12 Distributions

In physics and some areas of engineering it has become common to make use of certain 'functions' such as the *Dirac delta function* $\delta(x)$, having the property

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,\mathrm{d}x = f(0)$$

for all continuous functions $f(x)$. If we set $f(x)$ to be a continuous function that is everywhere zero except on a small interval $(a-\epsilon, a+\epsilon)$ on which $f > 0$, it follows that $\delta(a) = 0$ for all $a \neq 0$. However, setting $f(x) = 1$ implies $\int \delta(x)\,\mathrm{d}x = 1$, so we must assign an infinite value to $\delta(0)$,

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases} \tag{12.1}$$

As it stands this really won't do, since the δ-function vanishes a.e. and should therefore be assigned Lebesgue integral zero. Our aim in this chapter is to give a rigorous definition of such 'generalized functions', which avoids these contradictions.

In an intuitive sense we might think of the Dirac delta function as being the 'limit' of a sequence of functions (see Fig. 12.1) such as

$$\varphi_n(x) = \begin{cases} 2n & \text{if } |x| \leq 1/n \\ 0 & \text{if } |x| > 1/n \end{cases}$$

or of Gaussian functions

$$\psi_n(x) = \frac{1}{n\sqrt{\pi}} \mathrm{e}^{-x^2/n^2}.$$

Lebesgue's dominated convergence theorem does not apply to these sequences, yet the limit of the integrals is clearly 1, and for any continuous function $f(x)$ it is not difficult to show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\varphi_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\psi_n(x)\,\mathrm{d}x = f(0).$$

However, we will not attempt to define Dirac-like functions as limiting functions in some sense. Rather, following Laurent Schwartz, we define them as continuous linear functionals on a suitably defined space of regular test functions. This method is called the theory of *distributions* [1–6].

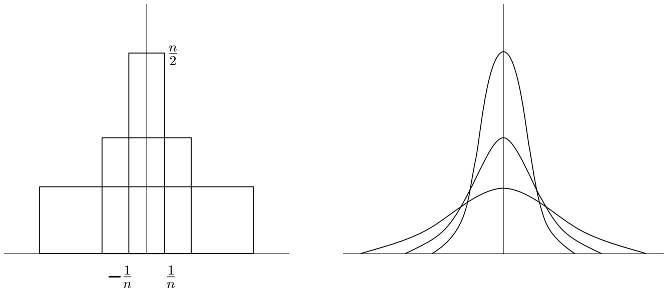

Figure 12.1 Dirac delta function as a limit of functions

12.1 Test functions and distributions

Spaces of test functions

The **support** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the closure of the region of $\mathbb{R}^n$ where $f(\mathbf{x}) \neq 0$. We will say a real-valued function f on $\mathbb{R}^n$ has **compact support** if the support is a closed bounded set; that is, there exists $R > 0$ such that $f(x_1, \dots, x_n) = 0$ for $|\mathbf{x}| \geq R$. A function f is said to be C^m if all partial derivatives of order m,

$$D_{\underline{m}} f = \frac{\partial^m f}{\partial x_1^{m_1}, \dots, \partial x_n^{m_n}},$$

exist and are continuous, where $\underline{m} = (m_1, \dots, m_n)$ and $m = |\underline{m}| \equiv \sum_{i=1}^n m_i$. We adopt the convention that $D_{(0,0,\dots,0)} f = f$. A function f is said to be C^∞, or **infinitely differentiable**, if it is C^m to all orders $m = 1, 2, \dots$ We set $\mathcal{D}^m(\mathbb{R}^n)$ to be the vector space of all C^m functions on $\mathbb{R}$ with compact support, called the space of **test functions of order** m.

Exercise: Show that $\mathcal{D}^m(\mathbb{R}^n)$ is a real vector space.

The space of infinitely differentiable test functions, $\mathcal{D}^\infty(\mathbb{R}^n)$, is often denoted simply as $\mathcal{D}(\mathbb{R}^n)$ and is called the space of **test functions**,

$$\mathcal{D}(\mathbb{R}^n) = \bigcap_{n=1}^{\infty} \mathcal{D}^m(\mathbb{R}^n).$$

To satisfy ourselves that this space is not empty, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

This function is infinitely differentiable everywhere, including the point $x = 0$ where all derivatives vanish both from the left and the right. Hence the function $\varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi(x) = f(-(x-a))f(x+a) = \begin{cases} \exp\left(\frac{2a}{x^2 - a^2}\right) & \text{if } |x| < a, \\ 0 & \text{if } |x| \geq a, \end{cases}$$

is everywhere differentiable and has compact support $[-a, a]$. There is a counterpart in $\mathbb{R}^n$,

$$\varphi(\mathbf{x}) = \begin{cases} \exp\left(\frac{2a}{|\mathbf{x}|^2 - a^2}\right) & \text{if } |\mathbf{x}| < a, \\ 0 & \text{if } |\mathbf{x}| \geq a, \end{cases}$$

where

$$|\mathbf{x}| = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}.$$

A sequence of functions $\varphi_n \in \mathcal{D}(\mathbb{R}^n)$ is said to **converge to order** m to a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ if the functions φ_n and φ all have supports within a common bounded set and

$$D_{\underline{k}}\varphi_n(\mathbf{x}) \to D_{\underline{k}}\varphi(\mathbf{x})$$

uniformly for all $\mathbf{x}$, for all $\underline{k}$ of orders $k = 0, 1, \ldots, m$. If we have convergence to order m for all $m = 0, 1, 2, \ldots$ then we simply say φ_n **converges** to φ, written $\varphi_n \to \varphi$.

Example 12.1 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any differentiable function having compact support on K in $\mathbb{R}$. The sequence of functions

$$\varphi_n(x) = \frac{1}{n}\varphi(x) \sin nx$$

are all differentiable and have common compact support K. Since $|\varphi(x) \sin nx| < 1$ it is evident that these functions approach the zero function uniformly as $n \to \infty$, but their derivatives

$$\varphi'_n(x) = \frac{1}{n}\varphi'(x) \sin nx + \varphi(x) \cos nx \nrightarrow 0.$$

This is an example of a sequence of functions that converge to order 0 to the zero function, but not to order 1.

To define this convergence topologically, we can proceed in the following manner. For every compact set $K \subset \mathbb{R}^n$ let $\mathcal{D}^m(K)$ be the space of C^m functions of compact support within K. This space is made into a topological space as in Example 10.25, by defining a norm

$$\|f\|_{K,m} = \sup_{\mathbf{x} \in K} \sum_{|\underline{k}| \leq m} \left| D_{\underline{k}} f(\mathbf{x}) \right|.$$

On $\mathcal{D}^m(\mathbb{R}^n)$ we define a set U to be open if for every $f \in U$ there exists a compact set K and a real $a > 0$ such that $f \in K$ and

$$\{g \in K \mid \|g - f\|_{K,m} < a\} \subseteq U.$$

It then follows that a sequence $f_k \in \mathcal{D}^m(\mathbb{R}^n)$ converges to order m to a function $f \in \mathcal{D}^m(\mathbb{R}^n)$

if and only if $f_n \to f$ with respect to this topology. A similar treatment gives a topology on $\mathcal{D}(\mathbb{R}^n)$ leading to convergence in all orders (see Problem 12.2).

Distributions

In this chapter, when we refer to 'continuity' of a functional S on a space such as $\mathcal{D}(\mathbb{R}^n)$, we will mean that whenever $f_n \to f$ in some specified sense on $\mathcal{D}(\mathbb{R}^n)$ we have $S(f_n) \to S(f)$. A **distribution of order** m on $\mathbb{R}^n$ is a linear functional T on $\mathcal{D}(\mathbb{R}^n)$,

$$T(a\varphi + b\psi) = aT(\varphi) + bT(\psi),$$

which is continuous to order m; that is, if $\varphi_k \to \varphi$ is any sequence of functions in $\mathcal{D}(\mathbb{R}^n)$ convergent to order m then $T(\varphi_k) \to T(\varphi)$. A linear functional T on $\mathcal{D}(\mathbb{R}^n)$ that is continuous with respect to sequences φ_i in $\mathcal{D}(\mathbb{R}^n)$ that are convergent to all orders will simply be referred to as a **distribution** on $\mathbb{R}^n$. In this sense of continuity, the space of distributions of order m on $\mathbb{R}^n$ is the dual space of $\mathcal{D}^m(\mathbb{R}^n)$ (see Section 10.9), and the space of distributions is the dual space of $\mathcal{D}(\mathbb{R}^n)$. Accordingly, these are denoted $\mathcal{D}'^m(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ respectively.

Note that a distribution T of order m is also a distribution of order m' for all $m' > m$. For, if φ_i is a convergent sequence of functions in $\mathcal{D}^{m'}(\mathbb{R}^n)$, then φ_i and all its derivatives up to order m' converge uniformly to a function $\varphi \in \mathcal{D}^{m'}(\mathbb{R}^n)$. In particular, it is also a sequence in $\mathcal{D}^m(\mathbb{R}^n)$ converging to order $m < m'$ to φ. Therefore a linear functional T of order m, having the property $T(\varphi_i) \to T(\varphi)$ for all convergent sequences φ_i in $\mathcal{D}^m(\mathbb{R}^n)$, automatically has this property for all convergent sequences in $\mathcal{D}^{m'}(\mathbb{R}^n)$. This is a curious feature, characteristic of dual spaces: given a function that is C^m we can only conclude that it is $C^{m'}$ for $m' \le m$, yet given a distribution of order m we are guaranteed that it is a distribution of order m' for all $m' \ge m$.

Regular distributions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **locally integrable** if it is integrable on every compact subset $K \subset \mathbb{R}^n$. Set $T_f : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$ to be the continuous linear functional defined by

$$T_f(\varphi) = \int_{\mathbb{R}^n} \varphi f \, \mathrm{d}\mu^n = \underset{\mathbb{R}^n}{\int \cdots \int} \varphi(\mathbf{x}) f(\mathbf{x}) \, \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

The integral always exists, since every test function φ vanishes outside some compact set. Linearity is straightforward by elementary properties of the integral operator,

$$T_f(a\varphi + b\psi) = aT_f(\varphi) + bT_f(\psi).$$

Continuity of T_f follows from the inequality (11.11) and Lebesgue's dominated convergence theorem 11.11,

$$\begin{aligned}\big|T_f(\varphi_i) - T_f(\varphi)\big| &= \Big|\int_{\mathbb{R}^n}\cdots\int f(\mathbf{x})(\varphi_i(\mathbf{x}) - \varphi(\mathbf{x}))\,\mathrm{d}^n x\Big| \\ &\le \int_{\mathbb{R}^n}\cdots\int \big|f(\mathbf{x})(\varphi_i(\mathbf{x}) - \varphi(\mathbf{x}))\big|\,\mathrm{d}^n x \\ &\le \int_{\mathbb{R}^n}\cdots\int |f(\mathbf{x})|\big|(\varphi_i(\mathbf{x}) - \varphi(\mathbf{x}))\big|\,\mathrm{d}^n x \\ &\to 0,\end{aligned}$$

since the sequence of integrable functions $f\varphi_i$ is dominated by the integrable function $\big(\sup_i|\varphi_i|\big)|f|$. Hence T_f is a distribution and the function f is called its **density**. In fact T_f is a distribution of order 0, since only convergence to order 0 is needed in its definition.

Two locally integrable functions f and g that are equal almost everywhere give rise to the same distribution, $T_f = T_g$. Conversely, if $T_f(\varphi) = T_g(\varphi)$ for all test functions φ then the density functions f and g are equal a.e. An outline proof is as follows: let I^n be any product of closed intervals $I^n = I_1 \times I_2 \times \cdots \times I_n$, and choose a test function φ arbitrarily close to the unit step function χ_{I^n}. Then $\int_{I^n}(f - g)\,\mathrm{d}\mu^n = 0$, which is impossible for all I^n if $f - g$ has non-vanishing positive part, $(f - g)^+ > 0$, on a set of positive measure. This argument may readily be refined to show that $f - g = 0$ a.e. Hence the density f is uniquely determined by T_f except on a set of measure zero. By identifying f with T_f, locally integrable functions can be thought of as distributions. Not all distributions, however, arise in this way; distributions having a density $T = T_f$ are sometimes referred to as **regular distributions**, while those not corresponding to any locally integrable function are called **singular**.

Example 12.2 Define the distribution δ_a on $\mathcal{D}(\mathbb{R})$ by

$$\delta_a(\varphi) = \varphi(a).$$

In particular, we write δ for δ_0, so that $\delta(\varphi) = \varphi(0)$. The map $\delta_a : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ is obviously linear, $\delta_a(b\varphi + c\psi) = b\varphi(a) + c\psi(a) = b\delta_a(\varphi) + c\delta_a(\psi)$, and is continuous since $\varphi_n \to \varphi \Longrightarrow \varphi_n(a) \to \varphi(a)$. Hence δ_a is a distribution, but by the reasoning at the beginning of this chapter it cannot correspond to any locally integrable function. It is therefore a singular distribution. Nevertheless, physicists and engineers often maintain the density notation and write

$$\delta(\varphi) \equiv \int_{-\infty}^{\infty} \varphi(x)\delta(x)\,\mathrm{d}x = \varphi(0). \tag{12.2}$$

In writing such an equation, the distribution δ is imagined to have the form T_δ for a density function $\delta(x)$ concentrated at the point $x = 0$ and having an infinite value there as in Eq. (12.1), such that

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1.$$

Using a similar convention, the distribution δ_a may be thought of as representing the density function $\delta_a(x)$ such that

$$\int_{-\infty}^{\infty} \varphi(x)\delta_a(x)\,\mathrm{d}x = \varphi(a)$$

for all test functions φ. It is common to write $\delta_a(x) = \delta(x-a)$, for on performing the 'change of variable' $x = y + a$,

$$\int_{-\infty}^{\infty} \varphi(x)\,\delta(x-a)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(y+a)\delta(y)\,\mathrm{d}y = \varphi(a).$$

The n-dimensional delta function may be similarly defined by

$$\delta^n_{\mathbf{a}}(\varphi) = \varphi(\mathbf{a}) = \varphi(a_1, \dots, a_n)$$

and can be written

$$\delta^n_{\mathbf{a}}(\varphi) \equiv T_{\delta^n_{\mathbf{a}}}(\varphi) = \underset{\mathbb{R}^n}{\int \cdots \int} \varphi(\mathbf{x})\delta^n(\mathbf{x}-\mathbf{a})\,\mathrm{d}^n x = \varphi(\mathbf{a})$$

where

$$\delta^n(\mathbf{x}-\mathbf{a}) = \delta(x_1-a_1)\delta(x_2-a_2)\dots\delta(x_n-a_n).$$

Although it is not in general possible to define the product of distributions, no problems arise in this instance because the delta functions on the right-hand side depend on separate and independent variables.

Problems

Problem 12.1 Construct a test function such that $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| \ge 2$.

Problem 12.2 For every compact set $K \subset \mathbb{R}^n$ let $\mathcal{D}(K)$ be the space of C^∞ functions of compact support within K. Show that if all integer vectors $\underline{k}$ are set out in a sequence where $N(\underline{k})$ denotes the position of $\underline{k}$ in the sequence, then

$$\|f\|_K = \sup_{\mathbf{x}\in K} \sum_{|\underline{k}|} \frac{1}{2^{N(\underline{k})}} \frac{|D_{\underline{k}} f(\mathbf{x})|}{1 + |D_{\underline{k}} f(\mathbf{x})|}$$

is a norm on $\mathcal{D}(K)$. Let a set U be defined as open in $\mathcal{D}(\mathbb{R}^n)$ if it is a union of open balls $\{g \in K \mid \|g - f\|_K < a\}$. Show that this is a topology and sequence convergence with respect to this topology is identical with convergence of sequences of functions of compact support to all orders.

Problem 12.3 Which of the following is a distribution?

(a) $T(\phi) = \sum_{n=1}^{m} \lambda_n \phi^{(n)}(0) \quad (\lambda_n \in \mathbb{R})$.

(b) $T(\phi) = \sum_{n=1}^{m} \lambda_n \phi(x_n) \quad (\lambda_n, x_n \in \mathbb{R})$.

(c) $T(\phi) = \big(\phi(0)\big)^2$.

(d) $T(\phi) = \sup \phi$.

(e) $T(\phi) = \int_{-\infty}^{\infty} |\phi(x)| \, dx$.

Problem 12.4 We say a sequence of distributions T_n converges to a distribution T, written $T_n \to T$, if $T_n(\phi) \to T(\phi)$ for all test functions $\phi \in \mathcal{D}$ (this is sometimes called *weak convergence*). If a sequence of continuous functions f_n converges uniformly to a function $f(x)$ on every compact subset of $\mathbb{R}$, show that the associated regular distributions $T_{f_n} \to T_f$.

In the distributional sense, show that we have the following convergences:

$$f_n(x) = \frac{n}{\pi(1 + n^2x^2)} \to \delta(x),$$

$$g_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \to \delta(x).$$

12.2 Operations on distributions

If T and S are distributions of order m on $\mathbb{R}^n$, then clearly $T + S$ and aT are distributions of this order for all $a \in \mathbb{R}$. Thus $\mathcal{D}'^m(\mathbb{R}^n)$ is a vector space.

Exercise: Prove that $T + S$ is linear and continuous. Similarly for aT.

The product ST of two distributions is not a distribution. For example, if we were to define $(ST)(\varphi) = S(\varphi)T(\varphi)$, this is not linear in φ. However, if α is a C^m function on $\mathbb{R}^n$ and T is a distribution of order m then αT can be defined as a distribution of order m, by setting

$$(\alpha T)(\varphi) = T(\alpha\varphi),$$

since $\alpha\varphi \in \mathcal{D}^m(\mathbb{R}^n)$ for all $\varphi \in \mathcal{D}^m(\mathbb{R}^n)$. Note that α need not be a test function for this construction – it works even if the function α does not have compact support.

If T is a regular distribution on $\mathbb{R}^n$, $T = T_f$, then $\alpha T_f = T_{\alpha f}$. For

$$\begin{aligned} \alpha T_f(\varphi) &= T_f(\alpha\varphi) \\ &= \underset{\mathbb{R}^n}{\int \cdots \int} \varphi\alpha f \, d^n x \\ &= T_{\alpha f}(\varphi). \end{aligned}$$

The operation of multiplying the regular distribution T_f by α is equivalent to simply multiplying the corresponding density function f by α. In this case α need only be a locally integrable function.

Example 12.3 The distribution δ defined in Example 12.2 is a distribution of order zero, since it is well-defined on the space of continuous test functions, $\mathcal{D}^0(\mathbb{R})$. For any continuous function $\alpha(x)$ we have

$$\alpha\delta(\varphi) = \delta(\alpha\varphi) = \alpha(0)\varphi(0) = \alpha(0)\delta(\varphi).$$

Thus

$$\alpha\delta = \alpha(0)\delta. \tag{12.3}$$

In terms of the 'delta function' this identity is commonly written as

$$\alpha(x)\delta(x) = \alpha(0)\delta(x),$$

since

$$\int_{-\infty}^{\infty} \alpha(x)\delta(x)\varphi(x)\,\mathrm{d}x = \alpha(0)\varphi(0) = \int_{-\infty}^{\infty} \alpha(0)\delta(x)\varphi(x)\,\mathrm{d}x.$$

For the delta function at an arbitrary point a, these identities are replaced by

$$\alpha\delta_a = \alpha(a)\delta_a, \qquad \alpha(x)\delta(x-a) = \alpha(a)\delta(x-a). \tag{12.4}$$

Setting $\alpha(x) = x$ results in the useful identities

$$x\delta = 0, \qquad x\delta(x) = 0. \tag{12.5}$$

Exercise: Extend these identities to the n-dimensional delta function,

$$\alpha\delta_{\mathbf{a}} = \alpha(\mathbf{a})\delta_{\mathbf{a}}, \qquad \alpha(\mathbf{x})\delta^n(\mathbf{x}-\mathbf{a}) = \alpha(\mathbf{a})\delta(\mathbf{x}-\mathbf{a}).$$

Differentiation of distributions

Let T_f be a regular distribution where f is a differentiable function. Standard results in real analysis ensure that the derivative $f' = \mathrm{d}f/\mathrm{d}x$ is a locally integrable function. Let φ be any test function from $\mathcal{D}^1(\mathbb{R})$. Using integration by parts

$$\begin{aligned} T_{f'}(\varphi) &= \int_{-\infty}^{\infty} \varphi(x)\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x \\ &= \Big[\varphi f\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi}{\mathrm{d}x} f(x)\,\mathrm{d}x \\ &= T_f(-\varphi'), \end{aligned}$$

since $\varphi(\pm\infty) = 0$. We can extend this identity to general distributions, by defining the **derivative of a distribution** T of order $m \geq 0$ on $\mathbb{R}$ to be the distribution T' of order $m+1$ given by

$$T'(\varphi) = T(-\varphi') = -T(\varphi'). \tag{12.6}$$

The derivative of a regular distribution then corresponds to taking the derivative of the density function. Note that the order of the distribution *increases* on differentiation, for $\varphi' \in \mathcal{D}^m(\mathbb{R})$ implies that $\varphi \in \mathcal{D}^{m+1}(\mathbb{R})$. In particular, if T is a distribution of order 0 then T' is a distribution of order 1.

To prove that T' is continuous (linearity is obvious), we use the fact that in the definition of convergence to order $m+1$ of a sequence of functions $\varphi_n \to \varphi$ it is required that all derivatives up to and including order $m+1$ converge uniformly on a compact subset K of

$\mathbb{R}$. In particular, $\varphi_n'(x) \to \varphi'(x)$ for all $x \in K$, and

$$T'(\varphi_n) = T(-\varphi_n') \to T(-\varphi') = T'(\varphi).$$

It follows that every distribution of any order is infinitely differentiable.

If T is a distribution of order greater or equal to 0 on $\mathbb{R}^n$, we may define its partial derivatives in a similar way,

$$\frac{\partial T}{\partial x_k}(\varphi) = -T\Big(\frac{\partial \varphi}{\partial x_k}\Big).$$

As for distributions on $\mathbb{R}$, any such distribution is infinitely differentiable. For higher derivatives it follows that

$$D_{\underline{m}} T(\varphi) = (-1)^m T(D_{\underline{m}}\varphi) \quad \text{where} \quad m = |\underline{m}| = \textstyle\sum_i m_i.$$

Exercise: Show that

$$\frac{\partial^2 T}{\partial x_i \partial x_j} = \frac{\partial^2 T}{\partial x_j \partial x_j}.$$

Example 12.4 Set $\theta(x)$ to be the **Heaviside step function**

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

This is evidently a locally integrable function, and generates a regular distribution T_θ. For any test function $\varphi \in \mathcal{D}^1(\mathbb{R})$

$$\begin{aligned} T_{\theta'}(\varphi) = T_\theta(-\varphi') &= -\int_{-\infty}^{\infty} \varphi'(x)\theta(x)\,\mathrm{d}x \\ &= -\int_0^\infty \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x \\ &= \varphi(0) \quad \text{since } \varphi(\infty) = 0 \\ &= \delta(\varphi). \end{aligned}$$

Thus we have the distributional equation, valid only over $\mathcal{D}^1(\mathbb{R})$,

$$T_\theta' = \delta.$$

This is commonly written in terms of ‘functions’ as

$$\delta(x) = \theta'(x) = \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}.$$

Intuitively, the step at $x = 0$ is ‘infinitely steep’.

Example 12.5 The derivative of the delta distribution is defined as the distribution δ' of order 1, which may be applied to any test function $\varphi \in \mathcal{D}^1(\mathbb{R})$:

$$\delta'(\varphi) = \delta(-\varphi') = -\varphi'(0).$$

Expressed in terms of the delta function, this reads

$$\int_{-\infty}^{\infty} \delta'(x)\varphi(x)\,\mathrm{d}x = -\varphi'(0),$$

for an arbitrary function differentiable on a neighbourhood of the origin $x = 0$. Continuing to higher derivatives, we have

$$\delta''(\varphi) = \varphi''(0)$$

or in Dirac's notation

$$\int_{-\infty}^{\infty} \delta''(x)\varphi(x)\,\mathrm{d}x = \varphi''(0).$$

For the mth derivative,

$$\delta^{(m)}(\varphi) = (-1)^m \varphi^{(m)}(0), \qquad \int_{-\infty}^{\infty} \delta^{(m)}(x)\varphi(x)\,\mathrm{d}x = (-1)^m \varphi^{(m)}(0).$$

For the product of a differentiable function α and a distribution T we obtain the usual Leibnitz rule,

$$(\alpha T)' = \alpha T' + \alpha' T,$$

for

$$\begin{aligned}(\alpha T)'(\varphi) &= \alpha T(-\varphi') \\ &= T(-\alpha\varphi') \\ &= T((-\alpha\varphi)' + \alpha'\varphi) \\ &= T'(\alpha\varphi) + \alpha' T(\varphi) \\ &= \alpha T'(\varphi) + \alpha' T(\varphi).\end{aligned}$$

Example 12.6 From Examples 12.3 and 12.5 we have that

$$(x\delta)' = 0' = 0$$

and

$$(x\delta)' = x\delta' + x'\delta = x\delta' + \delta.$$

Hence

$$x\delta' = -\delta.$$

We can also derive this equation by manipulating the delta function in natural ways,

$$x\delta'(x) = \big(x\delta(x)\big)' - x'\delta(x) = 0' - 1.\delta(x) = -\delta(x).$$

Exercise: Verify the identity $x\delta' = -\delta$ by applying both sides as distributions to an arbitrary test function $\phi(x)$.

Change of variable in δ-functions

In applications of the mathematics of delta functions it is common to consider 'functions' such as $\delta(f(x))$. While this is not an operation that generalizes to all distributions, there is a sense in which we can define this concept for the delta distribution for many functions f. Firstly, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous monotone increasing function such that $f(\pm\infty) = \pm\infty$ and we adopt Dirac's notation then, assuming integrals can be manipulated by the standard rules for change of variable,

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi(x)\delta\big(f(x)\big)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \varphi(x)\delta(y)\frac{\mathrm{d}y}{f'(x)} \quad \text{where } y = f(x)\\ &= \int_{-\infty}^{\infty} \frac{\varphi\big(f^{-1}(y)\big)}{f'\big(f^{-1}(y)\big)}\delta(y)\,\mathrm{d}y\\ &= \frac{\varphi(a)}{f'(a)} \quad \text{where } f(a) = 0.\end{aligned}$$

If $f(x)$ is monotone decreasing then the range of integration is inverted to $\int_{\infty}^{-\infty}$ resulting in a sign change. The general formula for a monotone function f of either direction, having a unique zero at $x = a$, is

$$\int_{-\infty}^{\infty} \varphi(x)\delta\big(f(x)\big)\,\mathrm{d}x = \frac{\varphi(a)}{|f'(a)|}. \tag{12.7}$$

Symbolically, we may write

$$\delta\big(f(x)\big) = \frac{1}{|f'(a)|}\delta(x - a),$$

or in terms of distributions,

$$\delta \circ f = \frac{1}{|f'(a)|}\delta_a. \tag{12.8}$$

Essentially this equation can be taken as the definition of the distribution $\delta \circ f$. Setting $f(x) = -x$, it follows that $\delta(x)$ is an even function, $\delta(-x) = \delta(x)$.

If two test functions φ and ψ agree on an arbitrary neighbourhood $[-\epsilon, \epsilon]$ of the origin $x = 0$ then

$$\delta(\varphi) = \delta(\psi) = \varphi(0) = \varphi(\psi).$$

Hence the distribution δ can be regarded as being a distribution on the space of functions $\mathcal{D}([-\epsilon, \epsilon])$, since essentially it only samples values of any test function φ in a neighbourhood of the origin. Thus it is completely consistent to write

$$\delta(\varphi) = \int_{-\epsilon}^{\epsilon} \varphi(x)\delta(x)\,\mathrm{d}x.$$

This just reiterates the idea that $\delta(x) = 0$ for all $x \neq 0$.

If $f(x)$ has zeros at $x = a_1, a_2, \ldots$ and f is a monotone function in the neighbourhood of each a_i, then a change of variable to $y = f(x)$ gives, on restricting integration to a small neighbourhood of each zero,

$$\int_{-\infty}^{\infty} \varphi(x)\delta\big(f(x)\big)\,\mathrm{d}x = \sum_i \frac{\varphi(a_i)}{|f'(a_i)|}.$$

Hence

$$\delta\big(f(x)\big) = \sum_i \frac{1}{|f'(a_i)|}\delta(x - a_i), \tag{12.9}$$

or equivalently

$$\delta \circ f = \sum_i \frac{1}{|f'(a_i)|}\delta_{a_i}.$$

Example 12.7 The function $f = x^2 - a^2 = (x - a)(x + a)$ is locally monotone at both its zeros $x = \pm a$, provided $a \neq 0$. In a small neighbourhood of $x = a$ the function f may be approximated by the monotone increasing function $2a(x - a)$, while in a neighbourhood of $x = -a$ it is monotone decreasing and approximated by $-2a(x + a)$. Thus

$$\delta(x^2 - a^2) = \delta\big(2a(x - a)\big) + \delta\big(-2a(x + a)\big) = \frac{1}{2a}\big(\delta(x - a) + \delta(x + a)\big),$$

in agreement with Eq. (12.9).

Problems

Problem 12.5 In the sense of convergence defined in Problem 12.4 show that if $T_n \to T$ then $T_n' \to T'$.

In the distributional sense, show that we have the following convergences:

$$f_n(x) = -\frac{2n^3 x}{\sqrt{\pi}}\mathrm{e}^{-n^2x^2} \to \delta'(x).$$

Problem 12.6 Evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{at}\,\sin bt\;\delta^{(n)}(t)\,\mathrm{d}t \quad \text{for } n = 0, 1, 2.$

(b) $\displaystyle\int_{-\infty}^{\infty} (\cos t + \sin t)\delta^{(n)}(t^3 + t^2 + t)\,\mathrm{d}t \quad \text{for } n = 0, 1.$

Problem 12.7 Show the following identities:

(a) $\displaystyle\delta((x - a)(x - b)) = \frac{1}{b - a}(\delta(x - a) + \delta(x - b)).$

(b) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\theta(x^2 - 1) = \delta(x - 1) - \delta(x + 1) = 2x\delta(x^2 - 1).$

(c) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\delta(x^2 - 1) = \tfrac{1}{2}(\delta'(x - 1) + \delta'(x + 1)).$

(d) $\delta'(x^2 - 1) = \tfrac{1}{4}(\delta'(x - 1) - \delta'(x + 1) + \delta(x - 1) + \delta(x + 1)).$

Problem 12.8 Show that for a monotone function $f(x)$ such that $f(\pm\infty) = \pm\infty$ with $f(a) = 0$

$$\int_{-\infty}^{\infty} \varphi(x)\delta'\big(f(x)\big)\,\mathrm{d}x = -\frac{1}{f'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{\varphi(x)}{|f'(x)|}\Big)\Big|_{x=a}.$$

For a general function $f(x)$ that is monotone on a neighbourhood of all its zeros, find a general formula for the distribution $\delta' \circ f$.

Problem 12.9 Show the identities

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\delta(f(x))\big) = f'(x)\delta'\big(f(x)\big)$$

and

$$\delta\big(f(x)\big) + f(x)\delta'\big(f(x)\big) = 0.$$

Hence show that $\phi(x, y) = \delta(x^2 - y^2)$ is a solution of the partial differential equation

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + 2\phi(x, y) = 0.$$

12.3 Fourier transforms

For any function $\varphi(x)$ its **Fourier transform** is the function $\mathcal{F}\varphi$ defined by

$$\mathcal{F}\varphi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-ixy}\varphi(x)\,\mathrm{d}x.$$

The **inverse Fourier transform** is defined by

$$\mathcal{F}^{-1}\varphi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{ixy}\varphi(x)\,\mathrm{d}x.$$

Fourier's integral theorem, applicable to all functions φ such that $|\varphi|$ is integrable over $[-\infty, \infty]$ and is of bounded variation, says that $\mathcal{F}^{-1}\mathcal{F}\varphi = \varphi$, expressed in integral form as

$$\begin{aligned}\varphi(a) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{e}^{iay}\int_{-\infty}^{\infty} \mathrm{e}^{-iyx}\varphi(x)\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\;\varphi(x)\int_{-\infty}^{\infty} \mathrm{e}^{iy(a-x)}\,\mathrm{d}y.\end{aligned}$$

The proof of this theorem can be found in many books on real analysis. The reader is referred to [6, chap. 7] or [2, p. 88].

Applying standard rules of integrals applied to delta functions, we expect

$$\delta_a(x) = \delta(x - a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{iy(a-x)}\,\mathrm{d}y, \tag{12.10}$$

or, on setting $a = 0$ and using $\delta(x) = \delta(-x)$,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-iyx}\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{iyx}\mathrm{d}y. \tag{12.11}$$

Similarly, the Fourier transform of the delta function should be

$$\mathcal{F}\delta(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-ixy}\delta(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \tag{12.12}$$

and Eq. (12.11) agrees with

$$\delta(x) = \mathcal{F}^{-1}\frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ixy}\,\mathrm{d}y. \tag{12.13}$$

Mathematical consistency can be achieved by defining the **Fourier transform of a distribution** T to be the distribution $\mathcal{F}T$ given by

$$\mathcal{F}T(\varphi) = T(\mathcal{F}\varphi), \tag{12.14}$$

for all test functions φ. For regular distributions we then have the desired result,

$$T_{\mathcal{F}f}(\varphi) = \mathcal{F}T_f(\varphi),$$

since

$$\begin{aligned}\mathcal{F}T_f(\varphi) &= T_f(\mathcal{F}\varphi)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\mathrm{e}^{-iyx}\varphi(x)\,\mathrm{d}x\Big)f(y)\,\mathrm{d}y\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)\Big(\int_{-\infty}^{\infty}\mathrm{e}^{-iyx}f(y)\,\mathrm{d}y\Big)\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty}\varphi(x)\mathcal{F}f(x)\,\mathrm{d}x\\ &= T_{\mathcal{F}f}(\varphi).\end{aligned}$$

If the inverse Fourier transform is defined on distributions by $\mathcal{F}^{-1}T(\varphi) = T(\mathcal{F}^{-1}\varphi)$, then

$$\mathcal{F}^{-1}\mathcal{F}T = T,$$

for

$$\mathcal{F}^{-1}\mathcal{F}T(\varphi) = \mathcal{F}T(\mathcal{F}^{-1}\varphi) = T(\mathcal{F}\mathcal{F}^{-1}\varphi) = T(\varphi).$$

There is, however, a serious problem with these definitions. If φ is a function of bounded support then $\mathcal{F}\varphi$ is generally an entire analytic function and cannot be of bounded support, since an entire function that vanishes on any open set must vanish everywhere. Hence the right-hand side of (12.14) is not in general well-defined. A way around this is to define a more general space of test functions $\mathcal{S}(\mathbb{R})$ called the space of **rapidly decreasing functions** – functions that approach 0 as $|x| \to \infty$ faster than any inverse power $|x|^{-n}$,

$$\mathcal{S}(\mathbb{R}) = \{\varphi \,|\, \sup_{x\in\mathbb{R}}|x^m\varphi^{(p)}(x)| < \infty \text{ for all integers } m, p > 0\}.$$

Convergence in $\mathcal{S}(\mathbb{R})$ is defined by $\varphi_n \to \varphi$ if and only if

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|x^m(\varphi^{(p)}(x) - \varphi^{(p)})| = 0 \text{ for all integers } m, p > 0.$$

The space of continuous linear functions on $\mathcal{S}(\mathbb{R})$ is denoted $\mathcal{S}'(\mathbb{R})$, and they are called **tempered distributions**. Since every test function is obviously a rapidly decreasing function, $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. If T is a tempered distribution in Eq. (12.14), the Fourier transform $\mathcal{F}T$ is well-defined, since the Fourier transform of any rapidly decreasing function may be shown to be a function of rapid decrease.

Example 12.8 The Fourier transform of the delta distribution is defined by

$$\begin{aligned}\mathcal{F}\delta_a(\varphi) &= \delta_a(\mathcal{F}\varphi)\\ &= \mathcal{F}\varphi(a)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-iax}\varphi(x)\,\mathrm{d}x\\ &= T_{(2\pi)^{-1/2}\mathrm{e}^{iax}}(\varphi).\end{aligned}$$

Similarly

$$\mathcal{F}^{-1}T_{\mathrm{e}^{-iax}} = \sqrt{2\pi}\,\delta_a.$$

The delta function versions of these distributional equations are

$$\mathcal{F}\delta_a(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-iyx}\,\delta(x-a)\,\mathrm{d}x = \frac{\mathrm{e}^{-iay}}{\sqrt{2\pi}}$$

and

$$\mathcal{F}^{-1}\mathrm{e}^{-iax} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{ixy}\mathrm{e}^{-iax}\,\mathrm{d}x = \sqrt{2\pi}\,\delta(x-a),$$

in agreement with Eqs. (12.10)–(12.13) above.

Problems

Problem 12.10 Find the Fourier transforms of the functions

$$f(x) = \begin{cases} 1 & \text{if} -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} 1 - \frac{|x|}{2} & \text{if} -a \le x \le a \\ 0 & \text{otherwise.} \end{cases}$$

Problem 12.11 Show that

$$\mathcal{F}\big(\mathrm{e}^{-a^2x^2/2}\big) = \frac{1}{|a|}\mathrm{e}^{-k^2/2a^2}.$$

Problem 12.12 Evaluate Fourier transforms of the following distributional functions:

(a) $\delta(x-a)$.
(b) $\delta'(x-a)$.
(c) $\delta^{(n)}(x-a)$.
(d) $\delta(x^2-a^2)$.
(e) $\delta'(x^2-a^2)$.

Problem 12.13 Prove that

$$x^m\delta^{(n)}(x) = (-1)^m\frac{n!}{(n-m)!}\delta^{(n-m)}(x) \qquad \text{for } n \ge m.$$

Hence show that the Fourier transform of the distribution

$$\sqrt{2\pi}\,\frac{k!}{(m+k)!}x^m\delta^{(m+k)}(-x) \quad (m, k \geq 0)$$

is $(-iy)^k$.

Problem 12.14 Show that the Fourier transform of the distribution

$$\delta_0 + \delta_a + \delta_{2a} + \cdots + \delta_{(2n-1)a}$$

is a distribution with density

$$\frac{1}{\sqrt{2\pi}}\frac{\sin(nay)}{\sin(\frac{1}{2}ay)}\mathrm{e}^{-(n-\frac{1}{2})iay}.$$

Show that

$$\mathcal{F}^{-1}(f(y)\mathrm{e}^{iby}) = (\mathcal{F}^{-1}f)(x+b).$$

Hence find the inverse Fourier transform of

$$g(y) = \frac{\sin nay}{\sin(\frac{1}{2}ay)}.$$

12.4 Green's functions

Distribution theory may often be used to find solutions of inhomogeneous linear partial differential equations by the technique of Green's functions. We give here two important standard examples.

Poisson's equation

To solve an inhomogeneous equation such as Poisson's equation

$$\nabla^2\phi = -4\pi\rho \tag{12.15}$$

we seek a solution to the distributional equation

$$\nabla^2 G(\mathbf{x}-\mathbf{x}') = \delta^3(\mathbf{x}-\mathbf{x}') = \delta(x-x')\delta(y-y')\delta(z-z'). \tag{12.16}$$

A solution of Poisson's equation (12.15) is then

$$\phi(\mathbf{x}) = -\iiint 4\pi\rho(\mathbf{x}')G(\mathbf{x}-\mathbf{x}')\,\mathrm{d}^3x',$$

for

$$\begin{aligned}\nabla^2\phi &= -\iiint 4\pi\rho(\mathbf{x}')\nabla^2 G(\mathbf{x}-\mathbf{x}')\,\mathrm{d}^3x' \\ &= -\iiint 4\pi\rho(\mathbf{x}')\delta^3(\mathbf{x}-\mathbf{x}')\,\mathrm{d}^3x' \\ &= -4\pi\rho(\mathbf{x}).\end{aligned}$$

To solve, set

$$g(\mathbf{k}) = \mathcal{F}G = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{y}} G(\mathbf{y}) \, \mathrm{d}^3 y.$$

By Fourier's theorem

$$G(\mathbf{y}) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{y}} g(\mathbf{k}) \, \mathrm{d}^3 k,$$

which implies that

$$\nabla^2 G(\mathbf{x} - \mathbf{x}') = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} -\mathbf{k}^2 \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} g(\mathbf{k}) \, \mathrm{d}^3 k.$$

But

$$\begin{aligned}\delta(\mathbf{y}) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \mathrm{e}^{ik_1 y_1} \, \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{e}^{ik_2 y_2} \, \mathrm{d}k_2 \int_{-\infty}^{\infty} \mathrm{e}^{ik_3 y_3} \, \mathrm{d}k_3 \\ &= \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{y}} \, \mathrm{d}^3 k,\end{aligned}$$

so

$$\delta^3(\mathbf{x} - \mathbf{x}') = \frac{1}{2\pi^3} \iiint_{-\infty}^{\infty} \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \, \mathrm{d}^3 k.$$

Substituting in Eq. (12.16) gives

$$g(\mathbf{k}) = -\frac{1}{(2\pi)^{3/2}\mathbf{k}^2},$$

and

$$G(\mathbf{x} - \mathbf{x}') = -\frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\mathrm{e}^{i\mathbf{k}\cdot\mathbf{y}}}{\mathbf{k}^2} \, \mathrm{d}^3 k. \tag{12.17}$$

The integration in **k**-space is best performed using polar coordinates (k, θ, ϕ) with the k_3-axis pointing along the direction $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ (see Fig. 12.2). Then

$$\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') = kR\cos\theta \quad (k = \sqrt{\mathbf{k}\cdot\mathbf{k}})$$

and

$$\mathrm{d}^3 k = k^2 \sin\theta \, \mathrm{d}k \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

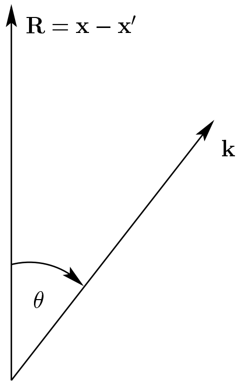

Figure 12.2 Change to polar coordinates in **k**-space.

This results in

$$\begin{aligned} G(\mathbf{R}) &= -\frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}k \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \frac{\mathrm{e}^{ikR\cos\theta}}{k^2} k^2 \sin\theta \\ &= -\frac{2\pi}{(2\pi)^3}\int_0^\infty \mathrm{d}k \int_0^\pi \mathrm{d}\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{-\mathrm{e}^{ikR\cos\theta}}{ikR}\right) \\ &= -\frac{1}{(2\pi)^2 R}\int_0^\infty \mathrm{d}k \frac{\mathrm{e}^{ikR} - \mathrm{e}^{-ikR}}{ik} \\ &= -\frac{1}{(2\pi)^2 R}\int_0^\infty \mathrm{d}k 2\frac{\sin kR}{k} \\ &= -\frac{1}{4\pi R}, \end{aligned}$$

on making use of the well-known definite integral

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

Hence

$$G(\mathbf{x} - \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}, \tag{12.18}$$

and a solution of Poisson's equation (12.15) is

$$\phi(\mathbf{x}) = \iiint \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3x',$$

where the integral is taken over all of the space, $-\infty < x', y', z' < \infty$. For a point charge, $\rho(\mathbf{x}) = q\delta^3(\mathbf{x} - \mathbf{a})$ the solution reduces to the standard coulomb solution

$$\phi(\mathbf{x}) = \frac{q}{|\mathbf{x} - \mathbf{a}|}.$$

Green's function for the wave equation

To solve the inhomogeneous wave equation

$$\Box\psi = -\frac{\partial^2}{c^2\partial t^2}\psi + \nabla^2\psi = f(\mathbf{x}, t) \tag{12.19}$$

it is best to adopt a relativistic 4-vector notation, setting $x^4 = ct$. The wave equation can then be written as in Section 9.4,

$$\Box\psi = g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\psi = f(x)$$

where μ and ν range from 1 to 4, $g^{\mu\nu}$ is the diagonal metric tensor having diagonal components $1, 1, 1, -1$ and the argument x in the last term is shorthand for $(\mathbf{x},\ x^4)$.
Again we look for a solution of the equation

$$\Box G(x - x') = \delta^4(x - x') \equiv \delta(x^1 - x'^1)\delta(x^2 - x'^2)\delta(x^3 - x'^3)\delta(x^4 - x'^4). \tag{12.20}$$

Every Green's function G generates a solution $\psi_G(x)$ of Eq. (12.19),

$$\psi_G(x) = \iiiint G(x - x')\, f(x')\, \mathrm{d}^4x'$$

for

$$\Box\psi_G = \iiiint \Box G(x - x') f(x')\, \mathrm{d}^4x' = \iiiint \delta^4(x - x') f(x')\, \mathrm{d}^4x' = f(x).$$

Exercise: Show that the general solution of the inhomogeneous wave equation (12.19) has the form $\psi_G(x) + \phi(x)$ where $\Box\phi = 0$.

Set

$$G(x - x') = \frac{1}{(2\pi)^2}\iiiint g(k)\,\mathrm{e}^{ik.(x-x')}\,\mathrm{d}^4k$$

where $k = (k_1, k_2, k_3, k_4)$, and

$$k.(x - x') = k_\mu(x^\mu - x'^\mu) = k_4(x^4 - x'^4) + \mathbf{k.}\cdot(\mathbf{x} - \mathbf{x}')$$

and $\mathrm{d}^4k = \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4$. Writing the four-dimensional δ function as a Fourier transform we have

$$\begin{aligned}\Box G(x - x') &= \frac{1}{(2\pi)^2}\iiiint -k^2 g(k)\,\mathrm{e}^{ik.(x-x')}\,\mathrm{d}^4k\\ &= \delta^4(x - x') = \frac{1}{(2\pi)^4}\iiiint \mathrm{e}^{ik.(x-x')}\,\mathrm{d}^4k,\end{aligned}$$

whence

$$g(k) = -\frac{1}{4\pi^2k^2}$$

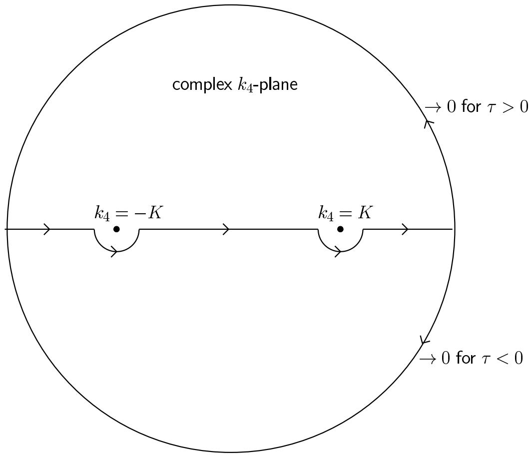

Figure 12.3 Green's function for the three-dimensional wave equation

where $k^2 \equiv k.k = k_\mu k^\mu$. The Fourier transform expression of the Green's function is thus

$$G(x - x') = -\frac{1}{(2\pi)^4} \iiiint \frac{e^{ik.(x-x')}}{k^2}\, d^4k. \tag{12.21}$$

To evaluate this integral set

$$\tau = x^4 - x'^4, \qquad \mathbf{R} = \mathbf{x} - \mathbf{x}', \qquad K = |\mathbf{k}| = \sqrt{\mathbf{k}\cdot\mathbf{k}},$$

whence $k^2 = K^2 - k_4^2$ and

$$G(x - x') = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} dk_4\, \frac{e^{ik_4\tau}}{k_4^2 - K^2} \iiint d^3k\, e^{i\mathbf{k}\cdot\mathbf{R}}.$$

Deform the path in the complex k_4-plane to avoid the pole singularities at $k_4 = \pm K$ as shown in Fig. 12.3 – convince yourself, however, that this has no effect on G satisfying Eq. (12.20).

For $\tau > 0$ the contour is completed in a counterclockwise sense by the upper half semicircle and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{ik_4\tau}}{k_4^2 - K^2}\, dk_4 &= 2\pi i \times \text{ sum of residues} \\ &= 2\pi i \Big(\frac{e^{iK\tau}}{2K} - \frac{e^{-iK\tau}}{2K}\Big).\end{aligned}$$

For $\tau < 0$ we complete the contour with the lower semicircle in a clockwise direction; no poles are enclosed and the integral vanishes. Hence

$$\int_{-\infty}^{\infty} \frac{e^{ik_4\tau}}{k_4^2 - K^2}\, dk_4 = -\frac{2\pi}{K}\theta(\tau)\sin K\tau$$

where $\theta(\tau)$ is the Heaviside step function.

This particular contour gives rise to a Green's function that vanishes for $\tau < 0$; that is, for $x^4 < x'^4$. It is therefore called the **outgoing wave condition** or **retarded Green's function**, for a source switched on at $(\mathbf{x}', x'^4)$ only affects field points at later times. If the contour had been chosen to lie above the poles, then the **ingoing wave condition** or **advanced Green's function** would have resulted.

To complete the calculation of G, use polar coodinates in $\mathbf{k}$-space with the k_3-axis parallel to $\mathbf{R}$. This gives

$$\begin{aligned}
G(x - x') &= -\frac{1}{(2\pi)^3}\theta(\tau)\int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}K \int_0^{\pi} \mathrm{d}\theta\ K^2 \sin\theta \mathrm{e}^{iKR\cos\theta}\frac{\sin K\tau}{K} \\
&= -\frac{\theta(\tau)}{2\pi^2 R}\int_0^{\infty} \mathrm{d}K \sin K\tau \ \sin KR \\
&= -\frac{\theta(\tau)}{2\pi^2 R}\int_0^{\infty} \mathrm{d}K \frac{(\mathrm{e}^{iK\tau} - \mathrm{e}^{-iK\tau})}{2i}\frac{(\mathrm{e}^{iKR} - \mathrm{e}^{-iKR})}{2i} \\
&= \frac{\theta(\tau)}{4\pi R}\big(\delta(\tau + R) - \delta(\tau - R)\big) \\
&= -\frac{\delta(\tau - R)}{4\pi R}.
\end{aligned}$$

The last step follows because the whole expression vanishes for $\tau < 0$ on account of the $\theta(\tau)$ factor, while for $\tau > 0$ we have $\delta(\tau + R) = 0$. Hence the Green's function may be written

$$G(x - x') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}\delta(x^4 - x'^4 - |\mathbf{x} - \mathbf{x}'|), \tag{12.22}$$

which is non-vanishing only on the future light cone of x'.

The solution of the inhomogeneous wave equation (12.19) generated by this Green's function is

$$\begin{aligned}
\psi(\mathbf{x}, t) &= \iiiint G(x - x')f(x')\,\mathrm{d}^4x' \\
&= -\frac{1}{4\pi}\iiint \frac{[f(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3x'
\end{aligned} \tag{12.23}$$

where $[f(\mathbf{x}', t')]_{\text{ret}}$ means f evaluated at the retarded time

$$t' = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}.$$

Problems

Problem 12.15 Show that the Green's function for the time-independent Klein–Gordon equation

$$(\nabla^2 - m^2)\phi = \rho(\mathbf{r})$$

can be expressed as the Fourier integral

$$G(\mathbf{x} - \mathbf{x}') = -\frac{1}{(2\pi)^3}\int\int\int \mathrm{d}^3k \frac{\mathrm{e}^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2 + m^2}.$$

Evaluate this integral and show that it results in

$$G(\mathbf{R}) = -\frac{e^{-mR}}{4\pi R} \quad \text{where} \quad \mathbf{R} = \mathbf{x} - \mathbf{x}', \quad R = |\mathbf{R}|.$$

Find the solution ϕ corresponding to a point source

$$\rho(\mathbf{r}) = q\delta^3(\mathbf{r}).$$

Problem 12.16 Show that the Green's function for the one-dimensional diffusion equation,

$$\frac{\partial^2 G(x,t)}{\partial x^2} - \frac{1}{\kappa}\frac{\partial G(x,t)}{\partial t} = \delta(x-x')\delta(t-t')$$

is given by

$$G(x-x', t-t') = -\theta(t-t')\sqrt{\frac{\kappa}{4\pi(t-t')}}\,e^{-(x-x')^2/4\kappa(t-t')},$$

and write out the corresponding solution of the inhomogeneous equation

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{1}{\kappa}\frac{\partial \psi(x,t)}{\partial t} = F(x,t).$$

Do the same for the two- and three-dimensional diffusion equations

$$\nabla^2 G(x,t) - \frac{1}{\kappa}\frac{\partial G(x,t)}{\partial t} = \delta^n(\mathbf{x}-\mathbf{x}')\delta(t-t') \quad (n = 2, 3).$$

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