experience an 'apparent' gravitational field.

The effect of a non-inertial frame in special relativity should be essentially indistinguishable from the effects of gravity. The metric interval of Minkowski space (see Chapter 9) in a general coordinate system $x'^{\mu'} = x'^{\mu'}(x^1, x^2, x^3, x^4)$, where x^ν are inertial coordinates, becomes

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = g'_{\mu'\nu'}\,\mathrm{d}x'^{\mu'}\mathrm{d}x'^{\nu'},$$

where

$$g'_{\mu'\nu'} = g_{\mu\nu}\frac{\partial x^\mu}{\partial x'^{\mu'}}\frac{\partial x^\nu}{\partial x'^{\nu'}}.$$

Expressed in general coordinates, the (geodesic) equations of motion of an inertial particle, $\mathrm{d}^2x^\mu/\mathrm{d}s^2 = 0$, are

$$\frac{\mathrm{d}^2x'^{\mu'}}{\mathrm{d}s^2} + \Gamma'^{\mu'}_{\alpha'\beta'}\frac{\mathrm{d}x'^{\alpha'}}{\mathrm{d}s}\frac{\mathrm{d}x'^{\beta'}}{\mathrm{d}s} = 0,$$

where

$$\Gamma'^{\mu'}_{\alpha'\beta'} = -\frac{\partial^2x'^{\mu'}}{\partial x^\mu\partial x^\nu}\frac{\partial x^\mu}{\partial x'^{\alpha'}}\frac{\partial x^\nu}{\partial x'^{\beta'}}.$$

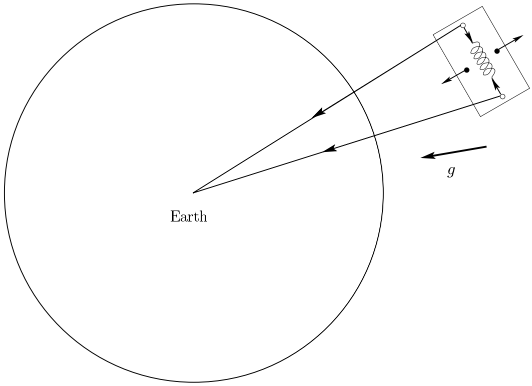

Figure 18.2 Tidal effects in a freely falling laboratory

The principle of equivalence is a purely local idea, and only applies to vanishingly small laboratories. A *real* gravitational field such as that due to the Earth cannot be totally transformed away in general. For example, if the freely falling Einstein elevator has significant size compared to the scale on which there is variation in the Earth's gravitational field, then particles at different positions in the lift will undergo different accelerations. Particles near the floor of the elevator will have larger accelerations than particles released from the ceiling, while particles released from the sides of the elevator will have a small horizontal acceleration relative to the central observer because the direction to the centre of the Earth is not everywhere parallel. These mutual accelerations or *tidal* forces can be measured in principle by connecting pairs of freely falling particles with springs (see Fig. 18.2).

Postulates of general relativity

The basic proposition of general relativity (Einstein, 1916) is the following: the world is a four-dimensional Minkowskian manifold (pseudo-Riemannian of index +2) called **space-time**. Its points are called **events**. The **world-line**, or space-time history of a material particle, is a parametrized curve $\gamma : \mathbb{R} \to M$ whose tangent $\dot{\gamma}$ is everywhere timelike. The **proper time**, or time as measured by a clock carried by the particle between parameter values $\lambda = \lambda_1$ and λ_2, is given by

$$\tau = \frac{1}{c}\int_{\lambda_1}^{\lambda_2} \sqrt{-g(\dot{\gamma}, \dot{\gamma})}\, \mathrm{d}\lambda = \frac{1}{c}\int_{\lambda_1}^{\lambda_2} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda = \frac{1}{c}\int_{s_1}^{s_2} \mathrm{d}s. \qquad (18.78)$$

A **test particle** is a particle of very small mass compared to the major masses in its neighbourhood and 'freely falling' in the sense that it is subject to no external forces. The world-line of a test particle is assumed to be a timelike geodesic. The world-line of a photon of small energy is a null geodesic. Both satisfy equations

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\rho}{\mathrm{d}s} = 0,$$

where the Christoffel symbols $\Gamma^{\mu}_{\nu\rho}$ are given by Eq. (18.40) with Greek indices subsituted. The affine parameter s is determined by

$$g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s} = \begin{cases} -1 & \text{for test particles,} \\ 0 & \text{for photons.} \end{cases}$$

An introduction to the theory of general relativity, together with many of its developments, can be found in [9, 11–16].

The principle of equivalence has a natural place in this postulate, since all particles have the same geodesic motion independent of their mass. Also, at each event p, geodesic normal coordinates may be found such that components of the metric tensor $g = g_{\mu\nu}\,\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu}$ have the Minkowski values $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and the Christoffel symbols vanish, $\Gamma^{\mu}_{\nu\rho}(p) = 0$. In such coordinates the space-time appears locally to be Minkowski space, and gravitational forces have been locally transformed away since any geodesic at p reduces to a 'local rectilinear motion' $\mathrm{d}^2x^{\mu}/\mathrm{d}s^2 = 0$. When it is possible to find coordinates that transform the metric to constant values on an entire chart, the metric is locally flat and all gravitational fields are 'fictitious' since they arise entirely from non-inertial effects. By Theorem 18.1 such coordinate transformations are possible if and only if the curvature tensor vanishes. Hence it is natural to identify the 'real' gravitational field with the curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$.

The equations determining the gravitational field are **Einstein's field equations**

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{18.79}$$

where $G_{\mu\nu}$ is the Einstein tensor defined above in Eqs. (18.61) and (18.62),

$$R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu} \quad \text{and} \quad R = R^{\alpha}_{\ \alpha} = g^{\mu\nu}R_{\mu\nu},$$

and $T_{\mu\nu}$ is the energy–stress tensor of all the matter fields present. The constant κ is known as **Einstein's gravitational constant**; we shall relate it to Newton's gravitational constant directly. These equations have the property that for weak fields, $g_{\mu\nu} \approx \eta_{\mu\nu}$, they reduce to the Poisson equation when appropriate identifications are made (see the discussion of the weak field approximation below), and by the contracted Bianchi identity (18.60) they guarantee a 'covariant' version of the conservation identities

$$T^{\mu\nu}{}_{;\nu} \equiv T^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\nu}T^{\alpha\nu} + \Gamma^{\nu}_{\alpha\nu}T^{\mu\alpha} = 0.$$

Measurement of the curvature tensor

The equation of geodesic deviation (18.47) can be used to give a physical interpretation of the curvature tensor. Consider a one-parameter family of timelike geodesics $\gamma(s, \lambda)$ with tangent vectors $U = U^{\mu}\partial_{x^{\mu}}$ when expressed in a coordinate chart $(A; x^{\mu})$, where

$$U^{\mu}(s) = \frac{\partial x^{\mu}}{\partial s}\Big|_{\lambda=0}, \qquad U^{\mu}U_{\mu} = -1, \qquad \frac{DU^{\mu}}{\partial s} = 0.$$

Suppose the connection vector $Y = Y^{\mu}\partial_{x^{\mu}}$, where $Y^{\mu} = \partial x^{\mu}/\partial\lambda$, is initially orthogonal to U at $s = s_0$,

$$Y^{\mu}U_{\mu}\Big|_{s=s_0} = g(U, Y)(s_0) = 0.$$

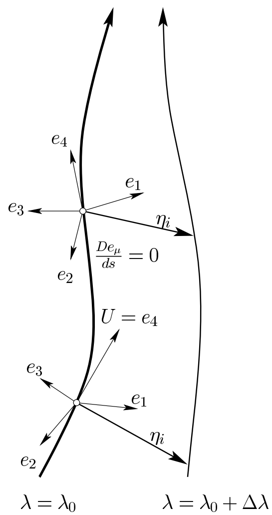

Figure 18.3 Physical measurement of curvature tensor

We then have $g(U, Y) = 0$ for all s, since $g(U, Y)$ is constant along each geodesic (see Section 18.5). Thus if $e_1,\ e_2,\ e_3$ are three mutually orthogonal spacelike vectors at $s = s_0$ on the central geodesic $\lambda = 0$ that are orthogonal to U, and they are parallel propagated along this geodesic, $De_i/\partial s = 0$, then they remain orthogonal to each other and U along this geodesic,

$$g(e_i, e_j) = \delta_{ij}, \qquad g(e_i, u) = 0.$$

In summary, if we set $e_4 = U$ then the four vectors $e_1, \dots, e_4$ are an orthonormal tetrad of vectors along $\lambda = \lambda_0$,

$$g(e_\mu, e_\nu) = \eta_{\mu\nu}.$$

The situation is depicted in Fig. 18.3. Let $\lambda = \lambda_0 + \delta\lambda$ be any neighbouring geodesic from the family, then since we are assuming δx^μ is orthogonal to U^μ, the equation of geodesic deviation in the form (18.48) can be written

$$\frac{D^2 \delta x^\mu}{ds^2} = \delta\lambda \frac{D^2 Y^\mu}{ds^2} = R^\mu{}_{\alpha\rho\nu} U^\alpha U^\rho \delta x^\nu. \tag{18.80}$$

Expanding δx^μ in terms of the basis e_μ we have, adopting a cartesian tensor summation convention,

$$\delta x^\mu = \eta_j e_j^\mu \equiv \sum_{j=1}^{3} \eta_j e_j^\mu,$$

where $\eta_j = \eta_j(s)$, so that

$$\frac{D\delta x^\mu}{\mathrm{d}s} = \frac{\mathrm{d}\eta_j}{\mathrm{d}s} e^\mu_j + \eta_j \frac{e^\mu_i}{\mathrm{d}s} = \frac{\mathrm{d}\eta_j}{\mathrm{d}s} e^\mu_j,$$
$$\frac{D^2\delta x^\mu}{\mathrm{d}s^2} = \frac{\mathrm{d}^2\eta_j}{\mathrm{d}s^2} e^\mu_j + \frac{\mathrm{d}\eta_j}{\mathrm{d}s}\frac{e^\mu_i}{\mathrm{d}s} = \frac{\mathrm{d}^2\eta_j}{\mathrm{d}s^2} e^\mu_j.$$

Substituting (18.80) results in

$$\frac{\mathrm{d}^2\eta_i}{\mathrm{d}s^2} = e_{i\mu}\frac{D^2\delta x^\mu}{\mathrm{d}s^2} = R^\mu{}_{\alpha\rho\nu} e_{i\mu} e^\alpha_4 e^\rho_4 e^\nu_j \eta_j,$$

which reads in any local coordinates at any point on $\lambda = \lambda_0$ such that $e^\mu_\alpha = \delta^\mu_\alpha$,

$$\frac{\mathrm{d}^2\eta_i}{\mathrm{d}s^2} = -R_{i4j4}\eta_j. \tag{18.81}$$

Thus $R_{i4j4} \equiv R_{\mu\alpha\rho\nu} e^\mu_i e^\alpha_4 e^\rho_j e^\nu_4$ measures the *relative accelerations* between neighbouring freely falling particles in the gravitational field. Essentially these are what are termed **tidal forces** in Newtonian physics, and could be measured by the strain on a spring connecting the two particles (see Fig. 18.2 and [17]).

The linearized approximation

Consider a one-parameter family of Minkowskian metrics having components $g_{\mu\nu} = g_{\mu\nu}(x^\alpha, \epsilon)$ such that $\epsilon = 0$ reduces to flat Minkowski space, $g_{\mu\nu}(0) = \eta_{\mu\nu}$. Such a family is known as a **linearized approximation** of general relativity. If we set

$$h_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial\epsilon}\bigg|_{\epsilon=0} \tag{18.82}$$

then for $|\epsilon| \ll 1$ we have 'weak gravitational fields' in the sense that the metric is only slightly different from Minkowski space,

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \epsilon h_{\mu\nu}.$$

From $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\mu$ it follows by differentiating with respect to ϵ at $\epsilon = 0$ that

$$\frac{\partial g^{\mu\rho}}{\partial\epsilon}\bigg|_{\epsilon=0}\eta_{\rho\nu} + \eta^{\mu\rho}h_{\rho\nu} = 0,$$

whence

$$\frac{\partial g^{\mu\nu}}{\partial\epsilon}\bigg|_{\epsilon=0} = -h^{\mu\nu} \equiv -\eta^{\mu\alpha}\eta\nu\beta. \tag{18.83}$$

In this equation and throughout the present discussion indices are raised and lowered with respect to the Minkowski metric, $\eta_{\mu\nu}$, $\eta^{\mu\nu}$. For $\epsilon \ll 1$ we evidently have $g^{\mu\nu} \approx \eta^{\mu\nu} - \epsilon h^{\mu\nu}$.

Assuming that partial derivatives with respect to x^μ and ϵ commute, it is straightforward to compute the linearization of the Christoffel symbols,

$$\begin{aligned}\Gamma^\mu_{\nu\rho} \equiv \frac{\partial}{\partial\epsilon}\Gamma^\mu_{\nu\rho}\bigg|_{\epsilon=0} &= -\tfrac{1}{2}h^{\mu\alpha}(\eta_{\alpha\nu,\rho} + \eta_{\alpha\rho,\nu} - \eta_{\nu\rho,\alpha}) + \tfrac{1}{2}\eta^{\mu\alpha}(h_{\alpha\nu,\rho} + h_{\alpha\rho,\nu} - h_{\nu\rho,\alpha})\\ &= \tfrac{1}{2}\eta^{\mu\alpha}(h_{\alpha\nu,\rho} + h_{\alpha\rho,\nu} - h_{\nu\rho,\alpha})\end{aligned}$$

and

$$\Gamma^{\mu}_{\nu\rho,\sigma} = \tfrac{1}{2}\eta^{\mu\alpha}(h_{\alpha\nu,\rho\sigma} + h_{\alpha\rho,\nu\sigma} - h_{\nu\rho,\alpha\sigma}).$$

Thus, from the component expansion of the curvature tensor (18.25), we have

$$r^{\mu}{}_{\nu\rho\sigma} \equiv \frac{\partial}{\partial\epsilon} R^{\mu}{}_{\nu\rho\sigma}\Big|_{\epsilon=0} = \Gamma^{\mu}_{\nu\sigma,\rho} - \gamma^{\mu}_{\nu\rho,\sigma}$$

since

$$\frac{\partial}{\partial\epsilon}\Gamma^{\alpha}_{\nu\sigma}\Gamma^{\mu}{}_{\alpha\rho}\Big|_{\epsilon=0} = \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\mu}{}_{\alpha\rho}\Big|_{\epsilon=0} + \Gamma^{\alpha}_{\nu\sigma}\Big|_{\epsilon=0}\Gamma^{\mu}{}_{\alpha\rho} = 0, \text{ etc.}$$

since $\Gamma^{\alpha}_{\nu\sigma}\big|_{\epsilon=0} = 0$. Thus

$$r^{\mu}{}_{\nu\rho\sigma} = \tfrac{1}{2}\eta^{\mu\alpha}\big(h_{\alpha\nu,\sigma\rho} + h_{\alpha\sigma,\nu\rho} - h_{\nu\sigma,\alpha\rho} - h_{\alpha\nu,\rho\sigma} - h_{\alpha\rho,\nu\sigma} + h_{\nu\rho,\alpha\sigma}\big)$$

and for small values of the parameter ϵ the Riemann curvature tensor is

$$R_{\mu\nu\rho\sigma} \approx \epsilon r_{\mu\nu\rho\sigma} = \frac{\epsilon}{2}\big(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}\big). \tag{18.84}$$

It is interesting to compare this equation with the expression in geodesic normal coordinates, Eq. (18.43).

The Newtonian tidal equation is derived by considering the motion of two neighbouring particles

$$\ddot{x}_i = -\phi_{,i} \quad \text{and} \quad \ddot{x}_i + \ddot{\eta}_i = -\phi_{,i}(\mathbf{x} + \boldsymbol{\eta}).$$

Since $\phi_{,i}(\mathbf{x} + \boldsymbol{\eta}) = \phi_{,i}(\mathbf{x}) + \phi_{ij}(\mathbf{x})\eta_j$ we have

$$\ddot{\eta}_i = -\phi_{,ij}(\mathbf{x})\eta_j.$$

Compare with the equation of geodesic deviation (18.81) with s replaced by ct, which is approximately correct for velocities $|\dot{\mathbf{x}}| \ll c$,

$$\ddot{\eta}_i = -c^2 R_{i4j4}\eta_j,$$

and we should have, by Eq. (18.84),

$$R_{i4j4} = \frac{\epsilon}{2}\big(h_{i4,4j} + h_{4j,i4} - h_{ij,44} - h_{44,ij}\big) = \frac{\phi_{,ij}}{c^2}.$$

This equation can only hold in a general way if

$$\epsilon h_{44} \approx -\frac{2\phi}{c^2} \quad \text{and} \quad h_{i4,4j}, h_{ij,44} \ll h_{44,ij},$$

and the Newtonian approximation implies that

$$g_{44} \approx -1 + \epsilon h_{44} \approx -1 - \frac{2\phi}{c^2} \quad (\phi \ll c^2). \tag{18.85}$$

Note that the Newtonian potential ϕ has the dimensions of a velocity square – the weak field slow motion approximation of general relativity arises when this velocity is small compared to the velocity of light c.

Multiplying Eq. (18.79) through by $g^{\mu\nu}$ we find

$$R - 2R = -R = \kappa T \quad \text{where} \quad T = T^{\mu}_{\ \mu} = g^{\mu\nu} T_{\mu\nu},$$

and Einstein's field equations can be written in the 'Ricci tensor form'

$$R_{\mu\nu} = \kappa\big(T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu}\big). \tag{18.86}$$

Hence

$$R_{44} = R^{i}{}_{4i4} \approx \frac{1}{c^2}\nabla^2\phi = \kappa\big(T_{44} + \tfrac{1}{2}T\big).$$

If we assume a perfect fluid, Example 9.4, for low velocities compared to c we have

$$T_{\mu\nu} = \Big(\rho + \frac{P}{c^2}\Big) V_\mu V_\nu + P g_{\mu\nu} \quad \text{where} \quad V_\mu \approx (v_1,\ v_2,\ v_3,\ -c),$$

so that

$$T = -c^2\Big(\rho + \frac{P}{c^2}\Big) + 4P = -\rho c^2 + 3P \approx -\rho c^2,$$

and

$$T_{44} \approx \Big(\rho + \frac{P}{c^2}\Big)c^2 + P\Big(-1 - \frac{\phi}{c^2}P\Big) \approx \rho c^2.$$

Substituting in the Ricci form of Einstein's equations we find

$$\frac{1}{c^2}\nabla^2\phi \approx \frac{1}{2}\kappa\rho c^2,$$

which is in agreement with the Newtonian equation (18.77) provided Einstein's gravitational constant has the form

$$\kappa = \frac{8\pi G}{c^4}. \tag{18.87}$$

Exercise: Show that the contracted Bianchi identity (18.60) implies that in geodesic coordinates at any point representing a local freely falling frame, the conservation identities (9.56) hold, $T^{\mu}{}_{\nu,\mu} = 0$.

Exercise: Show that if we had assumed field equations of the form $R_{\mu\nu} = \lambda T_{\mu\nu}$, there would have resulted the physically unsavoury result $T = \text{const}$.

Consider now the effect of a one-parameter family of coordinate transformations $x^\mu = x^\mu(y^\alpha, \epsilon)$ on a linearized approximation $g_{\mu\nu} = g_{\mu\nu}(\epsilon)$ and set

$$\xi^\mu = \frac{\partial x^\mu}{\partial \epsilon}\bigg|_{\epsilon=0}.$$

The transformation of components of the metric tensor results in

$$g_{\mu\nu}(\mathbf{x}, \epsilon) \to g'_{\mu\nu}(\mathbf{y}, \epsilon) = g_{\alpha\beta}(\mathbf{x}(\mathbf{y}, \epsilon), \epsilon)\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}$$

and taking $\partial/\partial\epsilon$ at $\epsilon = 0$ gives

$$h'_{\mu\nu}(\mathbf{y}) = h_{\mu\nu}(\mathbf{y}) + \xi_{\mu,\nu} + \xi_{\nu,\mu}. \tag{18.88}$$

These may be thought of as 'gauge transformations' for the weak fields $h_{\mu\nu}$, comparable with the gauge transformations (9.49), $A'_\mu = A_\mu + \psi_{,\mu}$, which leave the electromagnetic field $F_{\mu\nu}$ unchanged. In the present case, it is straightforward to verify that the transformations (18.88) leave the linearized Riemann tensor (18.84), or *real gravitational field*, invariant.

We define the quantities $\varphi_{\mu\nu}$ by

$$\varphi_{\mu\nu} = h_{\mu\nu} - h\eta_{\mu\nu} \quad \text{where} \quad h = h^\alpha_\alpha = \eta^{\alpha\beta}h_{\alpha\beta}.$$

The transformation of $\varphi^{\ \nu}_{\mu,\nu}$ under a gauge transformation is then

$$\varphi'^{\nu}_{\mu,\nu} = \varphi^{\ \nu}_{\mu,\nu} + \xi_{\mu,\nu}{}^{\nu} = \varphi^{\ \nu}_{\mu,\nu} + \Box\xi_\mu,$$

where indices are raised and lowered with the Minkowski metric, $\eta^{\mu\nu}$, $\eta_{\mu\nu}$. Just as done for the Lorentz gauge (9.51), it is possible (after dropping primes) to find ξ such that

$$\varphi^{\ \nu}_{\mu,\nu} = 0. \tag{18.89}$$

Such a gauge is commonly known as a **harmonic gauge**. There are still available gauge freedoms ξ_μ subject to solutions of the wave equation $\Box\xi_\mu = 0$.

A computation of the linearized Ricci tensor $r_{\mu\nu} = \eta^{\rho\sigma}r_{\rho\mu\sigma\nu}$ using Eq. (18.84) gives

$$r_{\mu\nu} = \tfrac{1}{2}\big(-\Box h_{\mu\nu} + \varphi^{\ \rho}_{\nu,\rho\mu} + \varphi^{\ \rho}_{\mu,\rho\nu}\big) = -\tfrac{1}{2}\Box h_{\mu\nu}$$

in a harmonic gauge. The Einstein tensor is thus $G_{\mu\nu} \approx -(\epsilon/2)\Box\varphi_{\mu\nu}$, and the linearized Einstein equation is

$$\epsilon\Box\varphi_{\mu\nu} = -\kappa T_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu},$$

having solution in terms of retarded Green's functions (12.23)

$$\epsilon\varphi_{\mu\nu}(\mathbf{x}, t) = -\frac{4G}{c^4}\iiint \frac{[T_{\mu\nu}(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3x'.$$

In vacuo, $T_{\mu\nu}$, Einstein's field equations can be written $R_{\mu\nu} = 0$, so that in the linearized approximation we have $\Box h_{\mu\nu} = 0$; these solutions are known as *gravitational waves* (see Problem 18.20 for further details).

The Schwarzschild solution

The vacuum Einstein field equations, $R_{\mu\nu} = 0$ are a non-linear set of 10 second-order equations for 10 unknowns $g_{\mu\nu}$ that can only be solved in a handful of special cases. The most important is that of *spherical symmetry* which, as we shall see in the next chapter, implies that the metric has the form in a set of coordinates $x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = ct$,

$$\mathrm{d}s^2 = \mathrm{e}^\lambda\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) - \mathrm{e}^\nu c^2\,\mathrm{d}t^2 \tag{18.90}$$

where θ and ϕ take the normal ranges of polar coordinates (r does not necessarily range from 0 to ∞), and λ and ν are functions of r and t. We will assume for simplicity that the solutions are *static* so that they are functions of the radial coordinate r alone, $\lambda = \lambda(r)$, $\nu = \nu(r)$. A remarkable theorem of Birkhoff assures us that all spherically symmetric vacuum solutions are in fact static for an appropriate choice of the coordinate t; a proof may be found in Synge [15].

We will perform calculations using Cartan's formalism. Many books prefer to calculate Christoffel symbols and do all computations in the coordinate system of Eq. (18.90). Let $e_1, \dots, e_4$ be the orthonormal basis

$$e_1 = \mathrm{e}^{-\frac{1}{2}\lambda}\partial_r, \quad e_2 = \frac{1}{r}\partial_\theta, \quad e_3 = \frac{1}{r\sin\theta}\partial_\phi, \quad e_4 = \mathrm{e}^{-\frac{1}{2}\nu}c^{-1}\partial_t$$

such that

$$g_{\mu\nu} = g(e_\mu, e_\nu) = \eta = \operatorname{diag}(1, 1, 1, -1)$$

and let $\varepsilon^1, \dots, \varepsilon^4$ be the dual basis

$$\varepsilon^1 = \mathrm{e}^{\frac{1}{2}\lambda}\,\mathrm{d}r, \quad \varepsilon^2 = r\,\mathrm{d}\theta, \quad \varepsilon^3 = r\sin\theta\,\mathrm{d}\phi, \quad \varepsilon^4 = \mathrm{e}^{\frac{1}{2}\nu}c\,\mathrm{d}t = \mathrm{e}^{\frac{1}{2}\nu}\,\mathrm{d}x^4.$$

We will write Cartan's structural relations in terms of the 'lowered' connection forms $\omega_{\mu\nu}$ since, by Eq. (18.73), we have $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Thus (18.64) can be written

$$\mathrm{d}\varepsilon^\mu = -\omega^\mu{}_\nu \wedge \varepsilon^\nu = -\eta^{\mu\rho}\omega_{\rho\nu}\varepsilon^\nu$$

and setting successively $\mu = 1, 2, 3, 4$ we have, writing derivatives with respect to r by a prime $'$,

$$\mathrm{d}\varepsilon^1 = \tfrac{1}{2}\mathrm{e}^{\frac{1}{2}\lambda}\lambda'\,\mathrm{d}r \wedge \mathrm{d}r = 0 = -\omega_{12}\wedge\varepsilon^2 - \omega_{13}\wedge\varepsilon^3 - \omega_{14}\wedge\varepsilon^4, \tag{18.91}$$

$$\mathrm{d}\varepsilon^2 = r^{-1}\mathrm{e}^{-\frac{1}{2}\lambda}\varepsilon^1\wedge\varepsilon^2 = \omega_{12}\wedge\varepsilon^1 - \omega_{23}\wedge\varepsilon^3 - \omega_{24}\wedge\varepsilon^4, \tag{18.92}$$

$$\mathrm{d}\varepsilon^3 = r^{-1}\mathrm{e}^{-\frac{1}{2}\lambda}\varepsilon^1\wedge\varepsilon^3 + r^{-1}\cot\theta\,\varepsilon^2\wedge\varepsilon^3 = \omega_{13}\wedge\varepsilon^1 + \omega_{23}\wedge\varepsilon^2 - \omega_{34}\wedge\varepsilon^4, \tag{18.93}$$

$$\mathrm{d}\varepsilon^4 = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}\lambda}\nu'\varepsilon^1\wedge\varepsilon^4 = -\omega_{14}\wedge\varepsilon^1 - \omega_{24}\wedge\varepsilon^2 - \omega_{34}\wedge\varepsilon^4. \tag{18.94}$$

From (18.93) it follows at once that $\omega_{34} = \Gamma_{344}\varepsilon^4$, and substituting in (18.94) we see that $\Gamma_{344} = 0$ since it is the sole coefficient of the 2-form basis element $\varepsilon^3\wedge\varepsilon^4$. Similarly, from (18.94), $\omega_{24} = \Gamma_{242}\varepsilon^2$ and

$$\omega_{14} = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}\lambda}\nu'\varepsilon^4 + \Gamma_{141}\varepsilon^1.$$

Continuing in this way we find the following values for the connection 1-forms:

$$\begin{aligned}
&\omega_{12} = -r^{-1}\mathrm{e}^{-\frac{1}{2}\lambda}\varepsilon^2, \qquad &&\omega_{13} = -r^{-1}\mathrm{e}^{-\frac{1}{2}\lambda}\varepsilon^3, \qquad &&\omega_{23} = -r^{-1}\cot\theta\,\varepsilon^3,\\
&\omega_{14} = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}\lambda}\nu'\varepsilon^4, &&\omega_{24} = 0, &&\omega_{34} = 0.
\end{aligned} \tag{18.95}$$

To obtain the curvature tensor it is now a simple matter of substituting these forms in the second Cartan structural equation (18.67), with indices lowered

$$\rho_{\mu\nu} = -\rho_{\nu\mu} = 2\,\mathrm{d}\omega_{\mu\nu} + 2\omega_{\mu\rho}\wedge\omega_{\sigma\nu}\eta^{\rho\sigma}. \tag{18.96}$$

For example

$$\begin{aligned}
\rho_{12} &= 2\big(\mathrm{d}\omega_{12} + \omega_{13}\wedge\omega_{32} + \omega_{14}\wedge\omega_{42}\big)\\
&= 2\,\mathrm{d}\big(-r^{-1}\mathrm{e}^{-\frac{1}{2}\lambda}\varepsilon^2\big) \quad \text{since } \omega_{13} \propto \omega_{23} \text{ and } \omega_{42} = 0\\
&= -2\mathrm{e}^{-\frac{1}{2}\lambda}\big(r^{-1}\mathrm{e}^{-\frac{1}{2}\lambda}\big)'\varepsilon^1\wedge\varepsilon^2 - 2r^{-1}\mathrm{e}^{-\frac{1}{2}\lambda}\,\mathrm{d}\varepsilon^2.
\end{aligned}$$

Substituting for $\mathrm{d}\varepsilon^2$ using Eq. (18.92) we find

$$\rho_{12} = r^{-1}\lambda' \mathrm{e}^{-\lambda}\varepsilon^1 \wedge \varepsilon^2.$$

Similarly,

$$\rho_{13} = r^{-1}\lambda' \mathrm{e}^{-\lambda}\varepsilon^1 \wedge \varepsilon^3, \qquad \rho_{23} = 2r^{-2}\big(1 - \mathrm{e}^{-\lambda}\big)\varepsilon^2 \wedge \varepsilon^3,$$
$$\rho_{14} = \mathrm{e}^{-\lambda}\big(\nu'' - \tfrac{1}{2}\lambda'\nu' + (\nu')^2\big)\varepsilon^1 \wedge \varepsilon^4,$$
$$\rho_{24} = r^{-1}\nu' \mathrm{e}^{-\lambda}\varepsilon^2 \wedge \varepsilon^4, \qquad \rho_{34} = r^{-1}\nu' \mathrm{e}^{-\lambda}\varepsilon^3 \wedge \varepsilon^4.$$

The components of the Riemann tensor in this basis are given by

$$R_{\mu\nu\rho\sigma} = \overline{R}(e_\mu, e_\nu, e_\rho, e_\sigma) = \rho_{\mu\nu}(e_\rho, e_\sigma).$$

The non-vanishing components are

$$\begin{aligned}
R_{1212} &= R_{1313} = \frac{\lambda'}{2r}\mathrm{e}^{-\lambda}, \\
R_{2323} &= \frac{1 - \mathrm{e}^{-\lambda}}{r^2}, \\
R_{1414} &= \frac{1}{4}\mathrm{e}^{-\lambda}\big(2\nu'' - \lambda'\nu' + (\nu')^2\big), \\
R_{2424} &= R_{3434} = \frac{\nu'}{2r}\mathrm{e}^{-\lambda}.
\end{aligned} \tag{18.97}$$

The Ricci tensor components

$$R_{\mu\nu} = \eta^{\rho\sigma} R_{\rho\mu\sigma\nu} = \sum_{i=1}^{3} R_{i\mu i\nu} - R_{4\mu 4\nu}$$

are therefore

$$R_{11} = \mathrm{e}^{-\lambda}\Big(-\frac{\nu''}{2} + \frac{\lambda'\nu'}{4} - \frac{(\nu')^2}{4} + \frac{\lambda'}{r}\Big), \tag{18.98}$$

$$R_{44} = \mathrm{e}^{-\lambda}\Big(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{(\nu')^2}{4} + \frac{\nu'}{r}\Big), \tag{18.99}$$

$$R_{22} = R_{33} = \mathrm{e}^{-\lambda}\Big(\frac{\lambda' - \nu'}{2r} - \frac{1}{r^2}\Big) + \frac{1}{r^2}. \tag{18.100}$$

To solve Einstein's vacuum equations $R_{\mu\nu} = 0$, we see by adding (18.98) and (18.99) that $\lambda' + \nu' = 0$, whence

$$\lambda = -\nu + C \quad (C = \text{const.})$$

A rescaling of the time coordinate, $t \to t' = \mathrm{e}^{C/2}t$, has the effect of making $C = 0$, which we now assume. By Eq. (18.99), $R_{44} = 0$ reduces the second-order differential equation to

$$\nu'' + (\nu')^2 + \frac{2\nu'}{r} = 0$$

and the substitution $\alpha = \mathrm{e}^\nu$ results in

$$\big(r^2\alpha'\big)' = 0,$$

whence

$$\alpha = e^{\nu} = A - \frac{2m}{r} \quad (m, A = \text{const.}).$$

If we substitute this into $R_{22} = 0$ we have, by (18.100), $(r\alpha)' = 1$ so that $A = 1$. The most general spherically symmetric solution of Einstein's vacuum equations is therefore

$$ds^2 = \frac{1}{1 - 2m/r} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) - \left(1 - \frac{2m}{r}\right) c^2 \, dt^2, \tag{18.101}$$

known famously as the **Schwarzschild solution**. Converting the polar coordinates to equivalent cartesian coordinates x, y, z we have, as $r \to \infty$,

$$ds^2 \approx dx^2 + dy^2 + dz^2 - c^2 \, dt^2 + \frac{2m}{r}(c^2 \, dt^2 + \cdots)$$

and $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ where

$$h_{44} = \frac{2m}{r} \approx \frac{-2\phi}{c^2},$$

assuming the Newtonian approximation with potential ϕ is applicable in this limit. Since the potential of a Newtonian mass M is given by $\phi = GM/r$, it is reasonable to make the identification

$$m = \frac{GM}{c^2}.$$

The constant m has dimensions of length and $2m = 2GM/c^2$, where the metric (18.101) exhibits singular behaviour, is commonly known as the **Schwarzschild radius**. For a solar mass, $M_\odot = 2 \times 10^{33}$ g, its value is about 3 km. However the Sun would need to collapse to approximately this size before strong corrections to Newtonian theory apply.

When paths of particles (timelike geodesics) and photons (null geodesics) are calculated in this metric, the following deviations from Newtonian theory are found for the solar system:

1. There is a slowing of clocks at a lower gravitational potential. At the surface of the Earth this amounts to a redshift from a transmitter to a receiver at a height h above it of

   $$z = \frac{GM_e h}{R_e^2 c^2}.$$

   This amounts to a redshift of about 10^{-15} m^{-1} and is measurable using the Mossbauer effect.
2. The perihelion of a planet in orbit around the Sun precesses by an amount

   $$\delta\varphi = \frac{6\pi M_\odot G}{c^2 a(1 - e^2)} \text{ per revolution.}$$

   For Mercury this comes out to 43 seconds of arc per century.

3. A beam of light passing the Sun at a closest distance r_0 is deflected an amount

$$\delta\varphi = \frac{4GM_\odot}{r_0 c^2}.$$

For a beam grazing the rim of the Sun $r_0 = R_\odot$ the deflection is 1.75 seconds of arc.

The limit $r \to 2m$ is of particular interest. Although it appears that the metric (18.101) is singular in this limit, this is really only a feature of the coordinates, not of the space-time as such. A clue that this may be the case is found by calculating the curvature components (18.97) for the Schwarzschild solution,

$$R_{1212} = R_{1313} = R_{2424} = R_{3434} = -\frac{m}{r^3}, \qquad R_{2424} = -R_{1414} = \frac{2m}{r^3}$$

all of which approach finite values as $r \to 2m$.

Exercise: Verify these expressions for components of the Riemann tensor.

More specifically, let us make the coordinate transformation from t to $v = ct + r + 2m \ln(r - 2m)$, sometimes referred to as *advanced time* since it can be shown to be constant on inward directed null geodesics, while leaving the spatial coordinates r, θ, ϕ unchanged. In these **Eddington–Finkelstein coordinates** the metric becomes

$$\mathrm{d}s^2 = -\Big(1 - \frac{2m}{r}\Big)\mathrm{d}v^2 + 2\mathrm{d}r\,\mathrm{d}v + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

As $r \to 2m$ the metric shows no abnormality in these coordinates. Inward directed timelike geodesics in the region $r > 2m$ reach $r = 2m$ in finite v-time (and also in finite proper time). However, after the geodesic particle crosses $r = 2m$ no light signals can be sent out from it into $r > 2m$ (see Fig. 18.4). The surface $r = 2m$ acts as a one-way membrane for light signals, called an **event horizon**. Observers with $r > 2m$ can never see any events inside $r = 2m$, an effect commonly referred to as a **black hole**.

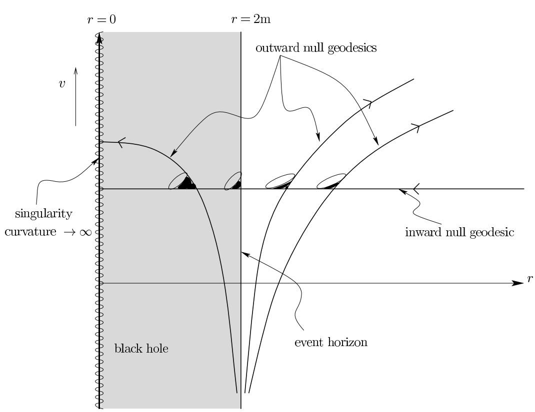

Figure 18.4 Schwarzschild solution in Eddington–Finkelstein coordinates

Problems

Problem 18.20 A **linearized plane gravitational wave** is a solution of the linearized Einstein equations $\Box h_{\mu\nu} = 0$ of the form $h_{\mu\nu} = h_{\mu\nu}(u)$ where $u = x^3 - x^4 = z - ct$. Show that the harmonic gauge condition (18.89) implies that, up to undefined constants,

$$h_{14} + h_{13} = h_{24} + h_{23} = h_{11} + h_{22} = 0, \qquad h_{34} = -\tfrac{1}{2}(h_{33} + h_{44}).$$

Use the remaining gauge freedom $\xi_\mu = \xi_\mu(u)$ to show that it is possible to transform $h_{\mu\nu}$ to the form

$$[h_{\mu\nu}] = \begin{pmatrix} \mathsf{H} & \mathsf{O} \\ \mathsf{O} & \mathsf{O} \end{pmatrix} \quad \text{where} \quad \mathsf{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix}.$$

Setting $h_{11} = \alpha(u)$ and $h_{12} = \beta(u)$, show that the equation of geodesic deviation has the form

$$\ddot{\eta}_1 = \frac{\epsilon}{2} c^2 \big(\alpha'' \eta_1 + \beta'' \eta_2\big), \qquad \ddot{\eta}_2 = \frac{\epsilon}{2} c^2 \big(\beta'' \eta_1 - \alpha'' \eta_2\big)$$

and $\ddot{\eta}_3 = 0$. Make a sketch of the distribution of neighbouring accelerations of freely falling particles about a geodesic observer in the two cases $\beta = 0$ and $\alpha = 0$. These results are central to the observational search for gravity waves.

Problem 18.21 Show that every two-dimensional space-time metric (signature 0) can be expressed locally in *conformal coordinates*

$$\mathrm{d}s^2 = \mathrm{e}^{2\phi}\big(\mathrm{d}x^2 - \mathrm{d}t^2\big) \quad \text{where} \quad \phi = \phi(x, t).$$

Calculate the Riemann curvature tensor component R_{1212}, and write out the two-dimensional Einstein vacuum equations $R_{ij} = 0$. What is their general solution?

Problem 18.22 (a) For a perfect fluid in general relativity,

$$T_{\mu\nu} = (\rho c^2 + P) U_\mu U_\nu + P g_{\mu\nu} \qquad (U^\mu U_\mu = -1)$$

show that the conservation identities $T^{\mu\nu}{}_{;\nu} = 0$ imply

$$\rho_{,\nu} U^\nu + (\rho c^2 + P) U^\nu_{;\nu},$$
$$(\rho c^2 + P) U^\mu_{;\nu} U^\nu + P_{,\nu}\big(g^{\mu\nu} + U^\mu U^\nu\big).$$

(b) For a pressure-free fluid show that the streamlines of the fluid (i.e. the curves $x^\mu(s)$ satisfying $\mathrm{d}x^\mu/\mathrm{d}s = U^\mu$) are geodesics, and ρU^μ is a covariant 4-current, $(\rho U^\mu)_{,\mu} = 0$.
(c) In the Newtonian approximation where

$$U_\mu = \Big(\frac{v_i}{c}, -1\Big) + O(\beta^2), \qquad P = O(\beta^2)\rho c^2, \qquad \Big(\beta = \frac{v}{c}\Big)$$

where $|\beta| \ll 1$ and $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$ with $\epsilon \ll 1$, show that

$$h_{44} \approx -\frac{2\phi}{c^2}, \qquad h_{ij} \approx -\frac{2\phi}{c^2}\delta_{ij} \quad \text{where} \quad \nabla^2\phi = 4\pi G\rho$$

and $h_{i4} = O(\beta) h_{44}$. Show in this approximation that the equations $T^{\mu\nu}{}_{;\nu} = 0$ approximate to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla P - \rho \nabla \phi.$$

Problem 18.23 (a) Compute the components of the Ricci tensor $R_{\mu\nu}$ for a space-time that has a metric of the form

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - 2\,\mathrm{d}u\,\mathrm{d}v + 2H\,\mathrm{d}v^2 \qquad (H = H(x, y, u, v)).$$

(b) Show that the space-time is a vacuum if and only if $H = \alpha(x, y, v) + f(v)u$ where $f(v)$ is an arbitrary function and α satisfies the two-dimensional Laplace equation

$$\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} = 0,$$

and show that it is possible to set $f(v) = 0$ by a coordinate transformation $u' = ug(v)$, $v' = h(v)$.
(c) Show that $R_{i4j4} = -H_{,ij}$ for $i, j = 1, 2$.

Problem 18.24 Show that a coordinate transformation $r = h(r')$ can be found such that the Schwarzschild solution has the form

$$\mathrm{d}s^2 = -\mathrm{e}^{\mu(r')}\,\mathrm{d}t^2 + \mathrm{e}^{\nu(r')}(\mathrm{d}r'^2 + r'^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)).$$

Evaluate the functions e^{μ} and e^{ν} explicitly.

Problem 18.25 Consider an oscillator at $r = r_0$ emitting a pulse of light (null geodesic) at $t = t_0$. If this is received by an observer at $r = r_1$ at $t = t_1$, show that

$$t_1 = t_0 + \int_{r_0}^{r_1} \frac{\mathrm{d}r}{c(1 - 2m/r)}.$$

By considering a signal emitted at $t_0 + \Delta t_0$, received at $t_1 + \Delta t_1$ (assuming the radial positions r_0 and r_1 to be constant), show that $t_0 = t_1$ and the **gravitational redshift** found by comparing *proper times* at emission and reception is given by

$$1 + z = \frac{\Delta\tau_1}{\Delta\tau_0} = \sqrt{\frac{1 - 2m/r_1}{1 - 2m/r_0}}.$$

Show that for two clocks at different heights h on the Earth's surface, this reduces to

$$z \approx \frac{2GM}{c^2}\frac{h}{R},$$

where M and R are the mass and radius of the Earth.

Problem 18.26 In the Schwarzschild solution show the only possible closed photon path is a circular orbit at $r = 3m$, and show that it is unstable.

Problem 18.27 (a) A particle falls radially inwards from rest at infinity in a Schwarzschild solution. Show that it will arrive at $r = 2m$ in a finite *proper time* after crossing some fixed reference position r_0, but that coordinate time $t \to \infty$ as $r \to 2m$.
(b) On an infalling extended body compute the tidal force in a radial direction, by parallel propagating a tetrad (only the radial spacelike unit vector need be considered) and calculating R_{1414}.
(c) Estimate the total tidal force on a person of height 1.8 m, weighing 70 kg, falling head-first into a solar mass black hole ($M_\odot = 2 \times 10^{30}$ kg), as he crosses $r = 2m$.

18.9 Cosmology

Cosmology is the study of the universe taken as a whole [18]. Generally it is assumed that on the broadest scale of observation the universe is homogeneous and isotropic – no particular positions or directions are singled out. Presuming that general relativity applies on this overall scale, the metrics that have homogeneous and isotropic spatial sections are known as **flat Robertson–Walker models**. The simplest of these are the so-called **flat** models

$$\begin{aligned} ds^2 &= a^2(t)(dx^2 + dy^2 + dz^2) - c^2\,dt^2 \\ &= a^2(t)\big(dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\big) - \big(dx^4\big)^2 \end{aligned} \tag{18.102}$$

where the word 'flat' refers to the 3-surfaces $t = $ const., not to the entire metric. Setting

$$\varepsilon^1 = a(t)\,dr, \qquad \varepsilon^2 = a(t)r\,d\theta, \qquad \varepsilon^3 = a(t)r\sin\theta\,d\phi, \qquad \varepsilon^4 = c\,dt,$$

the first structural relations imply, much as in spherical symmetry,

$$\begin{aligned} \omega_{12} &= -\frac{1}{ar}\varepsilon^2, \qquad \omega_{13} = -\frac{1}{ar}\varepsilon^3, \qquad \omega_{23} = -\frac{\cot\theta}{ar}\varepsilon^3 \\ \omega_{14} &= \frac{\dot a}{ca}\varepsilon^1, \qquad \omega_{24} = \frac{\dot a}{ca}\varepsilon^2, \qquad \omega_{34} = \frac{\dot a}{ca}\varepsilon^3 \end{aligned}$$

and substitution in the second structural relations gives, as in the Schwarzschild case,

$$\begin{aligned} \rho_{12} &= \frac{2\dot a^2}{c^2a^2}\varepsilon^1\wedge\varepsilon^2, \qquad \rho_{13} = \frac{2\dot a^2}{c^2a^2}\varepsilon^1\wedge\varepsilon^3, \qquad \rho_{23} = \frac{2\dot a^2}{c^2a^2}\varepsilon^2\wedge\varepsilon^3 \\ \rho_{14} &= -\frac{2\ddot a}{c^2a}\varepsilon^1\wedge\varepsilon^4, \qquad \rho_{24} = -\frac{2\ddot a}{c^2a}\varepsilon^2\wedge\varepsilon^4, \qquad \rho_{34} = -\frac{2\ddot a}{c^2a}\varepsilon^3\wedge\varepsilon^4. \end{aligned}$$

Hence the only non-vanishing curvature tensor components are

$$R_{1212} = R_{1313} = R_{2323} = \frac{\dot a^2}{c^2a^2}, \qquad R_{1414} = R_{2424} = R_{3434} = -\frac{\ddot a}{c^2a}.$$

The non-vanishing Ricci tensor components $R_{\mu\nu} = \sum_{i=1} 3R_{i\mu i\nu} - R_{4\mu 4\nu}$ are

$$R_{11} = R_{22} = R_{33} = \frac{1}{c^2}\Big(\frac{\ddot a}{a} + \frac{2\dot a^2}{a^2}\Big), \qquad R_{44} = -\frac{3\ddot a}{c^2a^2},$$

and the Einstein tensor is

$$G_{11} = G_{22} = G_{33} = -\frac{1}{c^2}\Big(\frac{2\ddot a}{a} + \frac{\dot a^2}{a^2}\Big), \qquad G_{44} = \frac{3\dot a^2}{c^2a^2}.$$

The **closed Robertson–Walker models** are defined in a similar way, but the spatial sections $t = $ const. are 3-spheres:

$$ds^2 = a^2(t)\big[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)\big] - c^2\,dt^2. \tag{18.103}$$

Combining the analysis in Example 18.4 and that given above, we set

$$\varepsilon^1 = a\,dx, \qquad \varepsilon^2 = a\sin\chi\,d\theta, \qquad \varepsilon^3 = a\sin\chi\sin\theta\,d\phi, \qquad \varepsilon^4 = c\,dt.$$

The sections $t = \text{const.}$ are compact spaces having volume

$$\begin{aligned} V(t) &= \int \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3 \\ &= \int a^3(t) \sin^2 \chi \sin\theta \, \mathrm{d}\chi \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi \\ &= a^3(t) \int_0^{\pi} \sin^2 \chi \, \mathrm{d}\chi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \\ &= 2\pi^2 a^3(t). \end{aligned} \tag{18.104}$$

We find that ω_{i4} are as in the flat case, while

$$\omega_{12} = -\frac{\cot\chi}{a}\varepsilon^2, \qquad \omega_{13} = -\frac{\cot\chi}{a}\varepsilon^3, \qquad \omega_{23} = -\frac{\cot\theta}{a\sin\chi}\varepsilon^3.$$

The second structural relations result in the same ρ_{i4} as for the flat case, while an additional term $\dot{a}^2/c^2a^2$ appears in the coefficients of the other curvature forms,

$$\rho_{12} = \Big(\frac{\dot{a}^2}{c^2a^2} + \frac{2\dot{a}^2}{c^2a^2}\Big), \text{ etc.}$$

Finally, the so-called **open Robertson–Walker models**, having the form

$$\mathrm{d}s^2 = a^2(t)\big[\mathrm{d}\chi^2 + \sinh^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\big] - c^2\,\mathrm{d}t^2, \tag{18.105}$$

give rise to similar expressions for ω_{ij} with hyperbolic functions replacing trigonometric, and

$$\rho_{12} = \Big(\frac{\dot{a}^2}{c^2a^2} - \frac{2\dot{a}^2}{c^2a^2}\Big), \text{ etc.}$$

In summary, the non-vanishing curvature tensor components in the three models may be written

$$\begin{aligned} R_{1212} = R_{1313} = R_{2323} &= \frac{\dot{a}^2}{c^2a^2} + \frac{k}{a^2}, \\ R_{1414} = R_{2424} = R_{3434} &= -\frac{\ddot{a}}{c^2a}, \end{aligned}$$

where $k = 0$ refers to the flat model, $k = 1$ the closed and $k = -1$ the open model. The Einstein tensor is thus

$$G_{11} = G_{22} = G_{33} = -\frac{1}{c^2}\Big(\frac{2\ddot{a}}{a} + \frac{k\dot{a}^2}{a^2}\Big), \qquad G_{44} = \frac{3\dot{a}^2}{c^2a^2},$$

and Einstein's field equations $G_{\mu\nu} = \kappa T_{\mu\nu}$ imply that the energy–stress tensor is that of a perfect fluid $T = P(t)\sum_{i=1}^3 \varepsilon^i \otimes \varepsilon^i + \rho(t)c^2\varepsilon^4 \otimes \varepsilon^4$ (see Example 9.4), where

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \tag{18.106}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\Big(\rho + 3\frac{P}{c^2}\Big). \tag{18.107}$$

Taking the time derivative of (18.106) and substituting (18.107) gives

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0. \tag{18.108}$$

Exercise: Show that (18.108) is equivalent to the Bianchi identity $T^{4\nu}{}_{;\nu} = 0$.

If we set $P = 0$, a form of matter sometimes known as **dust**, then Eq. (18.108) implies that $\mathrm{d}(\rho a^3)/\mathrm{d}t = 0$, and the density has evolution

$$\rho(t) = \rho_0 a^{-3} \quad (\rho_0 = \text{const.}). \tag{18.109}$$

This shows, using (18.104), that for a closed universe the total mass of the universe $M = \rho(t)V(t) = \rho_0 2\pi^2$ is finite and constant. Substituting (18.109) into Eq. (18.108) we have the **Friedmann equation**

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}a^{-1} - kc^2. \tag{18.110}$$

It is convenient to define rescaled variables

$$\alpha = \frac{3c^2}{8\pi G\rho_0}a, \qquad y = \frac{3c^3}{8\pi G\rho_0}t$$

and Eq. (18.110) becomes

$$\left(\frac{\mathrm{d}\alpha}{\mathrm{d}y}\right)^2 = \frac{1}{\alpha} - k. \tag{18.111}$$

The solutions are as follows.

$k = 0$: It is straightforward to verify that, up to an arbitrary origin of the time coordinate,

$$\alpha = \left(\frac{9}{4}\right)^{1/3} y^{2/3} \implies a(t) = (6\pi G\rho_0)^{1/3}t^{2/3}, \qquad \rho = \frac{1}{6\pi Gt^2}.$$

This solution is known as the **Einstein–de Sitter universe**.

$k = 1$: Equation (18.111) is best solved in parametric form

$$\alpha = \sin^2\eta, \qquad y = \eta - \sin\eta\cos\eta$$

a cycloid in the $\alpha - \eta$ plane, which starts at $\alpha = 0, \eta = 0$, rises to a maximum at $\eta = \pi/2$ then recollapses to zero at $\eta = \pi$. This behaviour is commonly referred to as an *oscillating universe*, but the term is not well chosen as there is no reason to expect that the universe can 'bounce' out of the singularity at $a = 0$ where the curvature and density are infinite.

$k = -1$: The solution is parametrically $\alpha = \sinh^2\eta$, $y = \sinh\eta\cosh\eta - \eta$, which expands indefinitely as $\eta \to \infty$.

Collectively these models are known as **Friedmann models**, the Einstein–de Sitter model acting as a kind of critical case dividing closed from open models (see Fig. 18.5).

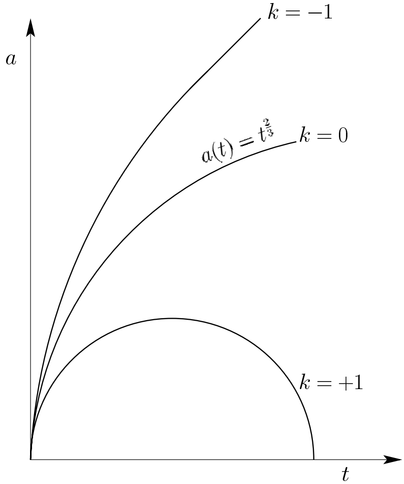

Figure 18.5 Friedmann cosmological models

Observational cosmology is still trying to decide which of these models is the closest representation of our actual universe, but most evidence favours the open model.

Problems

Problem 18.28 Show that for a closed Friedmann model of total mass M, the maximum radius is reached at $t = 2GM/3c^3$ where its value is $a_{\max} = 4GM/3\pi c^2$.

Problem 18.29 Show that the *radiation filled universe*, $P = \frac{1}{3}\rho$ has $\rho \propto a^{-4}$ and the time evolution for $k = 0$ is given by $a \propto t^{1/2}$. Assuming the radiation is black body, $\rho = a_S T^4$, where $a_S = 7.55 \times 10^{-15}$erg cm^{-3} K^{-4}, show that the temperature of the universe evolves with time as

$$T = \left(\frac{3c^2}{32\pi G a_S}\right)^{1/4} t^{-1/2} = \frac{1.52}{\sqrt{t}}\,\mathrm{K} \quad (t \text{ in seconds}).$$

Problem 18.30 Consider two radial light signals (null geodesics) received at the spatial origin of coordinates at times t_0 and $t_0 + \Delta t_0$, emitted from $\chi = \chi_1$ (or $r = r_1$ in the case of the flat models) at time $t = t_1 < t_0$. By comparing proper times between reception and emission show that the observer experiences a redshift in the case of an expanding universe ($a(t)$ increasing) given by

$$1 + z = \frac{\Delta t_0}{\Delta t_1} = \frac{a(t_0)}{a(t_1)}.$$

Problem 18.31 By considering light signals as in the previous problem, show that an observer at $r = 0$, in the Einstein–de Sitter universe can at time $t = t_0$ see no events having radial coordinate $r > r_H = 3ct_0$. Show that the mass contained within this radius, called the **particle horizon**, is given by $M_H = 6c^3 t_0/G$.

18.10 Variation principles in space-time

We conclude this chapter with a description of the variation principle approach to field equations, including an appropriate variational derivation of Einstein's field equations [13, 19]. Recall from Chapter 17 that for integration over an orientable four-dimensional space-time (M, g) we need a non-vanishing 4-form Ω. Let $\varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4$ be any o.n. basis of differential 1-forms, $g^{-1}(\varepsilon^\mu, \varepsilon^n) = \eta^{\mu\nu}$. The volume element $\Omega = \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3 \wedge \varepsilon^4$ defined by this basis is independent of the choice of orthonormal basis, provided they are related by a proper Lorentz transformation. Following the discussion leading to Eq. (8.32), we have that the components of this 4-form in an arbitrary coordinate $(U, \phi; x^\mu)$ are

$$\Omega_{\mu\nu\rho\sigma} = \frac{\sqrt{|g|}}{4!}\epsilon_{\mu\nu\rho\sigma} \quad \text{where} \quad g = \det[g_{\mu\nu}],$$

and we can write

$$\begin{aligned}\Omega &= \frac{\sqrt{|g|}}{4!}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma \\ &= \sqrt{-g}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4.\end{aligned}$$

Every 4-form Λ can be written $\Lambda = f\Omega$ for some scalar function f on M, and for every regular domain D contained in the coordinate domain U,

$$\int_D \Lambda = \int_{\phi(D)} f\sqrt{-g}\,\mathrm{d}^4x \quad (\mathrm{d}^4x \equiv \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3\,\mathrm{d}x^4).$$

If $f\sqrt{-g} = A^\mu{}_{,\mu}$ for some set of functions A^μ on M, then

$$\Lambda = A^\mu{}_{,\mu}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$$

and a simple argument, such as suggested in Problem 17.8, leads to

$$\Lambda = \mathrm{d}\alpha \quad \text{where} \quad \alpha = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}A^\mu\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma$$

and by Stokes' theorem (17.3),

$$\int_D \Lambda = \int_D \mathrm{d}\alpha = \int_{\partial D} \alpha. \tag{18.112}$$

Let $\Phi_A(x)$ be any set of fields on a neighbourhood V of regular domain $x \in D \subset M$, where the index $A = 1, \dots, N$ refers to all possible components (scalar, vector, tensor, etc.) that may arise. By a **variation** of these fields is meant a one-parameter family of fields $\tilde{\Phi}_A(x, \lambda)$ on D such that

1. $\tilde{\Phi}_A(x, 0) = \Phi_A(x)$ for all $x \in D$.
2. $\tilde{\Phi}_A(x, \lambda) = \Phi_A(x)$ for all λ and all $x \in V - D$.

The second condition implies that the condition holds on the boundary ∂D and also all derivatives of the variation field components agree there, $\tilde{\Phi}_{A,\mu}(x, \lambda) = \Phi_{A,\mu}(x)$ for all $x \in \partial D$. We define the variational derivatives

$$\delta\Phi_A = \frac{\partial}{\partial\lambda}\tilde{\Phi}_A(x, \lambda)\Big|_{\lambda=0}.$$

This vanishes on the boundary ∂D since $\tilde{\Phi}_A$ is independent of λ there. A **Lagrangian** is a function $L(\Phi_A, \Phi_{A,\mu})$ dependent on the fields and their derivatives. It defines a 4-form $\Lambda = L\Omega$ and an associated **action**

$$I = \int_D \Lambda = \int_{D\cap U} L\sqrt{-g}\, \mathrm{d}^4x.$$

Field equations arise by requiring that the action be stationary,

$$\delta I = \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_D \Lambda\Big|_{\lambda=0} = \int_{D\cap U} \delta(L\sqrt{-g})\, \mathrm{d}^4x,$$

called a **field action principle** that, as for path actions, can be evaluated by

$$\begin{aligned} 0 &= \int_{D\cap U} \frac{\partial L\sqrt{-g}}{\partial \Phi_A}\delta\Phi_A + \frac{\partial L\sqrt{-g}}{\partial \Phi_{A,\mu}}\delta\Phi_{A,\mu}\, \mathrm{d}^4x \\ &= \int_{D\cap U} \frac{\partial L\sqrt{-g}}{\partial \Phi_A}\delta\Phi_A + \frac{\partial}{\partial x^\mu}\Big(\frac{\partial L\sqrt{-g}}{\partial \Phi_{A,\mu}}\delta\Phi_A\Big) - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial L\sqrt{-g}}{\partial \Phi_{A,\mu}}\Big)\delta\Phi_A\, \mathrm{d}^4x. \end{aligned}$$

Using the version of Stokes' theorem given in (18.112), the middle term can be converted to an integral over the boundary

$$\int_{\partial D} \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\frac{\partial L}{\partial \Phi_{A,\mu}}\sqrt{-g}\delta\Phi_A\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma,$$

which vanishes since $\delta\Phi_A = 0$ on ∂D. Since $\delta\Phi_A$ are arbitrary functions (subject to this boundary constraint) on D, we deduce the Euler–Lagrange field equations

$$\frac{\delta L\sqrt{-g}}{\delta \Phi_A} \equiv \frac{\partial L\sqrt{-g}}{\partial \Phi_A} - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial L\sqrt{-g}}{\partial \Phi_{A,\mu}}\Big) = 0. \tag{18.113}$$

It is best to include the term $\sqrt{-g}$ within the derivatives since, as we shall see, the fields may depend specifically on the metric tensor components $g_{\mu\nu}$.

Hilbert action

Einstein's field equations may be derived from a variation principle, by setting the Lagrangian to be the Ricci scalar, $L = R$. This is known as the **Hilbert Lagrangian**. For independent variables it is possible to take either the metric tensor components $g_{\mu\nu}$ or those of the inverse tensor $g^{\mu\nu}$. We will adopt the latter, as it is slightly more convenient (the reader may try to adapt the analysis that follows for the variables $\Phi_A = g_{\mu\nu}$. We cannot use the Euler–Lagrange equations (18.113) as they stand since R is dependent on $g^{\mu\nu}$ and its first *and second* derivatives. While the Euler–Lagrange analysis can be extended to include Lagrangians that depend on second derivatives of the fields (see Problem 18.32), this would be a prohibitively complicated calculation. Proceeding directly, we have

$$\delta I_G = \delta\int_D R\Omega = \int_{D\cap U} \delta\big(R_{\mu\nu}g^{\mu\nu}\sqrt{-g}\big)\, \mathrm{d}^4x,$$

whence

$$\delta I_G = \int_{D\cap U} \big(\delta R_{\mu\nu}\, g^{\mu\nu}\sqrt{-g} + R_{\mu\nu}\delta g^{\mu\nu}\,\sqrt{-g} + R\delta\sqrt{-g}\big)\mathrm{d}^4x. \tag{18.114}$$

We pause at this stage to analyse the last term, $\delta\sqrt{-g}$. Forgetting temporarily that $g_{\mu\nu}$ is a symmetric tensor, and assuming that all components are independent, we see that the determinant is a homogeneous function of degree in the components,

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = G^{\mu\nu}\delta g_{\mu\nu}$$

where $G^{\mu\nu}$ is the cofactor of $g_{\mu\nu}$. We may therefore write

$$\delta g = g g^{\nu\mu}\delta g_{\mu\nu} = g g^{\mu\nu}\delta g_{\mu\nu} \tag{18.115}$$

since $g^{\mu\nu} = g^{\nu\mu}$. The symmetry of $g_{\mu\nu}$ may be imposed at this stage, without in any way altering this result. From $g^{\mu\nu}g_{\mu\nu} = \delta^{\mu}_{\mu} = 4$ it follows at once that we can write (18.115) as

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}. \tag{18.116}$$

Hence

$$\delta\big(\sqrt{-g}\big) = \frac{1}{2\sqrt{-g}}\delta(-g) = -\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}. \tag{18.117}$$

A similar analysis gives

$$\big(\sqrt{-g}\big)_{,\rho} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}g^{\mu\nu}{}_{,\rho} \tag{18.118}$$

and from the formula (18.40) for Christoffel symbols,

$$\Gamma^{\mu}_{\rho\mu} = \frac{1}{2}g^{\mu\nu}g_{\nu\mu,\rho} = \frac{1}{\sqrt{-g}}\big(\sqrt{-g}\big)_{,\rho}. \tag{18.119}$$

This identity is particularly useful in providing an equation for the covariant divergence of a vector field:

$$A^{\mu}{}_{;\mu} = A^{\mu}{}_{,\mu} + \Gamma^{\mu}_{\rho\mu}A^{\rho} = \frac{1}{\sqrt{-g}}\big(A^{\mu}\sqrt{-g}\big)_{,\mu}. \tag{18.120}$$

We are now ready to continue with our evaluation of δI_G. Using (18.117) we can write Eq. (18.114) as

$$\delta I_G = \int_{D\cap U} \big(R_{\mu\nu} - \tfrac{1}{2}R\big)\delta g^{\mu\nu}\,\sqrt{-g} + \delta R_{\mu\nu}\, g^{\mu\nu}\sqrt{-g}\mathrm{d}^4x. \tag{18.121}$$

To evaluate the last term, we write out the Ricci tensor components

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\mu\rho,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\rho}_{\alpha\rho} - \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\alpha\nu}.$$

Since $\delta\Gamma^{\rho}_{\mu\nu}$ is the limit as $\lambda \to 0$ of a difference of two connections, it is a tensor field and we find

$$\delta R_{\mu\nu} = \big(\delta\Gamma^{\rho}_{\mu\nu}\big)_{,\rho} - \big(\delta\Gamma^{\rho}_{\mu\rho}\big)_{,\nu},$$

as may be checked either directly or, more simply, in geodesic normal coordinates. Hence, using $g^{\mu\nu}{}_{;\rho} = 0$,

$$\delta R_{\mu\nu} g^{\mu\nu} = W^{\rho}{}_{;\rho} \quad \text{where} \quad W^{\rho} = \delta\Gamma^{\rho}_{\mu\nu}\, g^{\mu\nu} - \delta\Gamma^{\nu}_{\mu\nu}\, g^{\mu\rho}$$

and from Eq. (18.120) we see that

$$\delta R_{\mu\nu}\, g^{\mu\nu}\sqrt{-g} = \left(W^{\rho}\sqrt{-g}\right)_{,\rho}.$$

Since W^{ρ} depends on $\delta g^{\mu\nu}$ and $\delta g^{\mu\nu}{}_{,\sigma}$, it vanishes on the boundary ∂D, and the last term in Eq. (18.121) is zero by Stokes' theorem. Since $\delta g^{\mu\nu}$ is assumed arbitrary on D, the Hilbert action gives rise to Einstein's vacuum field equations,

$$G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0.$$

Energy–stress tensor of fields

With other fields Φ_A present we take the total Lagrangian to be

$$L = \frac{1}{2\kappa} + L_F(\Phi_A, \Phi_{A,\mu}, g^{\mu\nu}) \tag{18.122}$$

and we have

$$\begin{aligned} 0 = \delta I &= \delta \int_{D\cap U} \left(\frac{1}{2\kappa} R + L_F\right)\sqrt{-g}\, \mathrm{d}^4 x \\ &= \int_{D\cap U} \left(\frac{1}{2\kappa} G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g} + \frac{\partial L_F\sqrt{-g}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\delta L_F\sqrt{-g}}{\delta\Phi_A}\delta\Phi_A\right) \mathrm{d}^4 x. \end{aligned}$$

Variations with respect to field variables Φ_A give rise to the Euler–Lagrange field equations (18.113), while the coefficients of $\delta g^{\mu\nu}$ lead to the full Einstein's field equations

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad \text{where} \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial L_F\sqrt{-g}}{\partial g^{\mu\nu}}. \tag{18.123}$$

Example 18.5 An interesting example of a variation principle is the **Einstein–Maxwell theory**, where the field variables are taken to be components of a covector field A_μ – essentially the electromagnetic 4-potential given in Eq. (9.47) – and the field Lagrangian is taken to be

$$L_F = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} = \frac{1}{16\pi} F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} \tag{18.124}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. A straightforward way to compute the electromagnetic energy–stress tensor is to consider variations of $g^{\mu\nu}$,

$$\begin{aligned} T_{\mu\nu}\delta g^{\mu\nu} &= -\frac{2}{\sqrt{-g}}\delta\left(L_F\sqrt{-g}\right) \\ &= -\frac{2}{\sqrt{-g}}\left[\frac{-1}{8\pi}F_{\mu\nu}F_{\rho\sigma}\delta g^{\mu\rho}g^{\nu\sigma}\sqrt{-g} + L_F\delta\left(\sqrt{-g}\right)\right] \\ &= \tfrac{1}{4\pi}\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\right), \end{aligned} \tag{18.125}$$

on using Eq. (18.117). This expression agrees with that proposed in Example 9.5, Eq. (9.59).

Variation of the field variables A_μ gives

$$\begin{aligned}\delta \int_D L_F \Omega &= \frac{-1}{4\pi} \int_{D\cap U} \delta A_{\nu,\mu} F^{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x \\ &= \frac{-1}{4\pi} \int_{D\cap U} \delta A_{\nu;\mu} F^{\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x \\ &= \frac{-1}{4\pi} \int_{D\cap U} \big(\delta A_\nu F^{\mu\nu}\big)_{;\mu} \sqrt{-g} - \delta A_\nu F^{\mu\nu}{}_{;\mu} \sqrt{-g}\, \mathrm{d}^4 x \\ &= \frac{-1}{4\pi} \int_{D\cap U} \big(\delta A_\nu F^{\mu\nu} \sqrt{-g}\big)_{,\mu} - \delta A_\nu F^{\mu\nu}{}_{;\mu} \sqrt{-g}\, \mathrm{d}^4 x.\end{aligned}$$

As the first term in the integrand is an ordinary divergence its integral vanishes, and we arrive at the charge-free covariant Maxwell equations

$$F^{\mu\nu}{}_{;\mu} = 0. \tag{18.126}$$

The source-free equations follow automatically from $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$:

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0 \iff F_{\mu\nu;\rho} + F_{\nu\rho;\mu} + F_{\rho\mu;\nu} = 0. \tag{18.127}$$

Problems

Problem 18.32 If a Lagrangian depends on second and higher order derivatives of the fields, $L = L(\Phi_A,\ \Phi_{A,\mu},\ \Phi_{A,\mu\nu}, \dots)$ derive the generalized Euler–Lagrange equations

$$\frac{\delta L\sqrt{-g}}{\delta \Phi_A} \equiv \frac{\partial L\sqrt{-g}}{\partial \Phi_A} - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial L\sqrt{-g}}{\partial \Phi_{A,\mu}}\Big) + \frac{\partial^2}{\partial x^\mu \partial x^\nu}\Big(\frac{\partial L\sqrt{-g}}{\partial \Phi_{A,\mu\nu}}\Big) - \dots = 0.$$

Problem 18.33 For a skew symmetric tensor $F^{\mu\nu}$ show that

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\big(\sqrt{-g} F^{\mu\nu}\big)_{,\nu}.$$

Problem 18.34 Compute the Euler–Lagrange equations and energy–stress tensor for a scalar field Lagrangian in general relativity given by

$$L_S = -\psi_{,\mu}\psi_{,\nu} g^{\mu\nu} - m^2\psi^2.$$

Verify $T^{\mu\nu}{}_{;\nu} = 0$.

Problem 18.35 Prove the implication given in Eq. (18.127). Show that this equation and Eq. (18.126) imply $T^{\mu\nu}{}_{;\nu} = 0$ for the electromagnetic energy–stress tensor given in Eqn. (18.125).

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