7 Tensors

In Chapter 3 we saw that any vector space V gives rise to other vector spaces such as the dual space $V^* = L(V, \mathbb{R})$ and the space $L(V, V)$ of all linear operators on V. In this chapter we will consider a more general class of spaces constructed from a vector space V, known as *tensor spaces*, of which these are particular cases. In keeping with modern mathematical practice, tensors and their basic operations will be defined invariantly, but we will also relate it to the 'old-fashioned' multicomponented formulation that is often better suited to applications in physics [1].

There are two significantly different approaches to tensor theory. Firstly, the method of Section 7.1 defines the tensor product of two vector spaces as a factor space of a *free vector space* [2]. While somewhat abstract in character, this is an essentially constructive procedure. In particular, it can be used to gain a deeper understanding of associative algebras, and supplements the material of Chapter 6. Furthermore, it applies to infinite dimensional vector spaces. The second method defines tensors as *multilinear maps* [3–5]. Readers may find this second approach the easier to understand, and there will be no significant loss in comprehension if they move immediately to Section 7.2. For finite dimensional vector spaces the two methods are equivalent [6].

7.1 Free vector spaces and tensor spaces

Free vector spaces

If S is an arbitrary set, the concept of a *free vector space on S* over a field $\mathbb{K}$ can be thought of intuitively as the set of all 'formal finite sums'

$$a^1 s_1 + a^2 s_2 + \cdots + a^n s_n \quad \text{where} \quad n = 0, 1, 2, \ldots;\ a^i \in \mathbb{K},\ s_i \in S.$$

The word 'formal' means that if S is an algebraic structure that already has a concept of addition or scalar multiplication defined on it, then the scalar product and summation in the *formal* sum bears no relation to these.

More rigorously, the **free vector space** $F(S)$ on a set S is defined as the set of all functions $f : S \to \mathbb{K}$ that vanish at all but a finite number of elements of S. Clearly $F(S)$ is a vector space with the usual definitions,

$$(f + g)(s) = f(s) + g(s), \qquad (af)(s) = af(s).$$

It is spanned by the characteristic functions $\chi_t \equiv \chi_{\{t\}}$ (see Example 1.7)

$$\chi_t(s) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t, \end{cases}$$

since any function having non-zero values at just a finite number of places $s_1, s_2, \dots, s_n$ can be written uniquely as

$$f = f(s_1)\chi_{s_1} + f(s_2)\chi_{s_2} + \dots + f(s_n)\chi_{s_n}.$$

Evidently the elements of $F(S)$ are in one-to-one correspondence with the 'formal finite sums' alluded to above.

Example 7.1 The vector space $\hat{\mathbb{R}}^\infty$ defined in Example 3.10 is isomorphic with the free vector space on any countably infinite set $S = \{s_1, s_2, s_3, \dots\}$, since the map $\sigma : \hat{\mathbb{R}}^\infty \to F(S)$ defined by

$$\sigma(a^1, a^2, \dots, a^n, 0, 0, \dots) = \sum_{i=1}^{n} a^i \chi_{s_i}$$

is linear, one-to-one and onto.

The tensor product $V \otimes W$

Let V and W be two vector spaces over a field $\mathbb{K}$. Imagine forming a product $v \otimes w$ between elements of these two vector spaces, called their 'tensor product', subject to the rules

$$(av + bv') \otimes w = av \otimes w + bv' \otimes w, \qquad v \otimes (aw + bw') = av \otimes w + bv \otimes w'.$$

The main difficulty with this simple idea is that we have no idea of what space $v \otimes w$ belongs to. The concept of a free vector space can be used to give a proper definition for this product.

Let $F(V \times W)$ be the free vector space over $V \times W$. This vector space is, in a sense, much too 'large' since pairs such as $a(v, w)$ and (av, w), or $(v + v', w)$ and $(v, w) + (v', w)$, are totally unrelated in $F(V \times W)$. To reduce the vector space to sensible proportions, we define U to be the vector subspace of $F(V \times W)$ generated by all elements of the form

$$(av + bv', w) - a(v, w) - b(v', w) \quad \text{and} \quad (v, aw + bw') - a(v, w) - b(v, w')$$

where, for notational simplicity, we make no distinction between a pair (v, w) and its characteristic function $\chi_{(v,w)} \in F(V \times W)$. The subspace U contains essentially all vector combinations that are to be identified with the zero element. The **tensor product** of V and W is defined to be the factor space

$$V \otimes W = F(V \times W)/U.$$

The **tensor product** $v \otimes w$ **of a pair of vectors** $v \in V$ and $w \in W$ is defined as the equivalence class or coset in $V \otimes W$ to which (v, w) belongs,

$$v \otimes w = [(v, w)] = (v, w) + U.$$

This product is *bilinear*,

$$(av + bv') \otimes w = av \otimes w + bv' \otimes w,$$
$$v \otimes (aw + bw') = av \otimes w + bv \otimes w'.$$

To show the first identity,

$$\begin{aligned}(av + bv') \otimes w &= (av + bv',\ w) + U \\ &= (av + bv',\ w) - ((av + bv',\ w) - a(v, w) - b(v', w)) + U \\ &= a(v, w) + b(v', w) + U \\ &= av \otimes w + bv' \otimes w,\end{aligned}$$

and the second identity is similar.

If V and W are both finite dimensional let $\{e_i \mid i = 1, \dots, n\}$ be a basis for V, and $\{f_a \mid a = 1, \dots, m\}$ a basis of W. Every tensor product $v \otimes w$ can, through bilinearity, be written

$$v \otimes w = (v^i e_i) \otimes (w^a f_a) = v^i w^a (e_i \otimes f_a). \tag{7.1}$$

We will use the term **tensor** to describe the general element of $V \otimes W$. Since every tensor A is a finite sum of elements of the form $v \otimes w$ it can, on substituting (7.1), be expressed in the form

$$A = A^{ia} e_i \otimes f_a\,. \tag{7.2}$$

Hence the tensor product space $V \otimes W$ is spanned by the nm tensors $\{e_i \otimes f_a \mid i = 1, \dots, n,\ a = 1, \dots, m\}$.

Furthermore, these tensors form a basis of $V \otimes W$ since they are linearly independent. To prove this statement, let (ρ, φ) be any ordered pair of linear functionals $\rho \in V^*, \varphi \in W^*$. Such a pair defines a linear functional on $F(V \times W)$ by setting

$$(\rho, \varphi)(v, w) = \rho(v)\varphi(w),$$

and extending to all of $F(V \times W)$ by linearity,

$$(\rho, \varphi)\Big[\sum_r a^r (v_r, w_r)\Big] = \sum_r a^r \rho(v_r)\varphi(w_r).$$

This linear functional vanishes on the subspace U and therefore 'passes to' the tensor product space $V \otimes W$ by setting

$$(\rho, \varphi)\Big[\sum_r a^r v_r \otimes w_r\Big] = \sum_r a^r (\rho, \varphi)((v_r, w_r)).$$

Let ε^k, φ^b be the dual bases in V^* and W^* respectively; we then have

$$\begin{aligned}A^{ia} e_i \otimes f_a = 0 &\implies (\varepsilon^j, \varphi^b)(A^{ia} e_i \otimes f_a) = 0 \\ &\implies A^{ia} \delta^j_i \delta^b_a = 0 \\ &\implies A^{jb} = 0.\end{aligned}$$

Hence the tensors $e_i \otimes f_a$ are l.i. and form a basis of $V \otimes W$. The dimension of $V \otimes W$ is given by

$$\dim(V \otimes W) = \dim V \ \dim W.$$

Setting $V = W$, the elements of $V \otimes V$ are called **contravariant tensors of degree 2** on V. If $\{e_i\}$ is a basis of V every contravariant tensor of degree 2 has a unique expansion

$$T = T^{ij} e_i \otimes e_j,$$

and the real numbers T^{ij} are called the *components* of the tensor with respect to this basis. Similarly each element S of $V^* \otimes V^*$ is called a **covariant tensor of degree 2** on V and has a unique expansion with respect to the dual basis ε^i,

$$S = S_{ij}\, \varepsilon^i \otimes \varepsilon^j.$$

Dual representation of tensor product

Given a pair of vector spaces V_1 and V_2 over a field $\mathbb{K}$, a map $T : V_1 \times V_2 \to \mathbb{K}$ is said to be **bilinear** if

$$T(av_1 + bv'_1, v_2) = aT(v_1, v_2) + bT(v'_1, v_2),$$
$$T(v_1,\ av_2 + bv'_2,) = aT(v_1, v_2) + bT(v_1, v'_2),$$

for all $v_1, v'_1 \in V_1$, $v_2, v'_2 \in V_2$, and $a, b \in \mathbb{K}$. Bilinear maps can be added and multiplied by scalars in a manner similar to that for linear functionals given in Section 3.7, and form a vector space that will be denoted $(V_1, V_2)^*$.

Every pair of vectors (v, w) where $v \in V$, $w \in W$ defines a bilinear map $V^* \times W^* \to \mathbb{K}$, by setting

$$(v,\ w) : (\rho,\ \varphi) \mapsto \rho(v)\varphi(w).$$

We can extend this correspondence to all of $F(V \times W)$ in the obvious way by setting

$$\sum_r a^r (v_r,\ w_r) : (\rho,\ \varphi) \mapsto \sum_r a^r \rho(v_r)\varphi(w_r),$$

and since the action of any generators of the subspace U, such as $(av + bv',\ w) - a(v, w) - b(v', w)$, clearly vanishes on $V^* \times W^*$, the correspondence passes in a unique way to the tensor product space $V \otimes W = F(V, W)/U$. That is, every tensor $A = \sum_r a^r v_r \otimes w_r$ defines a bilinear map $V^* \times W^* \to \mathbb{K}$, by setting

$$A(\rho, \varphi) = \sum_r a^r \rho(v_r)\varphi(w_r).$$

This linear mapping from $V \otimes W$ into the space of bilinear maps on $V^* \times W^*$ is also one-to-one in the case of finite dimensional vector spaces. For, suppose $A(\rho, \varphi) = B(\rho, \varphi)$ for all $\rho \in V^*$, $\varphi \in W^*$. Let e_i, f_a be bases of V and W respectively, and ε^j, φ^a be the dual bases. Writing $A = A^{ia} e_i \otimes f_a = 0$, $B = B^{ia} e_i \otimes f_a = 0$, we have

$$A^{ia} \rho(e_i)\varphi(f_a) = B^{ia} \rho(e_i)\varphi(f_a)$$

for all linear functionals $\rho \in V^*$, $\varphi \in W^*$. If we set $\rho = \varepsilon^j$, $\varphi = \varphi^b$ then

$$A^{ia}\delta_i^j\delta_a^b = B^{ia}\delta_i^j\delta_a^b,$$

resulting in $A^{jb} = B^{jb}$. Hence $A = B$, and for finite dimensional vector spaces V and W we have shown that the linear correspondence between $V \otimes W$ and $(V^*, W^*)^*$ is one-to-one,

$$V \otimes W \cong (V^*, W^*)^*. \tag{7.3}$$

This isomorphism does not hold for infinite dimensional spaces.

A tedious but straightforward argument results in the associative law for tensor products of three vectors

$$u \otimes (v \otimes w) = (u \otimes v) \otimes w.$$

Hence the tensor product of three or more vector spaces is defined in a unique way,

$$U \otimes V \otimes W = U \otimes (V \otimes W) = (U \otimes V) \otimes W.$$

For finite dimensional spaces it may be shown to be isomorphic with the space of maps $A : U^* \times V^* \times W^* \to \mathbb{K}$ that are linear in each argument separately.

Free associative algebras

Let $\mathcal{F}(V)$ be the infinite direct sum of vector spaces

$$\mathcal{F}(V) = V^{(0)} \oplus V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus \dots$$

where $\mathbb{K} = V^{(0)}$, $V = V^{(1)}$ and

$$V^{(r)} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{r}.$$

The typical member of this infinite direct sum can be written as a *finite* formal sum of tensors from the tensor spaces $V^{(r)}$,

$$a + u + A_2 + \cdots + A_r.$$

To define a product rule on $\mathcal{F}(V)$ set

$$\underbrace{u_1 \otimes u_2 \otimes \cdots \otimes u_r}_{\in V^{(r)}}\,\underbrace{v_1 \otimes v_2 \otimes \cdots \otimes v_s}_{\in V^{(s)}} = \underbrace{u_1 \otimes \cdots \otimes u_r \otimes v_1 \otimes \cdots \otimes v_s}_{\in V^{(r+s)}}$$

and extend to all of $\mathcal{F}(V)$ by linearity. The distributive law (6.1) is automatically satisfied, making $\mathcal{F}(V)$ with this product structure into an associative algebra. The algebra $\mathcal{F}(V)$ has in essence no 'extra rules' imposed on it other than simple juxtaposition of elements from V and multiplication by scalars. It is therefore called the **free associative algebra** over V. All associative algebras can be constructed as a factor algebra of the free associative algebra over a vector space. The following example illustrates this point.

Example 7.2 If V is the one-dimensional free vector space over the reals on the singleton set $S = \{x\}$ then the free associative algebra over $\mathcal{F}(V)$ is in one-to-one correspondence with the algebra of real polynomials $\mathcal{P}$, Example 6.3, by setting

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \equiv a_0 + a_1 x + a_2 x \otimes x + \cdots + a_n \underbrace{x \otimes x \otimes \cdots \otimes x}_{n}.$$

This correspondence is an algebra isomorphism since the product defined on $\mathcal{F}(V)$ by the above procedure will be identical with multiplication of polynomials. For example,

$$\begin{aligned}(ax + bx \otimes x \otimes x)(c + dx \otimes x) &= acx + (ad + bc)x \otimes x \otimes x + bdx \otimes x \otimes x \otimes x \otimes x \\ &\equiv acx + (ad + bc)x^3 + bdx^5 \\ &= (ax + bx^3)(c + dx^2), \text{ etc.}\end{aligned}$$

Set $\mathcal{C}$ to be the ideal of $\mathcal{P}$ generated by $x^2 + 1$, consisting of all polynomials of the form $f(x)(x^2 + 1)g(x)$. By identifying i with the polynomial class $[x] = x + \mathcal{C}$ and real numbers with the class of constant polynomials $a \to [a]$, the algebra of complex numbers is isomorphic with the factor algebra $\mathcal{P}/\mathcal{C}$, for

$$i^2 \equiv [x]^2 = [x^2] = [x^2 - (x^2 + 1)] = [-1] \equiv -1.$$

Grassmann algebra as a factor algebra of free algebras

The definition of Grassmann algebra given in Section 6.4 is unsatisfactory in two key aspects. Firstly, in the definition of exterior product it is by no means obvious that the rules (EP1)–(EP3) produce a well-defined and unique product on $\Lambda(V)$. Secondly, the matter of linear independence of the basis vectors $e_{i_1 i_2 \dots i_r}$ $(i_1 < i_2 < \cdots < i_r)$ had to be postulated separately in Section 6.4. The following discussion provides a more rigorous foundation for Grassmann algebras, and should clarify these issues.

Let $\mathcal{F}(V)$ be the free associative algebra over a real vector space V, and let $\mathcal{S}$ be the ideal generated by all elements of $\mathcal{F}(V)$ of the form $u \otimes T \otimes v + v \otimes T \otimes u$ where $u, v \in V$ and $T \in \mathcal{F}(V)$. The general element of $\mathcal{S}$ is

$$S \otimes u \otimes T \otimes v \otimes U + S \otimes v \otimes T \otimes u \otimes U$$

where $u, v \in V$ and $S, T, U \in \mathcal{F}(V)$. The ideal $\mathcal{S}$ essentially identifies those elements of $\mathcal{F}(V)$ that will vanish when the tensor product $\otimes$ is replaced by the wedge product $\wedge$.

Exercise: Show that the ideal $\mathcal{S}$ is generated by all elements of the form $w \otimes T \otimes w$ where $w \in V$ and $T \in \mathcal{F}(V)$. [*Hint*: Set $w = u + v$.]

Define the *Grassmann algebra* $\Lambda(V)$ to be the factor algebra

$$\Lambda(V) = \mathcal{F}(V)/\mathcal{S}, \tag{7.4}$$

and denote the induced associative product by $\wedge$,

$$[A] \wedge [B] = [A \otimes B] \tag{7.5}$$

where $[A] \equiv A + \mathcal{S}$, $[B] \equiv B + \mathcal{S}$. As in Section 6.4, the elements $[A]$ of the factor algebra are called **multivectors**. There is no ambiguity in dropping the square brackets,

$A \equiv [A]$ and writing $A \wedge B$ for $[A] \wedge [B]$. The algebra $\Lambda(V)$ is the direct sum of subspaces corresponding to tensors of degree r,

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \dots,$$

where

$$\Lambda^r(V) = [V^{(r)}] = \{A + \mathcal{S} \mid A \in V^{(r)}\}$$

whose elements are called r**-vectors**. If A is an r-vector and B an s-vector then $A \wedge B$ is an $(r+s)$-vector.

Since, by definition, $u \otimes v + v \otimes u$ is a member of $\mathcal{S}$, we have

$$u \wedge v = [u \otimes v] = [-v \otimes u] = -v \wedge u,$$

for all $u, v \in V$. Hence $u \wedge u = 0$ for all $u \in V$.

Exercise: Prove that if A, B and C are any multivectors then for all $u, v \in V$

$$A \wedge u \wedge B \wedge v \wedge C + A \wedge v \wedge B \wedge u \wedge C = 0,$$

and $A \wedge u \wedge B \wedge u \wedge C = 0$.

Exercise: From the corresponding rules of tensor product show that exterior product is associative and distributive.

From the associative law

$$(u_1 \wedge u_2 \wedge \dots \wedge u_r) \wedge (v_1 \wedge v_2, \wedge \dots \wedge v_s) = u_1 \wedge \dots \wedge u_r \wedge v_1 \wedge \dots \wedge v_s,$$

in agreement with (EP2) of Section 6.4. This provides a basis-independent definition for exterior product on any finite dimensional vector space V, having the desired properties (EP1)–(EP3). Since every r-vector is the sum of simple r-vectors, the space of r-vectors $\Lambda^r(V)$ is spanned by

$$E_r = \{e_{i_1 i_2 \dots i_r} \mid i_1 < i_2 < \dots < i_r\},$$

where

$$e_{i_1 i_2 \dots i_r} = e_{i_1} \wedge e i_2 \cdots \wedge e_{i_r},$$

as shown in Section 6.4. It is left as an exercise to show that the set E_r does indeed form a basis of the space of r-vectors (see Problem 7.6). Hence, as anticipated in Section 6.4, the dimension of the space of r-vectors is

$$\dim \Lambda^r(V) = \binom{n}{r} = \frac{n!}{r!(n-r)!},$$

and the dimension of the Grassmann algebra $\Lambda(V)$ is 2^n.

Problems

Problem 7.1 Show that the direct sum $V \oplus W$ of two vector spaces can be defined from the free vector space as $F(V \times W)/U$ where U is a subspace generated by all linear combinations of the form

$$(av + bv', aw + bw') - a(v, w) - b(v', w').$$

Problem 7.2 Prove the so-called *universal property* of free vector spaces. Let $\varphi : S \to F(S)$ be the map that assigns to any element $s \in S$ its characteristic function $\chi_s \in F(S)$. If V is any vector space and $\alpha : S \to V$ any map from S to V, then there exists a unique linear map $T : F(S) \to V$ such that $\alpha = T \circ \varphi$, as depicted by the *commutative diagram*

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\alpha"'] & F(S) \arrow[d, "T"] \\ & V \end{tikzcd}$$

Show that this process is reversible and may be used to define the free vector space on S as being the unique vector space $F(S)$ for which the above commutative diagram holds.

Problem 7.3 Let $\mathcal{F}(V)$ be the free associative algebra over a vector space V.

(a) Show that there exists a linear map $I : V \to \mathcal{F}(V)$ such that if $\mathcal{A}$ is any associative algebra over the same field $\mathbb{K}$ and $S : V \to \mathcal{A}$ a linear map, then there exists a unique algebra homomorphism $\alpha : \mathcal{F}(V) \to \mathcal{A}$ such that $S = \alpha \circ I$.
(b) Depict this property by a commutative diagram.
(c) Show the converse: any algebra $\mathcal{F}$ for which there is a map $I : V \to \mathcal{F}$ such that the commutative diagram holds for an arbitrary linear map S is isomorphic with the free associative algebra over V.

Problem 7.4 Give a definition of quaternions as a factor algebra of the free algebra on a three-dimensional vector space.

Problem 7.5 The Clifford algebra $\mathcal{C}_g$ associated with an inner product space V with scalar product $g(u, v) \equiv u \cdot v$ can be defined in the following way. Let $\mathcal{F}(V)$ be the free associative algebra on V and $\mathcal{C}$ the two-sided ideal generated by all elements of the form

$$A \otimes (u \otimes v + v \otimes u - 2g(u, v)1) \otimes B \quad (A, B \in \mathcal{F}(V)).$$

The Clifford algebra in question is now defined as the factor space $\mathcal{F}(V)/\mathcal{C}$. Verify that this algebra is isomorphic with the Clifford algebra as defined in Section 6.3, and could serve as a basis-independent definition for the Clifford algebra associated with a real inner product space.

Problem 7.6 Show that E_2 is a basis of $\Lambda^2(V)$. In outline: define the maps $\varepsilon^{kl} : V^2 \to \mathbb{R}$ by

$$\varepsilon^{kl}(u, v) = \varepsilon^k(u)\varepsilon^l(v) - \varepsilon^l(u)\varepsilon^k(v) = u^k v^l - u^l v^k.$$

Extend by linearity to the tensor space $V^{(2,0)}$ and show there is a natural passage to the factor space, $\hat{\varepsilon}^{kl} : \Lambda^2(V) = V^{(2,0)}/\mathcal{S}^2 \to \mathbb{R}$. If a linear combination from E_2 were to vanish,

$$\sum_{i<j} A^{ij} e_i \wedge e_j = 0,$$

apply the map $\hat{\varepsilon}^{kl}$ to this equation, to show that all coefficients A^{ij} must vanish separately.

Indicate how the argument may be extended to show that if $r \leq n = \dim V$ then E_r is a basis of $\Lambda^r(V)$.

7.2 Multilinear maps and tensors

The dual representation of tensor product allows for an alternative definition of tensor spaces and products. Key to this approach is the observation in Section 3.7, that every finite dimensional vector space V has a natural isomorphism with V^{**} whereby a vector v acts as a linear functional on V^* through the identification

$$v(\omega) = \omega(v) = \langle v, \omega \rangle = \langle \omega, v \rangle.$$

Multilinear maps and tensor spaces of type (r, s)

Let $V_1, V_2, \dots, V_N$ be vector spaces over the field $\mathbb{R}$. A map

$$T : V_1 \times V_2 \times \cdots \times V_N \to \mathbb{R}$$

is said to be **multilinear** if it is linear in each argument separately,

$$\begin{aligned} T(v_1, \dots, v_{i-1}, av_i + bv'_i, \dots, v_N) \\ = aT(v_1, \dots, v_{i-1}, v_i, \dots, v_N) + bT(v_1, \dots, v_{i-1}, v'_i, \dots, v_N). \end{aligned}$$

Multilinear maps can be added and multiplied by scalars in the usual fashion,

$$(aT + bS)(v_1, v_2, \dots, v_s) = aT(v_1, v_2, \dots, v_s) + bS(v_1, v_2, \dots, v_s),$$

and form a vector space, denoted

$$V_1^* \otimes V_2^* \otimes \cdots \otimes V_N^*,$$

called the **tensor product of the dual spaces** $V_1^*, V_2^*, \dots, V_N^*$. When $N = 1$, the word 'multilinear' is simply replaced with the word 'linear' and the notation is consistent with the concept of the dual space V^* defined in Section 3.7 as the set of linear functionals on V. If we identify every vector space V_i with its double dual V_i^{**}, the **tensor product of the vector spaces** $V_1, V_2, \dots, V_N$, denoted $V_1 \otimes V_2 \otimes \cdots \otimes V_N$, is then the set of multilinear maps from $V_1^* \times V_2^* \times \cdots \times V_N^*$ to $\mathbb{R}$.

Let V be a vector space of dimension n over the field $\mathbb{R}$. Setting

$$V_1 = V_2 = \cdots = V_r = V^*, \qquad V_{r+1} = V_{r+2} = \cdots = V_{r+s} = V, \quad \text{where} \quad N = r + s,$$

we refer to any multilinear map

$$T : \underbrace{V^* \times V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times V \times \cdots \times V}_{s} \to \mathbb{R}$$

as a **tensor of type** (r, s) on V. The integer $r \geq 0$ is called the **contravariant degree** and $s \geq 0$ the **covariant degree** of T. The vector space of tensors of type (r, s) is denoted

$$V^{(r,s)} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{s}.$$

This definition is essentially equivalent to the dual representation of the definition in Section 7.1. Both definitions are totally 'natural' in that they do not require a choice of basis on the vector space V.

It is standard to set $V^{(0,0)} = \mathbb{R}$; that is, tensors of type (0, 0) will be identified as scalars. Tensors of type (0, 1) are linear functionals (covectors)

$$V^{(0,1)} \equiv V^*,$$

while tensors of type (1, 0) can be regarded as ordinary vectors

$$V^{(1,0)} \equiv V^{**} \equiv V.$$

Covariant tensors of degree 2

A tensor of type (0, 2) is a bilinear map $T : V \times V \to \mathbb{R}$. In keeping with the terminology of Section 7.1, such a tensor may be referred to as a **covariant tensor of degree 2** on V. Linearity in each argument reads

$$T(av + bw, u) = aT(v, u) + bT(w, u) \quad \text{and} \quad T(v, au + bw) = aT(v, u) + bT(v, w).$$

If $\omega, \rho \in V^*$ are linear functionals over V, let their **tensor product** $\omega \otimes \rho$ be the covariant tensor of degree 2 defined by

$$\omega \otimes \rho\,(u, v) = \omega(u)\,\rho(v).$$

Linearity in the first argument follows from

$$\begin{aligned}
\omega \otimes \rho\,(au + bv, w) &= \omega(au + bv)\rho(w) \\
&= (a\omega(u) + b\omega(v))\rho(w) \\
&= a\omega(u)\rho(w) + b\omega(v)\rho(w) \\
&= a\omega \otimes \rho\,(u, w) + b\omega \otimes \rho\,(v, w).
\end{aligned}$$

A similar argument proves linearity in the second argument v.

Example 7.3 Tensor product is not a commutative operation since in general $\omega \otimes \rho \neq \rho \otimes \omega$. For example, let e_1, e_2 be a basis of a two-dimensional vector space V, and let ε^1, ε^2 be the dual basis of V^*. If

$$\omega = 3\varepsilon^1 + 2\varepsilon^2, \qquad \rho = \varepsilon^1 - \varepsilon^2$$

then

$$\begin{aligned}
\omega \otimes \rho\,(u,\ v) &= \omega(u^1 e_1 + u^2 e_2)\,\rho(v^1 e_1 + v^2 e_2) \\
&= (3u^1 + 2u^2)(v^1 - v^2) \\
&= 3u^1 v^1 - 3u^1 v^2 + 2u^2 v^1 - 2u^2 v^2
\end{aligned}$$

and

$$\begin{aligned}\rho \otimes \omega\,(u, v) &= (u^1 - u^2)(3v^1 + 3v^2)\\ &= 3u^1v^1 + 2u^1v^2 - 3u^2v^1 - 2u^2v^2\\ &\neq \omega \otimes \rho\,(u,\ v).\end{aligned}$$

More generally, let $e_1, \dots, e_n$ be a basis of the vector space V and $\varepsilon^1, \dots, \varepsilon^n$ its dual basis, defined by

$$\varepsilon^i(e_j) = \langle \varepsilon^i, e_j\rangle = \delta^i_{\ j} \quad (i,\ j = 1, \dots, n). \tag{7.6}$$

Theorem 7.1 *The tensor products*

$$\varepsilon^i \otimes \varepsilon^j \quad (i,\ j = 1,\ \dots, n)$$

form a basis of the vector space $V^{(0,2)}$, which therefore has dimension n^2.

Proof: The tensors $\varepsilon^i \otimes \varepsilon^j$ $(i, j = 1,\ \dots, n)$ are linearly independent, for if

$$a_{ij}\varepsilon^i \otimes \varepsilon^j \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\varepsilon^i \otimes \varepsilon^j = 0,$$

then for each $1 \le k, l \le n$,

$$0 = a_{ij}\varepsilon^i \otimes \varepsilon^j\,(e_k,\ e_l) = a_{ij}\,\delta^i_{\ k}\,\delta^j_{\ l} = a_{kl}.$$

Furthermore, if T is any covariant tensor of degree 2 then

$$T = T_{ij}\varepsilon^i \otimes \varepsilon^j \quad \text{where} \quad T_{ij} = T(e_i,\ e_j) \tag{7.7}$$

since for any pair of vectors $u = u^ie_i,\ v = v^je_j$ in V,

$$\begin{aligned}(T - T_{ij}\varepsilon^i \otimes \varepsilon^j)(u, v) &= T(u, v) - T_{ij}\varepsilon^i(u)\varepsilon^j(v)\\ &= T(u^ie_i, v^je_j) - T_{ij}u^iv^j\\ &= u^iv^jT_{e_i,e_j} - T_{ij}u^iv^j\\ &= u^iv^jT_{ij} - T_{ij}u^iv^j = 0.\end{aligned}$$

Hence the n^2 tensors $\varepsilon^i \otimes \varepsilon^j$ are linearly independent and span $V^{(0,2)}$. They therefore form a basis of $V^{(0,2)}$. ■

The coefficients T_{ij} in the expansion $T = T_{ij}\varepsilon^i \otimes \varepsilon^j$ are uniquely given by the expression on the right in Eq. (7.7), for if $T = T'_{ij}\varepsilon^i \otimes \varepsilon^j$ then by linear independence of the $\varepsilon^i \otimes \varepsilon^j$

$$(T'_{ij} - T_{ij})\varepsilon^i \otimes \varepsilon^j = 0 \implies T'_{ij} - T_{ij} = 0 \implies T'_{ij} = T_{ij}.$$

They are called the **components of T with respect to the basis** $\{e_i\}$. For any vectors $u = u^i\,e_i,\ v = v^j\,e_j$

$$T(u, v) = T_{ij}u^iv^j. \tag{7.8}$$

Example 7.4 For the linear functional ω and ρ given in Example 7.3, we can write

$$\begin{aligned}\omega \otimes \rho &= (3\varepsilon^1 + 2\varepsilon^2) \otimes (\varepsilon^1 - \varepsilon^2)\\ &= 3\varepsilon^1 \otimes \varepsilon^1 - 3\varepsilon^1 \otimes \varepsilon^2 + 2\varepsilon^2 \otimes \varepsilon^1 - 2\varepsilon^2 \otimes \varepsilon^2\end{aligned}$$

and similarly

$$\rho \otimes \omega = 3\varepsilon^1 \otimes \varepsilon^1 + 2\varepsilon^1 \otimes \varepsilon^2 - 3\varepsilon^2 \otimes \varepsilon^1 - 2\varepsilon^2 \otimes \varepsilon^2.$$

Hence the components of the tensor products $\omega \otimes \rho$ and $\rho \otimes \omega$ with respect to the basis tensors $\varepsilon^i \otimes \varepsilon^j$ may be displayed as arrays,

$$[(\omega \otimes \rho)_{ij}] = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}, \qquad [(\rho \otimes \omega)_{ij}] = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}.$$

Exercise: Using the components $(\omega \otimes \rho)_{12}$, etc. in the preceding example verify the formula (7.8) for $\omega \otimes \rho\,(u, v)$. Do the same for $\rho \otimes \omega\,(u, v)$.

In general, if $\omega = w_i \varepsilon^i$ and $\rho = r_j \varepsilon^j$ then

$$\omega \otimes \rho = (w_i \varepsilon^i) \otimes (r_j \varepsilon^j) = w_i r_j \varepsilon^i \otimes \varepsilon^j,$$

and the components of $\omega \otimes \rho$ are

$$(\omega \otimes \rho)_{ij} = w_i r_j. \tag{7.9}$$

This $n \times n$ array of components is formed by taking all possible component-by-component products of the two linear functionals.

Exercise: Prove Eq. (7.9) by evaluating $(\omega \otimes \rho)_{ij} = (\omega \otimes \rho)(e_i, e_j)$.

Example 7.5 Let $(V, \cdot)$ be a real inner product space, as in Section 5.1. The map $g : V \times V \to \mathbb{R}$ defined by

$$g(u, v) = u \cdot v$$

is obviously bilinear, and is a covariant tensor of degree 2 called the **metric tensor** of the inner product. The components $g_{ij} = e_i \cdot e_j = g(e_i, e_j)$ of the inner product are the components of the metric tensor with respect to the basis $\{e_i\}$,

$$g = g_{ij}\, \varepsilon^i \otimes \varepsilon^j,$$

while the inner product of two vectors is

$$u \cdot v = g(u, v) = g_{ij} u^i v^j.$$

The metric tensor is **symmetric**,

$$g(u, v) = g(v, u).$$

Exercise: Show that a tensor T is symmetric if and only if its components form a symmetric array,

$$T_{ij} = T_{ji} \qquad \text{for all } i, j = 1, \dots, n.$$

Example 7.6 Let T be a covariant tensor of degree 2. Define the map $\bar{T} : V \to V^*$ by

$$\langle \bar{T} v, u \rangle = T(u, v) \qquad \text{for all } u, v \in V. \tag{7.10}$$

Here $\bar{T}(v)$ has been denoted more simply by $\bar{T}v$. The map $\bar{T}$ is clearly linear, $\bar{T}(au + bv) = a\bar{T}u + b\bar{T}v$, since

$$\langle \bar{T}(au + bv), w\rangle = T(w, au + bv) = aT(w, u) + bT(w, v) = \langle a\bar{T}u + b\bar{T}v, w\rangle$$

holds for all $w \in V$. Conversely, given a linear map $\bar{T} : V \to V^*$, Eq. (7.10) defines a tensor T since $T(u, v)$ so defined is linear both in u and v. Thus every covariant tensor of degree 2 can be identified with an element of $L(V, V^*)$.

Exercise: In components, show that $(\bar{T}v)_i = T_{ij}v^j$.

Contravariant tensors of degree 2

A **contravariant tensor of degree 2** on V, or tensor of type (2, 0), is a bilinear real-valued map S over V^*:

$$S : V^* \times V^* \to \mathbb{R}.$$

Then, for all $a,\ b \in \mathbb{R}$ and all $\omega,\ \rho,\ \theta \in V^*$

$$S(a\omega + b\rho, \theta) = aS(\omega, \theta) + bS(\rho, \theta) \quad \text{and} \quad S(\omega, a\rho + b\theta) = aS(\omega, \rho) + bS(\omega, \theta).$$

If u and v are any two vectors in V, then their **tensor product** $u \otimes v$ is the contravariant tensor of degree 2 defined by

$$u \otimes v\ (\omega, \rho) = u(\omega)\, v(\rho) = \omega(u)\, \rho(v). \tag{7.11}$$

If $e_1, \dots, e_n$ is a basis of the vector space V with dual basis $\{\varepsilon^j\}$ then, just as for $V^{(0,2)}$, the tensors $e_i \otimes e_j$ form a basis of the space $V^{(2,0)}$, and every contravariant tensor of degree 2 has a unique expansion

$$S = S^{ij} e_i \otimes e_j \quad \text{where} \quad S^{ij} = S(\varepsilon^i, \varepsilon^j).$$

The scalars S^{ij} are called the **components** of the tensor S with respect to the basis $\{e_i\}$.

Exercise: Provide detailed proofs of these statements.

Exercise: Show that the components of the tensor product of two vectors is given by

$$(u \otimes v)^{ij} = u^i v^j.$$

Example 7.7 It is possible to identify contravariant tensors of degree 2 with linear maps from V^* to V. If S is a tensor of type (0, 2) define a map $\bar{S} : V^* \to V$ by

$$(\bar{S}\rho)(\omega) \equiv \omega(\bar{S}\rho) = S(\omega, \rho) \qquad \text{for all } \omega, \rho \in V^*.$$

The proof that this correspondence is one-to-one is similar to that given in Example 7.6.

Example 7.8 Let $(V, \cdot)$ be a real inner product space with metric tensor g as defined in Example 7.5. Let $\bar{g} : V \to V^*$ be the map defined by g using Example 7.6,

$$\langle \bar{g}v, u\rangle = g(u, v) = u \cdot v \qquad \text{for all } u, v \in V. \tag{7.12}$$

From the non-singularity condition (SP3) of Section 5.1 the kernel of this map is $\{0\}$, from which it follows that it is one-to-one. Furthermore, because the dimensions of V and V^* are identical, $\bar{g}$ is onto and therefore invertible. As shown in Example 7.7 its inverse $\bar{g}^{-1} : V^* \to V$ defines a tensor g^{-1} of type (2, 0) by

$$g^{-1}(\omega, \rho) = \langle \omega, \bar{g}^{-1}\rho \rangle.$$

From the symmetry of the metric tensor g and the identities

$$\bar{g}\bar{g}^{-1} = \mathrm{id}_{V^*}, \qquad \bar{g}^{-1}\bar{g} = \mathrm{id}_V$$

it follows that g^{-1} is also a symmetric tensor, $g^{-1}(\omega, \rho) = g^{-1}(\rho, \omega)$:

$$\begin{aligned}
g^{-1}(\omega, \rho) &= \langle \omega, \bar{g}^{-1}\rho \rangle \\
&= \langle \bar{g}\bar{g}^{-1}\omega, \bar{g}^{-1}\rho \rangle \\
&= g(\bar{g}^{-1}\rho,\ \bar{g}^{-1}\omega) \\
&= g(\bar{g}^{-1}\omega,\ \bar{g}^{-1}\rho) \\
&= \langle \bar{g}\bar{g}^{-1}\rho, \bar{g}^{-1}\omega \rangle \\
&= \langle \rho, \bar{g}^{-1}\omega \rangle \\
&= g^{-1}(\rho, \omega).
\end{aligned}$$

It is usual to denote the components of the inverse metric tensor with respect to any basis $\{e_i\}$ by the symbol g^{ij}, so that

$$g^{-1} = g^{ij} e_i \otimes e_j \quad \text{where} \quad g^{ij} = g^{-1}(\varepsilon_i, \varepsilon_j) = g^{ji}. \tag{7.13}$$

From Example 7.6 we have

$$\langle \bar{g}e_i, e_j \rangle = g(e_j, e_i) = g_{ji},$$

whence

$$\bar{g}(e_i) = g_{ji}\varepsilon^j.$$

Similarly, from Example 7.7

$$\langle \bar{g}^{-1}\varepsilon^j, \varepsilon^k \rangle = g^{-1}(\varepsilon^k, \varepsilon^j) = g^{kj} \implies \bar{g}^{-1}(\varepsilon^j) = g^{kj} e_k.$$

Hence

$$e_i = \bar{g}^{-1} \circ \bar{g}(e_i) = \bar{g}^{-1}(g_{ji}\varepsilon^j) = g_{ji} g^{kj} e_k.$$

Since $\{e_i\}$ is a basis of V, or from Eq. (7.6), we conclude that

$$g^{kj} g_{ji} = \delta^k_i, \tag{7.14}$$

and the matrices $[g^{ij}]$ and $[g_{ij}]$ are inverse to each other.

Mixed tensors

A tensor R of covariant degree 1 and contravariant degree 1 is a bilinear map $R : V^* \times V \to \mathbb{R}$:

$$R(a\omega + b\rho, u) = aR(\omega, u) + bR(\rho, u)$$
$$R(\omega, au + bv) = aR(\omega, u) + bR(\omega, v),$$

sometimes referred to as a **mixed tensor**. Such tensors are of type (1, 1), belonging to the vector space $V^{(1,1)}$.

For a vector $v \in V$ and covector $\omega \in V^*$, define their tensor product $v \otimes \omega$ by

$$v \otimes \omega\,(\rho,\ u) = v(\rho)\,\omega(u).$$

As in the preceding examples it is straightforward to show that $e_i \otimes \varepsilon^j$ form a basis of $V^{(1,1)}$, and every mixed tensor R has a unique decomposition

$$R = R^i{}_j e_i \otimes \varepsilon^j \quad \text{where} \quad R^i{}_j = R(\varepsilon^i, e_j).$$

Example 7.9 Every tensor R of type (1, 1) defines a map $\bar{R} : V \to V$ by

$$\langle \bar{R}u, \omega\rangle \equiv \omega(\bar{R}u) = R(\omega, u).$$

The proof that there is a one-to-one correspondence between such maps and tensors of type (1, 1) is similar to that given in Examples 7.6 and 7.7. Operators on V and tensors of type (1, 1) can be thought of as essentially identical, $V^{(1,1)} \cong L(V, V)$.

If $\{e_i\}$ is a basis of V then setting

$$\bar{R}e_i = \bar{R}^j{}_i e_j$$

we have

$$\langle \bar{R}e_j, \varepsilon^i\rangle = R(\varepsilon^i, e_j) = R^i{}_j$$

and

$$\langle \bar{R}e_j, \varepsilon^i\rangle = \langle \bar{R}^k{}_j e_k, \varepsilon^i\rangle = \bar{R}^k{}_i \delta^i{}_k = \bar{R}^j{}_i.$$

Hence $R^k_i = \bar{R}^k_i$ and it follows that

$$\bar{R}e_i = R^j{}_i e_j.$$

On comparison with Eq. (3.6) it follows that the components of a mixed tensor R are the same as the matrix components of the associated operator on V.

Exercise: Show that a tensor R of type (1, 1) defines a map $\tilde{R} : V^* \to V^*$.

Example 7.10 Define the map $\delta : V^* \times V \to \mathbb{R}$ by

$$\delta(\omega,\ v) = \omega(v) = v(\omega) = \langle \omega, v\rangle.$$

This map is clearly linear in both arguments and therefore constitutes a tensor of type $(1, 1)$. If $\{e_i\}$ is any basis of V with dual basis $\{\varepsilon^j\}$, then it is possible to set

$$\delta = e_i \otimes \varepsilon^i \equiv e_1 \otimes \varepsilon^1 + e_2 \otimes \varepsilon^2 + \cdots + e_n \otimes \varepsilon^n$$

since

$$e_i \otimes \varepsilon^i(\omega, v) = e_i(\omega)\,\varepsilon^i(v) = w_i v^i \;=\; \omega(v).$$

An alternative expression for δ is

$$\delta = e_i \otimes \varepsilon^i = \delta^i{}_j e_i \otimes \varepsilon^j$$

from which the components of the mixed tensor δ are precisely the Kronecker delta $\delta^i{}_j$. As no specific choice of basis has been made in this discussion the components of the tensor δ are 'invariant', in the sense that they do not change under basis transformations.

Exercise: Show that the map $\bar{\delta} : V \to V$ that corresponds to the tensor δ according to Example 7.9 is the identity map

$$\bar{\delta} = \mathrm{id}_V.$$

Problems

Problem 7.7 Let $\bar{T}$ be the linear map defined by a covariant tensor T of degree 2 as in Example 7.6. If $\{e_i\}$ is a basis of V and $\{\varepsilon^j\}$ the dual basis, define the matrix of components of $\bar{T}$ with respect to these bases as $[\bar{T}_{ji}]$ where

$$\bar{T}(e_i) = \bar{T}_{ji}\varepsilon^j.$$

Show that the components of the tensor T in this basis are identical with the components as a map, $T_{ij} = \bar{T}_{ij}$.

Similarly if S is a contravariant tensor of degree 2 and $\bar{S}$ the linear map defined in Example 7.7, show that the components $\bar{S}^{ij}$ are identical with the tensor components S^{ij}.

Problem 7.8 Show that every tensor R of type $(1, 1)$ defines a map $\tilde{R} : V^* \to V^*$ by

$$\langle \tilde{R}\omega, u\rangle = R(\omega, u)$$

and show that for a natural definition of components of this map, $\tilde{R}^k{}_i = R^k{}_i$.

Problem 7.9 Show that the definition of tensor product of two vectors $u \times v$ given in Eq. (7.11) agrees with that given in Section 7.1 after relating the two concepts of tensor by isomorphism.

7.3 Basis representation of tensors

We now construct a basis of $V^{(r,s)}$ from any given basis $\{e_i\}$ of V and its dual basis $\{\varepsilon^j\}$, and display tensors of type (r, s) with respect to this basis. While the expressions that arise often turn out to have a rather complicated appearance as multicomponented objects, this is simply a matter of becoming accustomed to the notation. It is still the represention of tensors most frequently used by physicists.

Tensor product

If T is a tensor of type (r, s) and S is a tensor of type (p, q) then define $T \otimes S$, called their **tensor product**, to be the tensor of type $(r + p,\ s + q)$ defined by

$$\begin{aligned}(T \otimes S)(\omega^1, \dots, \omega^r, \rho^1, \dots, \rho^p, u_1, \dots, u_s, v_1, \dots, v_q) \\ = T(\omega^1, \dots, \omega^r, u_1, \dots, u_s)S(\rho^1, \dots, \rho^p, v_1, \dots, v_q).\end{aligned} \tag{7.15}$$

This product generalizes the definition of tensor products of vectors and covectors in the previous secion. It is readily shown to be associative

$$T \otimes (S \otimes R) = (T \otimes S) \otimes R,$$

so there is no ambiguity in writing expressions such as $T \otimes S \otimes R$.

If $e_1, \dots, e_n$ is a basis for V and $\varepsilon^1, \dots, \varepsilon^n$ the dual basis of V^* then the tensors

$$e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s} \quad (i_1,\ i_2, \dots, j_1, \dots, j_s = 1, 2, \dots, n)$$

form a basis of $V^{(r,s)}$, since every tensor T of type (r, s) has a unique expansion

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s} \tag{7.16}$$

where

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} = T(\varepsilon^{i_1}, \dots, \varepsilon^{i_r},\ e_{j_1}, \dots, e_{j_s}) \tag{7.17}$$

are called the **components** of the tensor T with respect to the basis $\{e_i\}$ of V.

Exercise: Prove these statements in full detail. Despite the apparent complexity of indices the proof is essentially identical to that given for the case of $V^{(0,2)}$ in Theorem 7.1.

The components of a linear combination of two tensors of the same type are given by

$$\begin{aligned}(T + aS)^{ij\dots}_{kl\dots} &= (T + aS)(\varepsilon^i, \varepsilon^j, \dots, e_k, e_l, \dots) \\ &= T(\varepsilon^i, \varepsilon^j, \dots, e_k, e_l, \dots) + aS(\varepsilon^i, \varepsilon^j, \dots, e_k, e_l, \dots) \\ &= T^{ij\dots}_{kl\dots} + aS^{ij\dots}_{kl\dots}\ .\end{aligned}$$

The components of the tensor product of two tensors T and S are given by

$$(T \otimes S)^{ij\dots pq\dots}_{kl\dots mn\dots} = T^{ij\dots}_{kl\dots} S^{pq\dots}_{mn\dots}$$

The proof follows from Eq. (7.15) on setting $\omega^1 = \varepsilon^i, \omega^2 = \varepsilon^j, \dots, \rho^1 = \varepsilon^k, \dots, u_1 = e_p$, $u_2 = e_q$, etc.

Exercise: Show that in components, a multilinear map T has the expression

$$T(\omega, \rho, \dots, u, v, \dots) = T^{ij\dots}_{kl\dots} w_i r_j \dots u^k v^l \dots \tag{7.18}$$

where $\omega = w_i \varepsilon^i$, $\rho = r_j \varepsilon^j$, $u = u^k e_k$, etc.

Change of basis

Let $\{e_i\}$ and $\{e'_j\}$ be two bases of V related by

$$e_i = A^j{}_i e'_j, \qquad e'_j = A'^k{}_j e_k \tag{7.19}$$

where the matrices $[A^i{}_j]$ and $[A'^i{}_j]$ are inverse to each other,

$$A'^k{}_j A^j{}_i = A^k{}_j A'^j{}_i = \delta^k_i. \tag{7.20}$$

As shown in Chapter 3 the dual basis transforms by Eq. (3.32),

$$\varepsilon'^j = A^j{}_k \varepsilon^k \tag{7.21}$$

and under the transformation laws components of vectors $v = v^i e_i$ and covectors $\omega = w_j \varepsilon^j$ are

$$v'^j = A^j{}_i v^i, \qquad w'_i = A'^j{}_i w_j. \tag{7.22}$$

The terminology 'contravariant' and 'covariant' transformation laws used in Chapter 3 is motivated by the fact that vectors and covectors are tensors of contravariant degree 1 and covariant degree 1 respectively.

If $T = T^{ij} e_i \otimes e_j$ is a tensor of type (2, 0) then

$$\begin{aligned} T &= T^{ij} e_i \otimes e_j \\ &= T^{ij} A^k{}_i e'_k \otimes A^l{}_j e'_l \\ &= T'^{kl} e'_k \otimes e'_l \end{aligned}$$

where

$$T'^{kl} = T^{ij} A^k{}_i A^l{}_j. \tag{7.23}$$

Exercise: Alternatively, show this result from Eq. (7.21) and

$$T'^{kl} = T(\varepsilon'^k, \varepsilon'^l).$$

Similarly the components of a covariant tensor $T = T_{ij} \varepsilon^i \otimes \varepsilon^j$ of degree 2 transform as

$$T'_{kl} = T_{ij} A'^i{}_k A'^j{}_l. \tag{7.24}$$

Exercise: Show (7.24) (i) by transformation of ε^i using Eq. (7.21), and (ii) from $T'_{ij} = T(e_i, e_j)$ using Eq. (7.19).

In the same way, the components of a mixed tensor $T = T^i{}_j e_i \otimes \varepsilon^j$ can be shown to have the transformation law

$$T'^k{}_l = T^i{}_j A^k{}_i A'^j{}_l. \tag{7.25}$$

Exercise: Show Eq. (7.25).

Before giving the transformation law of components of a general tensor it is useful to establish a convention known as the **kernel index notation**. In this notation we denote the indices on the transformed bases and dual bases by a primed index, $\{e'_{i'} \mid i' = 1, \dots, n\}$ and $\{\varepsilon'^{j'} \mid j' = 1, \dots, n\}$. The primes on the 'kernel' letters e and ε are essentially superfluous and little meaning is lost in dropping them, simply writing $e_{i'}$ and $\varepsilon^{j'}$ for the transformed bases. The convention may go even further and require that the primed indices range over an indexed set of natural numbers $i' = 1', \dots, n'$. These practices may seem a little bizarre and possibly confusing. Accordingly, we will only follow a 'half-blown' kernel index notation, with the key requirement that primed indices be used on transformed quantities. The main advantage of the kernel index notation is that it makes the transformation laws of tensors easier to commit to memory.

Instead of Eq. (7.19) we now write the basis transformations as

$$e_i = A^{i'}_{\ i} e'_{i'}, \qquad e'_{j'} = A'^{j}_{\ j'} e_j \tag{7.26}$$

where the matrix array $\mathsf{A} = [A^{j'}_{\ i}]$ is always written with the primed index in the superscript position, while its inverse $\mathsf{A}^{-1} = [A'^{k}_{\ j'}]$ has the primed index as a subscript. The relations (7.20) between these are now written

$$A'^{k}_{\ j'} A^{j'}_{\ i} = \delta^{k}_{i}, \qquad A^{j'}_{\ k} A'^{k}_{\ i'} = \delta^{j'}_{i'}, \tag{7.27}$$

which take the place of (7.20).

The dual basis satisfies

$$\varepsilon'^{i'}(e'_{j'}) = \delta^{i'}_{j'}$$

and is related to the original basis by Eq. (7.21), which reads

$$\varepsilon^{i} = A'^{i}_{\ j'} \varepsilon'^{j'}, \qquad \varepsilon'^{i'} = A^{i'}_{\ j} \varepsilon^{j}. \tag{7.28}$$

The transformation laws of vectors and covectors (7.22) are replaced by

$$v'^{i'} = A^{i'}_{\ j} v^{j}, \qquad w'_{j'} = A'^{i}_{\ j'} w_i. \tag{7.29}$$

Exercise: If $e_1 = e'_1 + e'_2$, $e_2 = e'_2$ are a basis transformation on a two-dimensional vector space V, write out the matrices $[A^{i'}_{\ j}]$ and $[A'^{i}_{\ j'}]$ and the transformation equation for the components of a contravariant vector v^i and a covariant vector w_j.

The tensor transformation laws (7.23), (7.24) and (7.25) can be replaced by

$$T'^{i'j'} = A^{i'}_{\ i} A^{j'}_{\ j} T^{ij},$$

$$T'_{i'j'} = A'^{i}_{\ i'} A'^{j}_{\ j'} T_{ij},$$

$$T'^{i'}_{\ k'} = A^{i'}_{\ i} A'^{k}_{\ k'} T^{i}_{\ k}.$$

When transformation laws are displayed in this notation the placement of the indices immediately determines whether $A^{i'}_{\ i}$ or $A'^{j}_{\ j'}$ is to be used, as only one of them will give rise to a formula obeying the conventions of summation convention and kernel index notation.

Exercise: Show that the components of the tensor δ are the same in all bases by

(a) showing $e_i \otimes \varepsilon^i = e'_{i'} \otimes \varepsilon'^{i'}$, and
(b) using the transformation law Eq. (7.25).

Now let T be a general tensor of type (r, s),

$$T = T^{i_1 \dots i_r}{}_{j_1 \dots j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \dots \otimes \varepsilon^{j_s}$$

where

$$T^{i_1 \dots i_r}{}_{j_1 \dots j_s} = T(\varepsilon^{i_1}, \dots, \varepsilon^{i_r}, e_{j_1}, \dots e_{j_s}).$$

The separation in spacing between contravariant indices and covariant indices is not strictly necessary but has been done partly for visual display and also to anticipate a further operation called 'raising and lowering indices', which is available in inner product spaces. The transformation of the components of T is given by

$$\begin{aligned} T'^{i'_1,\dots,i'_r}{}_{j'_1,\dots,j'_s} &= T(\varepsilon'^{i'_1}, \dots, \varepsilon'^{i'_r}, e'_{j'_1}, \dots, e'_{j'_s}) \\ &= T(A'^{i'_1}_{i_1} \varepsilon^{i_1}, \dots, A'^{i'_r}_{i_r} \varepsilon^{i_r}, A'^{j_1}_{j'_1} e_{j_1}, \dots, A'^{j_s}_{j'_s} e_{j_s}) \\ &= A'^{i'_1}_{i_1} \dots A'^{i'_r}_{i_r} A'^{j_1}_{j'_1} \dots A'^{j_s}_{j'_s} T^{i_1 \dots i_r}{}_{j_1 \dots j_s}. \end{aligned} \tag{7.30}$$

The general tensor transformation law of components merely replicates the contravariant and covarient transformation law given in (7.29) for each contravariant and covariant index separately. The final formula (7.30) compactly expresses a multiple summation that can represent an enormous number of terms, even in quite simple cases. For example in four dimensions a tensor of type $(3, 2)$ has $4^{3+2} = 1024$ components. Its transformation law therefore consists of 1024 separate formulae, each of which has in it a sum of 1024 terms that themselves are products of six indexed entities. Including all indices and primes on indices, the total number of symbols used would be that occurring in about 20 typical books.

Problems

Problem 7.10 Let e_1, e_2 and e_3 be a basis of a vector space V and $e'_{i'}$ a second basis given by

$$\begin{aligned} e'_1 &= e_1 - e_2, \\ e'_2 &= e_3, \\ e'_3 &= e_1 + e_2. \end{aligned}$$

(a) Display the transformation matrix $\mathsf{A}' = [A'^{i}_{i'}]$.
(b) Express the original basis e_i in terms of the $e'_{i'}$ and write out the transformation matrix $\mathsf{A} = [A^{j'}_{j}]$.
(c) Write the old dual basis ε^i in terms of the new dual basis $\varepsilon'^{i'}$ and conversely.
(d) What are the components of the tensors $T = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_3$ and $S = e_1 \otimes \varepsilon^1 + 3e_1 \otimes \varepsilon^3 - 2e_2 \otimes \varepsilon^3 - e_3 \otimes \varepsilon^1 + 4e_3 \otimes \varepsilon^2$ in terms of the basis e_i and its dual basis?
(e) What are the components of these tensors in terms of the basis $e'_{i'}$ and its dual basis?

Problem 7.11 Let V be a vector space of dimension 3, with basis e_1, e_2, e_3. Let T be the contravariant tensor of rank 2 whose components in this basis are $T^{ij} = \delta^{ij}$, and let S be the covariant tensor of rank 2 whose components are given by $S_{ij} = \delta_{ij}$ in this basis. In a new basis e'_1, e'_2, e'_3 defined by

$$e'_1 = e_1 + e_3$$
$$e'_2 = 2e_1 + e_2$$
$$e'_3 = 3e_2 + e_3$$

calculate the components $T'^{i'j'}$ and $S'_{i'j'}$.

Problem 7.12 Let $T : V \to V$ be a linear operator on a vector space V. Show that its components $T^i{}_j$ given by Eq. (3.6) are those of the tensor $\hat{T}$ defined by

$$\hat{T}(\omega,\ v) = \langle \omega, Tv \rangle.$$

Prove that they are also the components with respect to the dual basis of a linear operator $T^* : V^* \to V^*$ defined by

$$\langle T^*\omega,\ v \rangle = \langle \omega, Tv \rangle.$$

Show that tensors of type (r, s) are in one-to-one correspondence with linear maps from $V^{(s,0)}$ to $V^{(r,0)}$, or equivalently from $V^{(0,r)}$ to $V^{(0,s)}$.

Problem 7.13 Let $T : V \to V$ be a linear operator on a vector space V. Show that its components $T^i{}_j$ defined through Eq. (3.6) transform as those of a tensor of rank (1,1) under an arbitrary basis transformation.

Problem 7.14 Show directly from Eq. (7.14) and the transformation law of components g_{ij}

$$g'_{j'k'} = g_{jk} A'^j{}_{j'} A'^k{}_{k'},$$

that the components of an inverse metric tensor g^{ij} transform as a contravariant tensor of degree 2,

$$g'^{i'k'} = A^{i'}_l g^{lk} A^{k'}_k.$$

7.4 Operations on tensors

Contraction

The process of tensor product (7.15) creates tensors of higher degree from those of lower degrees,

$$\otimes : V^{(r,s)} \times V^{(p,q)} \to V^{(r+p,s+q)}.$$

We now describe an operation that lowers the degree of tensor. Firstly, consider a mixed tensor $T = T^i{}_j e_i \otimes \varepsilon^j$ of type $(1, 1)$. Its **contraction** is defined to be a scalar denoted $C^1_1 T$, given by

$$C^1_1 T = T(\varepsilon^i,\ e_i) = T(\varepsilon^1,\ e_1) + \cdots + T(\varepsilon^n,\ e_n).$$

Although a basis of V and its dual basis have been used in this definition, it is independent of the choice of basis, for if $e'_{i'} = A'^{i}{}_{i'} e_i$ is any other basis then

$$\begin{aligned} T(\varepsilon'^{i'}, e'_{i'}) &= T(A^{i'}_{i}\varepsilon^{i},\ A'^{k}{}_{i'}e_k) \\ &= A^{i'}_{i}A'^{k}{}_{i'}T(\varepsilon^{i},\ e_k) \\ &= \delta^{k}_{i}T(\varepsilon^{i},\ e_k) \quad \text{using Eq. (7.27)} \\ &= T(\varepsilon^{i},\ e_i). \end{aligned}$$

In components, contraction is written

$$C^1_1 T = T^i_i = T^1_1 + T^2_2 + \cdots T^n_n.$$

This is a basis-independent expression since

$$T'^{i'}{}_{i'} = T^{i}{}_{j} A^{i'}_{i} A'^{j}{}_{i'} = T^{i}{}_{j}\delta^{j}_{i} = T^{i}_{i}.$$

Exercise: If $T = u \otimes \omega$, show that its contraction is $C^1_1 T = \omega(u)$.

More generally, for a tensor T of type (r, s) with both $r > 0$ and $s > 0$ one can define its (p, q)**-contraction** $(1 \le p \le r,\ 1 \le q \le s)$ to be the tensor $C^p_q T$ of type $(r-1, s-1)$ defined by

$$\begin{aligned} (C^p_q T)(\omega^1, \dots, \omega^{r-1}, v_1, \dots, v_{s-1}) \\ = \textstyle\sum_{k=1}^{n} T(\omega^1, \dots, \omega^{p-1}, \varepsilon^k, \omega^{p+1}, \dots, \omega^{r-1}, \\ v_1, \dots, v_{q-1}, e_k, v_{q+1}, \dots, v_{s-1}). \end{aligned} \tag{7.31}$$

Exercise: Show that the definition of $C^p_q T$ is independent of choice of basis. The proof is essentially identical to that for the case $r = s = 1$.

On substituting $\omega^1 = \varepsilon^{i_1}, \dots, v_1 = e_{j_1}, \dots$, etc., we arrive at an expression for the (p, q)-contraction in terms of components,

$$(C^p_q T)^{i_1 \dots i_{r-1}}{}_{j_1 \dots j_{s-1}} = T^{i_1 \dots i_{p-1} k i_{p+1} \dots i_{r-1}}{}_{j_1 \dots j_{q-1} k j_{q+1} \dots j_{s-1}}. \tag{7.32}$$

Example 7.11 Let T be a tensor of type (2, 3) having components $T^{ij}{}_{klm}$. Set $A = C^1_2 T$, $B = C^2_3 T$ and $D = C^1_1 T$. In terms of the components of T,

$$\begin{aligned} A^{j}{}_{km} &= T^{ij}{}_{kim}, \\ B^{i}_{kl} &= T^{ij}{}_{klj}, \\ D^{j}{}_{lm} &= T^{ij}{}_{ilm}. \end{aligned}$$

Typical contraction properties of the special mixed tensor δ defined in Example 7.10 are illustrated in the following formulae:

$$\begin{aligned} \delta^{i}{}_{j} T^{j}{}_{kl} &= T^{i}_{kl}, \\ \delta^{i}{}_{j} S^{lm}{}_{ik} &= S^{lm}{}_{jk}, \\ \delta^{i}_{i} &= \underbrace{1 + 1 + \cdots + 1}_{n} = n = \dim V. \end{aligned}$$

Exercise: Write these equations in C^p_q form.

Raising and lowering indices

Let V be a real inner product space with metric tensor $g = g_{ij}\varepsilon^i \otimes \varepsilon^j$ such that

$$u \cdot v = g_{ij}u^i v^j = C^1_1 C^2_2 g \otimes u \otimes v.$$

By Theorem 5.1 the components g_{ij} are a *non-singular* matrix, so that $\det[g_{ij}] \neq 0$. As shown in Example 7.8 there is a tensor g^{-1} whose components, written $g^{ij} = g^{ji}$, form the inverse matrix G^{-1}. Given a vector $u = u^i e_i$, the components of the covector $C^1_2(g \otimes u)$ can be written

$$u_i = g_{ij}u^j,$$

a process that is called **lowering the index**. Conversely, given a covector $\omega = w_i \varepsilon^i$, the vector $C^2_1 g^{-1} \otimes \omega$ can be written in components

$$w^i = g^{ij}w_j,$$

and is called **raising the index**. Lowering and raising indices in succession, in either order, has no effect as

$$u^i = \delta^i{}_j u^j = g^{ik}g_{kj}u^j = g^{ik}u_k.$$

This is important, for without this property, the convention of retaining the same kernel letter u in a raising or lowering operation would be quite untenable.

Exercise: Show that lowering the index on a vector u is equivalent to applying the map $\bar{g}$ in Example 7.6 to u, while raising the index of a covector ω is equivalent to the map $\bar{g}^{-1}$ of Example 7.8.

The tensors g and g^{-1} can be used to raise and lower indices of tensors in general, for example

$$T_i{}^j = g_{ik}T^{kj} = g^{jk}T_{ik} = g_{ik}g^{jl}T^k{}_l, \text{ etc.}$$

It is strongly advised to space out the upper and lower indices of mixed tensors for this process, else it will not be clear which 'slot' an index should be raised or lowered into. For example

$$S_i{}^j{}_p{}^m = S^{kjq}{}_l g_{ik}g_{qp}g^{ml}.$$

If no metric tensor is specified there is no distinction in the relative ordering of covariant and contravariant indices and they can simply be placed one above the other or, as often done above, the contravariant indices may be placed first followed by the covariant indices. Given the capability to raise and lower indices, however, it is important to space all indices correctly. Indeed, by lowering all superscripts every tensor can be displayed in a purely covariant form. Alternatively, it can be displayed in a purely contravariant form by raising every subscript. However, unless the indices are correctly spaced we would not know where the different indices in either of these forms came from in the original 'unlowered' tensor.

Example 7.12 It is important to note that while $\delta^i{}_j$ are components of a mixed tensor the symbol δ_{ij} does *not* represent components of a tensor of covariant degree 2. We therefore try to avoid using this symbol in general tensor analysis. However, by Theorem 5.2, for a

Euclidean inner product space with positive definite metric tensor g it is always possible to find an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ such that

$$e_i \cdot e_j = g_{ij} = \delta_{ij}.$$

In this case a special restricted tensor theory called **cartesian tensors** is frequently employed in which only orthonormal bases are permitted and basis transformations are restricted to orthogonal transformations. In this theory δ_{ij} can be treated as a tensor. The inverse metric tensor then also has the same components $g^{ij} = \delta_{ij}$ and the lowered version of any component index is identical with its raised version,

$$T_{\dots i \dots} = g_{ij} T_{\dots}{}^{j}{}_{\dots} = \delta_{ij} T_{\dots}{}^{j}{}_{\dots} = T^{\dots i \dots}.$$

Thus every cartesian tensor may be written with all its indices in the lower position, $T_{ijk\dots}$, since raising an index has no effect on the values of the components of the tensor.

In cartesian tensors it is common to adopt the summation convention for repeated indices even when they are both subscripts. For example in the standard vector theory of three-dimensional Euclidean space commonly used in mechanics and electromagnetism, one adopts conventions such as

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i \equiv a_1 b_1 + a_2 b_2 + a_3 b_3,$$

and

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon_{ijk} a_j b_k,$$

where the alternating symbol ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123, \\ -1 & \text{if it is an odd permutation of } 123, \\ 0 & \text{if any pair of } ijk \text{ are equal.} \end{cases}$$

It will be shown in Chapter 8 that with respect to proper orthogonal transformations ϵ_{ijk} is a cartesian tensor of type (0, 3).

Symmetries

A tensor S of type (0, 2) is called **symmetric** if $S(u, v) = S(v, u)$ for all vectors u, v in V, while a tensor A of type (0, 2) is called **antisymmetric** if $A(u, v) = -A(v, u)$.

Exercise: In terms of components show that S is a symmetric tensor iff $S_{ij} = S_{ji}$ and A is antisymmetric iff $A_{ij} = -A_{ji}$.

Any tensor T of type (0, 2) can be decomposed into a symmetric and antisymmetric part, $T = \mathcal{S}(T) + \mathcal{A}(T)$, where

$$\mathcal{S}(T)(u, v) = \tfrac{1}{2}(T(u, v) + T(v, u)),$$
$$\mathcal{A}(T)(u, v) = \tfrac{1}{2}(T(u, v) - T(v, u)).$$

It is immediate that these tensors are symmetric and antisymmetric respectively. Setting $u = e_i$, $v = e_j$, this decomposition becomes

$$T_{ij} = T(e_i, e_j) = T_{(ij)} + T_{[ij]},$$

where

$$T_{(ij)} = \mathcal{S}(T)_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji})$$

and

$$T_{[ij]} = \mathcal{A}(T)_{ij} = \tfrac{1}{2}(T_{ij} - T_{ji}).$$

A similar discussion applies to tensors of type (2, 0), having components T^{ij}, but one cannot talk of symmetries of a mixed tensor.

Exercise: Show that $T^i_j = T^j_i$ is not a tensor equation, since it is not invariant under basis transformations.

If A is an antisymmetric tensor of type $(0, 2)$ and S a symmetric tensor of type $(2, 0)$ then their total contraction vanishes,

$$C^1_1 C^2_2\, A \otimes S \equiv A_{ij} S^{ij} = 0 \tag{7.33}$$

since

$$A_{ij} S^{ij} = -A_{ji} S^{ji} = -A_{ij} S^{ij}.$$

Problems

Problem 7.15 Let g_{ij} be the components of an inner product with respect to a basis u_1, u_2, u_3

$$g_{ij} = u_i \cdot u_j = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Find an orthonormal basis of the form $e_1 = u_1$, $e_2 = u_2$, $e_3 = au_1 + bu_2 + cu_3$ such that $a > 0$, and find the index of this inner product.

(b) If $v = u_1 + \frac{1}{2}u_3$ find its lowered components v_i.

(c) Express v in terms of the orthonormal basis found above, and write out its lowered components with respect to that basis.

Problem 7.16 Let g be a metric tensor on a vector space V and define T to be the tensor

$$T = ag^{-1} \otimes g + \delta \otimes u \otimes \omega$$

where u is a non-zero vector of V and ω is a covector.

(a) Write out the components $T^{ij}{}_{kl}$ of the tensor T.

(b) Evaluate the components of the following four contractions:

$$A = C^1_1 T, \qquad B = C^1_2 T, \qquad C = C^2_1 T, \qquad D = C^2_2 T$$

and show that $B = C$.

(c) Show that $D = 0$ iff $\omega(u) = -a$. Hence show that if $T^{ij}_{kl} u^l u_j = 0$, then $D = 0$.

(d) Show that if $n = \dim V > 1$ then $T^{ij}_{kl} u^l u_j = 0$ if and only if $a = \omega(u) = 0$ or $u_i u^i = 0$.

Problem 7.17 On a vector space V of dimension n let T be a tensor of rank $(1, 1)$, S a symmetric tensor of rank $(0, 2)$ and δ the usual 'invariant tensor' of rank $(1, 1)$. Write out the components $R^{ij}{}_{klmr}$ of the tensor

$$R = T \otimes S \otimes \delta + S \otimes \delta \otimes T + \delta \otimes T \otimes S.$$

Perform the contraction of this tensor over i and k, using any available contraction properties of $\delta^i{}_j$. Perform a further contraction over the indices j and r.

Problem 7.18 Show that covariant symmetric tensors of rank 2, satisfying $T_{ij} = T_{ji}$, over a vector space V of dimension n form a vector space of dimension $n(n + 1)/2$.

(a) A tensor S of type $(0, r)$ is called *totally symmetric* if $S_{i_1 i_2 \dots i_r}$ is left unaltered by any interchange of indices. What is the dimension of the vector space spanned by the totally symmetric tensors on V?

(b) Find the dimension of the vector space of covariant tensors of rank 3 having the cyclic symmetry

$$T(u, v, w) + T(v, w, u) + T(w, u, v) = 0.$$

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