8 Exterior algebra

In Section 6.4 we gave an intuitive introduction to the concept of Grassmann algebra $\Lambda(V)$ as an associative algebra of dimension 2^n constructed from a vector space V of dimension n. Certain difficulties, particularly those relating to the definition of exterior product, were cleared up by the more formal approach to the subject in Section 7.1. In this chapter we propose a definition of Grassmann algebra entirely of tensors [1–5]. This presentation has a more 'concrete'constructive character, and to distinguish it from the previous treatments we will use the term *exterior algebra* over V to describe it from here on.

8.1 *r*-Vectors and *r*-forms

A tensor of type $(r, 0)$ is said to be **antisymmetric** if, as a multilinear function, it changes sign whenever any pair of its arguments are interchanged,

$$A(\alpha^1, \dots, \alpha^i, \dots, \alpha^j, \dots, \alpha^r) = -A(\alpha^1, \dots, \alpha^j, \dots, \alpha^i, \dots, \alpha^r). \tag{8.1}$$

Equivalently, if π is any permutation of $1, \dots, r$ then

$$A(\alpha^{\pi(1)}, \alpha^{\pi(2)}, \dots, \alpha^{\pi(r)}) = (-1)^\pi A(\alpha^1, \alpha^2, \dots, \alpha^r).$$

To express these conditions in components, let $\{e_i\}$ be any basis of V and $\{\varepsilon^j\}$ its dual basis. Setting $\alpha^1 = \varepsilon^{i_1},\ \alpha^2 = \varepsilon^{i_2}, \dots$ in (8.1), a tensor A is antisymmetric if it changes sign whenever any pair of component indices is interchanged,

$$A^{i_1 \dots j \dots k \dots i_r} = -A^{i_1 \dots k \dots j \dots i_r}.$$

For any permutation π of $1, \dots, r$ we have

$$A^{i_{\pi(1)} \dots i_{\pi(r)}} = (-1)^\pi A^{i_1 \dots i_r}.$$

Antisymmetric tensors of type $(r, 0)$ are also called r**-vectors**, forming a vector space denoted $\Lambda^r(V)$. Ordinary vectors of V are 1-vectors and scalars will be called 0-vectors.

A similar treatment applies to antisymmetric tensors of type $(0, r)$, called r**-forms**. These are usually denoted by Greek letters α, β, etc., and satisfy

$$\alpha(v_{\pi(1)}, \dots, v_{\pi(r)}) = (-1)^\pi \alpha(v_1, \dots, v_r),$$

or in terms of components

$$\alpha_{i_{\pi(1)}\dots i_{\pi(r)}} = (-1)^\pi \alpha_{i_1\dots i_r}.$$

Linear functionals, or covectors, are called 1-forms and scalars are 0-forms. The vector space of r-forms is denoted $\Lambda^{*r}(V) \equiv \Lambda^r(V^*)$.

As shown in Eq. (7.33), the total contraction of a 2-form α and a symmetric tensor S of type (2, 0) vanishes,

$$\alpha_{ij} S^{ij} = 0.$$

The same holds true of more general contractions such as that between an r-form α and a tensor S of type $(s, 0)$ that is symmetric in any pair of indices; for example, if $S^{ikl} = S^{lki}$ then

$$\alpha_{ijkl} S^{ikl} = 0.$$

The antisymmetrization operator $\mathcal{A}$

Let T be any totally contravariant tensor of degree r; that is, of type $(r, 0)$. Its **antisymmetric part** is defined to be the tensor $\mathcal{A}T$ given by

$$\mathcal{A}T\big(\omega^1, \omega^2, \dots, \omega^r\big) = \frac{1}{r!}\sum_\sigma (-1)^\sigma T\big(\omega^{\sigma(1)}, \omega^{\sigma(2)}, \dots, \omega^{\sigma(r)}\big), \tag{8.2}$$

where the summation on the right-hand side runs through all permutations σ of $1, 2, \dots, r$. If π is any permutation of $1, 2, \dots, r$ then

$$\begin{aligned}
\mathcal{A}T\big(\alpha^{\pi(1)}, \alpha^{\pi(2)}, \dots, \alpha^{\pi(r)}\big) &= \frac{1}{r!}\sum_\sigma (-1)^\sigma T\big(\alpha^{\pi\sigma(1)}, \alpha^{\pi\sigma(2)}, \dots, \alpha^{\pi\sigma(r)}\big) \\
&= \frac{1}{r!}\sum_{\sigma'} (-1)^\pi (-1)^{\sigma'} T\big(\alpha^{\sigma'(1)}, \alpha^{\sigma'(2)}, \dots, \alpha^{\sigma'(r)}\big) \\
&= (-1)^\pi \mathcal{A}T\big(\alpha^1, \alpha^2, \dots, \alpha^r\big),
\end{aligned}$$

since $\sigma' = \pi\sigma$ runs through all permutations of $1, 2, \dots, r$ and $(-1)^{\sigma'} = (-1)^\pi(-1)^\sigma$. Hence $\mathcal{A}T$ is an antisymmetric tensor.

The **antisymmetrization operator** $\mathcal{A} : V^{(r,0)} \to \Lambda^r(V) \subseteq V^{(r,0)}$ is clearly a linear operator on $V^{(r,0)}$,

$$\mathcal{A}(aT + bS) = a\mathcal{A}(T) + b\mathcal{A}(S),$$

and since the antisymmetric part of an r-vector A is always A itself, it is **idempotent**

$$\mathcal{A}^2 = \mathcal{A}.$$

Thus $\mathcal{A}$ is a *projection operator* (see Problem 3.6). This property generalizes to the following useful theorem:

Theorem 8.1 *If T is a tensor of type $(r, 0)$ and S a tensor of type $(s, 0)$, then*

$$\mathcal{A}(\mathcal{A}T \otimes S) = \mathcal{A}(T \otimes S), \qquad \mathcal{A}(T \otimes \mathcal{A}S) = \mathcal{A}(T \otimes S).$$

Proof: We will prove the first equation, the second being essentially identical. Let $\omega^1, \omega^2, \dots, \omega^{r+s}$ be any $r+s$ covectors. Then

$$\mathcal{A}T \otimes S\big(\omega^1, \omega^2, \dots, \omega^{r+s}\big) = \frac{1}{r!}\sum_{\sigma}(-1)^{\sigma}\, T\big(\omega^{\sigma(1)}, \dots, \omega^{\sigma(r)}\big) S\big(\omega^{r+1}, \dots, \omega^{r+s}\big).$$

Treating each permutation σ in this sum as a permutation σ' of $1, 2, \dots, r+s$ that leaves the last s numbers unchanged, this equation can be written

$$\mathcal{A}T \otimes S\big(\omega^1, \omega^2, \dots, \omega^{r+s}\big) = \frac{1}{r!}\sum_{\sigma'}(-1)^{\sigma'}\, T\big(\omega^{\sigma'(1)}, \dots, \omega^{\sigma'(r)}\big) S\big(\omega^{\sigma'(r+1)}, \dots, \omega^{\sigma'(r+s)}\big).$$

Now for each permutation σ', as ρ ranges over all permutations of $1, 2, \dots, r+s$, the product $\pi = \rho\sigma'$ also ranges over all such permutations, and $(-1)^{\pi} = (-1)^{\rho}(-1)^{\sigma'}$. Hence

$$\begin{aligned}
&\mathcal{A}(\mathcal{A}T \otimes S)\big(\omega^1, \omega^2, \dots, \omega^{r+s}\big)\\
&= \frac{1}{(r+s)!}\sum_{\rho}(-1)^{\rho}\frac{1}{r!}\sum_{\sigma'}(-1)^{\sigma'}\, T\big(\omega^{\rho\sigma'(1)}, \dots, \omega^{\rho\sigma'(r)}\big) S\big(\omega^{\rho\sigma'(r+1)}, \dots, \omega^{\rho\sigma'(r+s)}\big)\\
&= \frac{1}{r!}\sum_{\sigma'}\frac{1}{(r+s)!}\sum_{\pi}(-1)^{\pi}\, T\big(\omega^{\pi(1)}, \dots, \omega^{\pi(r)}\big) S\big(\omega^{\pi(r+1)}, \dots, \omega^{\pi(r+s)}\big),
\end{aligned}$$

since there are $r!$ permutations of type σ', each making an identical contribution. Hence

$$\begin{aligned}
&\mathcal{A}(\mathcal{A}T \otimes S)\big(\omega^1, \omega^2, \dots, \omega^{r+s}\big)\\
&= \frac{1}{(r+s)!}\sum_{\pi}(-1)^{\pi}\, T\big(\omega^{\pi(1)}, \dots, \omega^{\pi(r)}\big) S\big(\omega^{\pi(r+1)}, \dots, \omega^{\pi(r+s)}\big)\\
&= \mathcal{A}(T \otimes S)\big(\omega^1, \omega^2, \dots, \omega^{r+s}\big),
\end{aligned}$$

as required. ■

The same symbol $\mathcal{A}$ can also be used to represent the projection operator $\mathcal{A} : V^{(0,r)} \to \Lambda^{*r}$ defined by

$$\mathcal{A}T(u_1, u_2, \dots, u_r) = \frac{1}{r!}\sum_{\sigma}(-1)^{\sigma}\, T(u_{\sigma(1)}, u_{\sigma(2)}, \dots, u_{\sigma(r)}).$$

Theorem 8.1 has a natural counterpart for tensors T of type $(0, r)$ and S of type $(0, s)$.

8.2 Basis representation of *r*-vectors

Let $\{e_i\}$ be any basis of V with dual basis $\{\varepsilon^j\}$, then setting $\omega^1 = \varepsilon^{i_1}, \omega^1 = \varepsilon^{i_1}, \dots, \omega^r = \varepsilon^{i_r}$ in Eq. (8.2) results in an equation for the components of any tensor T of type $(r, 0)$

$$(\mathcal{A}T)^{i_1 i_2 \dots i_r} = T^{[i_1 i_2 \dots i_r]} \equiv \frac{1}{r!}\sum_{\sigma}(-1)^{\sigma}\, T^{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(r)}}.$$

From the properties of the antisymmetrization operator, the square bracketing of any set of indices satisfies

$$T^{[i_1 i_2 \dots i_r]} = (-1)^\pi T^{[i_{\pi(1)} i_{\pi(2)} \dots i_{\pi(r)}]}, \quad \text{for any permutation } \pi$$
$$T^{[[i_1 i_2 \dots i_r]]} = T^{[i_1 i_2 \dots i_r]}.$$

If A is an r-vector then $\mathcal{A}A = A$, or in components,

$$A^{i_1 i_2 \dots i_r} = A^{[i_1 i_2 \dots i_r]}.$$

Similar statements apply to tensors of covariant type, for example

$$T_{[ij]} = \tfrac{1}{2}(T_{ij} - T_{ji}),$$
$$T_{[ijk]} = \tfrac{1}{6}(T_{ijk} + T_{jki} + T_{kij} - T_{ikj} - T_{jik} - T_{kji}).$$

Theorem 8.1 can be expressed in components as

$$T^{[[i_1 i_2 \dots i_r]} S^{j_{r+1} j_{r+2} \dots j_{r+s}]} = T^{[i_1 i_2 \dots i_r} S^{j_{r+1} j_{r+2} \dots j_{r+s}]},$$

or, with a slight generalization, square brackets occurring anywhere within square brackets may always be eliminated,

$$T^{[i_1 \dots [i_k \dots i_{k+l}] \dots i_r]} = T^{[i_1 \dots i_k \dots i_{k+l} \dots i_r]}.$$

By the antisymmetry of its components every r-vector A can be written

$$\begin{aligned} A &= A^{i_1 i_2 \dots i_r} e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r} \\ &= A^{i_1 i_2 \dots i_r} e_{i_1 i_2 \dots i_r} \end{aligned} \tag{8.3}$$

where

$$e_{i_1 i_2 \dots i_r} = \frac{1}{r!} \sum_\sigma (-1)^\sigma e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)a}} \otimes \dots \otimes e_{i_{\sigma(r)a}}. \tag{8.4}$$

For example

$$e_{12} = \tfrac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1),$$
$$\begin{aligned} e_{123} = \tfrac{1}{6}\big(&e_1 \otimes e_2 \otimes e_3 - e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_3 \otimes e_1 \\ &- e_2 \otimes e_1 \otimes e_3 + e_3 \otimes e_1 \otimes e_2 - e_3 \otimes e_2 \otimes e_1\big), \text{ etc.} \end{aligned}$$

The r-vectors $e_{i_1 \dots i_r}$ have the property

$$e_{i_1 \dots i_r} = \begin{cases} 0 & \text{if any pair of indices are equal,} \\ (-1)^\pi e_{i_{\pi(1)} \dots i_{\pi(r)}} & \text{for any permutation } \pi \text{ of } 1, 2, \dots, r. \end{cases} \tag{8.5}$$

Hence the expansion (8.3) can be reduced to one in which every term has $i_1 < i_2 < \dots < i_r$,

$$A = r! \underbrace{\sum \dots \sum}_{i_1 < i_2 < \dots < i_r} A^{i_1 i_2 \dots i_r} e_{i_1 i_2 \dots i_r}. \tag{8.6}$$

Hence $\Lambda^r(V)$ is spanned by the set

$$E_r = \{e_{i_1 i_2 \dots i_r} \mid i_1 < i_2 < \dots < i_r\}.$$

Furthermore this set is linearly independent, for if there were a linear relation

$$0 = \underbrace{\sum \cdots \sum}_{i_1 < i_2 < \cdots < i_r} B^{i_1 i_2 \dots i_r} \, e_{i_1 i_2 \dots i_r},$$

application of this multilinear function to arguments $\varepsilon^{j_1}, \varepsilon^{j_2}, \dots, \varepsilon^{j_r}$ with $j_1 < j_2 \cdots < j_r$ gives

$$0 = \underbrace{\sum \cdots \sum}_{i_1 < i_2 < \cdots < i_r} B^{i_1 i_2 \dots i_r} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \dots \delta^{j_r}_{i_r} = B^{j_1 j_2 \dots j_r}.$$

Hence E_r forms a basis of $\Lambda^r(V)$.

The dimension of the vector space $\Lambda^r(V)$ is the number of subsets $\{i_1 < i_2 < \cdots < i_r\}$ occurring in the first n integers $\{1, 2, \dots, n\}$,

$$\dim \Lambda^r(V) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

In particular $\dim \Lambda^n(V) = 1$, while $\dim \Lambda^{(n+k)}(V) = 0$ for all $k > 0$. The latter follows from the fact that if $r > n$ then all r-vectors $e_{i_1 i_2 \dots i_r}$ vanish, since some pair of indices must be equal.

An analogous argument shows that the set of r-forms

$$\mathcal{E}_r = \{\varepsilon^{i_1 \dots i_r} \mid i_i < \cdots < i_r\}, \tag{8.7}$$

where

$$\varepsilon^{i_1 \dots i_r} = \frac{1}{r!} \sum_\pi (-1)^\pi \varepsilon^{i_{\pi(1)}} \otimes \varepsilon^{i_{\pi(2)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(r)}}, \tag{8.8}$$

is a basis of $\Lambda^{*r}(V)$ and the dimension of the space of r-forms is also $\binom{n}{r}$.

8.3 Exterior product

The vector space $\Lambda(V)$ is defined to be the direct sum

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V).$$

Elements of $\Lambda(V)$ are called **multivectors**, written

$$A = A_0 + A_1 + \cdots + A_n \quad \text{where} \quad A_r \in \Lambda^r(V).$$

As shown in Section 6.4,

$$\dim(\Lambda(V)) = \sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

For any r-vector A and s-vector B we define their **exterior product** or **wedge product** $A \wedge B$ to be the $(r+s)$-vector

$$A \wedge B = \mathcal{A}(A \otimes B), \tag{8.9}$$

and extend to all of $\Lambda(V)$ by linearity,

$$(aA + bB) \wedge C = aA \wedge C + bB \wedge C, \qquad A \wedge (aB + bC) = aA \wedge B + bA \wedge C.$$

The wedge product of a 0-vector, or scalar, a with an r-vector A is simply scalar multiplication, since

$$a \wedge A = \mathcal{A}(a \otimes A) = \mathcal{A}(aA) = a\mathcal{A}A = aA.$$

For general multivectors $(\sum_a A_a)$ and $(\sum_b B_b)$ we have

$$\Big(\sum_a A_a\Big) \wedge \Big(\sum_b B_b\Big) = \sum_a \sum_b A_a \wedge B_b.$$

The associative law holds by Theorem 8.1 and the associative law for tensor products,

$$\begin{aligned} A \wedge (B \wedge C) &= \mathcal{A}(A \otimes \mathcal{A}(B \otimes C)) \\ &= \mathcal{A}(A \otimes (B \otimes C)) \\ &= \mathcal{A}((A \otimes B) \otimes C) \\ &= \mathcal{A}(\mathcal{A}(A \otimes B) \otimes C) \\ &= (A \wedge B) \wedge C. \end{aligned}$$

The space $\Lambda(V)$ with wedge product $\wedge$ is therefore an associative algebra, called the **exterior algebra** over V. There is no ambiguity in writing expressions such as $A \wedge B \wedge C$. Since the exterior product has the property

$$\wedge \,:\, \Lambda^r(V) \times \Lambda^s(V) \to \Lambda^{r+s}(V),$$

it is called a *graded product* and the exterior algebra $\Lambda(V)$ is called a *graded algebra*.

Example 8.1 If u and v are vectors then their exterior product has the property

$$\begin{aligned} (u \wedge v)(\omega, \rho) &= \mathcal{A}(u \otimes v)(\omega, \rho) \\ &= \mathcal{A}(u \otimes v)(\omega, \rho) \\ &= \tfrac{1}{2}(u(\omega)v(\rho) - u(\rho)v(\omega)), \end{aligned}$$

whence

$$u \wedge v = \tfrac{1}{2}(u \otimes v - v \otimes u) = -v \wedge u. \tag{8.10}$$

Obviously $u \wedge u = 0$. Setting $\omega = \varepsilon^i$ and $\rho = \varepsilon^j$ in the derivation of (8.10) gives

$$(u \wedge v)^{ij} = \tfrac{1}{2}(u^i v^j - u^j v^i).$$

In many textbooks exterior product $A \wedge B$ is defined as $\dfrac{(r+s)!}{r!s!}\mathcal{A}(A \otimes B)$, in which case the factor $\frac{1}{2}$ does not appear in these formulae.

The anticommutation property (8.10) is easily generalized to show that for any permutation π of $1, 2, \dots, r$,

$$u_{\pi(1)} \wedge u_{\pi(2)} \wedge \cdots \wedge u_{\pi(r)} = (-1)^{\pi} u_1 \wedge u_2 \wedge \cdots \wedge u_r. \tag{8.11}$$

The basis r-vectors $e_{i_1\dots i_r}$ defined in Eq. (8.4) can clearly be written

$$e_{i_1 i_2 \dots i_r} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}, \tag{8.12}$$

and the permutation property (8.5) is equivalent to Eq. (8.11).

For any pair of basis elements $e_{i_1\dots i_r} \in \Lambda^r(V)$ and $e_{j_1\dots j_s} \in \Lambda^s(V)$ it follows immediately from Eq. (8.12) that

$$e_{i_1\dots i_r} \wedge e_{j_1\dots j_s} = e_{i_1\dots i_r j_1\dots j_s}. \tag{8.13}$$

These expressions permit us to give a unique expression for the exterior products of arbitrary multivectors, for if A is an r-vector and B an s-vector,

$$A = r! \underbrace{\sum \cdots \sum}_{i_1 < i_2 < \cdots < i_r} A^{i_1\dots i_r} e_{i_1\dots i_r}, \qquad B = s! \underbrace{\sum \cdots \sum}_{j_1 < j_2 < \cdots < j_s} B^{j_1\dots j_s} e_{j_1\dots j_s},$$

then

$$A \wedge B = r!s! \underbrace{\sum \cdots \sum}_{i_1 < \cdots < i_r} \underbrace{\sum \cdots \sum}_{j_1 < \cdots < j_s} A^{i_1\dots i_r} B^{j_1\dots j_s} e_{i_1\dots i_r j_1\dots j_s}. \tag{8.14}$$

Alternatively, the formula for wedge product can be written

$$\begin{aligned} A \wedge B &= A^{i_1 i_2 \dots i_r} e_{i_1 i_2 \dots i_r} \wedge B^{j_1 j_2 \dots j_s} e_{j_1 j_2 \dots j_s} \\ &= A^{i_1\dots i_r} B^{j_1\dots j_s} e_{i_1\dots i_r j_1\dots j_s} \\ &= A^{[i_1\dots i_r} B^{j_1\dots j_s]} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{j_1} \otimes \cdots \otimes e_{j_s} \end{aligned}$$

on using Eq. (8.4). The tensor components of $A \wedge B$ are thus

$$(A \wedge B)^{i_1\dots i_r j_1\dots j_s} = A^{[i_1\dots i_r} B^{j_1\dots j_s]}. \tag{8.15}$$

Example 8.2 If u and v are 1-vectors, then

$$(u \wedge v)^{ij} = u^{[i} v^{j]} = \tfrac{1}{2}(u^i v^j - u^j v^i)$$

as in Example 8.1. For exterior product of a 1-vector u and a 2-vector A we find, using the skew symmetry $A^{jk} = -A^{kj}$,

$$\begin{aligned} (u \wedge A)^{ijk} &= u^{[i} A^{jk]} \\ &= \tfrac{1}{6}\big(u^i A^{jk} - u^i A^{kj} + u^j A^{ki} - u^j A^{ik} + u^k A^{ij} - u^k A^{ji}\big) \\ &= \tfrac{1}{3}\big(u^i A^{jk} + u^j A^{ki} + u^k A^{ij}\big). \end{aligned}$$

The wedge product of three vectors is

$$\begin{aligned} u \wedge v \wedge w = \tfrac{1}{6}(&u \otimes v \otimes w + v \otimes w \otimes u + w \otimes u \otimes v \\ &- u \otimes w \otimes v - w \otimes v \otimes u - v \otimes u \otimes w). \end{aligned}$$

In components,

$$\begin{aligned} (u \wedge v \wedge w)^{ijk} &= u^{[i} v^j w^{k]} \\ &= \tfrac{1}{6}\big(u^i v^j w^k - u^i v^k w^j + u^j v^k w^i - u^j v^i w^k + u^k v^i w^j - u^k v^j w^i\big). \end{aligned}$$

Continuing in this way, the wedge product of any r vectors $u_1, u_2, \dots, u_r$ results in the r-vector

$$u_1 \wedge u_2 \wedge \dots \wedge u_r = \frac{1}{r!} \sum_{\pi} (-1)^{\pi} u_{\pi(1)} \otimes u_{\pi(2)} \otimes \dots \otimes u_{\pi(r)}$$

which has components

$$(u_1 \wedge u_2 \wedge \dots \wedge u_r)^{i_1 i_2 \dots i_r} = u_1{}^{[i_1} u_2{}^{i_2} \dots u_r{}^{i_r]}.$$

The anticommutation rule for vectors, $u \wedge v = -v \wedge u$, generalizes for an r-vector A and s-vector B to

$$A \wedge B = (-1)^{rs} B \wedge A. \tag{8.16}$$

This result has been proved in Section 6.4, Eq. (6.20). It follows from

$$\begin{aligned} A \wedge B &= A^{i_1 \dots i_r} B^{j_1 \dots j_s} e_{i_1 \dots i_r j_1 \dots j_s} \\ &= (-1)^{rs} B^{j_1 \dots j_s} A^{i_1 \dots i_r} e_{j_1 \dots j_s i_1 \dots i_r} \end{aligned}$$

since rs interchanges are needed to bring the indices $j_1, \dots, j_r$ in front of the indices $i_1, \dots, i_s$.

If r is even then A commutes with all multivectors, while a pair of odd degree multivectors always anticommute. For example if u is a 1-vector and A a 2-vector, then $u \wedge A = A \wedge u$, since

$$(u \wedge A)^{ijk} = u^{[i} A^{jk]} = A^{[jk} u^{i]} = A^{[ij} u^{k]} = (A \wedge u)^{ijk}.$$

The space of *multiforms* is defined in a totally analagous manner,

$$\Lambda^*(V) = \Lambda(V^*) = \Lambda_0^*(V) \oplus \Lambda_1^*(V) \oplus \Lambda_2^*(V) \oplus \dots \oplus \Lambda_n^*(V),$$

with an exterior product

$$\alpha \wedge \beta = \mathcal{A}(\alpha \otimes \beta) \tag{8.17}$$

having identical properties to the wedge product on multivectors,

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma \quad \text{and} \quad \alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha.$$

The basis r-forms defined in Eq. (8.7) can be written as

$$\varepsilon^{i_1 \dots i_r} = \varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_r},$$

and the component expression for exterior product of a pair of forms is

$$(\alpha \wedge \beta)_{i_1 \dots i_r j_1 \dots j_s} = \alpha_{[i_1 \dots i_r} \beta_{j_1 \dots j_s]}.$$

Simple p-vectors and subspaces

A **simple p-vector** is one that can be written as a wedge product of 1-vectors,

$$A = v_1 \wedge v_2 \wedge \dots \wedge v_p, \quad v_i \in \Lambda^1(V) = V.$$

Similarly a **simple** p-form α is one that is decomposable into a wedge product of 1-forms,

$$\alpha = \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^p, \quad \alpha^i \in \Lambda^{*1}(V) = V^*.$$

Let W be a p-dimensional subspace of V. For any basis $e_1, e_2, \dots, e_p$ of W, define the p-vector $E_W = e_1 \wedge e_2 \wedge \cdots \wedge e_p$. If $e'_1, e'_2, \dots, e'_p$ is a second basis then for some coefficients $B^i_{i'}$

$$e'_{i'} = \sum_{i=1}^{p} B^i_{i'} e_i,$$

and the p-vector corresponding to this basis is

$$\begin{aligned} E'_W &= e'_1 \wedge \cdots \wedge e'_p \\ &= \sum_{i_1} \cdots \sum_{i_p} B^{i_1}_1 B^{i_2}_2 \dots B^{i_p}_p e_{i_1} \wedge e_{i_2} \wedge e_{i_p} \\ &= \sum_{\pi} (-1)^{\pi} B^{i_{\pi(1)}}_1 B^{i_{\pi(2)}}_2 \dots B^{i_{\pi(p)}}_p \, e_1 \wedge e_2 \wedge \cdots \wedge e_p \\ &= \det[B^i_{j'}] E_W. \end{aligned}$$

Hence the subspace W corresponds uniquely, up to a multiplying factor, to a simple p-vector E_W.

Theorem 8.2 *A vector u belongs to W if and only if $u \wedge E_W = 0$.*

Proof: This statement is an immediate corollary of Theorem 6.2. ■

Problems

Problem 8.1 Express components of the exterior product of two 2-vectors $A = A^{ij} e_{ij}$ and $B = B^{kl} e_{kl}$ as a sum of six terms,

$$(A \wedge B)^{ijkl} = \tfrac{1}{6}(A^{ij} B^{kl} + A^{ik} B^{lj} + \dots).$$

How many terms would be needed for a product of a 2-vector and a 4-vector? Show that in general the components of the exterior product of an r-vector and an s-vector can be expressed as a sum of $\dfrac{(r+s)!}{r!s!}$ terms.

Problem 8.2 Let V be a four-dimensional vector space with basis $\{e_1, e_2, e_3, e_4\}$, and A a 2-vector on V.

(a) Show that a vector u satisfies the equation

$$A \wedge u = 0$$

if and only if there exists a vector v such that

$$A = u \wedge v.$$

[*Hint*: Pick a basis such that $e_1 = u$.]

(b) If

$$A = e_2 \wedge e_1 + a e_1 \wedge e_3 + e_2 \wedge e_3 + c e_1 \wedge e_4 + b e_2 \wedge e_4$$

write out explicitly the equations $A \wedge u = 0$ where $u = u^1 e_1 + u^2 e_2 + u^3 e_3 + u^4 e_4$ and show that they have a solution if and only if $c = ab$. In this case find two vectors u and v such that $A = u \wedge v$.

(c) In general show that the 4-vector $A \wedge A = 8\alpha e_1 \wedge e_2 \wedge e_3 \wedge e_4$ where

$$\alpha = A^{12}A^{34} + A^{23}A^{14} + A^{31}A^{24},$$

and

$$\det[A^{ij}] = \alpha^2.$$

(d) Show that A is the wedge product of two vectors $A = u \wedge v$ if and only if $A \wedge A = 0$.

Problem 8.3 Prove **Cartan's lemma**, that if $u_1, u_2, \dots, u_p$ are linearly independent vectors and $v_1, \dots, v_p$ are vectors such that

$$u_1 \wedge v_1 + u_2 \wedge v_2 + \dots + u_p \wedge v_p = 0,$$

then there exists a symmetric set of coefficients $A_{ij} = A_{ji}$ such that

$$v_i = \sum_{j=1}^{p} A_i j u_j.$$

[*Hint*: Extend the u_i to a basis for the whole vector space V.]

Problem 8.4 If V is an n-dimensional vector space and A a 2-vector, show that there exists a basis $e_1, e_2, \dots, e_n$ of V such that

$$A = e_1 \wedge e_2 + e_3 \wedge e_4 + \dots e_{2r-1} \wedge e_{2r},$$

for some number $2r$, called the *rank* of A.
(a) Show that the rank only depends on the 2-vector A, not on the choice of basis, by showing that $A^r \neq 0$ and $A^{r+1} = 0$ where

$$A^p = \underbrace{A \wedge A \wedge \dots \wedge A}_{p}.$$

(b) If $f_1, f_2, \dots, f_n$ is any basis of V and $A = A^{ij} f_i \otimes f_j$ where $A^{ij} = -A^{ji}$, show that the rank of the matrix of components $\mathsf{A} = [A^{ij}]$ coincides with the rank as defined above.

Problem 8.5 Let V be an n-dimensional space and A an arbitrary $(n-1)$-vector.
(a) Show that the subspace V_A of vectors u such that $u \wedge A = 0$ has dimension $n - 1$.
(b) Show that every $(n-1)$-vector A is decomposable, $A = v_1 \wedge v_2 \wedge \dots \wedge v_{n-1}$ for some vectors $v_1, \dots, v_{n-1} \in V$. [*Hint*: Take a basis for $e_1, \dots, e_n$ of V such that the first $n-1$ vectors span the subspace V_A, which is always possible by Theorem 3.7, and expand A in terms of this basis.]

8.4 Interior product

Let u be a vector in V and α an r-form. We define the **interior product** $\mathrm{i}_u\alpha$ to be an $(r-1)$-form defined by

$$(\mathrm{i}_u\alpha)(u_2, \dots, u_r) = r\alpha(u, u_2, \dots, u_r). \tag{8.18}$$

The interior product of a vector with a scalar is assumed to vanish, $\mathrm{i}_u a = 0$ for all $a \in \Lambda^{*0}(V) \equiv \mathbb{R}$. The component expression with respect to any basis $\{e_i\}$ of the interior product of a vector with an r-form is given by

$$\begin{aligned}(\mathrm{i}_u\alpha)_{i_2\dots i_r} &= (\mathrm{i}_u\alpha)(e_{i_2}, \dots, e_{i_r})\\ &= r\alpha(u^i e_i,\ e_{i_2}, \dots, e_{i_r})\\ &= r u^i \alpha_{i i_2\dots i_r}.\end{aligned}$$

Hence

$$\mathrm{i}_u\alpha = rC^1_1(u \otimes \alpha),$$

where C^1_1 is the $(1, 1)$ contraction operator.

Performing the interior product with two vectors in succession on any r-form α has the property

$$\mathrm{i}_u(\mathrm{i}_v\alpha) = -\mathrm{i}_v(\mathrm{i}_u\alpha), \tag{8.19}$$

for

$$\begin{aligned}(\mathrm{i}_u(\mathrm{i}_v\alpha))(u_3, \dots, u_r) &= (r-1)(\mathrm{i}_v\alpha)(u,\ u_3, \dots, u_r)\\ &= (r-1)r\alpha(v,\ u,\ u_3, \dots, u_r)\\ &= -(r-1)r\alpha(u,\ v,\ u_3, \dots, u_r)\\ &= -(\mathrm{i}_v(\mathrm{i}_u\alpha))(u_3, \dots, u_r).\end{aligned}$$

It follows immediately that $(\mathrm{i}_u)^2 \equiv \mathrm{i}_u\mathrm{i}_u = 0$.

Another important identity, for an arbitrary r-form α and s-form β, is the **antiderivation law**

$$\mathrm{i}_u(\alpha \wedge \beta) = (\mathrm{i}_u\alpha) \wedge \beta + (-1)^r \alpha \wedge (\mathrm{i}_u\beta). \tag{8.20}$$

Proof: Let $u_1, u_2, \dots, u_{r+s}$ be arbitrary vectors. By Eqs. (8.18) and (8.17)

$$\begin{aligned}\big(\mathrm{i}_{u_1}(\alpha \wedge \beta)\big)(u_2, \dots, u_{r+s}) &= (r+s)\mathcal{A}(\alpha \otimes \beta)(u_1,\ u_2, \dots,\ u_{r+s})\\ &= \frac{1}{(r+s-1)!}\sum_\sigma (-1)^\sigma \alpha(u_{\sigma(1)}, \dots, u_{\sigma(r)})\beta(u_{\sigma(r+1)}, \dots, u_{\sigma(r+s)}).\end{aligned}$$

For each $1 \le a \le r+s$ let γ_a be the cyclic permutation $(1\,2\,\dots\,a)$. If σ is any permutation such that $\sigma(a) = 1$ then $\sigma = \sigma'\gamma_a$ where $\sigma'(1) = \sigma(a) = 1$. The signs of the permutations σ and σ' are related by $(-1)^\sigma = (-1)^{\sigma'}(-1)^{a+1}$, and the sum of permutations in the above equation may be written as

$$\begin{aligned}&\frac{1}{(r+s-1)!}\bigg(\sum_{a=1}^{r}\sum_{\sigma'}(-1)^{\sigma'}(-1)^{a+1}\alpha\big(u_{\sigma'(2)}, \dots, u_{\sigma'(a)}, u_1 \dots, u_{\sigma'(r)}\big)\\ &\quad \times \beta\big(u_{\sigma'(r+1)}, \dots, u_{\sigma'(r+s)}\big) + \sum_{b=1}^{s}\sum_{\sigma'}(-1)^{\sigma'}(-1)^{r+b+1}\alpha\big(u_{\sigma'(2)}, \dots, u_{\sigma'(r+1)}\big)\\ &\quad \times \beta\big(u_{\sigma'(r+2)}, \dots, u_{\sigma'(r+b)}, u_1, \dots, u_{\sigma'(r+s)}\big)\bigg).\end{aligned}$$

By cyclic permutations u_1 can be brought to the first argument of α and β respectively, introducing factors $(-1)^{a+1}$ and $(-1)^{b+1}$ in the two sums, to give

$$\frac{1}{(r+s-1)!}\Big(r\sum_{\sigma'}(-1)^{\sigma'}\alpha\big(u_1, u_{\sigma'(2)}, \dots, u_{\sigma'(a-1)}, u_{\sigma'(a+1)}, \dots, u_{\sigma'(r)}\big)$$
$$\times\,\beta\big(u_{\sigma'(r+1)}, \dots, u_{\sigma'(r+s)}\big) + s(-1)^r\sum_{\sigma'}(-1)^{\sigma'}\alpha\big(u_{\sigma'(1)}, \dots, u_{\sigma'(r)}\big)$$
$$\times\,\beta\big(u_1,\ u_{\sigma'(r+1)}, \dots, u_{\sigma'(r+b-1)}, u_{\sigma'(r+b+1)}, \dots, u_{\sigma'(r+s)}\big)\Big),$$

where σ' ranges over all permutations of $(2, 3, \dots, r+s)$. Thus

$$\begin{aligned} \mathrm{i}_{u_1}(\alpha\wedge\beta)(u_2, \dots, u_{r+s}) &= \mathcal{A}\big((\mathrm{i}_{u_1}\alpha)\otimes\beta + (-1)^r\alpha\otimes(\mathrm{i}_{u_1}\beta)\big)\big(u_2, \dots, u_{r+s}\big) \\ &= \big(\mathrm{i}_{u_1}\alpha\big)\wedge\beta + (-1)^r\alpha\wedge\big(\mathrm{i}_{u_1}\beta\big)(u_2, \dots, u_{r+s}). \end{aligned}$$

Equation (8.20) follows on setting $u = u_1$. ■

8.5 Oriented vector spaces

n-Vectors and n-forms

Let V be a vector space of dimension n with basis $\{e_1, \dots, e_n\}$ and dual basis $\{\varepsilon^1, \dots, \varepsilon^n\}$. Since the spaces $\Lambda^n(V)$ and $\Lambda^{*n}(V)$ are both one-dimensional, the n-vector

$$E = e_{12\dots n} = e_1\wedge e_2\wedge\dots\wedge e_n$$

forms a basis of $\Lambda^n(V)$, while the n-form

$$\Omega = \varepsilon^{12\dots n} = \varepsilon^1\wedge\dots\wedge\varepsilon^n$$

is a basis of $\Lambda^{*n}(V)$. These will sometimes be referred to as **volume elements** associated with this basis.

Exercise: Show that every non-zero n-vector is the volume element associated with some basis of V.

Given a basis $\{e_1, \dots, e_n\}$, every n-vector A has a unique expansion

$$\begin{aligned} A &= aE = a\,e_1\wedge e_2\wedge\dots\wedge e_n \\ &= \frac{a}{n!}\sum_{\sigma}(-1)^{\sigma}e_{\sigma(1)}\otimes e_{\sigma(2)}\otimes\dots\otimes e_{\sigma(n)} \\ &= \frac{a}{n!}\epsilon^{i_1 i_2\dots i_n}e_{i_1}\otimes e_{i_2}\otimes\dots\otimes e_{i_n} \end{aligned}$$

where the **ϵ-symbols**, or **Levi–Civita symbols**, $\epsilon^{i_1 i_2\dots i_n}$ and $\epsilon_{i_1 i_2\dots i_n}$ are defined by

$$\epsilon^{i_1 i_2\dots i_n} = \epsilon_{i_1 i_2\dots i_n} = \begin{cases} 1 & \text{if } i_1\dots i_n \text{ is an even permutation of } 1, 2, \dots, n, \\ -1 & \text{if } i_1\dots i_n \text{ is an odd permutation of } 1, 2, \dots, n, \\ 0 & \text{if any pair of indices are equal.} \end{cases} \tag{8.21}$$

The ϵ-symbols are clearly antisymmetric in any pair of indices.

Exercise: Show that any n-form β has a unique expansion

$$\beta = b\Omega = \frac{b}{n!}\epsilon_{i_1\dots i_n}\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_n}.$$

Every n-vector and n-form has tensor components proportional to the ϵ-symbols,

$$A^{i_1\dots i_n} = \frac{a}{n!}\epsilon^{i_1\dots i_n},$$
$$\beta_{i_1\dots i_n} = \frac{b}{n!}\epsilon_{i_1\dots i_n},$$

and setting $a = b = 1$ we have

$$E^{i_1\dots i_n} = \frac{1}{n!}\epsilon^{i_1\dots i_n}, \qquad \Omega_{i_1\dots i_n} = \frac{1}{n!}\epsilon_{i_1\dots i_n}. \tag{8.22}$$

Transformation laws of n-vectors and n-forms

The transformation matrix $\mathsf{A} = [A^{i'}_i]$ appearing in Eq. (7.26) satisfies

$$A^{i'_1}_{i_1} A^{i'_2}_{i_2} \dots A^{i'_n}_{i_n} \epsilon^{i_1 i_2 \dots i_n} = \det[A^{i'}_i]\, \epsilon^{i'_1 i'_2 \dots i'_n}. \tag{8.23}$$

Proof: If $i'_1 = 1, i'_2 = 2, \dots, i'_n = n$ then the left-hand side is the usual expansion of the determinant of the matrix A as a sum of products of its elements taken from different rows and columns with appropriate $\pm$ signs, while the right-hand side is $\det \mathsf{A}\epsilon^{12\dots n} = \det \mathsf{A}$. From the antisymmetry of the epsilon symbol in any pair of indices i and j we have

$$\dots A^{i'}_i \dots A^{j'}_j \dots \epsilon^{\dots i \dots j \dots} = -\dots A^{i'}_j \dots A^{j'}_i \dots \epsilon^{\dots i \dots j \dots} = -\dots A^{j'}_i \dots A^{i'}_j \dots \epsilon^{\dots i \dots j \dots},$$

and the whole expression is antismmetric in any pair of indices i', j'. In particular, it vanishes if $i' = j'$. Hence if π is any permutation of $(1, 2, \dots, n)$ such that $i'_1 = \pi(1)$, $i'_2 = \pi(2), \dots, i'_n = \pi(n)$ then both sides of Eq. (8.23) are multiplied by the sign of the permutation $(-1)^\pi$, while if any pair of indices $i'_1 \dots i'_n$ are equal, both sides of the equation vanish. ■

If $A = aE$ is any n-vector, then from the law of transformation of tensor components, Eq. (7.30),

$$\begin{aligned} A^{i'_1\dots i'_n} &= A^{i_1\dots i_n} A^{i'_1}_{i_1} \dots A^{i'_n}_{i_n} \\ &= \frac{a}{n!}\epsilon^{i_1\dots i_n} A^{i'_1}_{i_1} \dots A^{i'_n}_{i_n} \\ &= \det[A^{j'}_i]\,\frac{a}{n!}\epsilon^{i'_1\dots i'_n}. \end{aligned}$$

Setting $A = a'E'$, the factor a is seen to transform as

$$a' = a \det[A^{i'}_j] = a \det \mathsf{A}. \tag{8.24}$$

If $a = 1$ we arrive at the transformation law of volume elements,

$$E = \det \mathsf{A}\, E', \qquad E' = \det \mathsf{A}^{-1}\, E. \tag{8.25}$$

A similar formula to Eq. (8.23) holds for the inverse matrix $\mathsf{A}' = [A'^{i}{}_{j'}]$,

$$A^{i_1}_{i'_1}\, A^{i_2}_{i'_2} \ldots A^{i_n}_{i'_n}\, \epsilon_{i_1 i_2 \ldots i_n} = \det[A^{i}_{i'}]\, \epsilon_{i'_1 i'_2 \ldots i'_n}, \tag{8.26}$$

and the transformation law for an n-form $\beta = b\Omega = \frac{b}{n!}\epsilon^{12\ldots n} = b'\Omega'$ is

$$b' = b \det[A'^{i}{}_{j'}] = b \det \mathsf{A}^{-1} \quad \text{that is} \quad b = b' \det \mathsf{A}, \tag{8.27}$$

$$\Omega = \det \mathsf{A}^{-1}\, \Omega', \quad \Omega' = \det \mathsf{A}\, \Omega. \tag{8.28}$$

Exercise: Prove Eqs. (8.26)–(8.28).

Note, from Eqs. (8.23) and (8.26), that the ϵ-symbols do not transform as components of tensors under general basis transformations. They do however transform as tensors with respect to the restricted set of unimodular transformations, having $\det \mathsf{A} = \det[A^{j'}_{i}] = 1$. In particular, for cartesian tensors they transform as tensors provided only proper orthogonal transformations, rotations, are permitted. The term 'tensor density' is sometimes used to refer to entities that include determinant factors like those in (8.23) and (8.26), while scalar quantities that transform like a or b in (8.24) and (8.27) are referred to as 'densities'.

Oriented vector spaces

Two bases $\{e_i\}$ and $\{e'_{i'}\}$ are said to have the **same orientation** if the transformation matrix $\mathsf{A} = [A^{i'}_{j}]$ in Eq. (7.26) has positive determinant, $\det \mathsf{A} > 0$; otherwise they are said to be **oppositely oriented**. Writing $\{e_i\} o \{e'_{i'}\}$ iff $\{e_i\}$ and $\{e'_{i'}\}$ have the same orientation, it is straightforward to show that o is an equivalence relation and divides the set of all bases on V into two equivalence classes, called **orientations**. A vector space V together with a choice of orientation is called an **oriented vector space**. Any basis belonging to the selected orientation will be said to be **positively oriented**, while oppositely oriented bases will be called **negatively oriented**.

Example 8.3 Euclidean three-dimensional space $\mathbb{E}^3$ together with choice of a right-handed orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an oriented vector space. The orientation consists of the set of all bases related to $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ through a positive determinant transformation. A left-handed set of axes has opposite orientation since the basis transformation will involve a reflection, having negative determinant.

Example 8.4 Let V be an n-dimensional vector space. Denote the set of all volume elements on V by $\dot{\Lambda}^n(V)$ and $\dot{\Lambda}^{*n}(V)$. Two non-zero n-vectors A and B can be said to **have the same orientation** if $A = cB$ with $c > 0$. This clearly provides an equivalence relation on $\dot{\Lambda}^n(V)$, dividing it into two non-intersecting equivalence classes. A selection of one of these two classes is an alternative way of specifying an orientation on a vector space V, for we may stipulate that a basis $\{e_i\}$ has positive orientation if $A = aE$ with $a > 0$ for all volume elements A in the chosen class. By Eqs. (8.24) and (8.25), this is equivalent to dividing the set of bases on V into two classes.

Now let V be an oriented n-dimensional real inner product space having index $t = r - s$ where $n = r + s$. If $\{e_1, e_2, \dots, e_n\}$ is any positively oriented orthonormal frame such that

$$e_i \cdot e_j = \eta_{ij} = \begin{cases} \eta_i = \pm 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

then r is the number of $+1$'s and s the number of -1's among the η_i. As pseudo-orthogonal transformations all have determinant ± 1, those relating positively oriented orthonormal frames must have $\det = 1$. Hence, by Eq. (8.25), the volume element $E = e_1 \wedge e_2 \wedge \dots \wedge e_n$ is independent of the choice of positively oriented orthonormal basis and is entirely determined by the inner product and the orientation on V.

By Eq. (8.22), the components of the volume element E with respect to any positively oriented orthonormal basis are

$$E^{i_1 i_2 \dots i_n} = \frac{1}{n!} \epsilon^{i_1 i_2 \dots i_n}.$$

With respect to an arbitrary positively oriented basis e'_i, not necessarily orthonormal, the components of E are, by Eqs. (7.30) and (8.23),

$$E'^{i'_1 i'_2 \dots i'_n} = \det[A^{i'}_{i}] \frac{1}{n!} \epsilon^{i'_1 i'_2 \dots i'_n}. \tag{8.29}$$

Take note that these are the components of the volume element E determined by the original orthonormal basis expressed with respect to the new basis, *not* the components of the volume element $e'_1 \wedge e'_2 \wedge \dots \wedge e'_n$ determined by the new basis. It is possible to arrive at a formula for the components on the right-hand side of Eq. (8.29) that is independent of the transformation matrix A. Consider the transformation of components of the metric tensor, defined by $u \cdot v = g_{ij} u^i v^j$,

$$g'_{i'j'} = g_{ij} A'^{i}_{i'} A'^{j}_{j'},$$

which can be written in matrix form

$$\mathsf{G}' = (\mathsf{A}^{-1})^T \mathsf{G} \mathsf{A}^{-1} \quad \text{where} \quad \mathsf{G} = [g_{ij}], \ \mathsf{G}' = [g'_{i'j'}].$$

On taking determinants

$$g' = g(\det \mathsf{A})^{-2} \quad \text{where} \quad g = \det \mathsf{G} = \pm 1, \ g' = \det \mathsf{G}',$$

and subsituting in (8.29) we have

$$E'^{i'_1 i'_2 \dots i'_n} = \frac{1}{n! \sqrt{|g'|}} \epsilon^{i'_1 i'_2 \dots i'_n}.$$

Eliminating the primes, it follows that the components of the volume element E defined by the inner product can be written in an arbitrary positively oriented basis as

$$E^{i_1 i_2 \dots i_n} = \frac{1}{n! \sqrt{|g|}} \epsilon^{i_1 i_2 \dots i_n}. \tag{8.30}$$

On lowering the indices of E we have

$$\begin{aligned} E_{i_1 i_2 \dots i_n} &= g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_n j_n} E^{j_1 j_2 \dots j_n} \\ &= \frac{1}{n! \sqrt{|g|}} g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_n j_n} \epsilon^{j_1 j_2 \dots j_n} \\ &= \frac{1}{n! \sqrt{|g|}} g \epsilon_{i_1 i_2 \dots i_n}. \end{aligned}$$

Since the sign of the determinant g is equal to $(-1)^s$ we have, in any positively oriented basis,

$$E_{i_1 i_2 \dots i_n} = (-1)^s \frac{\sqrt{|g|}}{n!} \epsilon_{i_1 i_2 \dots i_n}. \tag{8.31}$$

Exercise: Show that the components of the n-form $\Omega = \varepsilon^{12 \dots n}$ defined by a positively oriented o.n. basis are

$$\Omega_{i_1 i_2 \dots i_n} = \frac{\sqrt{|g|}}{n!} \epsilon_{i_1 i_2 \dots i_n} = (-1)^s E_{i_1 i_2 \dots i_n}. \tag{8.32}$$

ϵ-*Symbol identities*

The ϵ-symbols satisfy a number of fundamental identities, the most general of which is

$$\epsilon_{i_1 \dots i_n} \epsilon^{j_1 \dots j_n} = \delta^{j_1 \dots j_n}_{i_1 \dots i_n}, \tag{8.33}$$

where the generalized δ-symbol is defined by

$$\delta^{j_1 \dots j_r}_{i_1 \dots i_r} = \begin{cases} 1 & \text{if } j_1 \dots j_r \text{ is an even permutation of } i_1 \dots i_r, \\ -1 & \text{if } j_1 \dots j_r \text{ is an odd permutation of } i_1 \dots i_r, \\ 0 & \text{otherwise.} \end{cases} \tag{8.34}$$

Total contraction of (8.33) over all indices gives

$$\epsilon_{i_1 \dots i_n} \epsilon^{i_1 \dots i_n} = n!. \tag{8.35}$$

Contracting (8.33) over the first $n - 1$ indices gives

$$\epsilon_{i_1 \dots i_{n-1} j} \epsilon^{i_1 \dots i_{n-1} k} = (n-1)! \delta^k_j, \tag{8.36}$$

for if $k \neq j$ each term in the summation $\delta^{i_1 \dots i_{n-1} k}_{i_1 \dots i_{n-1} j}$ vanishes since in every summand either one pair of superscripts or one pair of subscripts must be equal, while if $k = j$ the expression is a sum of $(n-1)!$ terms each of value $+1$.

The most general contraction identity arising from (8.34) is

$$\epsilon_{i_1 \dots i_{n-r} j_1 \dots j_r} \epsilon^{i_1 \dots i_{n-r} k_1 \dots k_r} = (n-r)! \delta^{k_1 \dots k_r}_{j_1 \dots j_r}, \tag{8.37}$$

where the δ-symbol on the right-hand side can be expressed in terms of Kronecker deltas,

$$\delta^{k_1\dots k_r}_{j_1\dots j_r} = \sum_\sigma (-1)^\sigma \delta^{k_1}_{j_{\sigma(1)}} \delta^{k_2}_{j_{\sigma(2)}} \dots \delta^{k_r}_{j_{\sigma(r)}}$$
$$= \delta^{k_1}_{j_1}\delta^{k_2}_{j_2}\dots\delta^{k_r}_{j_r} - \delta^{k_1}_{j_2}\delta^{k_2}_{j_1}\dots\delta^{k_r}_{j_r} + \dots, \tag{8.38}$$

a sum of $r!$ terms in which the j_i indices run over every permutation of $j_1, j_2, \dots, j_r$.

Example 8.5 In three dimensions we have

$$\epsilon_{ijk}\epsilon^{lmn} = \delta^l_i\delta^m_j\delta^n_k - \delta^l_i\delta^n_j\delta^m_k + \delta^m_i\delta^n_j\delta^l_k - \delta^m_i\delta^l_j\delta^n_k + \delta^n_i\delta^l_j\delta^m_k - \delta^n_i\delta^m_j\delta^l_k,$$
$$\epsilon_{ijk}\epsilon^{imn} = \delta^m_j\delta^n_k - \delta^m_k\delta^n_j,$$
$$\epsilon_{ijk}\epsilon^{ijn} = 2\delta^n_k,$$
$$\delta_{ijk}\epsilon^{ijk} = 6.$$

The last three identities are particularly useful in cartesian tensors where the summation convention is used on repeated subscripts, giving

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}, \qquad \epsilon_{ijk}\epsilon_{ijn} = 2\delta_{nk}, \qquad \delta_{ijk}\epsilon_{ijk} = 6.$$

For example, the *vector product* $\mathbf{u}\times\mathbf{v}$ of two vectors $\mathbf{u} = u_i\mathbf{e}_i$ and $\mathbf{v} = v_i\mathbf{e}_i$ is defined as the vector whose components are given by

$$(\mathbf{u}\times\mathbf{v})_k = \epsilon_{ijk}(u\wedge v)^{ij} = \epsilon_{kij}u_iv_j.$$

The vector identity

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

follows from

$$\big(\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\big)_i = \epsilon_{ijk}u_j\epsilon_{klm}v_lw_m$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})u_jv_lw_m = (u_jw_j)v_i - (u_jv_j)w_i = \big((\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\big)_i.$$

Exercise: Show the identity $(\mathbf{u}\times\mathbf{v})^2 = \mathbf{u}^2\,\mathbf{v}^2 - (\mathbf{u}\cdot\mathbf{v})^2$.

8.6 The Hodge dual

Inner product of p-vectors

The coupling between linear functionals (1-forms) and vectors, denoted

$$\langle u, \omega\rangle \equiv \langle\omega, u\rangle = \omega(u) = u^iw_i,$$

can be extended to define a product between p-vectors A and p-forms β,

$$\langle A, \beta\rangle = p!C^1_1C^2_2\dots C^p_p A\otimes\beta = p!A^{i_1i_2\dots i_p}\beta_{i_1i_2\dots i_p}. \tag{8.39}$$

For each fixed p-form β the map $A \mapsto \langle A, \beta\rangle$ clearly defines a linear functional on the vector space $\Lambda^p(V)$.

Exercise: Show that

$$\langle u_1 \wedge u_2 \wedge \cdots \wedge u_p, \beta\rangle = p!\beta(u_1, u_2, \ldots, u_p). \tag{8.40}$$

Theorem 8.3 *If A is a p-vector, β a $(p+1)$-form and u an arbitrary vector, then*

$$\langle A, \mathrm{i}_u\beta\rangle = \langle u \wedge A, \beta\rangle. \tag{8.41}$$

Proof: For a simple p-vector, $A = u_1 \wedge u_2 \wedge \cdots \wedge u_p$, using Eqs. (8.40) and (8.18),

$$\begin{aligned}\langle u_1 \wedge u_2 \wedge \cdots \wedge u_p, \mathrm{i}_u\beta\rangle &= p!(\mathrm{i}_u\beta)(u_1, u_2, \ldots, u_p)\\ &= (p+1)!\beta(u, u_1, \ldots, u_p)\\ &= \langle u \wedge u_1 \wedge \cdots \wedge u_p, \beta\rangle.\end{aligned}$$

Since every p-vector A is a sum of simple p-vectors, this generalizes to arbitrary p-vectors by linearity. ■

If V is an inner product space it is possible to define the **inner product** of two p-vectors A and B to be

$$(A, B) = \langle A, \beta\rangle \tag{8.42}$$

where β is the tensor formed from B by lowering indices,

$$\beta_{i_1 i_2 \ldots i_p} = B_{i_1 i_2 \ldots i_p} = g_{i_1 j_1} g_{i_1 j_2} \ldots g_{i_p j_p} B^{j_1 j_2 \ldots j_p}.$$

Lemma 8.4 *Let X and Y be simple p-vectors, $X = x_1 \wedge x_2 \wedge \cdots \wedge x_p$ and $Y = y_1 \wedge y_2 \wedge \cdots \wedge y_p$, then*

$$(X, Y) = \det[x_i \cdot y_j]. \tag{8.43}$$

Proof: With respect to a basis $\{e_i\}$

$$\begin{aligned}(X, Y) &= p!x_1^{[i_1} x_2^{i_2} \ldots x_p^{i_p]} g_{i_1 j_1} g_{i_1 j_2} \ldots g_{i_p j_p} y_1^{[j_1} y_2^{j_2} \ldots y_p^{j_p]}\\ &= p!x_1^{[i_1} x_2^{i_2} \ldots x_p^{i_p]} y_{1\,i_1} y_{2\,i_2} \ldots y_{p\,i_p}\\ &= \sum_\sigma (-1)^\sigma x_1^{i_{\sigma(1)}} x_2^{i_{\sigma(2)}} \ldots x_p^{i_{\sigma(p)}} y_{1\,i_1} y_{2\,i_2} \ldots y_{p\,i_p}\\ &= \det[\langle x_i, \gamma_j\rangle] \quad \text{where } \gamma_j = y_{j\,k}\varepsilon^k\\ &= \det[g_{kl} x_i^k y_j^l]\\ &= \det[x_i \cdot y_j].\end{aligned}$$
■

Theorem 8.5 *The map $A, B \mapsto (A, B)$ makes $\Lambda^p(V)$ into a real inner product space.*

Proof: From Eqs. (8.42) and (8.39)

$$(A, B) = p!A^{i_1 i_2 \ldots i_p} g_{i_1 j_1} g_{i_1 j_2} \ldots g_{i_p j_p} B^{j_1 j_2 \ldots j_p} = (B, A)$$

so that (A, B) is a symmetric bilinear function of A and B. It remains to show that $(.,.)$ is non-singular, satisfying (SP3) of Section 5.1.

Let $e_1, e_2, \dots, e_n$ be an orthonormal basis,

$$e_i \cdot e_j = \eta_i \delta_{ij}, \quad \eta_i = \pm 1,$$

and for any arbitrary increasing sequences of indices $\mathbf{h} = h_1 < h_2 < \dots < h_p$ set $e_{\mathbf{h}}$ to be the basis p-vector

$$e_{\mathbf{h}} = e_{h_1} \wedge e_{h_2} \wedge \dots \wedge e_{h_p}.$$

If $\mathbf{h} = h_1 < h_2 < \dots < h_p$ and $\mathbf{k} = k_1 < k_2 < \dots < k_p$ are any pair of increasing sequences of indices and $h_i \notin \{k_1, k_2, \dots, k_p\}$ for some i, then, by Lemma 8.4

$$(e_{\mathbf{h}}, e_{\mathbf{k}}) = \det[e_{h_i} \cdot e_{k_j}] = 0$$

since the ith row of the determinant vanishes completely. On the other hand, if $\mathbf{h} = \mathbf{k}$ we have

$$(e_{\mathbf{h}}, e_{\mathbf{h}}) = \eta_{h_1} \eta_{h_2} \dots \eta_{h_p}.$$

In summary,

$$(e_{\mathbf{h}}, e_{\mathbf{k}}) = \det[e_{h_i} \cdot e_{k_j}] = \begin{cases} 0 & \text{if } \mathbf{h} \neq \mathbf{k}, \\ \eta_{h_1} \eta_{h_2} \dots \eta_{h_p} & \text{if } \mathbf{h} = \mathbf{k}, \end{cases} \tag{8.44}$$

so $E_p = \{e_{\mathbf{h}} \mid h_1 < h_2 < \dots < h_p\}$ forms an orthonormal basis with respect to the inner product on $\Lambda^p(V)$. The matrix of the inner product is non-singular with respect to this basis since it is diagonal with ± 1's along the diagonal. ■

Exercise: Show that $(E, E) = (-1)^s$ where $E = e_{12\dots n}$ and s is the number of $-$ signs in g_{ij}.

An inner product can of course be defined on $\Lambda^{*p}(V)$ in exactly the same way as for $\Lambda^p(V)$,

$$(\alpha, \beta) = \langle A, \beta \rangle = p!\, A^{i_1 i_2 \dots i_p} \beta_{i_1 i_2 \dots i_p} \tag{8.45}$$

where A is the tensor formed from α by raising indices,

$$A^{i_1 i_2 \dots i_p} = \alpha^{i_1 i_2 \dots i_p} = g^{i_1 j_1} g^{i_1 j_2} \dots g^{i_p j_p} \alpha_{j_1 j_2 \dots j_p}.$$

Exercise: Show that Theorem 8.3 can be expressed in the alternative form: for any p-form α, $(p+1)$-form β and vector u

$$(\alpha, \mathrm{i}_u \beta) = (\bar{g}u \wedge \alpha, \beta) \tag{8.46}$$

where $\bar{g}u$ is the 1-form defined in Example 7.8 by lowering the index of u^i.

The Hodge star operator

Let V be an oriented inner product space and $\{e_1, e_2, \dots, e_n\}$ be a positively oriented orthonormal basis, $e_i \cdot e_j \delta_{ij} \eta_i$, with associated volume element $E = e_{12\dots n} = e_1 \wedge e_2 \wedge$

$\cdots \wedge e_n$. For any $A \in \Lambda^p(V)$ the map $f_A : \Lambda^{n-p}(V) \to \mathbb{R}$ defined by $A \wedge B = f_A(B)E$ is linear since

$$f_A(B + aC)E = A \wedge (B + aC) = A \wedge B + aA \wedge C = \big(f_A(B) + af_A(C)\big)E.$$

Thus f_A is a linear functional on $\Lambda^{n-p}(V)$, and as the inner product $(.\,,.)$ on $\Lambda^{n-p}(V)$ is non-singular there exists a unique $(n-p)$-vector $*A$ such that $f_A(B) = (*A, B)$. The $(n-p)$-vector $*A$ is uniquely determined by this equation, and is frequently referred to as the **(Hodge) dual** of A,

$$A \wedge B = (*A, B)E \qquad \text{for all } B \in \Lambda^{(n-p)}(V). \tag{8.47}$$

The one-to-one map $* : \Lambda^p(V) \to \Lambda^{n-p}(V)$ is called the **Hodge star operator**; it assigns an $(n-p)$-vector to each p-vector, and vice versa. This reciprocity is only possible because the dimensions of these two vector spaces are identical,

$$\dim \Lambda^p(V) = \dim \Lambda^{n-p}(V) = \binom{n}{p} = \binom{n}{n-p} = \frac{n!}{p!(n-p)!}.$$

Since E is independent of the choice of positively oriented orthonormal basis, the Hodge dual is a basis-independent concept.

To calculate the components of the dual with respect to a positively oriented orthonormal basis $e_1, \dots, e_n$, set $\mathbf{i} = i_1 < i_2 < \cdots < i_p$ and $\mathbf{j} = j_1 < j_2 < \cdots < j_{n-p}$. Since

$$e_{\mathbf{i}} \wedge e_{\mathbf{j}} = \epsilon_{i_1 i_2 \dots i_p j_i \dots j_{n-p}} E,$$

we have from Eq. (8.47) that

$$(*e_{\mathbf{i}}, e_{\mathbf{j}}) = \epsilon_{i_1 i_2 \dots i_p j_i \dots j_{n-p}},$$

and Eq. (8.44) gives

$$*e_{\mathbf{i}} = \epsilon_{i_1 i_2 \dots i_p j_i \dots j_{n-p}} (e_{\mathbf{j}}, e_{\mathbf{j}}) e_{\mathbf{j}} \tag{8.48}$$

where $\mathbf{j} = j_1 < j_2 < \cdots < j_{n-p}$ is the complementary set of indices to $i_1 < \cdots < i_p$. By a similar argument

$$*e_{\mathbf{j}} = \epsilon_{j_1 j_2 \dots j_{n-p} i_1 \dots i_p} (e_{\mathbf{i}}, e_{\mathbf{i}}) e_{\mathbf{i}},$$

and, temporarily suspending the summation convention,

$$\begin{aligned} * * e_{\mathbf{i}} &= \epsilon_{i_1 i_2 \dots i_p j_i \dots j_{n-p}} \epsilon_{j_1 j_2 \dots j_{n-p} i_1 \dots i_{n-p}} (e_{\mathbf{i}}, e_{\mathbf{i}})(e_{\mathbf{j}}, e_{\mathbf{j}}) e_{\mathbf{i}} \\ &= (-1)^{p(n-p)} \eta_{i_1} \eta_{i_2} \dots \eta_{i_p} \eta_{j_1} \eta_{j_2} \dots \eta_{j_{n-p}} e_{\mathbf{i}} \\ &= (-1)^{p(n-p)+s} e_{\mathbf{i}} \end{aligned}$$

where s is the number of -1's in g_{ij}. The coefficient s can also be written as $s = \frac{1}{2}(n-t)$ where t is the index of the metric. As any p-vector A is a linear combination of the $e_{\mathbf{i}}$, we have the identity

$$* * A = (-1)^{p(n-p)+s} A. \tag{8.49}$$

Theorem 8.6 *For any p-vectors A and B we also have the following identities*:

$$A \wedge *B = B \wedge *A = (-1)^s (A, B) E, \tag{8.50}$$

and

$$(*A, *B) = (-1)^s (A, B). \tag{8.51}$$

Proof: The first part of (8.50) follows from

$$A \wedge *B = (*A, *B)E = (*B, *A)E = B \wedge *A.$$

The second part of (8.50) follows on using Eqs. (8.16) and (8.49),

$$\begin{aligned} A \wedge *B &= (-1)^{p(n-p)} * B \wedge A \\ &= (-1)^{p(n-p)} * B \wedge A \\ &= (-1)^{p(n-p)} (* * B, A)E \\ &= (-1)^s (A, B)E. \end{aligned}$$

Using Eqs. (8.47) and (8.50) we have

$$(*A, *B)E = A \wedge *B = (-1)^s (A, B)E$$

and (8.51) follows at once since $E \neq 0$. ■

The component prescription for the Hodge dual in an o.n. basis is straightforward, and is left as an exercise

$$*A^{j_1 \dots j_{n-p}} = \frac{(-1)^s}{(n-p)!} \epsilon^{i_1 \dots i_p j_1 \dots j_{n-p}} A_{i_1 \dots i_p}. \tag{8.52}$$

Writing this equation as the component form of the tensor equation,

$$*A^{j_1 \dots j_{n-p}} = \frac{(-1)^s n!}{(n-p)!} E^{i_1 \dots i_p j_1 \dots j_{n-p}} A_{i_1 \dots i_p},$$

and using Eq. (8.30), we can express the Hodge dual in an arbitrary basis:

$$*A^{j_1 \dots j_{n-p}} = \frac{(-1)^s}{(n-p)! \sqrt{|g|}} \epsilon^{i_1 \dots i_p j_1 \dots j_{n-p}} A_{i_1 \dots i_p}. \tag{8.53}$$

Exercise: Show that on lowering indices, (8.53) can be written

$$*A_{j_1 \dots j_{n-p}} = \frac{\sqrt{|g|}}{(n-p)!} \epsilon_{i_1 \dots i_p j_1 \dots j_{n-p}} A^{i_1 \dots i_p}. \tag{8.54}$$

Example 8.6 Treating 1 as the basis 0-vector, we obtain from (8.48) that

$$*1 = \epsilon_{12 \dots n} (E, E) E = (-1)^s E.$$

Conversely the dual of the volume element E is 1,

$$*E = \epsilon_{12 \dots n} 1 = 1.$$

These two formulae agree with the double star formula (8.49) on setting $p = n$ or $p = 0$.

Example 8.7 In three-dimensional cartesian tensors, $s = 0$ and all indices are in the subscript position:

$$\begin{aligned}(*1)_{ijk} &= \epsilon_{ijk},\\ A = A_i e_i &\implies (*A)_{ij} = \tfrac{1}{2!}\epsilon_{kij}A_k = \tfrac{1}{2}\epsilon_{ijk}A_k,\\ A = A_{ij}e_i\otimes e_j &\implies (*A)_i = \epsilon_{jki}A_{jk} = \epsilon_{ijk}A_{jk},\\ *E &= \tfrac{1}{3!}\epsilon_{ijk}\epsilon_{ijk} = 1.\end{aligned}$$

The concept of *vector product* of any two vectors $\mathbf{u}\times\mathbf{v}$ is the dual of the wedge product, for

$$\mathbf{u}\equiv u = u_i e_i,\ \mathbf{v}\equiv v = v_j e_j \implies (u\wedge v)_{ij} = \tfrac{1}{2}(u_i v_j - u_j v_i),$$

whence

$$\begin{aligned}*(u\wedge v)_i &= \epsilon_{ijk}(u\wedge v)_{jk}\\ &= \tfrac{1}{2}\epsilon_{ijk}(u_j v_k - u_k v_j)\\ &= \epsilon_{ijk}u_j v_k\\ &= (\mathbf{u}\times\mathbf{v})_i.\end{aligned}$$

Example 8.8 In four-dimensional Minkowski space, with $s = 1$, the formulae for duals of various p-vectors are

$$\begin{aligned}(*1)^{ijkl} &= -\frac{1}{4!}\epsilon^{ijkl} = -\frac{1}{24}\epsilon^{ijkl},\\ A = A^i e_i &\implies (*A)^{ijk} = \frac{-1}{3!}\epsilon^{lijk}A_l = \frac{1}{6}\epsilon^{ijkl}A_l,\\ B = B^{ijk}e_i\otimes e_j\otimes e_k &\implies (*B)^i = -\epsilon^{jkli}B_{jkl} = \epsilon^{ijkl}B_{jkl},\\ F = F^{ij}e_i\otimes e_j &\implies (*F)^{ij} = \frac{-1}{2!}\epsilon^{klij}F_{kl} = -\frac{1}{2}\epsilon^{ijkl}F_{kl},\\ *E &= -\epsilon^{ijkl}\frac{-1}{4!}\epsilon_{ijkl} = \frac{4!}{4!} = 1.\end{aligned}$$

Note that if the components F^{ij} are written out in the following 'electromagnetic form', the significance of which will become clear in Chapter 9,

$$[F^{ij}] = \begin{pmatrix} 0 & B_3 & -B_2 & -E_1\\ -B_3 & 0 & B_1 & -E_2\\ B_2 & -B_1 & 0 & -E_3\\ E_1 & E_2 & E_3 & 0\end{pmatrix} \implies [F_{ij}] = \begin{pmatrix} 0 & B_3 & -B_2 & E_1\\ -B_3 & 0 & B_1 & E_2\\ B_2 & -B_1 & 0 & E_3\\ -E_1 & -E_2 & -E_3 & 0\end{pmatrix}$$

then the dual tensor essentially permutes electric and magnetic fields, up to a sign change,

$$*F^{ij} = \begin{pmatrix} 0 & -E_3 & E_2 & -B_1\\ E_3 & 0 & -E_1 & -B_2\\ -E_2 & E_1 & 0 & -B_3\\ B_1 & B_2 & B_3 & 0\end{pmatrix}.$$

The index lowering map $\bar{g} : V \to V^*$ can be uniquely extended to a linear map from p-vectors to p-forms,

$$\bar{g} : \Lambda^p(V) \to \Lambda^{*p}(V)$$

by requiring that

$$\bar{g}(u \wedge v \wedge \cdots \wedge w) = \bar{g}(u) \wedge \bar{g}(v) \wedge \cdots \wedge \bar{g}(w).$$

In components it has the expected effect

$$(\bar{g}A)_{i_1 i_2 \dots i_p} = g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_p j_p} A^{j_1 j_2 \dots j_p}$$

and from Eqs. (8.42) and (8.45) it follows that

$$(\bar{g}(A), \bar{g}(B)) = (A, B). \tag{8.55}$$

If the Hodge star operator is defined on forms by requiring that it commutes with the lowering operator $\bar{g}$,

$$*\bar{g}(A) = \bar{g}(*A) \tag{8.56}$$

then we find the defining relation for the Hodge star of a p-form α in a form analogous to Eq. (8.47),

$$\alpha \wedge \beta = (-1)^s (*\alpha, \beta)\Omega. \tag{8.57}$$

The factor $(-1)^s$ that enters this equation is essentially due to the fact that the 'lowered' basis vectors $\bar{g}(e_i)$ have a different orientation to the dual basis ε^i if s is an odd number, while they have the same orientation when s is even.

Problems

Problem 8.6 Show that the quantity $\langle A, \beta \rangle$ defined in Eq. (8.39) vanishes for all p-vectors A if and only if $\beta = 0$. Hence show that the correspondence between linear functionals on $\Lambda(V)$ and p-forms is one-to-one,

$$(\Lambda^p(V))^* \cong \Lambda^{*p}(V).$$

Problem 8.7 Show that the interior product between basis vectors e_i and $\varepsilon^{i_1 i_2 \dots i_r}$ is given by

$$\mathrm{i}_{e_i} \varepsilon^{i_1 \dots i_r} = \begin{cases} 0 & \text{if } i \notin \{i_1, \dots i_r\}, \\ (-1)^{a-1} \varepsilon^{i_1 \dots i_{a-1} i_{a+1} \dots i_r} & \text{if } i = i_a. \end{cases}$$

Problem 8.8 Prove Eq. (8.52).

Problem 8.9 Every p-form α can be regarded as a linear functional on $\Lambda^p(V)$ through the action $\alpha(A) = \langle A, \alpha \rangle$. Show that the basis $\varepsilon^{\mathbf{i}}$ is dual to the basis $e_{\mathbf{j}}$ where $\mathbf{i} = i_1 < i_2 < \cdots < i_p$, $\mathbf{j} = j_1 < j_2 < \cdots < j_p$,

$$\langle e_{\mathbf{j}}, \varepsilon^{\mathbf{i}} \rangle = \delta^{\mathbf{i}}_{\mathbf{j}} \equiv \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \dots \delta^{i_p}_{j_p}.$$

Verify that

$$\sum_{i_1 < i_2 < \cdots < i_p} \!\!\cdots \sum e_{i_1 i_2 \dots i_p} \varepsilon^{i_1 i_2 \dots i_p} = \dim \Lambda^p(V).$$

Problem 8.10 Show that if u, v and w are vectors in an n-dimensional real inner product space then

(a) $(u \wedge v, u \wedge v) = (u \cdot u)(v \cdot v) - (u \cdot v)^2$.
(b) $u \wedge *(v \wedge w) = (u \cdot w) * v - (u \cdot v) * w$.
(c) Which identities do these equations reduce to in three-dimensional cartesian vectors?

Problem 8.11 Let g_{ij} be the Minkowski metric on a four-dimensional space, having index 2 (so that there are three + signs and one − sign).

(a) By calculating the inner products $(e_{i_1 i_2}, e_{j_1, j_2})$, using (8.44) show that there are three +1's −1's in these inner products, and the index of the inner product defined on the six-dimensional space of bivectors $\Lambda^2(V)$ is therefore 0.
(b) What is the index of the inner product on $\Lambda^2(V)$ if V is n-dimensional and g_{ij} has index t? [*Ans.*: $\frac{1}{2}(t^2 - n)$.]

Problem 8.12 Show that in an arbitrary basis the component representation of the dual of a p-form α is

$$(*\alpha)_{j_1 j_2 \dots j_{n-p}} = \frac{\sqrt{|g|}}{(n-p)!} \epsilon_{i_1 \dots i_p j_1 j_2 \dots j_{n-p}} \alpha^{i_1 \dots i_p}. \tag{8.58}$$

Problem 8.13 If u is any vector, and α any p-form show that

$$\mathrm{i}_u * \alpha = *(\alpha \wedge \bar{g}(u)).$$

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