9 Special relativity

In Example 7.12 we saw that a Euclidean inner product space with positive definite metric tensor g gives rise to a restricted tensor theory called **cartesian tensors**, wherein all bases $\{e_i\}$ are required to be orthonormal and basis transformations $e_i = A^{i'}_{\ i} e_{i'}$ are restricted to orthogonal transformations. Cartesian tensors may be written with all their indices in the lower position, $T_{ijk\dots}$ and it is common to adopt the summation convention for repeated indices even though both are subscripts.

In a general pseudo-Euclidean inner product space we may also restrict ourselves to orthonormal bases wherein

$$g_{ij} = \begin{cases} \pm 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

so that only pseudo-orthogonal transformation matrices $\mathsf{A} = [A^{i'}_{\ i}]$ are allowed. The resulting tensor theory is referred to as a *restricted tensor theory*. For example, in a four-dimensional Minkowskian vector space the metric tensor in an orthonormal basis is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and the associated restricted tensors are commonly called **4-tensors**. In 4-tensor theory there is a simple connection between covariant and contravariant indices, for example

$$U^1 = U_1, \quad U^2 = U_2, \quad U^3 = U_3, \quad U^4 = -U_4,$$

but the distinction between the two types of indices must still be maintained. In this chapter we give some applications of 4-tensor theory in Einstein's special theory of relativity [1–3].

9.1 Minkowski space-time

In Newtonian mechanics an **inertial frame** is a one-to-one correspondence between physical events and points of $\mathbb{R}^4$, each event being assigned coordinates (x, y, z, t) such that the motion of any free particle is represented by a *rectilinear* path $\mathbf{r} = \mathbf{u}t + \mathbf{r}_0$. This is Newton's first law of motion. Coordinate transformations $(x, y, z, t) \to (x', y', z', t')$

between inertial frames are called **Galilean transformations**, shown in Example 2.29 to have the form

$$t' = t + a, \qquad \mathbf{r}' = \mathsf{A}\mathbf{r} - \mathbf{v}t + \mathbf{b} \tag{9.1}$$

where a is a real constant, $\mathbf{v}$ and $\mathbf{b}$ are constant vectors, and A is a 3×3 orthogonal matrix, $\mathsf{A}^T\mathsf{A} = \mathsf{I}$.

If there is no rotation, $\mathsf{A} = \mathsf{I}$ in (9.1), then a rectilinear motion $\mathbf{r} = \mathbf{u}t + \mathbf{r}_0$ is transformed to

$$\mathbf{r}' = \mathbf{u}'t' + \mathbf{r}'_0$$

where $\mathbf{r}'_0 = \mathbf{r}_0 + \mathbf{b} - a(\mathbf{u} - \mathbf{v})$ and

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}.$$

This is known as the law of **transformation of velocities** and its inverse form,

$$\mathbf{u} = \mathbf{u}' + \mathbf{v}$$

is called the Newtonian law of **addition of velocities**.

In 1888 the famous Michelson–Morley experiment, using light beams oppositely directed at different points of the Earth's orbit, failed to detect any motion of the Earth relative to an 'aether' postulated to be an absolute rest frame for the propagation of electromagnetic waves. The apparent interpretation that the speed of light be constant under transformations between inertial frames in relative motion is clearly at odds with Newton's law of addition of velocities. Eventually the resolution of this problem came in the form of **Einstein's principle of relativity** (1905). This is essentially an extension of Galileo's and Newton's ideas on invariance of mechanics, made to include electromagnetic fields (of which light is a particular manifestation). The geometrical interpretation due to Hermann Minkowski (1908) is the version we will discuss in this chapter.

Poincaré and Lorentz transformations

In classical mechanics we assume that events (x, y, z, t) form a Galilean space-time, as described in Example 2.29. In relativity the structure is somewhat different. Instead of separate spatial and temporal intervals there is a single interval defined between pairs of events, written

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2$$

where c is the velocity of light ($c \approx 3 \times 10^8$ m s^{-1}). This singles out events connected by a light signal as satisfying $\Delta s^2 = 0$. Setting $(x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = ct)$, the interval reads

$$\Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu, \tag{9.2}$$

where

$$\mathsf{G} = [g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Throughout this chapter, Greek indices μ, ν, etc. will range from 1 to 4 while indices $i, j, k \dots$ range from 1 to 3. The set $\mathbb{R}^4$ with this interval structure is called **Minkowski space-time**, or simply **Minkowski space**. The geometrical version of the principle of relativity says that the set of **events** forms a Minkowski space-time. The definition of Minkoswki space as given here is not altogether satisfactory. We will give a more precise definition directly, in terms of an *affine space*.

The restricted class of coordinate systems for which the space-time interval has the form (9.2) will be called **inertial frames**. We will make the assumption, as in Newtonian mechanics, that free particles have rectilinear paths with respect to inertial frames in Minkowski space-time. As shown in Example 2.30, transformations preserving (9.2) are of the form

$$x'^{\mu'} = L^{\mu'}_{\nu} x^{\nu} + a^{\mu'} \tag{9.3}$$

where the coefficients $L^{\mu'}_{\nu}$ satisfy

$$g_{\rho\sigma} = g_{\mu'\nu'} L^{\mu'}_{\rho} L^{\nu'}_{\sigma}. \tag{9.4}$$

Equation (9.3) is known as a **Poincaré transformation**, while the linear transformations $x'^{\mu'} = L^{\mu'}_{\rho} x^{\rho}$ that arise on setting $a^{\mu'} = 0$ are called **Lorentz transformations**.

We define the **light cone** C_p at an event $p = (x, y, z, t)$ to be the set of points connected to p by *light signals*,

$$C_p = \{p' = (x', y', z', ct') \mid \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 = 0\}$$

where $\Delta x = x' - x$, $\Delta y = y' - y$, etc. Events p' on C_p can be thought of either as a receiver of light signals from p, or as a transmitter of signals that arrive at p. Poincaré transformations clearly preserve the light cone C_p at any event p.

As for Eq. (5.9), the matrix version of (9.4) is (see also Example 2.30)

$$\mathsf{G} = \mathsf{L}^T \mathsf{G} \mathsf{L}, \tag{9.5}$$

where $\mathsf{G} = [g_{\mu\nu}]$ and $\mathsf{L} = [L^{\mu'}_{\nu}]$. Taking determinants, we have $\det \mathsf{L} = \pm 1$. It is further possible to subdivide Lorentz transformations into those having $L^{4'}_{4} \geq 1$ and those having $L^{4'}_{4} \leq -1$ (see Problem 9.2). Those Lorentz tansformations for which both $\det \mathsf{L} = +1$ and $L^{4'}_{4} \geq 1$ are called **proper Lorentz transformations**. They are analogous to rotations about the origin in Euclidean space. All other Lorentz transformations are called **improper**.

Affine geometry

There is an important distinction to be made between Minkowski space and a Minkowskian vector space as defined in Section 5.1. Most significantly, Minkowski space is not a vector space since events do not combine linearly in any natural sense. For example, consider two events q and p, having coordinates q^μ and p^μ with respect to some inertial frame. If the linear combination $q + bp$ is defined in the obvious way as being the event having coordinates $(q + bp)^\mu = q^\mu + bp^\mu$, then under a Poincaré transformation (9.3)

$$q'^{\mu'} + bp'^{\mu'} = L^{\mu'}_{\ \nu}(q^\nu + bp^\nu) + (1 + b)a^{\mu'} \neq L^{\mu'}_{\ \nu}(q + bp)^\nu + a^{\mu'}.$$

In particular, the origin $q^\mu = 0$ of Minkowski space has no invariant meaning since it is transformed to a non-zero point under a general Poincaré transformation. The difference of any pair of points, $q^\mu - p^\mu$, does however always undergo a linear transformation

$$q'^{\mu'} - p'^{\mu'} = L^{\mu'}_{\ \nu}(q^\nu - p^\nu)$$

and can be made to form a genuine vector space. Loosely speaking, a structure in which *differences* of points are defined and form a vector space is termed an *affine space*.

More precisely, we define an **affine space** to be a pair (M, V) consisting of a set M and a vector space V, such that V acts freely and transitively on M as an abelian group of transformations. The operation of V on M is written $+ : M \times V \to M$, and is required to satisfy

$$p + (u + v) = (p + u) + v, \qquad p + 0 = p$$

for all $p \in M$. There is then no ambiguity in writing expressions such as $p + u + v$. Recall from Section 2.6 that a *free* action means that if $p + u = p$ $(p \in M)$ then $u = 0$, while the action is *transitive* if for any pair of points $p, q \in M$ there exists a vector $u \in V$ such that $q = p + u$. The vector u in this equation is necessarily unique, for if $q = p + u = p + u'$ then $p + u - u' = p$, and since the action is free it follows that $u = u'$.

Let p_0 be a fixed point of M. For any point $p \in M$ let $x(p) \in V$ be the unique vector such that $p = p_0 + x(p)$. This establishes a one-to-one correspondence between the underlying set M of an affine space and the vector space V acting on it. If e_i is any basis for V then the real functions $p \mapsto x^i(p)$ where $x(p) = x^i(p)e_i$ are said to be **coordinates on** M determined by the basis e_i and the **origin** p_0.

As anticipated above, in an affine space it is always possible to define the difference of any pair of points $q - p$. Given a fixed point $p_0 \in M$ let $x(p)$ and $x(q)$ be the unique vectors in V such that $p = p_0 + x(p)$ and $q = p_0 + x(q)$, and define the **difference** of two points of M to be the vector $q - p = x(q) - x(p) \in V$. This definition is independent of the choice of fixed point p_0, for if p'_0 is a second fixed point such that $p_0 = p'_0 + v$ then

$$p = p'_0 + v + x(p) = p'_0 + x'(p),$$
$$q = p'_0 + v + x(q) = p'_0 + x'(q),$$

and

$$x'(q) - x'(p) = v + x(q) - v - x(p) = x(q) - x(p) = q - p.$$

Minkowski space and 4-tensors

Minkowski space can now be defined as an affine space (M, V) where V is a four-dimensional Minkowskian vector space having metric tensor g, acting freely and transitively on the set M. If $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of V such that

$$g_{\mu\nu} = g(e_\mu, e_\nu) = \begin{cases} 1 & \text{if } \mu = \nu < 4 \\ -1 & \text{if } \mu = \nu = 4 \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

we say an **inertial frame** is a choice of fixed point $p_0 \in M$, called the **origin**, together with the coordinates $x^\mu(p)$ on M defined by

$$p = p_0 + x^\mu(p)e_\mu \quad (p \in M).$$

The **interval** between any two events q and p in M is defined by

$$\Delta s^2 = g(p - q, p - q).$$

This is independent of the choice of fixed point p_0 or orthonormal frame e_μ, since it depends only on the vector difference between p and q and the metric tensor g. In an inertial frame the interval may be expressed in terms of coordinates

$$\Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu \quad \text{where} \quad \Delta x^\mu = x^\mu(q) - x^\mu(p) = x^\mu(q - p). \tag{9.6}$$

Under a Lorentz transformation $e_\nu = L^{\mu'}_{\ \nu} e'_{\mu'}$ and a change of origin $p_0 = p'_0 + a^{\mu'} e'_{\mu'}$ we have for an arbitrary point p

$$\begin{aligned} p &= p_0 + x^\nu(p)e_\nu \\ &= p'_0 + a^{\mu'} e'_{\mu'} + x^\nu(q) L^{\mu'}_{\ \nu} e'_{\mu'} \\ &= p'_0 + x'^{\mu'}(p) e'_{\mu'} \end{aligned}$$

where $x'^{\mu'}(p)$ is given by the Poincaré transformation

$$x'^{\mu'}(p) = L^{\mu'}_{\ \nu} x^\nu(p) + a^{\mu'}. \tag{9.7}$$

It is a simple matter to verify that the coordinate expression (9.6) for Δs^2 is invariant with respect to Poincaré transformations (9.7).

Elements $v = v^\mu e_\mu$ of V will be termed **4-vectors**. With respect to a Poincaré transformation (9.7) the components v^μ transform as

$$v'^{\mu'} = L^{\mu'}_{\ \nu} v^\nu,$$

where $L^{\mu'}_{\ \nu}$ satisfy (9.4). The inverse transformations are

$$v^\nu = L'^{\nu}_{\ \rho'} v'^{\rho'} \quad \text{where} \quad L'^{\nu}_{\ \rho'} L^{\rho'}_{\ \mu} = \delta^\nu_{\ \mu}.$$

Elements of $V(r, s)$, defined in Chapter 7, are termed **4-tensors of type** (r, s). Since we restrict attention to orthonormal bases of V, the components $T^{\sigma\tau\dots}{}_{\mu\nu\dots}$ of a 4-tensor are only required to transform as a tensor with respect to the Lorentz transformations,

$$T'^{\sigma'\rho'\dots}{}_{\mu'\nu'\dots} = T^{\sigma\rho\dots}{}_{\mu\nu\dots} L'^{\sigma'}_{\ \sigma} L'^{\rho'}_{\ \rho} \dots L^{\mu}_{\ \mu'} L^{\nu}_{\ \nu'} \dots$$

4-tensors of type (0, 1) are called **4-covectors**, and 4-tensors of type (0, 0) will be termed **4-scalars** or simply **scalars**. The important thing about 4-tensors, as for general tensors, is that if a 4-tensor equation can be shown to hold in one particular frame it holds in all frames. This is an immediate consequence of the homogeneous transformation law of components.

By Eq. (9.4) g is a covariant 4-tensor of rank 2 since its components $g_{\mu\nu}$ transform as

$$g'_{\mu'\nu'} = g_{\mu\nu} L'^{\mu}{}_{\mu'} L'^{\nu}{}_{\nu'}$$

where $g'_{\mu'\nu'} = g_{\mu'\nu'}$. The inverse metric $g^{\mu\nu}$, defined by

$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu},$$

has identical components to $g_{\mu\nu}$ and is a contravariant tensor of rank 2,

$$g'^{\mu'\nu'} = g^{\mu\nu} L^{\mu'}_{\mu} L^{\nu'}_{\nu}, \qquad g^{\mu\nu} = g'^{\mu'\nu'} L'^{\mu}{}_{\mu'} L'^{\nu}{}_{\nu'}.$$

We will use $g^{\mu\nu}$ and $g_{\mu\nu}$ to raise and lower indices of 4-tensors; for example,

$$U^{\mu} = g^{\mu\nu} U_{\nu}, \quad W_{\mu} = g_{\mu\nu} W^{\nu}, \quad T_{\mu\nu}{}^{\rho} = g_{\mu\alpha} g^{\rho\beta} T^{\alpha}{}_{\nu\beta}.$$

Given two 4-vectors $A = A^{\mu} e_{\mu}$, $B = B^{\nu} e_{\nu}$, define their **inner product** to be the scalar

$$g(A, B) = A^{\mu} B_{\mu} = g_{\mu\nu} A^{\mu} B^{\nu} = A_{\mu} B^{\mu} = g^{\mu\nu} A_{\mu} B_{\nu}.$$

We say the vectors are **orthogonal** if $A^{\mu} B_{\mu} = 0$. The **magnitude** of a 4-vector A^{μ} is defined to be $g(A, A) = A^{\mu} A_{\mu}$. A non-zero 4-vector A^{μ} is called

spacelike if $g(A, A) = A^{\mu} A_{\mu} > 0$,
timelike if $g(A, A) = A^{\mu} A_{\mu} < 0$,
null if $g(A, A) = A^{\mu} A_{\mu} = 0$.

The set of all null 4-vectors is called the **null cone**. This is a subset of the vector space V of 4-vectors. The concept of a **light cone at** $p \in M$, defined in Section 2.30, is the set of points of M that are connected to p by a null vector, $C_p = \{q \mid g(q - p, q - p) = 0\} \subset M$. Figure 9.1 shows how the null cone separates 4-vectors into the various classes. Timelike or null vectors falling within or on the upper half of the null cone are called **future-pointing**, while those in the lower half are **past-pointing**.

Spacelike vectors, however, lie outside the null cone and form a continuously connected region of V, making it impossible to define invariantly the concept of a future-pointing or past-pointing spacelike vector – see Problem 9.2.

Problems

Problem 9.1 Show that

$$[L^{\mu'}_{\nu}] = \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\gamma & \gamma \\ \alpha & \beta & -\gamma & 1+\gamma \end{pmatrix} \quad \text{where} \quad \gamma = \tfrac{1}{2}(\alpha^2 + \beta^2)$$

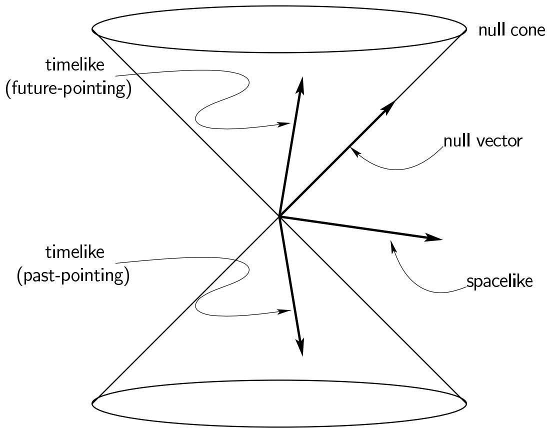

Figure 9.1 The null cone in Minkowski space

is a Lorentz transformation for all values of α and β. Find those 4-vectors V^μ whose components are unchanged by all Lorentz transformations of this form.

Problem 9.2 Show that for *any* Lorentz transformation $L^{\mu'}_{\nu}$ one must have either

$$L^4_{\ 4} \geq 1 \quad \text{or} \quad L^4_{\ 4} \leq -1.$$

(a) Show that those transformations having $L^4_{\ 4} \geq 1$ have the property that they preserve the concept of 'before' and 'after' for timelike separated events by demonstrating that they preserve the sign of Δx^4.
(b) What is the effect of a Lorentz transformation having $L^4_{\ 4} \leq -1$?
(c) Is there any meaning, independent of the inertial frame, to the concepts of 'before' and 'after' for spacelike separated events?

Problem 9.3 Show that (i) if T^α is a timelike 4-vector it is always possible to find a Lorentz transformation such that $T'^{\alpha'}$ will have components $(0, 0, 0, a)$ and (ii) if N^α is a null vector then it is always possible to find a Lorentz transformation such that $N'^{\alpha'}$ has components $(0, 0, 1, 1)$.

Let U^α and V^α be 4-vectors. Show the following:

(a) If $U^\alpha V_\alpha = 0$ and U^α is timelike, then V^α is spacelike.
(b) If $U^\alpha V_\alpha = 0$ and U^α and V^α are both null vectors, then they are proportional to each other.
(c) If U^α and V^α are both timelike future-pointing then $U_\alpha V^\alpha < 0$ and $U^\alpha + V^\alpha$ is timelike.
(d) Find other statements similar to the previous assertions when U^α and V^α are taken to be various combinations of null, future-pointing null, timelike future-pointing, spacelike, etc.

Problem 9.4 If the 4-component of a 4-vector equation $A^4 = B^4$ is shown to hold in all inertial frames, show that all components are equal in all frames, $A^\mu = B^\mu$.

9.2 Relativistic kinematics

Special Lorentz transformations

From time to time we will call upon specific types of Lorentz transformations. The following two examples present the most commonly used types.

Example 9.1 Time-preserving Lorentz transformations have $t' = t$, or equivalently $x'^4 = x^4$. Such transformations have $L^4_4 = 1$, $L^4_i = 0$ for $i = 1, 2, 3$, and substituting in Eq. (9.4) with $\rho = \sigma = 4$ gives

$$\sum_{i'=1}^{3}\left(L^{i'}_4\right)^2 - \left(L^4_4\right)^2 = -1 \implies \sum_{i'=1}^{3}(L^{i'}_4)^2 = 0.$$

This can only hold if $L^{i'}_4 = 0$ for $i' = 1,\ 2,\ 3$. Hence

$$\mathsf{L} = [L^{\mu'}_{\nu}] = \begin{pmatrix} & & & 0 \\ & [a_{ij}] & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (i, j, \dots = 1, 2, 3) \tag{9.8}$$

where $\mathsf{A} = [a_{ij}]$ is an orthogonal 3×3 matrix, $\mathsf{A}^T\mathsf{A} = \mathsf{I}$, which follows on substituting L in Eq. (9.5). If $\det \mathsf{L} = \det \mathsf{A} = +1$ these transformations are spatial rotations, while if $\det \mathsf{A} = -1$ they are space reflections.

Example 9.2 Lorentz transformations that leave the y and z coordinates unchanged are of the form

$$\mathsf{L} = [L^{\mu'}_{\nu}] = \begin{pmatrix} L^1_1 & 0 & 0 & L^1_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ L^4_1 & 0 & 0 & L^4_4 \end{pmatrix}.$$

Substituting in Eq. (9.4) gives

$$L^1_1 L^1_1 - L^4_1 L^4_1 = g_{11} = 1, \tag{9.9}$$
$$L^1_1 L^1_4 - L^4_1 L^4_4 = g_{14} = g_{41} = 0, \tag{9.10}$$
$$L^1_4 L^1_4 - L^4_4 L^4_4 = g_{44} = -1. \tag{9.11}$$

From (9.11), we have $(L^4_4)^2 = 1 + \sum_{i=1}^{3}(L^i_4)^2 \geq 1$, and assuming $L^4_4 \geq 1$ it is possible to set $L^4_4 = \cosh\alpha$ for some real number α. Then $L^1_4 = \pm\sqrt{\cosh^2\alpha - 1} = \sinh\alpha$ on choosing α with the appropriate sign. Similarly, (9.9) implies that $L^1_1 = \cosh\beta$, $L^4_1 = \sinh\beta$ and (9.10) gives

$$0 = \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta = \sinh(\alpha - \beta) \implies \alpha = \beta.$$

Let v be the unique real number defined by

$$\tanh\alpha = -\frac{v}{c},$$

then trigonometric identities give that $|v| < c$ and

$$\cosh\alpha = \gamma, \qquad \sinh\alpha = -\gamma\frac{v}{c}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{9.12}$$

The resulting Lorentz transformations have the form

$$\mathsf{L} = [L^{\mu'}{}_{\nu}] = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix}, \tag{9.13}$$

and are known as **boosts** with velocity v in the x-direction. Written out explicitly in x, y, z, t coordinates they read

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\Big(t - \frac{v}{c^2}x\Big). \tag{9.14}$$

The inverse transformation is obtained on replacing v by $-v$

$$x = \gamma(x' + vt'), \qquad y = y', \qquad z = z', \qquad t = \gamma\Big(t' + \frac{v}{c^2}x'\Big). \tag{9.15}$$

The parameter v plays the role of a relative velocity between the two frames since the spatial origin ($x' = 0, y' = 0, z' = 0$) in the primed frame satisfies the equation $x = vt$ in the unprimed frame. As the relative velocity v must always be less than c we have the first indication that according to relativity theory, the velocity of light c is a limiting velocity for material particles.

Exercise: Verify that performing two Lorentz transformations with velocities v_1 and v_2 in the x-directions in succession is equivalent to a single Lorentz transformation with velocity

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}.$$

Relativity of time, length and velocity

Two events $p = (x_1, y_1, z_1, ct_1)$ and $q = (x_2, y_2, z_2, ct_2)$ are called *simultaneous* with respect to an inertial frame K if $\Delta t = t_2 - t_1 = 0$. Consider a second frame K' related to K by a boost, (9.14). These equations are linear and therefore apply to coordinate differences,

$$\Delta x' = \gamma(\Delta x - v\Delta t), \qquad \Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x).$$

Hence,

$$\Delta t = 0 \implies \Delta t' = -\gamma\frac{v}{c^2}\Delta x \neq 0 \quad \text{if} \quad x_1 \neq x_2,$$

demonstrating the effect known as **relativity of simultaneity**: simultaneity of spatially separated points is not an absolute concept.

Consider now a clock at rest in K' marking off successive 'ticks' at events (x', y', z', ct'_1) and (x', y', z', ct'_2). The time difference according to K is given by (9.15),

$$\Delta t = \gamma\Big(\Delta t' + \frac{v}{c^2}\Delta x'\Big) = \gamma\,\Delta t' \quad \text{if} \quad \Delta x' = 0.$$

That is,

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \geq \Delta t', \tag{9.16}$$

an effect known as **time dilatation** – a moving clock appears to slow down. Equivalently, a stationary clock in K appears to run slow according to the moving observer K'.

Now consider a rod of length $\ell = \Delta x$ at rest in K. Again, using the inverse boost transformation (9.15) we have

$$\ell = \Delta x = \gamma(\Delta x' + v\Delta t') = \gamma\,\Delta x' \quad \text{if} \quad \Delta t' = 0.$$

The rod's length with respect to K' is determined by considering simultaneous moments $t'_1 = t'_2$ at the end points,

$$\ell' = \Delta x' = \frac{\ell}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}\,\ell \leq \ell. \tag{9.17}$$

The common interpretation of this result is that the length of a rod is contracted when viewed by a moving observer, an effect known as the **Lorentz–Fitzgerald contraction**. By reversing the roles of K and K' it is similarly found that a moving rod is contracted in the direction of its motion. The key to this effect is that, by the relativity of simultaneity, pairs of events on the histories of the ends of the rod that are simultaneous with respect to K' differ from simultaneous pairs in the frame K. Since there is no contraction perpendicular to the motion, a moving volume V will undergo a contraction

$$V' = \sqrt{1 - \frac{v^2}{c^2}}\,V. \tag{9.18}$$

This is the most useful application of the Lorentz–Fitzgerald contraction.

Exercise: Give the reverse arguments to the above; that a clock at rest runs slow relative to a moving observer, and that a moving rod appears contracted.

Let a particle have velocity $\mathbf{u} = (u_x, u_y, u_z)$ with respect to K, and $\mathbf{u}' = (u'_x, u'_y, u'_z)$ with respect to K'. Setting

$$u_x = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad u'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'}, \quad u_y = \frac{\mathrm{d}y}{\mathrm{d}t}, \text{ etc.}$$

and using the Lorentz transformations (9.14), we have

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}, \qquad u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)}, \qquad u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)}. \tag{9.19}$$

Comparing with the Newtonian discussion at the beginning of this chapter it is natural to call this the **relativistic law of transformation of velocities**. Similarly on using the inverse Lorentz transformations (9.15), we arrive at the **relativistic law of addition of velocities**:

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}, \qquad u_y = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}, \qquad u_z = \frac{u'_z}{\gamma(1 + u'_x v/c^2)}. \tag{9.20}$$

The same result can be obtained from (9.19) by replacing v by $-v$ and interchanging primed and unprimed velocities.

For a particle moving in the x–y plane set $u_x = u\cos\theta$, $u_y = u\sin\theta$, $u_z = 0$ and $u'_x = u'\cos\theta'$, $u'_y = u'\sin\theta'$, $u'_z = 0$. If $u' = c$ it follows from Eq. (9.20) that $u = c$, and the velocity of light is independent of the motion of the observer as required by Einstein's principle of relativity. The second equation of (9.20) gives a relation between the θ and θ', the angles the light beam subtends with the x- and x'-directions respectively:

$$\sin\theta = \frac{\sin\theta'}{1 + (v/c)\cos\theta'}\sqrt{1 - \frac{v^2}{c^2}}. \tag{9.21}$$

This formula is known as the **relativistic aberration of light**. If $\dfrac{v}{c} \ll 1$ then

$$\delta\theta = \theta - \theta' \approx -\frac{v}{c}\sin\theta',$$

a Newtonian formula for aberration of light, which follows simply from the triangle addition law of velocities and was used by the astronomer Bradley nearly 300 years ago to estimate the velocity of light.

Problems

Problem 9.5 From the law of transformation of velocities, Eq. (9.19), show that the velocity of light in an arbitrary direction is invariant under boosts.

Problem 9.6 If two intersecting light beams appear to be making a non-zero angle ϕ in one frame K, show that there always exists a frame K' whose motion relative to K is in the plane of the beams such that the beams appear to be directed in opposite directions.

Problem 9.7 A source of light emits photons uniformly in all directions in its own rest frame.

(a) If the source moves with velocity v with respect to an inertial frame K, show the 'headlight effect': half the photons seem to be emitted in a forward cone whose semi-angle is given by $\cos\theta = v/c$.

(b) In films of the *Star Wars* genre, star fields are usually seen to be swept backwards around a rocket as it accelerates towards the speed of light. What would such a rocketeer really see as his velocity $v \to c$?

Problem 9.8 If two separate events occur at the same time in some inertial frame S, prove that there is no limit on the time separations assigned to these events in other frames, but that their space separation varies from infinity to a minimum that is measured in S. With what speed must an observer travel in order that two simultaneous events at opposite ends of a 10-metre room appear to differ in time by 100 years?

Problem 9.9 A supernova is seen to explode on Andromeda galaxy, while it is on the western horizon. Observers A and B are walking past each other, A at 5 km/h towards the east, B at 5 km/h towards the west. Given that Andromeda is about a million light years away, calculate the difference in time attributed to the supernova event by A and B. Who says it happened earlier?

Problem 9.10 Twin A on the Earth and twin B who is in a rocketship moving away from him at a speed of $\frac{1}{2}c$ separate from each other at midday on their common birthday. They decide to each blow out candles exactly four years from B's departure.

(a) What moment in B's time corresponds to the event P that consists of A blowing his candle out? And what moment in A's time corresponds to the event Q that consists of B blowing her candle out?
(b) According to A which happened earlier, P or Q? And according to B?
(c) How long will A have to wait before he *sees* his twin blowing her candle out?

9.3 Particle dynamics

World-lines and proper time

Let $I = [a, b]$ be any closed interval of the real line $\mathbb{R}$. A continuous map $\sigma : I \to M$ is called a **parametrized curve** in Minkowski space (M, V). In an inertial frame generated by a basis e_μ of V such a curve may be written as four real functions $x^\mu \circ \sigma : I \to \mathbb{R}$. We frequently write these functions as $x^\mu(\lambda)$ $(a \leq \lambda \leq b)$ in place of $x^\mu(\sigma(\lambda))$, and generally assume them to be differentiable.

If the parametrized curve σ passes through the event p having coordinates p^μ, so that $p^\mu = x^\mu(\lambda_0) \equiv x^\mu(\sigma(\lambda_0))$ for some $\lambda_0 \in I$, define the **tangent 4-vector** to the curve at p to be the 4-vector U given by

$$U = U^\mu e_\mu \in V \quad \text{where} \quad U^\mu = \left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_0}.$$

This definition is independent of the choice of orthonormal basis e_μ on V, for if $e'_{\mu'}$ is a second o.n. basis related by a Lorentz transformation $e_\nu = L^{\mu'}{}_\nu e'_{\mu'}$, then

$$U' = U'^{\mu'} e'_{\mu'} = \left.\frac{\mathrm{d}x'^{\mu'}(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_0} L^{\mu}{}_{\mu'} e_\mu = \left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_0} e_\mu = U.$$

The parametrized curve σ is called **timelike, spacelike** or **null** at p if its tangent 4-vector at p is timelike, spacelike or null, respectively. The path of a material particle will be assumed to be timelike at all events through which it passes, and is frequently referred to as the particle's **world-line** (see Fig. 9.2). This assumption amounts to the requirement that the particle's velocity is always less than c, for

$$0 > g(U(\lambda), U(\lambda)) = g_{\mu\nu}\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu(\lambda)}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\left(\sum_{i=1}^{3}\left(\frac{\mathrm{d}x^i(t)}{\mathrm{d}t}\right)^2 - c^2\right),$$

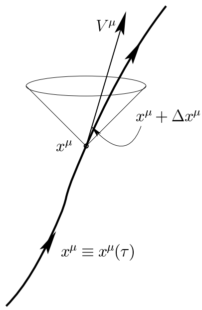

Figure 9.2 World-line of a material particle

on setting $t = x^4/c = t(\lambda)$. Hence

$$v^2 = \sum_{i=1}^{3} \left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)^2 < c^2.$$

For two neighbouring events on the world-line, $x^\mu(\lambda)$ and $x^\mu(\lambda + \Delta\lambda)$, set

$$\Delta\tau^2 = -\frac{1}{c^2}\Delta s^2 = -\frac{1}{c^2} g_{\mu\nu} \Delta x^\mu \Delta x^\nu > 0,$$

where

$$\Delta x^\mu = x^\mu(\lambda + \Delta\lambda) - x^\mu(\lambda).$$

In the limit $\Delta\lambda \to 0$

$$\Delta\tau^2 \to -\frac{1}{c^2} g_{\mu\nu} \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu(\lambda)}{\mathrm{d}\lambda} \big(\Delta\lambda\big)^2 = -\frac{1}{c^2}(v^2 - c^2)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 \Delta\lambda^2.$$

Hence

$$\Delta\tau \to \sqrt{1 - \frac{v^2}{c^2}}\,\Delta t = \frac{1}{\gamma}\Delta t. \tag{9.22}$$

Since the velocity of the particle is everywhere less than c, the relativistic law of transformation of velocities (9.19) can be used to find a combination of rotation and boost, (9.8) and (9.14), which transforms the particle's velocity to zero at any given point $p = \sigma(\lambda_0)$ on the particle's path. Any such inertial frame in which the particle is momentarily at rest is known as an **instantaneous rest frame** or i.r.f. at p. The i.r.f will of course vary from point to point on a world-line, unless the velocity is constant along it. Since $\mathbf{v} = \mathbf{0}$ in an i.r.f. we have from (9.22) that $\Delta\tau/\Delta t \to 1$ as $\Delta t \to 0$. Thus $\Delta\tau$ measures the time interval registered on an inertial clock instantaneously comoving with the particle. It is generally interpreted as the time measured on a clock *carried* by the particle from x^μ to $x^\mu + \Delta x^\mu$.

The factor $1/\gamma$ in Eq. (9.22) represents the time dilatation effect of Eq. (9.16) on such a clock due to its motion relative to the external inertial frame. The total time measured on a clock carried by the particle from event p to event q is given by

$$\tau_{pq} = \int_p^q \mathrm{d}\tau = \int_{t_p}^{t_q} \frac{\mathrm{d}t}{\gamma},$$

and is called the **proper time** from p to q. If we fix the event p and let q vary along the curve then proper time can be used as a parameter along the curve,

$$\tau = \int_{t_p}^{t} \frac{\mathrm{d}t}{\gamma} = \tau(t). \tag{9.23}$$

The tangent 4-vector $V = V^\mu e_\mu$ calculated with respect to this special parameter is called the **4-velocity** of the particle,

$$V^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \gamma(\mathbf{v},\ c). \tag{9.24}$$

Unlike coordinate time t, proper time τ is a true scalar parameter independent of inertial frame; hence the components of 4-velocity V^μ transform as a contravariant 4-vector

$$V'^{\mu'} = L^{\mu'}_{\ \nu} V^\nu.$$

From Eq. (9.24) the magnitude of the 4-velocity always has constant magnitude $-c^2$,

$$g(V, V) = V^\mu V_\mu = \left(v^2 - c^2\right)\gamma^2 = -c^2. \tag{9.25}$$

The **4-acceleration** of a particle is defined to be the contravariant 4-vector $A = A^\mu e_\mu$ with components

$$A^\mu = \frac{\mathrm{d}V^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu(\tau)}{\mathrm{d}\tau^2}. \tag{9.26}$$

Expressing these components in terms of the coordinate time parameter t gives

$$A^\mu = \gamma\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v} + \gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t},\ c\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right). \tag{9.27}$$

The 4-vectors A and V are orthogonal to each other since

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big(g(V, V)\big) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-c^2\right) = 0,$$

and expanding the left-hand side gives $g(A, V) + g(V, A) = 2g(A, V)$, so that

$$g(A, V) = A^\mu V_\mu = 0. \tag{9.28}$$

Exercise: Show that in an i.r.f. the components of 4-velocity and 4-acceleration are given by

$$V^\mu = (\mathbf{0},\ c), \qquad A^\mu = (\mathbf{a},\ 0) \quad \text{where} \quad \mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t},$$

and verify that the 4-vectors A and V are orthogonal to each other.

Relativistic particle dynamics

We assume each particle has a constant scalar m attached to it, called its **rest mass**. This may be thought of as the Newtonian mass in an instantaneous rest frame of the particle, satisfying Newton's second law $\mathbf{F} = m\mathbf{a}$ for any imposed force $\mathbf{F}$ in that frame. The **4-momentum** of the particle is defined to be the 4-vector having components $P^\mu = mV^\mu$ where $V = V^\mu e_\mu$ is the 4-velocity of the particle,

$$P^\mu = \left(\mathbf{p},\ \frac{E}{c}\right) \tag{9.29}$$

where

$$\mathbf{p} = m\gamma\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \text{momentum}, \tag{9.30}$$

$$E = m\gamma c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \text{energy}. \tag{9.31}$$

For $v \ll c$ the momentum reduces to the Newtonian formula $\mathbf{p} = m\mathbf{v}$ and the energy can be written as $E \approx mc^2 + \frac{1}{2}mv^2 + \dots$ The energy contribution $E = mc^2$, which arises even when the particle is at rest, is called the particle's **rest-energy**.

Exercise: Show the following identities:

$$g(P, P) = P^\mu P_\mu = -m^2c^2, \qquad E = \sqrt{p^2c^2 + m^2c^2}, \qquad \mathbf{p} = \frac{E\mathbf{v}}{c^2}. \tag{9.32}$$

The relations (9.32) make sense even in the limit $v \to c$ provided the particle has zero rest mass, $m = 0$. Such particles will be termed **photons**, and satisfy the relations

$$E = pc, \qquad \mathbf{p} = \frac{E}{c}\mathbf{n} \quad \text{where} \quad \mathbf{n}\cdot\mathbf{n} = 1. \tag{9.33}$$

Here $\mathbf{n}$ is called the **direction of propagation** of the photon. The 4-momentum of a photon has the form

$$P^\mu = \left(\mathbf{p},\ \frac{E}{c}\right) = \frac{E}{c}(\mathbf{n},\ 1),$$

and is clearly a null vector, $P^\mu P_\mu = 0$.

In analogy with Newton's law $\mathbf{F} = m\mathbf{a}$, it is sometimes useful to define a **4-force** $F = F^\mu e_\mu$ having components

$$F^\mu = \frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = mA^\mu. \tag{9.34}$$

By Eq. (9.28) the 4-force is always orthogonal to the 4-velocity. Defining **3-force f** in the usual way by

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

and using $\dfrac{\mathrm{d}}{\mathrm{d}\tau} = \gamma \dfrac{\mathrm{d}}{\mathrm{d}t}$ we obtain

$$F^\mu = \gamma \left(\mathbf{f}, \ \frac{1}{c} \frac{\mathrm{d}E}{\mathrm{d}t} \right). \tag{9.35}$$

Problems

Problem 9.11 Using the fact that the 4-velocity $V^\mu = \gamma(u)(u_x, u_y, u_z, c)$ transforms as a 4-vector, show from the transformation equation for V'^4 that the transformation of u under boosts is

$$\frac{\gamma(u')}{\gamma(u)} = \gamma(v) \left(1 - \frac{v u_x}{c^2}\right).$$

From the remaining transformation equations for $V'^{i'}$ derive the law of transformation of velocities (9.19).

Problem 9.12 Let K' be a frame with velocity v relative to K in the x-direction.

(a) Show that for a particle having velocity u', acceleration a' in the x'-direction relative to K', its acceleration in K is

$$a = \frac{a'}{[\gamma(1 + v u'/c^2)]^3}.$$

(b) A rocketeer leaves Earth at $t = 0$ with constant acceleration g at every moment relative to his instantaneous rest frame. Show that his motion relative to the Earth is given by

$$x = \frac{c^2}{g} \left(\sqrt{1 + \frac{g^2}{c^2} t^2} - 1 \right).$$

(c) In terms of his own proper time τ show that

$$x = \frac{c^2}{g} \left(\cosh \frac{g}{c} \tau - 1 \right).$$

(d) If he proceeds for 10 years of his life, decelerates with $g = 9.80 \text{ m s}^{-2}$ for another 10 years to come to rest, and returns in the same way, taking 40 years in all, how much will people on Earth have aged on his return? How far, in light years, will he have gone from Earth?

Problem 9.13 A particle is in hyperbolic motion along a world-line whose equation is given by

$$x^2 - c^2 t^2 = a^2, \qquad y = z = 0.$$

Show that

$$\gamma = \frac{\sqrt{a^2 + c^2 t^2}}{a}$$

and that the proper time starting from $t = 0$ along the path is given by

$$\tau = \frac{a}{c} \cosh^{-1} \frac{ct}{a}.$$

Evaluate the particle's 4-velocity V^μ and 4-acceleration A^μ. Show that A^μ has constant magnitude.

Problem 9.14 For a system of particles it is generally assumed that the **conservation of total 4-momentum** holds in any localized interaction,

$$\sum_a P^\mu_{(a)} = \sum_b Q^\mu_{(b)}.$$

Use Problem 9.4 to show that the law of conservation of 4-momentum holds for a given system provided the law of energy conservation holds in all inertial frames. Also show that the law of conservation of momentum in all frames is sufficient to guarantee conservation of 4-momentum.

Problem 9.15 A particle has momentum $\mathbf{p}$, energy E in a frame K.

(a) If K' is an inertial frame having velocity $\mathbf{v}$ relative to K, use the transformation law of the momentum 4-vector $P^\mu = \left(\mathbf{p}, \dfrac{E}{c}\right)$ to show that

$$E' = \gamma(E - \mathbf{v}\cdot\mathbf{p}), \qquad \mathbf{p}'_\perp = \mathbf{p}_\perp \quad \text{and} \quad \mathbf{p}'_\parallel = \gamma\left(\mathbf{p}_\parallel - \frac{E}{c^2}\mathbf{v}\right),$$

where $\mathbf{p}_\perp$ and $\mathbf{p}_\parallel$ are the components of $\mathbf{p}$ respectively perpendicular and parallel to $\mathbf{v}$.

(b) If the particle is a photon, use these transformations to derive the aberration formula

$$\cos\theta' = \frac{\cos\theta - v/c}{1 - \cos\theta\,(v/c)}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{v}$.

Problem 9.16 Use $F^\mu V_\mu = 0$ to show that

$$\mathbf{f}\cdot\mathbf{v} = \frac{\mathrm{d}E}{\mathrm{d}t}.$$

Also show this directly from the definitions (9.30) and (9.31) of $\mathbf{p}$, E.

9.4 Electrodynamics

4-Tensor fields

A **4-tensor field** of type (r, s) consists of a map $T : M \to V^{(r,s)}$. We can think of this as a 4-tensor assigned at each point of space-time. The components of a 4-tensor field are functions of space-time coordinates

$$T^{\mu\nu\dots}{}_{\rho\sigma\dots} = T^{\mu\nu\dots}{}_{\rho\sigma\dots}(x^\alpha).$$

Define the **gradient** of a 4-tensor field T to be the 4-tensor field of type $(r, s+1)$, having components

$$T^{\mu\nu\dots}{}_{\rho\sigma\dots,\tau} = \frac{\partial}{\partial x^\tau} T^{\mu\nu\dots}{}_{\rho\sigma\dots}.$$

This is a 4-tensor field since a Poincaré transformation (9.7) induces the transformation

$$\begin{aligned} T^{\mu' \dots}{}_{\rho' \dots, \tau'} &= \frac{\partial}{\partial x'^{\tau'}} \left(T^{\alpha \dots}{}_{\beta \dots} L^{\mu'}_{\ \alpha} \dots L'^{\beta}_{\ \rho'} \dots \right) \\ &= \frac{\partial x^{\gamma}}{\partial x'^{\tau'}} \frac{\partial}{\partial x^{\gamma}} \left(T^{\alpha \dots}{}_{\beta \dots} L^{\mu'}_{\ \alpha} \dots L'^{\beta}_{\ \rho'} \dots \right) \\ &= T^{\alpha \dots}{}_{\beta \dots, \gamma} L'^{\gamma}_{\ \tau'} L^{\mu'}_{\ \alpha} \dots L'^{\beta}_{\ \rho'} \dots \end{aligned}$$

For example, if $f : M \to \mathbb{R}$ is a scalar field, its gradient is a 4-covector field,

$$f_{,\mu} = \frac{\partial f(x^{\alpha})}{\partial x^{\mu}}.$$

Example 9.3 A 4-vector field, $J = J^{\mu}(x^{\alpha}) e_{\mu}$, is said to be **divergence-free** if

$$J^{\mu}_{\ ,\mu} = 0.$$

Setting $j_i = J^i$ $(i = 1, 2, 3)$ and $\rho = \dfrac{1}{c} J^4$, the divergence-free condition reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{9.36}$$

known both in hydrodynamics and electromagnetism as the **equation of continuity**. Interpreting ρ as the *charge density* or charge per unit volume, $\mathbf{j}$ is the *current density*. The charge per unit time crossing unit area normal to the unit vector $\mathbf{n}$ is given by $\mathbf{j} \cdot \mathbf{n}$. Equation (9.36) implies *conservation of charge* – the rate of increase of charge in a volume $\mathcal{V}$ equals the flux of charge entering through the boundary surface $\mathcal{S}$:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}V = - \int_{\mathcal{V}} \nabla \cdot \mathbf{j} \, \mathrm{d}V = - \int_{\mathcal{S}} \mathbf{j} \cdot \mathrm{d}\mathbf{S}.$$

Electromagnetism

As in Example 9.3, let there be a continuous distribution of electric charge present in Minkowski space-time, having **charge density** $\rho(\mathbf{r}, t)$ and **charge flux density** or **current density** $\mathbf{j} = \rho \mathbf{v}$, where $\mathbf{v}(\mathbf{r}, t)$ is the velocity field of the fluid. The total charge of a system is a scalar quantity – else an unionized gase would not generally be electrically neutral. Charge density in a local instantaneous rest frame of the fluid at any event p is denoted $\rho_0(p)$ and is known as **proper charge density**. It may be assumed to be a scalar quantity, since it is defined in a specific inertial frame at p. On the other hand, the charge density ρ is given by

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V}$$

where, by the length–volume contraction effect (9.18),

$$\Delta V = \frac{1}{\gamma} \Delta V_0.$$

Since charge is a scalar quantity, $\Delta q = \Delta q_0$, charge density and proper charge density are related by

$$\rho = \lim_{\Delta V_0 \to 0} \frac{\Delta q_0}{V_0/\gamma} = \gamma \rho_0.$$

If the charged fluid has a 4-velocity field $V = V^\mu(x^\alpha) e_\mu$, define the **4-current** J to be the 4-vector field having components

$$J^\mu = \rho_0 V^\mu.$$

From Eq. (9.24) together with the above we have

$$J^\mu = (\mathbf{j},\ \rho c),$$

and by Example 9.3, conservation of charge is equivalent to requiring the 4-current be divergence-free,

$$J^\mu{}_{,\mu} = 0 \iff \nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.$$

In **electrodynamics** we are given a 4-current field $J = J^\mu e_\mu$ representing the charge density and current of the electric charges present, also known as the **source field**, and an antisymmetric 4-tensor field $F = F_{\mu\nu}(x^\alpha)\varepsilon^\mu \otimes \varepsilon^\nu$ such that $F_{\mu\nu} = -F_{\nu\mu}$, known as the **electromagnetc field**, satisfying the **Maxwell equations**:

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0, \tag{9.37}$$

$$F^{\mu\nu}{}_{,\nu} = \frac{4\pi}{c} J^\mu, \tag{9.38}$$

where $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$. Units adopted here are the Gaussian units, which are convenient for the formal presentation of the subject.

The first set (9.37) is known as the **source-free Maxwell equations**, while the second set (9.38) relates electromagnetic field and sources. It is common to give explicit symbols for the components of the electromagnetic field tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}, \quad \text{i.e. set } F_{12} = B_3, \text{ etc.} \tag{9.39}$$

The 3-vector fields $\mathbf{E} = (E_1,\ E_2,\ E_3)$ and $\mathbf{B} = (B_1,\ B_2,\ B_3)$ are called the **electric** and **magnetic** fields, respectively. The source-free Maxwell equations (9.37) give non-trivial equations only when all three indices μ, ν and ρ are unequal, giving four independent equations

$$(\mu,\ \nu,\ \rho) = (1, 2, 3) \qquad \Longrightarrow \nabla . \mathbf{B} = \mathbf{0}, \tag{9.40}$$

$$(\mu,\ \nu,\ \rho) = (2, 3, 4),\ \text{etc.} \Longrightarrow \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}. \tag{9.41}$$

The second set of Maxwell equations (9.38) imply charge conservation for, on commuting partial derivatives and using the antisymmetry of $F^{\mu\nu}$, we have

$$J^{\mu}{}_{,\mu} = \frac{c}{4\pi} F^{\mu\nu}{}_{,\nu\mu} = \frac{c}{8\pi}(F^{\mu\nu} - F^{\nu\mu})_{,\mu\nu} = 0.$$

Using $F_{i4} = -F_{4i} = E_i$ and $F_{ij} = \epsilon_{ijk} B_k$, Eqs. (9.38) reduce to the vector form of Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{9.42}$$

$$-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}. \tag{9.43}$$

Exercise: Show Eqs. (9.42) and (9.43).

There are essentially two independent **invariants** that can be constructed from an electromagnetic field,

$$F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad *F_{\mu\nu}F^{\mu\nu}$$

where the **dual electromagnetic tensor** $*F_{\mu\nu}$ is given in Example 8.8. Substituting electric and magnetic field components we find

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2) \quad \text{and} \quad *F_{\mu\nu}F^{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}. \tag{9.44}$$

Exercise: Show that the source-free Maxwell equations (9.37) can be written in the dual form

$$*F^{\mu\nu}{}_{,\nu} = 0.$$

The equation of motion of a charged particle, charge q, is given by the **Lorentz force equation**

$$\frac{\mathrm{d}}{\mathrm{d}\tau} P_\mu = \frac{q}{c} F_{\mu\nu} V^\nu = F_\mu \tag{9.45}$$

where the 4-momentum P_μ has components $(\mathbf{p}, -\mathcal{E}/c)$. Energy is written here as $\mathcal{E}$ so that no confusion with the magnitude of electric field can arise. Using Eq. (9.34) for components of the 4-force F_μ we find that

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\Big(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\Big) \tag{9.46}$$

and taking $\cdot\mathbf{v}$ of this equation gives rise to the energy equation (see Problem 9.16)

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \mathbf{f}\cdot\mathbf{v} = q\mathbf{E}\cdot\mathbf{v}.$$

Potentials and gauge transformations

The source-free equations (9.37) are true if and only if in a neighbourhood of any event there exists a 4-covector field $A_\mu(x^\alpha)$, called the **4-potential**, such that

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{9.47}$$

The *if* part of this statement is simple, for (9.47) implies, on commuting partial derivatives,

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = A_{\nu,\mu\rho} - A_{\mu,\nu\rho} + A_{\rho,\nu\mu} - A_{\nu,\rho\mu} + A_{\mu,\rho\nu} - A_{\rho,\mu\nu} = 0.$$

The converse will be postponed till Chapter 17, Theorem 17.5.

Exercise: Setting $A_\mu = (A_1, A_2, A_3, -\phi) = (\mathbf{A}, -\phi)$, show that Eq. (9.47) reads

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{9.48}$$

A is known as the *vector potential*, and ϕ as the **scalar potential**.

If the 4-vector potential of an electromagnetic field is altered by addition of the gradient of a scalar field ψ

$$\tilde{A}_\mu = A_\mu + \psi_{,\mu} \tag{9.49}$$

then the electromagnetic tensor $F_{\mu\nu}$ remains unchanged

$$\tilde{F}_{\mu\nu} = \tilde{A}_{\nu,\mu} - \tilde{A}_{\mu,\nu} = A_{\nu,\mu} + \psi_{,\nu\mu} - A_{\mu,\nu} - \psi_{,\mu\nu} = F_{\mu\nu}.$$

A transformation (9.49), which has no effect on the electromagnetic field, is called a **gauge transformation**.

Exercise: Write the gauge transformation (9.49) in terms of the vector and scalar potential,

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\psi, \qquad \tilde{\phi} = \phi - \frac{1}{c}\frac{\partial \psi}{\partial t},$$

and check that **E** and **B** given by Eq. (9.48) are left unchanged by these transformations.

Under a gauge transformation, the divergence of A^μ transforms as

$$\tilde{A}^\mu{}_{,\mu} = A^\mu{}_{,\mu} + \Box\psi$$

where

$$\Box\psi = \psi^{,\mu}{}_{\mu} = g^{\mu\nu}\psi_{,\mu\nu} = \nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}.$$

The operator $\Box$ is called the **wave operator** or **d'Alembertian**. If we choose ψ to be any solution of the inhomogeneous wave equation

$$\Box\psi = -A^\mu{}_{,\mu} \tag{9.50}$$

then $\tilde{A}^\mu{}_{,\mu} = 0$. Ignoring the tilde over A, any choice of 4-potential A^μ that satisfies

$$A^\mu{}_{,\mu} = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \tag{9.51}$$

is called a **Lorentz gauge**. Since solutions of the inhomogeneous wave equation (9.50) are always locally available, we may always adopt a Lorentz gauge if we wish. It should, however, be pointed out that the 4-potential A_μ is not uniquely determined by the Lorentz gauge condition (9.51), for it is still possible to add a further gradient $\bar{\psi}_{,\mu}$ provided $\bar{\psi}$ is a solution of the wave equation, $\Box\bar{\psi} = 0$. This is said to be the available **gauge freedom** in the Lorentz gauge.

In terms of a 4-potential, the source-free part of the Maxwell equations (9.37) is automatically satisfied, while the source-related part (9.38) reads

$$F^{\mu\nu}{}_{,\nu} = A^{\nu,\mu}{}_{,\nu} - A^{\mu,\nu}{}_{,\nu} = \frac{4\pi}{c} J^{\mu}.$$

If A^{μ} is in a Lorentz gauge (9.51), then the first term in the central expression vanishes and the Maxwell equations reduce to inhomogeneous wave equations,

$$\Box A^{\mu} = -\frac{4\pi}{c} J^{\mu}, \qquad A^{\mu}{}_{,\mu} = 0, \tag{9.52}$$

or in terms of vector and scalar potentials

$$\Box \mathbf{A} = -\frac{4\pi}{c}\mathbf{j}, \quad \Box\phi = -4\pi\rho, \quad \nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0. \tag{9.53}$$

In the case of a vacuum, $\rho = 0$ and $\mathbf{j} = \mathbf{0}$, the Maxwell equations read

$$\Box\mathbf{A} = \mathbf{0}, \qquad \Box\phi = 0.$$

Problems

Problem 9.17 Show that with respect to a rotation (9.8) the electric and magnetic fields **E** and **B** transform as 3-vectors,

$$E'_i = a_{ij}E_j, \qquad B'_i = a_{ij}B_j.$$

Problem 9.18 Under a boost (9.13) show that the 4-tensor transformation law for $F_{\mu\nu}$ or $F^{\mu\nu}$ gives rise to

$$E'_1 = F'_{14} = E_1, \qquad E'_2 = \gamma\Big(E_2 - \frac{v}{c}B_3\Big), \qquad E'_3 = \gamma\Big(E_2 + \frac{v}{c}B_2\Big),$$

$$B'_1 = F'_{23} = B_1, \qquad B'_2 = \gamma\Big(B_2 + \frac{v}{c}E_3\Big), \qquad B'_3 = \gamma\Big(B_2 - \frac{v}{c}E_2\Big).$$

Decomposing **E** and **B** into components parallel and perpendicular to $\mathbf{v} = (v, 0, 0)$, show that these transformations can be expressed in vector form:

$$\mathbf{E}'_{\|} = \mathbf{E}_{\|}, \qquad \mathbf{E}'_{\perp} = \gamma\Big(\mathbf{E}_{\perp} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\Big),$$

$$\mathbf{B}'_{\|} = \mathbf{B}_{\|}, \qquad \mathbf{B}'_{\perp} = \gamma\Big(\mathbf{B}_{\perp} - \frac{1}{c}\mathbf{v}\times\mathbf{E}\Big).$$

Problem 9.19 It is possible to use transformation of **E** and **B** under boosts to find the field of a uniformly moving charge. Consider a charge q travelling with velocity **v**, which without loss of generality may be taken to be in the x-direction. Let $\mathbf{R} = (x - vt,\ y,\ z)$ be the vector connecting charge to field point $\mathbf{r} = (x,\ y,\ z)$. In the rest frame of the charge, denoted by primes, suppose the field is the coulomb field

$$\mathbf{E}' = \frac{q\mathbf{r}'}{r'^3}, \qquad \mathbf{B}' = \mathbf{0}$$

where

$$\mathbf{r}' = (x',\ y',\ z') = \left(\frac{x - vt}{\sqrt{1 - v^2/c^2}},\ y,\ z\right).$$

Apply the transformation law for **E** and **B** derived in Problem 9.18 to show that

$$\mathbf{E} = \frac{q\mathbf{R}(1 - v^2/c^2)}{R^3(1 - (v^2/c^2)\sin^2\theta)^{3/2}} \quad \text{and} \quad \mathbf{B} = \frac{1}{c}\mathbf{v} \times \mathbf{E},$$

where θ is the angle between **R** and **v**. At a given distance R where is most of the electromagnetic field concentrated for highly relativistic velocities $v \approx c$?

Problem 9.20 A particle of rest mass m, charge q is in motion in a uniform constant magnetic field $\mathbf{B} = (0, 0, B)$. Show from the Lorentz force equation that the energy $\mathcal{E}$ of the particle is constant, and its motion is a helix about a line parallel to **B**, with angular frequency

$$\omega = \frac{qcB}{\mathcal{E}}.$$

Problem 9.21 Let **E** and **B** be perpendicular constant electric and magnetic fields, $\mathbf{E} \cdot \mathbf{B} = 0$.

(a) If $B^2 > E^2$ show that a transformation to a frame K' having velocity $\mathbf{v} = k\mathbf{E} \times \mathbf{B}$ can be found such that $\mathbf{E}'$ vanishes.
(b) What is the magnitude of $\mathbf{B}'$ after this transformation?
(c) If $E^2 > B^2$ find a transformation that makes $\mathbf{B}'$ vanish.
(d) What happens if $E^2 = B^2$?
(e) A particle of charge q is in motion in a crossed constant electric and magnetic field $\mathbf{E} \cdot \mathbf{B} = 0$, $B^2 > E^2$. From the solution of Problem 9.20 for a particle in a constant magnetic field, describe its motion.

Problem 9.22 An electromagnetic field $F_{\mu\nu}$ is said to be of 'electric type' at an event p if there exists a unit timelike 4-vector U_μ at p, $U_\alpha U^\alpha = -1$, and a spacelike 4-vector field E_μ orthogonal to U^μ such that

$$F_{\mu\nu} = U_\mu E_\nu - U_\nu E_\mu, \qquad E_\alpha U^\alpha = 0.$$

(a) Show that any purely electric field, i.e. one having $\mathbf{B} = \mathbf{0}$, is of electric type.
(b) If $F_{\mu\nu}$ is of electric type at p, show that there is a velocity **v** such that

$$\mathbf{B} = \frac{\mathbf{v}}{c} \times \mathbf{E} \quad (|\mathbf{v}| < c).$$

Using Problem 9.18 show that there is a Lorentz transformation that transforms the electromagnetic field to one that is purely electric at p.
(c) If $F_{\mu\nu}$ is of electric type everywhere with U^μ a constant vector field, and satisfies the Maxwell equations *in vacuo*, $J^\mu = 0$, show that the vector field E^μ is divergence-free, $E^\nu{}_{,\nu} = 0$.

Problem 9.23 Use the gauge freedom $\Box\psi = 0$ in the Lorentz gauge to show that it is possible to set $\phi = 0$ and $\nabla \cdot \mathbf{A} = 0$. This is called a *radiation gauge*.

(a) What gauge freedoms are still available to maintain the radiation gauge?
(b) Suppose **A** is independent of coordinates x and y in the radiation gauge. Show that the Maxwell equations have solutions of the form

$$\mathbf{E} = (E_1(u),\ E_2(u),\ 0), \qquad \mathbf{B} = (-E_2(u),\ E_1(u),\ 0)$$

where $u = ct - z$ and $E_i(u)$ are arbitrary differentiable functions.
(c) Show that these solutions may be interpreted as right-travelling electromagnetic waves.

9.5 Conservation laws and energy–stress tensors

Conservation of charge

Consider a general four-dimensional region Ω of space-time with boundary 3-surface $\partial\Omega$. The **four-dimensional Gauss theorem** (see Chapter 17) asserts that for any vector field A^α

$$\iiiint\limits_{\Omega} A^\alpha{}_{,\alpha}\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3\, \mathrm{d}x^4 = \iiint\limits_{\partial\Omega} A^\alpha \mathrm{d}S_\alpha. \tag{9.54}$$

If $\partial\Omega$ has the parametric form $x^\alpha = x^\alpha(\lambda_1, \lambda_2, \lambda_3)$, the vector 3-volume element $\mathrm{d}S_\alpha$ is defined by

$$\mathrm{d}S_\alpha = \epsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\beta}{\partial \lambda_1} \frac{\partial x^\gamma}{\partial \lambda_2} \frac{\partial x^\delta}{\partial \lambda_3}\, \mathrm{d}\lambda_1\, \mathrm{d}\lambda_2\, \mathrm{d}\lambda_3,$$

with the four-dimensional epsilon symbol $\epsilon_{\alpha\beta\gamma\delta}$ defined by Eq. (8.21). Since the epsilon symbol transforms as a tensor with respect to basis transformations having determinant 1, it is a 4-tensor if we restrict ourselves to proper Lorentz transformations, and it follows that $\mathrm{d}S_\alpha$ is a 4-vector. Furthermore, $\mathrm{d}S_\alpha$ is orthogonal to the 3-surface $\partial\Omega$, for any 4-vector X^α tangent to the 3-surface has a linear decomposition

$$X^\alpha = \sum_{i=1}^{3} c_i \frac{\partial x^\delta}{\partial \lambda_i},$$

and by the total antisymmetry of $\epsilon_{\alpha\beta\gamma\delta}$ it follows that

$$\mathrm{d}S_\alpha X^\alpha = \sum_{i=1}^{3} c_i \epsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\alpha}{\partial \lambda_i} \frac{\partial x^\beta}{\partial \lambda_1} \frac{\partial x^\gamma}{\partial \lambda_2} \frac{\partial x^\delta}{\partial \lambda_3} = 0.$$

The four-dimensional Gauss theorem is a natural generalization of the well-known three-dimensional result. In Chapter 17, it will become clear that this theorem is independent of the choice of parametrization λ_i on $\partial\Omega$.

A 3-surface $\mathcal{S}$ is called **spacelike** if its orthogonal 3-volume element $\mathrm{d}S_\alpha$ is a timelike 4-covector. The reason for this terminology is that a 4-vector orthogonal to three linearly independent spacelike 4-vectors must be timelike. The archetypal spacelike 3-surface is given by the equation $t = \text{const.}$ in a given inertial frame. Parametrically the surface may be given by $x = \lambda_1$, $y = \lambda_2$, $z = \lambda_3$ and its 3-volume element is

$$\mathrm{d}S_\alpha = \epsilon_{\alpha 123}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = (0,\ 0,\ 0,\ -\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z).$$

Given a current 4-vector $J^\alpha = (\mathbf{j},\ c\rho)$, satisfying the divergence-free condition $J^\alpha{}_{,\alpha} = 0$, it is natural to define the 'total charge' over an arbitrary spacelike 3-surface $\mathcal{S}$ to be

$$Q = -\frac{1}{c} \iiint\limits_{\mathcal{S}} J^\alpha \mathrm{d}S_\alpha, \tag{9.55}$$

as this gives the expected $Q = \iiint \rho\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ when $\mathcal{S}$ is a surface of type $t = \text{const.}$

Let Ω be a 4-volume enclosed by two spacelike surfaces $\mathcal{S}$ and $\mathcal{S}'$ having infinite extent. Using the four-dimensional Gauss theorem and the divergence-free condition $J^\alpha_{\ ,\alpha} = 0$ we obtain the law of **conservation of charge**,

$$Q' - Q = \frac{1}{c}\Big(\iiint_{\mathcal{S}} J^\alpha \, dS_\alpha - \iiint_{\mathcal{S}'} J^\alpha \, dS_\alpha\Big) = \frac{1}{c}\iiiint_{\Omega} J^\alpha_{\ ,\alpha} \, dx^1 \, dx^2 \, dx^3 \, dx^4 = 0$$

where the usual physical assumption is made that the 4-current J^α vanishes at spatial infinity $|\mathbf{r}| \to \infty$. This implies that there are no contributions from the timelike 'sides at infinity' to the 3-surface integral over ∂S. Note that in Minkowski space, dS^α is required to be 'inwards-pointing' on the spacelike parts of the boundary, $\mathcal{S}$ and $\mathcal{S}'$, as opposed to the more usual outward pointing requirement in three-dimensional Euclidean space.

As seen in Example 9.3 and Section 9.4 there is a converse to this result: given a conserved quantity Q, generically called 'charge', then $J^\alpha = (\mathbf{j},\ c\rho)$, where ρ is the charge density and $\mathbf{j}$ the charge flux density, form the components of a divergence-free 4-vector field, $J^\alpha_{\ ,\alpha} = 0$.

Energy–stress tensors

Assume now that the total 4-momentum P^μ of a system is conserved. Treating its components as four separate conserved 'charges', we are led to propose the existence of a quantity $T^{\mu\nu}$ such that

$$T^{\mu\nu}_{\ \ \ ,\nu} = 0 \tag{9.56}$$

and the total 4-momentum associated with any spacelike surface S is given by

$$P^\mu = -\frac{1}{c}\iiint_S T^{\mu\nu} \, dS_\nu. \tag{9.57}$$

In order to ensure that Eq. (9.56) be a tensorial equation it is natural to postulate that $T^{\mu\nu}$ is a 4-tensor field, called the **energy–stress tensor** of the system. This will also guarantee that the quantity P^μ defined by (9.57) is a 4-vector. For a surface $t = \text{const.}$ we have

$$P^\mu = \Big(\mathbf{p},\ \frac{E}{c}\Big) = \frac{1}{c}\int_{t=\text{const.}} T^{\mu 4} \, d^3x$$

and the physical interpretation of the components of the energy–stress tensor $T^{\mu\nu}$ are

$$\begin{aligned}
T^{44} &= \text{energy density,}\\
T^{4i} &= \frac{1}{c} \times \text{energy flux density,}\\
T^{i4} &= c \times \text{momentum density,}\\
T^{ij} &= j\text{th component of flux of } i\text{th component of momentum} = \text{stress tensor.}
\end{aligned}$$

It is usual to require that $T^{\mu\nu}$ are components of a symmetric tensor, $T^{\mu\nu} = T^{\nu\mu}$. The argument for this centres around the concept of **angular 4-momentum**, which for a continuous distribution of matter is defined to be

$$M^{\mu\nu} = \iiint_S x^\mu \, dP^\nu - x^\nu \, dP^\mu \equiv -\frac{1}{c}\iiint_S (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}) \, dS_\rho = -M^{\nu\mu}.$$

Conservation of angular 4-momentum $M^{\mu\nu}$ is equivalent to

$$0 = (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho})_{,\rho} = \delta^\mu_{\ \rho} T^{\nu\rho} - \delta^\nu_{\ \rho} T^{\mu\rho} = T^{\nu\mu} - T^{\mu\nu}.$$

Example 9.4 Consider a fluid having 4-velocity $V^\mu = \gamma(\mathbf{v}, c)$ where $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$. Let the local rest mass density (as measured in the i.r.f.) be $\rho(\mathbf{r}, t)$. In the i.r.f. at any point of the fluid the energy density is given by ρc^2, and since there is no energy flux in the i.r.f. we may set $T^{4i} = 0$. By the symmetry of $T^{\mu\nu}$ there will also be no momentum density T^{i4} and the energy–stress tensor has the form

$$T^{\mu\nu} = \begin{pmatrix} & & & 0 \\ & T_{ij} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \rho c^2 \end{pmatrix} = \begin{pmatrix} P_1 & & & 0 \\ & P_2 & & 0 \\ & & P_3 & 0 \\ 0 & 0 & 0 & \rho c^2 \end{pmatrix},$$

where the diagonalization of the 3×3 matrix $[T_{ij}]$ can be achieved by a rotation of axes. The P_i are called the **principal pressures** at that point. If they are all equal, $P_1 = P_2 = P_3 = P$, then the fluid is said to be a **perfect fluid** and P is simply called the **pressure**. In that case

$$T^{\mu\nu} = \left(\rho + \frac{1}{c^2}P\right) V^\mu V^\nu + P g^{\mu\nu}, \tag{9.58}$$

as may be checked by verifying that this equation holds in the i.r.f. at any point, in which frame $V^\mu = (0, 0, 0, c)$. Since (9.58) is a 4-tensor equation it must hold in all inertial frames.

Exercise: Verify that the conservation laws $T^{\mu\nu}{}_{,\nu} = 0$ reduce for $v \ll c$ to the equation of continuity and Euler's equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0,$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla P.$$

Example 9.5 The energy–stress tensor of the electromagnetic field is given by

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^\mu{}_\rho F^{\nu\rho} - \tfrac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right) = T^{\nu\mu}. \tag{9.59}$$

The energy density of the electromagnetic field is thus

$$\epsilon = T^{44} = \frac{1}{16\pi}\left(4F^4{}_i F^{4i} - g^{44} F_{\rho\sigma} F^{\rho\sigma}\right) = \frac{1}{16\pi}\left(4\mathbf{E}^2 + 2(\mathbf{B}^2 - \mathbf{E}^2)\right) = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}$$

and the energy flux density has components

$$cT^{4i} = \frac{c}{4\pi} F^4{}_j F^{ij} = \frac{c}{4\pi} E_j \epsilon_{ijk} B_k = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{B})_i.$$

The vector $\mathbf{S} = \dfrac{c}{4\pi}(\mathbf{E}\times\mathbf{B})$ is known as the **Poynting vector**. The spatial components T_{ij}

are known as the *Maxwell stress tensor*

$$T^{ij} = T_{ij} = \frac{1}{16\pi}\big(4(F_{ik}F_j^{\ k} + F_{i4}F_j^{\ 4}) - \delta_{ij}2(\mathbf{B}^2 - \mathbf{E}^2)\big)$$
$$= \frac{1}{4\pi}\big(-E_i E_j - B_i B_j + \tfrac{1}{2}\delta_{ij}(\mathbf{E}^2 + \mathbf{B}^2)\big).$$

The total 4-momentum of an electromagnetic field over a spacelike surface S is calculated from Eq. (9.57).

Exercise: Show that the average pressure $P = \frac{1}{3}\sum_i T_{ii}$ of an electromagnetic field is equal to $\frac{1}{3}\times$ energy density. Show that this also follows from the fact that $T^{\mu\nu}$ is trace-free, $T^{\mu}{}_{\mu} = 0$.

For further developments in relativistic classical field theory the reader is referred to [4, 5].

Problems

Problem 9.24 Show that as a consequence of the Maxwell equations,

$$T_{\alpha}{}^{\beta}{}_{,\beta} = -\frac{1}{c}F_{\alpha\gamma}J^{\gamma}$$

where T_{α}^{β} is the electromagnetic energy–stress tensor (9.59), and when no charges and currents are present it satisfies Eq. (9.56). Show that the $\alpha = 4$ component of this equation has the form

$$\frac{\partial\epsilon}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{j}\cdot\mathbf{E}$$

where ϵ = energy density and $\mathbf{S}$ = Poynting vector. Interpret this equation physically.

Problem 9.25 For a plane wave, Problem 9.23, show that

$$T_{\alpha\beta} = \epsilon\, n_{\alpha}\, n_{\beta}$$

where $\epsilon = E^2/4\pi$ and $n^{\alpha} = (\mathbf{n}, 1)$ is the null vector pointing in the direction of propagation of the wave. What pressure does the wave exert on a wall placed perpendicular to the path of the wave?

References

[1] W. Kopczyński and A. Trautman. *Spacetime and Gravitation*. Chichester, John Wiley & Sons, 1992.

[2] W. Rindler. *Introduction to Special Relativity*. Oxford, Oxford University Press, 1991.

[3] R. K. Sachs and H. Wu. *General Relativity for Mathematicians*. New York, Springer-Verlag, 1977.

[4] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*. Reading, Mass., Addison-Wesley, 1971.

[5] W. Thirring. *A Course in Mathematical Physics, Vol. 2: Classical Field Theory*. New York, Springer-Verlag, 1979.