17 Integration on manifolds

The theory of integration over manifolds is only available for a restricted class known as *oriented manifolds*. The general theory can be found in [1–11]. An n-dimensional differentiable manifold M is called **orientable** if there exists a differential n-form ω that vanishes at no point $p \in M$. The n-form ω is called a **volume element** for M, and the pair (M, ω) is an **oriented manifold**. Since the space $\big(\Lambda_n\big)_p(M) \equiv \big(\Lambda^{*n}\big)_p(M)$ is one-dimensional at each $p \in M$, any two volume elements are proportional to each other, $\omega' = f\omega$, where $f : M \to \mathbb{R}$ is a non-vanishing smooth function on M. If the manifold is a connected topological space it has the same sign everywhere; if $f(p) > 0$ for all $p \in M$, the two n-forms ω and ω' are said to assign the same **orientation** to M, otherwise they are **oppositely oriented**. Referring to Example 8.4, a manifold is orientable if each cotangent space $T^*_p(M)$ $(p \in M)$ is oriented by assigning a non-zero n-form $\Omega = \omega_p$ at p and the orientations are assigned in a smooth and continuous way over the manifold.

With respect to a coordinate chart $(U, \phi; x^i)$ the volume element ω can be written

$$\omega = g(x^1, \dots, x^n)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \dots \wedge \mathrm{d}x^n = \frac{g}{n!}\epsilon_{i_1 i_2 \dots i_n}\,\mathrm{d}x^{i_1} \otimes \mathrm{d}x^{i_2} \otimes \dots \otimes \mathrm{d}x^{i_n}.$$

If $(U', \phi'; x'^{i'})$ is a second coordinate chart then, in the overlap region $U \cap U'$,

$$\omega = g'(x'^{i_1}, \dots, x'^{i_n})\,\mathrm{d}x'^1 \wedge \mathrm{d}x'^2 \wedge \dots \wedge \mathrm{d}x'^n$$

where

$$g'(\mathbf{x}') = g(\mathbf{x}) \det\Big[\frac{\partial x^i}{\partial x'^{j'}}\Big].$$

The sign of the component function g thus remains unchanged if and only if the Jacobian determinant of the coordinate transformation is positive throughout $U \cap V$, in which case the charts are said to have the **same orientation**. A differentiable manifold is in fact orientable if and only if there exists an atlas of charts (U_i, ϕ_i) covering M, such that any two charts (U_i, ϕ_i) and (U_j, ϕ_j) have the same orientation on their overlap $U_i \cap U_j$, but the proof requires the concept of a *partition of unity*.

Problems

Problem 17.1 Show that in spherical polar coordinates

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2 \sin\theta\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi,$$

and that $a^2 \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi$ is a volume element on the 2-sphere $x^2 + y^2 + z^2 = a^2$.

Problem 17.2 Show that the 2-sphere $x^2 + y^2 + z^2 = 1$ is an orientable manifold.

17.1 Partitions of unity

Given an open covering $\{U_i \mid i \in I\}$ of a topological space S, an open covering $\{V_a\}$ of S is called a **refinement** of $\{U_i\}$ if each $V_a \subseteq U_i$ for some $i \in I$. The refinement is said to be **locally finite** if every point p belongs to at most a finite number of V_a's. The topological space S is said to be **paracompact** if for every open covering U_i $(i \in I)$ there exists a locally finite refinement. As it may be shown that every locally compact Hausdorff second-countable space is paracompact [12], we will from now on restrict attention to manifolds that are paracompact topological spaces.

Given a locally finite open covering $\{V_a\}$ of a manifold M, a **partition of unity subordinate to the covering** $\{V_a\}$ consists of a family of differentiable functions $g_a : M \to \mathbb{R}$ such that

(1) $0 \leq g_a \leq 1$ on M for all α,
(2) $g_a(p) = 0$ for all $p \notin V_a$,
(3) $\sum_a g_a(p) = 1$ for all $p \in M$.

It is important that the covering be locally finite, so that the sum in (3) reduces to a finite sum.

Theorem 17.1 *For every locally finite covering $\{V_a\}$ of a paracompact manifold M there exists a partition of unity $\{g_a\}$ subordinate to this refinement.*

Proof: For each $p \in M$ let B_p be an open neighbourhood of p such that its closure $\overline{B_p}$ is compact and contained in some V_a – for example, take B_p to be the inverse image of a small coordinate ball in $\mathbb{R}^n$. As the sets $\{B_p\}$ form an open covering of M, they have a locally finite refinement $\{B'_\alpha\}$. For each a let V'_a be the union of all B'_α whose closure $\overline{B'_\alpha} \subset V_a$. Since every $\overline{B'_\alpha} \subset \overline{B_p} \subset V_a$ for some p and a, it follows that the sets V'_a are an open covering of M. For each a the closure of V'_a is compact and, by the local finiteness of the covering $\{B'_\alpha\}$,

$$\overline{V'_a} = \bigcup \overline{B'_\alpha} \subset V_a.$$

As seen in Lemma 16.1 for any point $p \in \overline{V'_a}$ it is possible to find a differentiable function $h_p : M \to \mathbb{R}$ such that $h_p(p) = 1$ and $h_p = 0$ on $M - V_a$. For each point $p \in \overline{V'_a}$ let U_p be the open neighbourhood $\{q \in V_a \mid f_p(q) > \frac{1}{2}\}$. Since $\overline{V'_a}$ is compact, there exists a finite subcover $\{U_{p_1}, \dots, U_{p_k}\}$. The function $h_a = h_{p_1} + \dots + h_{p_k}$ has the following three properties: (i) $h_a \geq 0$ on M, (ii) $h > 0$ on $\overline{V'_a}$, and (iii) $h_a = 0$ outside V_a. As $\{V_a\}$ is a locally finite covering of M, the function $h = \sum_a h_a$ is well-defined, and positive everywhere on M. The functions $g_a = h_a/h$ satisfy all requirements for a partition of unity subordinate to V_a. ■

We can now construct a non-vanishing n-form on a manifold M from any atlas of charts $(U_\alpha, \phi_\alpha; x^i_\alpha)$ having positive Jacobian determinants on all overlaps. Let $\{V_a\}$ be a locally finite refinement of $\{U_\alpha\}$ and g_a a partition of unity subordinate to $\{V_a\}$. The charts $(V_a, \phi_a; x^i_a)$ where $\phi_a = \phi_\alpha\big|_{V_a}$ and $x^i_a = x^i_\alpha\big|_{V_a}$ form an atlas on M, and

$$\omega = \sum_a g_a \,\mathrm{d}x^1_a \wedge \mathrm{d}x^2_a \wedge \cdots \wedge \mathrm{d}x^n_a$$

is a differential n-form on M that nowhere vanishes.

Example 17.1 The *Möbius band* can be thought of as a strip of paper with the two ends joined together after giving the strip a twist, as shown in Fig. 17.1. For example, let $M = \{(x, y) \mid -2 \le x \le 2, -1 < y < 1\}$ where the end edges are identified in opposing directions, $(2, y) \equiv (-2, -y)$. This manifold can be covered by two charts

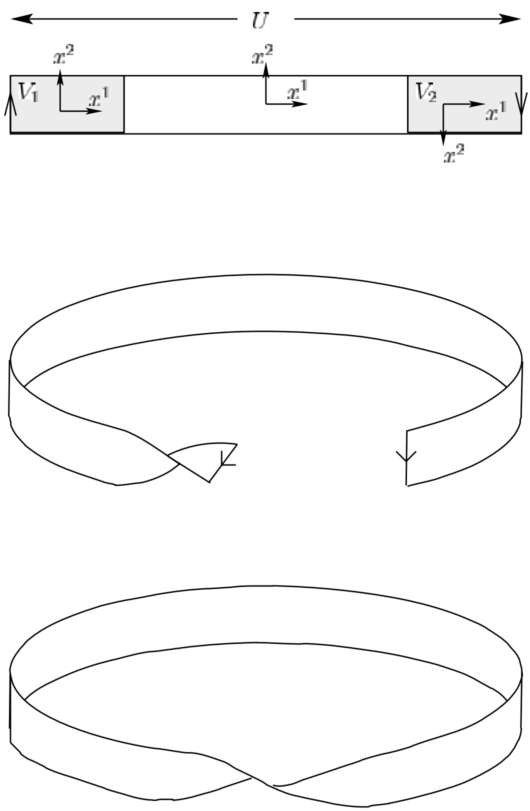

Figure 17.1 Möbius band

$(U = \{(x, y) \mid -2 < x < 2,\ -1 < y < 1\}, \phi = \mathrm{id}_U)$ and $(V = V_1 \cup V_2, \psi)$ where

$$V_1 = \{(x, y) \mid -2 \le x < -1,\ -1 < y < 1\},$$
$$V_2 = \{(x, y) \mid 1 \le x < 2,\ -1 < y < 1\},$$
$$(x', y') = \psi(x, y) = \begin{cases} (x + 2, y) & \text{if } (x, y) \in V_1, \\ (x - 2, -y) & \text{if } (x, y) \in V_2 . \end{cases}$$

The Jacobian is $+1$ on $U \cap V_1$ and -1 on $U \cap V_2$, so these two charts do not have the same orientation everywhere. The Möbius band is non-orientable, for if there existed a non-vanishing 2-form ω, we would have $\omega = f\,\mathrm{d}x \wedge \mathrm{d}y$ with $f(x, y) > 0$ or $f(x, y) < 0$ everywhere on U. Setting $\omega = f'\,\mathrm{d}x' \wedge \mathrm{d}y'$ we have $f' = f$ on V_1 and $f' = -f$ on V_2. Hence $f'(x', y')$ must vanish on the line $x = \pm 2$, which contradicts ω being non-vanishing everywhere.

17.2 Integration of *n*-forms

There is no natural way to define the integral of a scalar function $f : M \to \mathbb{R}$ over a compact region D. For, if $D \subset U$ where U is the domain of a coordinate chart $(U, \phi; x^i)$ the multiple integral

$$\int_D f = \int_{\phi(D)} f \circ \phi^{-1}(x^1, \dots, x^n)\,\mathrm{d}x^1 \dots \mathrm{d}x^n$$

will have a different expression in a second chart $(V, \psi; y^i)$ such that $D \subset U \cap V$

$$\begin{aligned} \int_{\psi(D)} f \circ \psi^{-1}(\mathbf{y})\,\mathrm{d}y^1 \dots \mathrm{d}y^n &= \int_{\phi(D)} f \circ \phi^{-1}(\mathbf{x}) \left| \det\left[\frac{\partial y^i}{\partial x^j}\right] \right| \mathrm{d}x^1 \dots \mathrm{d}x^n \\ &\ne \int_{\phi(D)} f \circ \phi^{-1}(\mathbf{x})\,\mathrm{d}x^1 \dots \mathrm{d}x^n . \end{aligned}$$

As seen above, n-forms absorb a Jacobian determinant in their coordinate transformation, and it turns out that these are the ideal objects for integration. However, it is necessary that the manifold be orientable so that the *absolute value* of the Jacobian occurring in the integral transformation law can be omitted.

Let (M, ω) be an n-dimensional oriented differentiable manifold, and $(U, \phi; x^i)$ a positively oriented chart. On U we can write $\omega = g(\mathbf{x})\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n$ where $g > 0$. The **support** of an n-form α is defined as the closure of the set on which $\alpha \ne 0$,

$$\operatorname{supp} \alpha = \overline{\{p \in M \mid \alpha_p \ne 0\}}.$$

If α has compact support contained in U, and $\alpha = f\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \dots \wedge \mathrm{d}x^n$ on U we define its **integral** over M to be

$$\int_M \alpha = \int_{\phi(U)} f(x^1, \dots, x^n)\,\mathrm{d}x^1 \dots \mathrm{d}x^n = \int_{\phi(\operatorname{supp} \alpha)} f\,\mathrm{d}x^1 \dots \mathrm{d}x^n$$

where $f(x^1, \dots, x^n)$ is commonly written in place of $\hat{f} = f \circ \phi^{-1}$. If $(V, \psi; x'^i)$ is a second positively oriented chart also containing the support of α and $f'(x'^1, \dots, x'^n) \equiv f \circ \psi^{-1}$, we have by the change of variable formula in multiple integration

$$\begin{aligned}\int_{\psi(V)} f'(x'^1, \dots, x'^n)\, \mathrm{d}x'^1 \dots \mathrm{d}x'^n &= \int_{\psi(\operatorname{supp}\alpha)} f'(\mathbf{x}')\, \mathrm{d}x'^1 \dots \mathrm{d}x'^n \\ &= \int_{\phi(\operatorname{supp}\alpha)} f'(\mathbf{x}') \left|\det\left[\frac{\partial x'^i}{\partial x^j}\right]\right| \mathrm{d}x^1 \dots \mathrm{d}x^n \\ &= \int_{\phi(U)} f(\mathbf{x})\, \mathrm{d}x^1 \dots \mathrm{d}x^n\end{aligned}$$

since

$$\alpha = f \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n = f' \mathrm{d}x'^1 \wedge \dots \wedge'^{\,n} \quad \text{where} \quad f = f' \det\left[\frac{\partial x'^i}{\partial x^j}\right]$$

and the Jacobian determinant is everywhere positive. The definition of the integral is therefore independent of the coordinate chart, provided the support lies within the domain of the chart.

For an arbitrary n-form α with compact support and atlas (U_a, ϕ_a), assumed to be locally finite, let g_a be a partition of unity subordinate to the open covering $\{U_a\}$. Evidently

$$\alpha = \sum_a g_a \alpha$$

and each of the summands $g_a \alpha$ has compact support contained in U_a. We define the integral of α over M to be

$$\int_M \alpha = \sum_a \int_M g_a \alpha. \tag{17.1}$$

Exercise: Prove that $\int_M$ is a linear operator, $\int_M \alpha + c\beta = \int_M \alpha + c \int_M \beta$.

If α is a differential k-form with compact support on M and $\varphi : N \to M$ is a regular embedding of a k-dimensional manifold N in M (see Section 15.4), define the integral of α on $\varphi(N)$ to be

$$\int_{\varphi(N)} \alpha = \int_N \varphi^* \alpha.$$

The right-hand side is well-defined since $\varphi^*\alpha$ is a differential k-form on N with compact support, since $\varphi : M \to N$ is a homeomorphism from N to $\varphi(N)$ in the relative topology with respect to M.

Problems

Problem 17.3 Show that the definition of the integral of an n-form over a manifold M given in Eq. (17.1) is independent of the choice of partition of unity subordinate to $\{U_a\}$.

17.3 Stokes' theorem

Stokes' theorem requires the concept of a submanifold with boundary. This is not an easy notion in general, but for most practical purposes it is sufficient to restrict ourselves to regions made up of 'coordinate cubical regions'. Let Γ_k be the standard unit k-cube,

$$\Gamma_k = \{\mathbf{x} \in \mathbb{R}^k \mid 0 \le x^i \le 1 \ (i = 1, \dots, k)\} \subset \mathbb{R}^k.$$

The unit 0-cube is taken to be the singleton $\Gamma_0 = \{0\} \subset \mathbb{R}$. A **$k$-cell** in a manifold M is a smooth map $\sigma : U \to M$ where U is an open neighbourhood U of Γ_k in $\mathbb{R}^k$ (see Fig. 17.2), and its **support** is defined as the image of the standard k-cube, $\sigma(\Gamma_k)$. A **cubical k-chain** in M consists of a formal sum

$$C = c^1\sigma_1 + c^2\sigma_2 + \cdots + c^r\sigma_r \quad \text{where} \quad c^i \in \mathbb{R},\ r \text{ a positive integer.}$$

The set of all cubical k-chains is denoted $\mathcal{C}_k$; it forms an abelian group under addition of k-chains defined in the obvious way. It is also a vector space if we define scalar multiplication as $aC = \sum_i ac^i\sigma_i$ where $a \in \mathbb{R}$.

For each $i = 1, \dots, k$ and $\epsilon = 0, 1$ define the maps $\varphi_i^\epsilon : \Gamma_{k-1} \to \Gamma_k$ by

$$\varphi_i^\epsilon(y^1, \dots, y^{k-1}) = (y^1, \dots, y^{i-1}, \epsilon, y^i, \dots, y^{k-1}).$$

These maps can be thought of as the $(i, 0)$-face and $(i, 1)$-face respectively of Γ_k. If the interior of the standard k-cube Γ_k is oriented in the natural way, by assigning the k-form $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$ to be positively oriented over Γ_k, then for each face map $x^i = 0$ or 1 the orientation on Γ_{k-1} is assigned according to the following rule: set the k-form $\mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^k$ to be positively oriented if x^i is increasing outwards at the face, else it is negatively oriented. According to this rule the $(k-1)$-form $\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^{k-1}$ has orientation $(-1)^i$ on the $(i, 0)$-face, while on the $(i, 1)$-face it has orientation $(-1)^{i+1}$. This is sometimes called the *outward normal rule* – the orientation on the boundary surface must be chosen such that at every point there exist local positively oriented coordinates such that the first coordinate x^1 points outwards from the surface.

Exercise: On the two-dimensional square, verify that the outward normal rule implies that the direction of increasing x or y coordinate on each side proceeds in an anticlockwise fashion around the square.

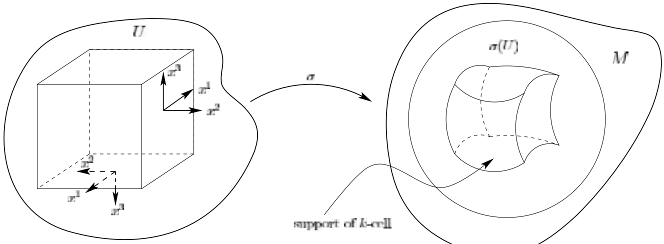

Figure 17.2 k-Cell on a manifold M

This gives the rationale for the **boundary map** $\partial : \mathcal{C}_k \to (C)_{k-1}$, defined by:

(i) for a k-cell $\sigma \quad (k > 0)$, set

$$\partial\sigma = \sum_{i=1}^{k} (-1)^i \big(\sigma \circ \varphi_i^0 - \sigma \circ \varphi_i^1\big),$$

(ii) for a cubical k-chain $C = \sum c^i \sigma_i$ set

$$\partial C = \sum_i c^i \partial\sigma_i .$$

An important identity is $\partial^2 = 0$. For example if σ is a 2-cell, its boundary is given by

$$\partial\sigma = -\varphi_1^0 + \varphi_1^1 + \varphi_2^0 - \varphi_2^1.$$

The boundary of the face $\varphi_1^0(z) = (0, z)$ is $\partial\varphi_1^0 = -\rho^{01} + \rho^{00}$, where $\rho^{ab} : \{0\} \to \mathbb{R}$ is the map $\rho^{ab}(0) = (a, b)$. Hence

$$\partial \circ \partial\sigma = -\rho^{01} + \rho^{00} + \rho^{11} - \rho^{10} + \rho^{10} - \rho^{00} - \rho^{11} + \rho^{01} = 0.$$

For a k-cell,

$$\partial^2\sigma = \sum_{\epsilon=0}^{1}\sum_{\epsilon'=0}^{1}\Big[\sum_i \sum_{j<i} (-1)^{i+\epsilon+j+\epsilon'} \sigma \circ \varphi_{ij}^{\epsilon\epsilon'} + \sum_i \sum_{j\geq i} (-1)^{i+\epsilon+j+\epsilon'+1} \sigma \circ \varphi_{ij}^{\epsilon\epsilon'}\Big]$$

where

$$\varphi_{ij}^{\epsilon\epsilon'}(z^1, \dots, z^{k-2}) = \begin{cases} (z^1, \dots, z^{j-1}, \epsilon', z^j, \dots, z^{i-1}, \epsilon, z^i, \dots, z^{k-2}) & \text{if } j < i, \\ (z^1, \dots, z^{i-1}, \epsilon, z^i, \dots, z^{j-1}, \epsilon', z^j, \dots, z^{k-2}) & \text{if } j \geq i. \end{cases}$$

It follows from this equation that all terms cancel in pairs, so that $\partial^2\sigma = 0$. The identity extends by linearity to all k-chains.

Exercise: Write out $\partial^2\sigma$ for a 3-cube, and verify the cancellation property.

For a k-form α on M and a k-chain $C = \sum c^i \sigma_i$ we define the integral

$$\int_C \alpha = \sum c^i \int_{\sigma_i} \alpha = \sum c^i \int_{\Gamma_k} \sigma_i^* \alpha.$$

Theorem 17.2 (Stokes' theorem) *For any $(k+1)$-chain C, and differential k-form on M,*

$$\int_C \mathrm{d}\alpha = \int_{\partial C} \alpha. \tag{17.2}$$

Proof: By linearity, it is only necessary to prove the theorem for a $(k+1)$-cell σ. The left-hand side of (17.2) can be written, using Theorem 16.2,

$$\int_\sigma \mathrm{d}\alpha = \int_{\Gamma_{k+1}} \sigma^* \mathrm{d}\alpha = \int_{\Gamma_{k+1}} \mathrm{d}(\sigma^*\alpha)$$

while the right-hand side is

$$\int_{\partial\sigma} \alpha = \sum_{\epsilon=0}^{1} \sum_{i=1}^{k+1} (-1)^{i+\epsilon} \int_{\sigma\circ\varphi_i^\epsilon} \alpha$$
$$= \sum_{\epsilon=0}^{1} \sum_{i=1}^{k+1} (-1)^{i+\epsilon} \int_{\Gamma_k} \left(\varphi_i^\epsilon\right)^* \circ \sigma^*\alpha.$$

Since $\sigma^*\alpha$ is a differential k-form on $\mathbb{R}^{k+1}$ it can be written as

$$\sigma^*\alpha = \sum_{i=1}^{k+1} A_i \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^{k+1}$$

where the A_i are differentiable functions $A_i : V \to \mathbb{R}$ on an open neighbourhood V of Γ_{k+1}. Hence

$$\mathrm{d}(\sigma^*\alpha) = \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \frac{\partial A_i}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^{k+1}$$
$$= \sum_{i=1}^{k+1} (-1)^{i+1} \frac{\partial A_i}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^j \wedge \cdots \wedge \mathrm{d}x^{k+1}.$$

Substituting in the left-hand integral of Eq. (17.2) we have

$$\int_{\Gamma_{k+1}} \mathrm{d}(\sigma^*\alpha) = \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{\partial A_i}{\partial x^i} \mathrm{d}x^1 \, \mathrm{d}x^2 \ldots \mathrm{d}x^{k+1}$$
$$= \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 \int_0^1 \cdots \int_0^1 \mathrm{d}x^1 \ldots \mathrm{d}x^{i-1} \mathrm{d}x^{i+1} \ldots \mathrm{d}x^{k+1}$$
$$\times \big[A_i(x^1, \ldots, x^{i-1}, 1, x^{i+1}, \ldots, x^{k+1})$$
$$- A_i(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^{k+1})\big]$$
$$= \sum_{\epsilon=0}^{1} \sum_{i=1}^{k+1} (-1)^{i+\epsilon} \int_0^1 \cdots \int_0^1 \mathrm{d}x^1 \ldots \mathrm{d}x^{i-1} \, \mathrm{d}x^{i+1} \ldots$$
$$\mathrm{d}x^{k+1} A_i(x^1, \ldots, x^{i-1}, \epsilon, x^{i+1}, \ldots, x^{k+1})$$
$$= \sum_{\epsilon=0}^{1} \sum_{i=1}^{k+1} (-1)^{i+\epsilon} \int_{\Gamma_k} \left(\varphi_i^\epsilon\right)^* \circ \sigma^*\alpha$$
$$= \int_{\partial\sigma} \alpha,$$

as required. ■

Example 17.2 In Example 15.9 we defined the integral of a differential 1-form ω over a curve with end points $\gamma : [t_1, t_2] \to M$ to be

$$\int_\gamma \omega = \int_{t_1}^{t_2} \langle \omega, \dot{\gamma} \rangle \mathrm{d}t = \int_{t_1}^{t_2} \langle \gamma^*\omega, \frac{\mathrm{d}}{\mathrm{d}t} \rangle \mathrm{d}t.$$

A 1-cell $\sigma : U \to M$ is a curve with parameter range $U = (-a, 1 + a) \supset \Gamma_1 = [0, 1]$, and can be made to cover an arbitrary range $[t_1, t_2]$ by a change of parameter $t \to t' =$

$t_1 + (t_2 - t_1)t$. The integral of $\omega = w_i(x^j)\,\mathrm{d}x^i$ over the support of the 1-cell is

$$\int_\sigma \omega = \int_{\Gamma_1} \sigma * \omega = \int_0^1 w_i(\mathbf{x}(t))\frac{\mathrm{d}x^i}{\mathrm{d}t}\mathrm{d}t,$$

which agrees with the definition of the integral given in Example 15.9.

The boundary of the 1-cell σ is

$$\partial\sigma = \sigma \circ \varphi_1^1 - \sigma \circ \varphi_1^0$$

where the two terms on the right-hand side are the 0-cells $\Gamma_0 = \{0\} \to \sigma(1)$ and $\Gamma_0 = \{0\} \to \sigma(0)$, respectively. Setting $\omega = \mathrm{d}f$ where f is a differentiable function on M, Stokes' theorem gives

$$\int_\sigma \mathrm{d}f = \int_{\partial\sigma} f = f(\sigma(1)) - f(\sigma(0)).$$

If $M = \mathbb{R}$ and the 1-cell σ is defined by $x = \sigma(t) = a + t(b-a)$, Stokes' theorem reduces to the fundamental theorem of calculus,

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \int_0^1 \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{d}t = \int_\sigma \mathrm{d}f = f(b) - f(a).$$

Regular domains

In the above discussion, the only requirement made concerning the cell maps σ was that they be differentiable on a neighbourhood of the unit k-cube Γ_k. For example, they could be completely degenerate and map the entire set Γ_k into a single point of M. For this reason, the chains are sometimes called **singular**. A **fundamental n-chain** on an n-dimensional manifold M has the form

$$C = \sigma_1 + \sigma_2 + \dots + \sigma_N$$

where each $\sigma_i : U \to \sigma_i(U)$ is a diffeomorphism and the interiors of the supports $\sigma_i(\Gamma_k)$ of different cells are non-intersecting:

$$\sigma_i\big((\Gamma_n)^o\big) \cap \sigma_j\big((\Gamma_n)^o\big) = \emptyset \quad \text{for} \quad i \neq j.$$

A **regular domain** $D \subseteq M$ is a closed set of the form

$$D = \bigcup_{i=1}^n \sigma_i(\Gamma_n)$$

where σ_i are the n-cells of a fundamental chain C (see Fig. 17.3). We may think of a regular domain as subdivided into cubical cells or a *region with boundary* – the 'boundary' consisting of boundary points of the chain that are not on the common faces of any pair of cells.

Theorem 17.3 *If D is a regular domain 'cubulated' in two different ways by fundamental chains $C = \sigma_i + \dots + \sigma_N$ and $C' = \tau_1 + \dots + \tau_M$ then $\int_C \omega = \int_{C'} \omega$ for every differential n-form ω.*

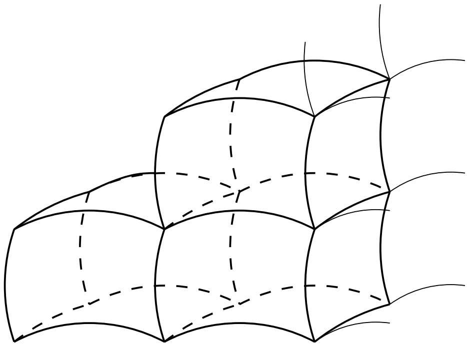

Figure 17.3 A regular domain on a manifold

Proof: Let $A_{ij} = \sigma_i(\Gamma_n) \cap \tau_j(\Gamma_n)$ and

$$B_{ij} = (\sigma_i)^{-1}(A_{ij}), \qquad C_{ij} = (\tau_j)^{-1}(A_{ij}).$$

The maps $\tau_j^{-1} \circ \sigma_i : B_i j \to C_{ij}$ are all diffeomorphisms and

$$\begin{aligned}\int_{B_{ij}} \sigma_i^* \omega &= \int_{B_{ij}} \sigma_i^* \circ (\tau_j^{-1})^* \circ \tau_j^* \omega \\ &= \int_{B_{ij}} \big((\tau_j^{-1}) \circ \sigma_i\big)^* \circ \tau_j^* \omega \\ &= \int_{C_{ij}} \tau_j^* \omega\,.\end{aligned}$$

Hence

$$\int_C \omega = \sum_i \int_{\sigma_i} \omega = \sum_{i,j} \int_{B_{ij}} \sigma_i^* \omega = \sum_{i,j} \int_{C_{ij}} \tau_j^* \omega = \int_{C'} \omega. \qquad \blacksquare$$

If α is a differential $(n-1)$-form,

$$\int_{\partial D} \alpha = \int_{\partial C} \alpha = \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{\epsilon=0}^{1} \int_{\Gamma_{n-1}} (\varphi_j^\epsilon)^* \circ \sigma_i^* \alpha.$$

Since the outward normals on common faces of adjoining cells are oppositely directed, the faces will be oppositely oriented and the integrals will cancel, leaving only an integral on the ‘free’ parts of the boundary of D. This results in Stokes’ theorem for a regular domain

$$\int_D \mathrm{d}\alpha = \int_{\partial D} \alpha.$$

A **regular k-domain** D_k is defined as the image of a regular domain $D \subset K$ of a k-dimensional manifold K under a regular embedding $\varphi : K \to M$, and for any k-form

β and $(k-1)$-form α we set

$$\int_{D_k} \beta = \int_{\varphi(D)} \beta = \int_D \varphi * \beta$$
$$\int_{\partial D_k} \alpha = \int_{\partial\varphi(D)} \alpha = \int_{\partial D} \varphi * \alpha.$$

The general form of Stokes' theorem asserts that for any $(k-1)$-form α and regular k-domain D_k,

$$\int_{D_k} \mathrm{d}\alpha = \int_{\partial D_k} \alpha. \tag{17.3}$$

Example 17.3 In low dimensions, Stokes' theorem reduces to a variety of familiar forms. For example, let D be a regular 2-domain in $\mathbb{R}^2$ bounded by a circuit $C = \partial D$ having induced orientation according to the 'right hand rule'. By this we mean that if the outward normal is taken locally in the direction of the first coordinate x^1, and the tangent to C in the direction x^2, then $\mathrm{d}x^1 \wedge \mathrm{d}x^2$ is positively oriented. If $\alpha = P\,\mathrm{d}x + Q\,\mathrm{d}y$, then

$$\mathrm{d}\alpha = \frac{\partial P}{\partial y}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial Q}{\partial x}\mathrm{d}x \wedge \mathrm{d}y = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\mathrm{d}x \wedge \mathrm{d}y,$$

and Stokes' theorem is equivalent to **Green's theorem**

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,\mathrm{d}x\,\mathrm{d}y = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

If D is a bounded region in $\mathbb{R}^3$ whose boundary is a surface $S = \partial D$, the induced orientation is such that if $(\mathbf{e}, \mathbf{f})$ are a correctly ordered pair of tangent vectors to S and $\mathbf{n}$ the outward normal to S, then $(\mathbf{n}, \mathbf{e}, \mathbf{f})$ is a positively oriented basis of vectors in $\mathbb{R}^3$. Let $\alpha = A_1\,\mathrm{d}y \wedge \mathrm{d}z + A_2\,\mathrm{d}z \wedge \mathrm{d}x + A_3\,\mathrm{d}x \wedge \mathrm{d}z$ be a 2-form, then

$$\mathrm{d}\alpha = \big(A_{1,1} + A_{2,2} + A_{3,3}\big)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \nabla \cdot \mathbf{A}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

where $\mathbf{A} = (A_1, A_2, A_3)$. Stokes' theorem reads

$$\int_D \mathrm{d}\alpha = \iiint_D \nabla \cdot \mathbf{A}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{\partial D} \alpha = \iint_S A_1\,\mathrm{d}y\,\mathrm{d}z + A_2\,\mathrm{d}z\,\mathrm{d}x + A_3\,\mathrm{d}x\,\mathrm{d}z.$$

If the bounding surface is locally parametrized by two parameters, $x = x(\lambda_1, \lambda_2),\ y = y(\lambda_1, \lambda_2),\ z = z(\lambda_1, \lambda_2)$, then we can write

$$\iint_S \alpha = \iint_S \epsilon_{ijk} A_i \frac{\partial x^j}{\partial \lambda_1}\frac{\partial x^k}{\partial \lambda_2}\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2$$

and it is common to write Stokes' theorem in the standard **Gauss theorem** form

$$\iiint_D \nabla \cdot \mathbf{A}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_S \mathbf{A} \cdot \mathrm{d}\mathbf{S},$$

where **S** is the vector area normal to S, having components

$$\mathrm{d}S_i = \epsilon_{ijk}\frac{\partial x^j}{\partial \lambda_1}\frac{\partial x^k}{\partial \lambda_2}\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2.$$

Let Σ be an oriented 2-surface in $\mathbb{R}^3$, with boundary an appropriately oriented circuit $C = \partial\Sigma$, and α a differential 1-form $\alpha = A_1\,\mathrm{d}x + A_2\,\mathrm{d}y + A_3\,\mathrm{d}z = A_i\,\mathrm{d}x^i$. Then

$$\begin{aligned}\int_\Sigma \mathrm{d}\alpha &= \int_\Sigma A_{i,j}\,\mathrm{d}x^j \wedge \mathrm{d}x^i \\ &= \iint\limits_\Sigma A_{k,j}\Big(\frac{\partial x^j}{\partial\lambda_1}\frac{\partial x^k}{\partial\lambda_2} - \frac{\partial x^k}{\partial\lambda_1}\frac{\partial x^j}{\partial\lambda_2}\Big)\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \\ &= \iint\limits_\Sigma \epsilon_{ijk}A_{k,j}\,\mathrm{d}S_i\end{aligned}$$

and

$$\int_{\partial\Sigma}\alpha = \oint_C A_i\,\mathrm{d}x^i = \oint_C A_1\,\mathrm{d}x + A_2\,\mathrm{d}y + A_3\,\mathrm{d}z.$$

This can be expressed in the familiar form of Stokes' theorem

$$\iint\limits_\Sigma (\nabla \times \mathbf{A})\cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{A}\cdot \mathrm{d}\mathbf{r}.$$

Exercise: Show that $\mathrm{d}S_i$ is 'normal' to the surface S in the sense that $\mathrm{d}S_i\,\partial x^i/\partial\lambda_a = 0$ for $a = 1, 2$.

Problems

Problem 17.4 Let $\alpha = y^2\,\mathrm{d}x + x^2\,\mathrm{d}y$. If γ_1 is the stretch of y-axis from $(x = 0, y = -1)$ to $(x = 0, y = 1)$, and γ_2 the unit right semicircle connecting these points, evaluate

$$\int_{\gamma_1}\alpha, \quad \int_{\gamma_2}\alpha \quad \text{and} \quad \int_{S^1}\alpha.$$

Verify Stokes' theorem for the unit circle and the unit right semicircular region encompassed by γ_1 and γ_2.

Problem 17.5 If $\alpha = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y$ compute $\int_{\partial\Omega}\alpha$ where Ω is (i) the unit cube, (ii) the unit ball in $\mathbb{R}^3$. In each case verify Stokes' theorem,

$$\int_{\partial\Omega}\alpha = \int_\Omega \mathrm{d}\alpha.$$

Problem 17.6 Let S be the surface of a cylinder of elliptical cross-section and height $2h$ given by

$$x = a\cos\theta, \quad y = b\sin\theta \quad (0 \le \theta < 2\pi), \qquad -h \le z \le h.$$

(a) Compute $\int_S \alpha$ where $\alpha = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x - 2z\,\mathrm{d}x \wedge \mathrm{d}y$.
(b) Show $\mathrm{d}\alpha = 0$, and find a 1-form ω such that $\alpha = \mathrm{d}\omega$.
(c) Verify Stokes' theorem $\int_S \alpha = \int_{\partial S}\omega$.

Problem 17.7 A torus in $\mathbb{R}^3$ may be represented parametrically by

$$x = \cos\phi(a + b\cos\psi), \qquad y = \sin\phi(a + b\cos\psi), \qquad z = b\sin\psi$$

where $0 \le \phi < 2\pi$, $0 \le \psi < 2\pi$. If b is replaced by a variable ρ that ranges from 0 to b, show that

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \rho(a + \rho\cos\psi)\,\mathrm{d}\phi \wedge \mathrm{d}\psi \wedge \mathrm{d}\rho.$$

By integrating this 3-form over the region enclosed by the torus, show that the volume of the solid torus is $2\pi^2 ab^2$. Can you see this by a simple geometrical argument?

Evaluate the volume by performing the integral of the 2-form $\alpha = x\,\mathrm{d}y \wedge \mathrm{d}z$ over the surface of the torus and using Stokes' theorem.

Problem 17.8 Show that in n dimensions, if V is a regular n-domain with boundary $S = \partial V$, and we set α to be an $(n-1)$-form with components

$$\alpha = \sum_{i=1}^{n} (-1)^{i+1} A^i\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \dots \wedge \mathrm{d}x^n,$$

Stokes' theorem can be reduced to the *n-dimensional Gauss theorem*

$$\int \!\!\cdots_{V}\!\! \int A^i{}_{,i}\,\mathrm{d}x^1 \dots \mathrm{d}x^n = \int \!\!\cdots_{S}\!\! \int A^i\,\mathrm{d}S_i$$

where $\mathrm{d}S_i = \mathrm{d}x^1 \dots \mathrm{d}x^{i-1}\,\mathrm{d}x^{i+1} \dots \mathrm{d}x^n$ is a 'vector volume element' normal to S.

17.4 Homology and cohomology

In the previous section we considered regions that could be subdivided into 'cubical' parts. While this has practical advantages when it comes to integration, and makes the proof of Stokes' theorem relatively straightforward, the subject of *homology* is more standardly based on triangular cells. There is no essential difference in this change since any k-cube is readily triangulated, as well as the converse. For example, a triangle in two dimensions is easily divided into squares (see Fig. 17.4). Dividing a tetrahedron into four cubical regions is harder to visualize, and is left as an excercise for the reader.

Ordered simplices and chains in Euclidean space

A set of points $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\}$ in Euclidean space $\mathbb{R}^n$ is said to be **independent** if the p vectors $\mathbf{x}_i - \mathbf{x}_0$ $(i = 1, \dots, p)$ are linearly independent. The **ordered p-simplex** with these points as **vertices** consists of their convex hull,

$$\langle \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p \rangle = \left\{ \mathbf{y} = \sum_{i=0}^{p} t^i \mathbf{x}_i \;\middle|\; \text{all } t^j \ge 0 \text{ and } \sum_{i=0}^{p} t^i = 1 \right\},$$

together with a specific ordering $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p)$ of the vertex points. We will often denote an ordered r-simplex by a symbol such as Δ or Δ_p. Two ordered p-simplices with the

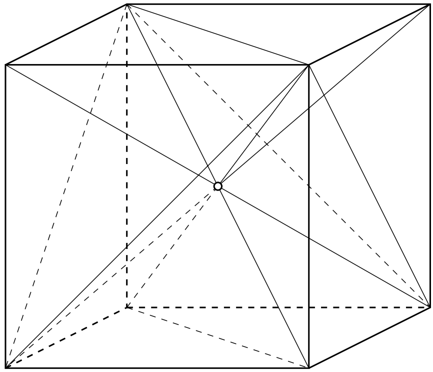

Figure 17.4 Dividing the triangular into 'cubical' cells

same set of vertices will be taken to be identical if the orderings are related by an even permutation, else they are given the opposite sign. For example,

$$\langle \mathbf{x}, \mathbf{y}, \mathbf{z}\rangle = -\langle \mathbf{x}, \mathbf{z}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{z}, \mathbf{x}\rangle.$$

The **standard n-simplex** on $\mathbb{R}^n$ is $\bar{\Delta}_n = \langle \mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_n\rangle$ where $\mathbf{e}_i$ is the ith basis vector $\mathbf{e}_i = (0, 0, \dots, 1, \dots, 0)$, i.e.

$$\bar{\Delta}_n = \{(t_1, t_2, \dots, t_n) \mid \textstyle\sum_{j=1}^n t_j = 1 \text{ and } 0 \le t_i \le 1\ (i = 1, \dots, n)\} \subset \mathbb{R}^n.$$

A 0-simplex $\langle \mathbf{x}_0\rangle$ is a single point $\mathbf{x}_0$ together with a plus or minus sign.

A 1-simplex $\langle \mathbf{x}_0, \mathbf{x}_1\rangle$ is a closed directed line from $\mathbf{x}_0$ to $\mathbf{x}_1$.

A 2-simplex $\langle \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\rangle$ is an oriented triangle, where the vertices are taken in a definite order.

A 3-simplex is a tetrahedron in which the vertices are again given a specific order up to even permutations. These examples are depicted in Fig. 17.5.

A p-**chain** is a formal sum $C = \sum_{\mu=1}^M c^\mu \Delta_\mu$ where a_μ are real numbers and Δ_μ are p-simplices in $\mathbb{R}^n$. The set of all p-chains on $\mathbb{R}^n$ is obviously a vector space, denoted $\mathcal{C}_p(\mathbb{R}^n)$.

The ith face of a p-simplex $\Delta = \langle \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\rangle$ is defined as the ordered $(p-1)$-simplex

$$\Delta_i = (-1)^i \langle \mathbf{x}_0, \dots, \widehat{\mathbf{x}_i}, \dots, \mathbf{x}_p\rangle \equiv (-1)^i \langle \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_p\rangle,$$

and the **boundary** of a p-simplex Δ is defined as the $(p-1)$-chain

$$\partial\Delta = \sum_{i=0}^{p} \Delta_i = \sum_{i=0}^{p} (-1)^i \langle \mathbf{x}_0, \dots, \widehat{\mathbf{x}_i}, \dots, \mathbf{x}_p\rangle,$$

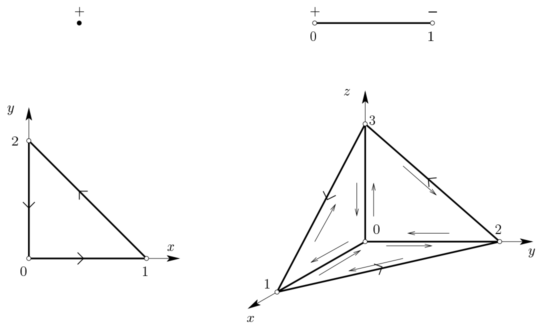

Figure 17.5 Standard low dimensional p-simplexes

and extends to all p-chains by linearity. For example,

$$\begin{aligned}
\partial\langle \mathbf{x}_0, \mathbf{x}_1\rangle &= \langle \mathbf{x}_1\rangle - \langle \mathbf{x}_0\rangle,\\
\partial\langle \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\rangle &= \langle \mathbf{x}_1, \mathbf{x}_2\rangle - \langle \mathbf{x}_0, \mathbf{x}_2\rangle + \langle \mathbf{x}_0, \mathbf{x}_1\rangle,\\
\partial\langle \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\rangle &= \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\rangle - \langle \mathbf{x}_0, \mathbf{x}_2, \mathbf{x}_3\rangle + \langle \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_3\rangle - \langle \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\rangle.
\end{aligned}$$

For each $p = 0, \dots, n$ the boundary operator generates a linear map $\partial : \mathcal{C}_p(\mathbb{R}^n) \to \mathcal{C}_{p-1}(\mathbb{R}^n)$ by setting

$$\partial\left(\sum_\mu c^\mu \Delta_\mu\right) = \sum_\mu c^\mu \partial \Delta_\mu.$$

If we set the boundary of any 0-form to be the zero chain, $\partial\langle \mathbf{x}\rangle = 0$ it is trivial to see that two successive applications of the boundary operator on any 1-simplex vanishes,

$$\partial^2\langle \mathbf{x}_0, \mathbf{x}_1\rangle = \partial(\langle \mathbf{x}_1\rangle - \langle \mathbf{x}_0\rangle) = 0 - 0 = 0.$$

This identity generalizes to arbitrary p-simplices, for

$$\begin{aligned}
\partial\partial\langle \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\rangle &= \sum_{i=0}^{p} \partial(-1)^i \langle \mathbf{x}_0, \dots, \widehat{\mathbf{x}_i}, \dots, \mathbf{x}_p\rangle\\
&= \sum_{i=0}^{p}(-1)^i \Big[\sum_{j<i}(-1)^j \langle \mathbf{x}_0, \dots, \widehat{\mathbf{x}_j}, \dots, \widehat{\mathbf{x}_i}, \dots, \mathbf{x}_p\rangle\\
&\qquad + \sum_{j>i}(-1)^{j+1}\langle \mathbf{x}_0, \dots, \widehat{\mathbf{x}_i}, \dots, \widehat{\mathbf{x}_j}, \dots, \mathbf{x}_p\rangle\Big]\\
&= 0
\end{aligned}$$

since all terms cancel in pairs. The identity $\partial^2 = 0$ follows from the linearity of the boundary operator ∂ on $\mathcal{C}_p(\mathbb{R}^n)$.

Exercise: Write out the cancellation of terms in this argument explicitly for a 2-simplex $\langle \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2 \rangle$ and 3-simplex $\langle \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$.

A p-chain C is said to be a **cycle** if it has no boundary, $\partial C = 0$. It is said to be a **boundary** if there exists a $(p+1)$-chain C' such that $C = \partial C'$. Clearly every boundary is a p-chain since $\partial C = \partial^2 C' = 0$, but the converse need not be true.

Simplicial homology on manifolds

Let M be an n-dimensional differentiable manifold. A **(singular)** p**-simplex** σ_p on M is a smooth map $\phi : U \to M$ where U is an open subset of $\mathbb{R}^p$ containing the standard p-simplex $\bar{\Delta}_p$. A p**-chain on** M is a formal linear combination of p-simplices on M,

$$C = \sum_{\mu=1}^{M} c^\mu \sigma_{p\mu} \quad (c^\mu \in \mathbb{R}),$$

and let $\mathcal{C}_p(M)$ be the real vector space generated by all p-simplices on M.

For each $i = 1, 2, \dots, p-1$ denote by $\varphi_i : \mathbb{R}^{p-1} \to \mathbb{R}^p$ the map that embeds $\mathbb{R}^{p-1}$ into the plane $x^i = 0$ of $\mathbb{R}^p$,

$$\varphi_i(x^1, \dots, x^{p-1}) = (x^1, \dots, x^i, 0, x^{i+1}, \dots, x^{p-1}),$$

and for $i = 0$ set

$$\varphi_0(x^1, \dots, x^{p-1}) = \big(1 - \textstyle\sum_{i=1}^{p-1} x^i,\ x^1, \dots, x^{p-1}\big).$$

The maps $\varphi_0, \varphi_1, \dots, \varphi_{p-1}$ are $(p-1)$-simplices in $\mathbb{R}^p$, whose supports are the various faces of the standard p-simplex, $\bar{\Delta}_p$,

$$\varphi_i(\bar{\Delta}_{p-1}) = \bar{\Delta}_{pi} \quad (i = 0, 1, \dots, p-1).$$

If σ is a p-simplex in M, define its ith face to be the $(p-1)$-simplex

$$\sigma_i = \sigma \circ \varphi_i : \bar{\Delta}_{p-1} \to M,$$

and its **boundary** to be the $(p-1)$-chain

$$\partial_p \sigma = \sum_{i=0}^{p-1} (-1)^i \sigma_i .$$

Extend by linearity to all chains $C \in \mathcal{C}_p(M)$,

$$\partial_p C = \partial \sum_{\mu=1}^{M} c^\mu \sigma_{p\mu} = \sum_{\mu=1}^{M} c^\mu \partial \sigma_{p\mu}.$$

A p**-boundary** B is a singular p-cycle on M that is the boundary of a $(p+1)$-chain, $B = \partial_{p+1} C$. A p**-cycle** C is a singular p-chain on M whose boundary vanishes, $\partial_p C = 0$. Since $\partial^2 = 0$ it is clear that every p-boundary is a p-cycle.

If we let $B_p(M)$ be the set of all p-boundaries on M, and $Z_p(M)$ all p-cycles, these are both vector subspaces of $\mathcal{C}_p(M)$:

$$B_p(M) = \operatorname{im} \partial_{p+1},$$
$$Z_p(M) = \ker \partial_p \supseteq B_p(M).$$

We define the p**th homology space** to be the factor space

$$H_p(M) = Z_p(M)/B_p(M).$$

Commonly this is called the *pth homology group*, only the abelian group property being relevant. Two cycles C_1 and C_2 are said to be **homologous** if they belong to the same homology class – that is, if there exists a chain C such that $C_1 - C_2 = \partial C$. The dimension of the pth homology space is known as the p**th Betti number**,

$$b_p = \dim H_p(M),$$

and the quantity

$$\chi(M) = \sum_{p=0}^{n} (-1)^p b_p$$

is known as the **Euler characteristic** of the manifold M. A non-trivial result that we shall not attempt to prove is that the Betti numbers are topological invariants – two manifolds that are topologically homeomorphic have the same Betti numbers and Euler characteristic [13].

Example 17.4 Since $\partial\langle\mathbf{0}\rangle = \mathbf{0}$ every 0-simplex in a manifold M has boundary 0. Hence every 0-chain in M is a 0-cycle, and $Z_0(M) = \mathcal{C}_0(M)$. The zeroth homology space $H_0(M) = Z_0(M)/B_0(M)$ counts the number of 0-chains that are not boundaries of 1-chains. Since a 1-simplex is essentially a smooth curve $\sigma : [0, 1] \to M$ it has boundary $\sigma(1) - \sigma(0)$, where we represent the 0-simplex map $\mathbf{0} \to p \in M$ simply by its image point p. Two 0-simplices p and q are homologous if $p - q$ is a boundary; that is, if they are the end points of a smooth curve connecting them. This is true if and only if they belong to the same connected component of M. Thus $H_0(M)$ is spanned by a set of simplices $\{p_0, p_1, \dots\}$, one from each connected component of M, and the zeroth Betti number β_0 is the number of connected components of the topological space M.

De Rham cohomology groups and duality

Let $\mathcal{C}^r(M) = \Lambda_r(M)$ be the real vector space consisting of all differential r-forms on M. Its elements are also known as r**-cochains** on M. The exterior derivative d is a linear operator $\mathrm{d} : \mathcal{C}^r(M) \to \mathcal{C}^{r+1}(M)$ for each $r = 0, 1, \dots, n$, with the property $\mathrm{d}^2 = 0$. We write its restriction to $\mathcal{C}^r(M)$ as d_r.

A differential r-form α is said to be **closed** if $\mathrm{d}\alpha = 0$, and it is said to be **exact** if there exists an $(r-1)$-form β such that $\alpha = \mathrm{d}\beta$. Clearly every exact r-form is closed since $\mathrm{d}^2\beta = 0$. In the language of cochains these definitions can be expressed as follows: an r-cochain α is called an r**-cocycle** if it is a closed differential form, while it is an r**-coboundary** if it is exact. We denoted the vector subspace of r-cocycles by $Z^r(M)$, and the subspace of r-coboundaries by $B^r(M)$

$$Z^r(M) = \{\alpha \in \mathcal{C}^r(M) \mid \mathrm{d}\alpha = 0\} = \ker \mathrm{d}_r \subset \mathcal{C}^r(M)$$
$$B^r(M) = \{\alpha \in \mathcal{C}^r(M) \mid \alpha = \mathrm{d}\beta,\ \beta \in \mathcal{C}^{r-1}(M)\} = \operatorname{im} \mathrm{d}_{r-1} \subset Z^r(M).$$

The r**th de Rham cohomology space (group)** is defined as the factor space $H^r(M) = Z^r(M)/B^r(M)$, and any two r-cocycles α and β are said to be **cohomologous** if they belong to the same coset, $\alpha - \beta = \mathrm{d}\gamma$ for some $(r-1)$-cochain γ. The dimensions of the vector spaces $H^r(M)$ are denoted b^r.

Example 17.5 Since there are no differential forms of degree -1 we always set $B^0(M) = 0$. Hence $H^0(M) = Z^0(M)$. A 0-form f is closed and belongs to $Z^0(M)$ if and only if $\mathrm{d}f = 0$. Hence $f = \text{const.}$ on each connected component of M, and $H^0(M) = \mathbb{R} + \mathbb{R} + \cdots + \mathbb{R}$, one contribution from each such component. Hence $b^0 = b_0$ is the number of connected components of M (see Example 17.4).

Example 17.6 If $M = \mathbb{R}$, then from the previous example $H^0(\mathbb{R}) = \mathbb{R}$. A 1-form $\omega \in \mathcal{C}^1(M)$ is closed if $\mathrm{d}\omega = 0$. Setting $\omega = f(x)\,\mathrm{d}x$ we can clearly always write

$$\omega = \mathrm{d}f = \frac{\mathrm{d}F}{x} \quad \text{where} \quad F(x) = \int_0^x f(y)\,\mathrm{d}y.$$

Hence every closed 1-form is exact and $H^1(\mathbb{R}) = 0$, $b^1 = 0$. It is not difficult to verify that this is also the value of the Betti numbers, $b_1 = 0$.

Define a bracket $\langle\, , \rangle : \mathcal{C}_r(M) \times \mathcal{C}^r(M) \to \mathbb{R}$ by setting

$$\langle C, \alpha \rangle = \int_C \alpha.$$

for every r-chain $C \in \mathcal{C}_r(M)$ and r-cochain $\alpha \in \mathcal{C}^r(M) = \Lambda_r(M)$. For every α the map $C \mapsto \langle C, \alpha \rangle$ is evidently linear on $\mathcal{C}_r(M)$ and for every $C \in \mathcal{C}_r(M)$ the map $\alpha \mapsto \langle C, \alpha \rangle$ is linear on $\mathcal{C}^r(M)$. By Stokes' theorem the exterior derivative d is the adjoint of the boundary operator ∂ with respect to this bracket, in the sense that

$$\langle C, \mathrm{d}\alpha \rangle = \langle \partial C, \alpha \rangle.$$

The bracket induces a bracket $\langle\, , \rangle$ on $H_r(M) \times H^r(M)$ by setting

$$\langle [C], [\alpha] \rangle = \langle C, \alpha \rangle = \int_C \alpha$$

for any pair $[C] \in H_r(M)$ and $[\alpha] \in H^r(M)$. It is independent of the choice of representative from the homology and cohomology classes, for if $C' = C + \partial C_1$ and $\alpha' = \alpha + \mathrm{d}\alpha_1$, where

$\partial C = 0$ and $d\alpha = 0$, then

$$\begin{aligned}\langle C', \alpha' \rangle &= \int_{C+\partial C_1} \alpha + d\alpha_1 \\ &= \int_C \alpha + \int_{\partial C_1} \alpha + \int_C d\alpha_1 + \int_{\partial C_1} d\alpha_1 \\ &= \int_C \alpha + \int_{C_1} d\alpha + \int_{\partial C} \alpha_1 + \int_{\partial^2 C_1} \alpha_1 \quad \text{by Stokes' theorem} \\ &= \int_C \alpha = \langle C, \alpha \rangle.\end{aligned}$$

For any fixed r-cohomology class $[\alpha]$, the map $f_{[\alpha]} : H_p(M) \to \mathbb{R}$ given by

$$f_{[\alpha]}([C]) = \langle [C], [\alpha] \rangle$$

is a well-defined linear functional on $H_r(M)$. De Rham's theorem asserts that this correspondence between linear functionals on $H_r(M)$ and cohomology classes $[\alpha] \in H^r(M)$ is bijective.

Theorem 17.4 (de Rham) *The bilinear map on $H_r(M) \times H^r(M) \to \mathbb{R}$ defined by $([C], [\alpha]) \mapsto \langle [C], [\alpha] \rangle$ is non-degenerate in both arguments. That is, every linear functional on $H_r(M)$ has the form $f_{[\alpha]}$ for a uniquely defined r-cohomology class $[\alpha]$.*

The proof lies beyond the scope of this book, and may be found in [6, 10]. There are a variety of ways of expressing de Rham's theorem. Essentially it says that the rth cohomology group is isomorphic with the dual space of the rth homology group,

$$H^r(M) \cong \big(H_r(M)\big)^*.$$

If the Betti numbers are finite then $b^r = b_r$.

The integral of a closed r-form α over an r-cycle C

$$\langle C, \alpha \rangle = \int_C \alpha$$

is sometimes called a **period** of α. By Stokes' theorem all periods of α vanish if α is an exact form, and the period of any closed r-form vanishes over a boundary r-cycle $C = \partial C'$. Let $C_1, \dots, C_k$ be $k = b_r$ linearly independent cycles in $Z_r(M)$, such that $[C_i] \neq [C_j]$ for $i \neq j$. De Rham's theorem implies that an r-form α is exact if and only if all the periods $\int_{C_i} \alpha = 0$. If α is exact then we have already remarked that all its periods vanish. The converse follows from the fact that $\langle [C], [\alpha] \rangle = 0$ for every $[C] \in H_r(M)$, since $[C]$ can be expanded to $[C] = \sum_{i=1}^k [C_i]$. By non-degeneracy of the product $\langle\, ,\, \rangle$ we must have $[\alpha] = 0$, so that $\alpha = d\beta$ for some $(r-1)$-form β.

Problems

Problem 17.9 Show that any tetrahedron may be divided into 'cubical' regions. Describe a procedure for achieving the same result for a general k-simplex.

Problem 17.10 For any pair of subspaces H and K of the exterior algebra $\Lambda^*(M)$, set $H \wedge K$ to be the vector subspace spanned by all $\alpha \wedge \beta$ where $\alpha \in H, \beta \in K$. Show that

(a) $Z^p(M) \wedge Z^q(M) \subseteq Z^{p+q}(M)$,

(b) $Z^p(M) \wedge B^q(M) \subseteq B^{p+q}(M)$,

(c) $B^p(M) \wedge B^q(M) \subseteq B^{p+q}(M)$.

Problem 17.11 Show that for any set of real numbers $a_1, \dots, a_k$ there exists a closed r-form α whose periods $\int_{C_i} \alpha = a_i$.

Problem 17.12 If S^1 is the unit circle, show that $b^0 = b^1 = 1$.

17.5 The Poincaré lemma

The fact that every exact differential form is closed has a kind of local converse.

Theorem 17.5 (Poincaré lemma) *On any open set $U \subseteq M$ homeomorphic to $\mathbb{R}^n$, every closed differential form of degree $k \geq 1$ is exact: if $\mathrm{d}\alpha = 0$ on U where $\alpha \in \Lambda_k(U)$, then there exists a $(k-1)$-form β on U such that $\alpha = \mathrm{d}\beta$.*

Proof: We prove the theorem on $\mathbb{R}^n$ itself, with coordinates $x^1, \dots, x^n$, and set $\alpha = \alpha_{i_1 i_2 \dots i_k}(\mathbf{x})\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \dots \wedge \mathrm{d}x^{i_k}$. Let α^t $(0 \leq t \leq 1)$ be the one-parameter family of k-forms

$$\alpha^t = \alpha_{i_1 \dots i_k}(t\mathbf{x})\, \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}.$$

The map $h_k : \Lambda_k(\mathbb{R}^n) \to \Lambda_{k-1}(\mathbb{R}^n)$ defined by

$$h_k \alpha = \int_0^1 t^{k-1} \mathrm{i}_X \alpha^t \, \mathrm{d}t \quad \text{where} \quad X = x^i \frac{\partial}{\partial x^i}$$

satisfies the key identity

$$(\mathrm{d} \circ h_k + h_{k+1} \circ \mathrm{d})\alpha = \alpha \tag{17.4}$$

for any k-form α on $\mathbb{R}^n$. To prove (17.4) write out the left-hand side,

$$(\mathrm{d} \circ h_k + h_{k+1} \circ \mathrm{d})\alpha = \int_0^1 t^{k-1} \mathrm{d}\mathrm{i}_X \alpha^t + t^k \mathrm{i}_X (\mathrm{d}\alpha)^t \mathrm{d}t,$$

and

$$\begin{aligned} \mathrm{d}\alpha^t &= \frac{\partial \alpha_{i_1 \dots i_k}(t\mathbf{x})}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k} \\ &= t(\mathrm{d}\alpha)^t. \end{aligned}$$

Using the Cartan identity, Eq. (16.13),

$$\begin{aligned} (\mathrm{d} \circ h_k + h_{k+1} \circ \mathrm{d})\alpha &= \int_0^1 t^{k-1} \big(\mathrm{d} \circ \mathrm{i}_X + \mathrm{i}_X \circ \mathrm{d}\big) \alpha^t \mathrm{d}t \\ &= \int_0^1 t^{k-1} \mathcal{L}_X \alpha^t \mathrm{d}t \end{aligned}$$

and from the component formula for the Lie derivative (15.39),

$$\begin{aligned}(\mathcal{L}_X\alpha^t)_{i_1\dots i_k} &= \frac{\partial \alpha_{i_1\dots i_k}(t\mathbf{x})}{\partial x^j}x^j + \frac{\partial x^j}{\partial x^{i_1}}\alpha_{ji_2\dots i_k}(t\mathbf{x}) + \dots + \frac{\partial x^j}{\partial x^{i_k}}\alpha_{i_1\dots j}(t\mathbf{x}) \\ &= t\frac{\partial \alpha_{i_1\dots i_k}(t\mathbf{x})}{\partial t x^j}\frac{\mathrm{d}tx^j}{\mathrm{d}t} + \delta^j_{i_1}\alpha_{j\dots i_k}(t\mathbf{x}) + \dots + \delta^j_{x^{i_k}}\alpha_{i_1\dots j}(t\mathbf{x}) \\ &= t\frac{\mathrm{d}\alpha_{i_1\dots i_k}(t\mathbf{x})}{\mathrm{d}t} + k\alpha_{i_1\dots i_k}(t\mathbf{x}).\end{aligned}$$

Hence

$$\mathcal{L}_X\alpha^t = t\frac{\mathrm{d}\alpha^t}{\mathrm{d}t} + k\alpha^t,$$

and Eq. (17.4) follows from

$$(\mathrm{d}\circ h_k + h_{k+1}\circ \mathrm{d})\alpha = \int_0^1 t^k\frac{\mathrm{d}\alpha^t}{\mathrm{d}t} + kt^{k-1}\alpha^t\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t^k\alpha^t}{\mathrm{d}t}\mathrm{d}t = \alpha.$$

If $\mathrm{d}\alpha = 0$ we have $\alpha = \mathrm{d}\beta$ where $\beta = h_k\alpha$, and the theorem is proved. ■

An immediate corollary of this theorem and de Rham's theorem is that all homology groups $H_k(\mathbb{R}^n)$ are trivial for $k \geq 1$; that is, all Betti numbers for $k \geq 1$ vanish in Euclidean space, $b_k = b^k = 0$. Of course $b_0 = b^0 = 1$ since there is a single connected component.

Example 17.7 In $\mathbb{R}^3$ let α be the 1-form $\alpha = A_1\,\mathrm{d}x^1 + A_2\,\mathrm{d}x^2 + A_3\,\mathrm{d}x^3$. Its exterior derivative is

$$\mathrm{d}\alpha = (A_{2,1} - A_{1,2})\,\mathrm{d}x^1\wedge\mathrm{d}x^2 + (A_{1,3} - A_{3,1})\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + (A_{3,2} - A_{2,3})\,\mathrm{d}x^2\wedge\mathrm{d}x^3$$

and Poincaré's lemma asserts that $\mathrm{d}\alpha = 0$ if and only if there exists a function f on $\mathbb{R}^3$ such that $\alpha = \mathrm{d}f$. In components,

$$A_{2,1} - A_{1,2} = A_{1,3} - A_{3,1} = A_{3,2} - A_{2,3} = 0 \iff A_1 = f_{,1},\quad A_2 = f_{,2},\quad A_3 = f_{,3},$$

or in standard 3-vector language, with $\mathbf{A} = (A_1,\ A_2,\ A_3)$,

$$\nabla\times\mathbf{A} = 0 \iff \mathbf{A} = \nabla f.$$

If α is the differential 2-form $\alpha = A_3\,\mathrm{d}x^1\wedge\mathrm{d}x^2 + A_2\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + A_1\,\mathrm{d}x^2\wedge\mathrm{d}x^3$, then

$$\mathrm{d}\alpha = (A_{1,1} + A_{2,2} + A_{3,3})\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3.$$

The Poincaré lemma says

$$\mathrm{d}\alpha = 0 \iff \alpha = \mathrm{d}\beta \quad\text{where}\quad \beta = B_1\,\mathrm{d}x^1 + B_2\,\mathrm{d}x^2 + B_3\,\mathrm{d}x^3,$$

or in components

$$A_{1,1} + A_{2,2} + A_{3,3} = 0 \iff A_1 = B_{3,2} - B_{2,3},\quad A_2 = B_{1,3} - B_{3,1},\quad A_3 = B_{2,1} - B_{1,2},$$

which reduces to the familiar 3-vector statement

$$\nabla\cdot\mathbf{A} = 0 \iff \text{there exists } \mathbf{B} \text{ such that } \mathbf{A} = \nabla\times\mathbf{B}.$$

Example 17.8 On the manifold $M = \mathbb{R}^2 - \{0\}$ with coordinates $x^1 = x,\ x^2 = y$, let ω be the differential 1-form

$$\omega = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2},$$

which cannot be extended smoothly to a 1-form on all of $\mathbb{R}^2$ because of the singular behaviour at the origin. On M, however, it is closed since

$$\begin{aligned}\mathrm{d}\omega &= \frac{-\mathrm{d}y \wedge \mathrm{d}x}{x^2 + y^2} + \frac{2y^2\,\mathrm{d}y \wedge \mathrm{d}x}{(x^2 + y^2)^2} + \frac{\mathrm{d}x \wedge \mathrm{d}y}{x^2 + y^2} - \frac{2x^2\,\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2)^2}\\ &= \frac{2\mathrm{d}x \wedge \mathrm{d}y}{x^2 + y^2} - \frac{2(x^2 + y^2)}{(x^2 + y^2)^2}\mathrm{d}x \wedge \mathrm{d}y = 0.\end{aligned}$$

Locally it is possible everywhere to find a function f such that $\omega = \mathrm{d}f$. For example, it is straightforward to verify that the pair of differential equations

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2 + y^2}, \qquad \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}$$

has a solution $f = \arctan(y/x)$. However f is not globally defined on M, since it is essentially the polar angle given by $x = r\cos f,\ y = r\sin f$ and increases by 2π on any circuit of the origin beginning at the positive branch of the y-axis. This demonstrates that Poincaré's lemma does not in general hold on manifolds not homeomorphic with $\mathbb{R}^n$.

Electrodynamics

An electromagnetic field is represented by an antisymmetric 4-tensor field F in Minkowski space, having components $F_{\mu\nu}(x^\alpha)$ $(\mu, \nu = 1, \dots, 4)$ (see Chapter 9). Define the **Maxwell 2-form** φ as having components $F_{\mu\nu}$,

$$\begin{aligned}\varphi &= F_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\\ &= 2\big(B_3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + B_2\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_1\,\mathrm{d}x^2 \wedge \mathrm{d}x^3\\ &\qquad + E_1\,\mathrm{d}x^1 \wedge \mathrm{d}x^4 + E_2\,\mathrm{d}x^2 \wedge \mathrm{d}x^4 + E_3\,\mathrm{d}x^3 \wedge \mathrm{d}x^4\big)\end{aligned}$$

where $\mathbf{E} = (E_1,\ E_2,\ E_3)$ is the electric field, $\mathbf{B} = (B_1,\ B_2,\ B_3)$ the magnetic field and $x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = ct$ are inertial coordinates. The source-free Maxwell equations (9.37) can be written

$$\begin{aligned}\mathrm{d}\varphi = 0 \iff\ & F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0\\ \iff\ & \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0.\end{aligned}$$

By the Poincaré lemma, there exists a 1-form α, known as the 4-vector potential, such that $\varphi = \mathrm{d}\alpha$. Writing the components of α as $(A_1,\ A_2,\ A_3,\ -\phi)$ this equation reads

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi.$$

To express the equations relating the electromagnetic field to its sources in terms of differential forms we must define the **dual Maxwell 2-form** $*\varphi = *F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ where

$*F_{\mu\nu}$ is defined as in Example 8.8,

$$*\varphi = 2\big(-E_3\,\mathrm{d}x^1\wedge\mathrm{d}x^2 - E_2\,\mathrm{d}x^3\wedge\mathrm{d}x^1 - E_1\,\mathrm{d}x^2\wedge\mathrm{d}x^3 \\ + B_1\,\mathrm{d}x^1\wedge\mathrm{d}x^4 + B_2\,\mathrm{d}x^2\wedge\mathrm{d}x^4 + B_3\,\mathrm{d}x^3\wedge\mathrm{d}x^4\big).$$

The distribution of electric charge present is represented by a 4-current vector field $J = J^\mu e_\mu$ having components $J^\mu = (\mathbf{j},\ \rho c)$ where $\rho(\mathbf{r},\ t)$ is the charge density and $\mathbf{j}$ the charge flux density (see Section 9.4).

$$\begin{aligned}\vartheta &= *J_{\mu\nu\rho}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho \\ &= -\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}J^\sigma\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho \\ &= -c\rho\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + J^1\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4 \\ &\quad + J^2\,\mathrm{d}x^3\wedge\mathrm{d}x^1\wedge\mathrm{d}x^4 + J^3\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^4.\end{aligned}$$

Equations (9.38) may then be written as

$$\mathrm{d}*\varphi = -\vartheta \iff \nabla\cdot\mathbf{E} = 4\pi\rho, \qquad -\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j}.$$

Charge conservation follows from

$$\mathrm{d}\vartheta = -\mathrm{d}^2*\varphi = 0 \iff \nabla\cdot\mathbf{j} + \frac{1}{c}\frac{\partial\rho}{\partial t} = 0.$$

Example 17.9 Although Maxwell's vacuum equations take on the deceptively symmetrical form

$$\mathrm{d}\varphi = 0, \qquad \mathrm{d}*\varphi = 0$$

we cannot assume that $*\varphi = \mathrm{d}\beta$ for a globally defined 1-form β. For example, the coulomb field

$$\mathbf{B} = \mathbf{0}, \qquad \mathbf{E} = \Big(\frac{qx}{r^3},\ \frac{qy}{r^3},\ \frac{qz}{r^3}\Big)$$

corresponds to the Maxwell 2-form

$$\varphi = \frac{2q}{r^3}\big(x\,\mathrm{d}x\wedge\mathrm{d}x^4 + y\,\mathrm{d}y\wedge\mathrm{d}x^4 + z\,\mathrm{d}z\wedge\mathrm{d}x^4\big) = \frac{2q}{r^2}\mathrm{d}r\wedge\mathrm{d}x^4$$

where $r^2 = x^2 + y^2 + z^2$, with dual 2-form

$$*\varphi = -\frac{2q}{r^3}\big(z\,\mathrm{d}x\wedge\mathrm{d}y + y\,\mathrm{d}z\wedge\mathrm{d}x + x\,\mathrm{d}y\wedge\mathrm{d}z\big).$$

This 2-form is, however, only defined on the subspace $M = \mathbb{R}^3 - \{0\}$. A short calculation in spherical polar coordinates results in

$$*\varphi = -2q\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi = \mathrm{d}(2q\cos\theta\,\mathrm{d}\phi) = \mathrm{d}(2q(\cos\theta - 1)\,\mathrm{d}\phi).$$

Either of the choices $\beta = 2q\cos\theta\,\mathrm{d}\phi$ or $\beta' = 2q(\cos\theta - 1)\mathrm{d}\phi$ will act as a potential 1-form for $*\varphi$, but neither is defined on all of M since the angular coordinate ϕ is not well-defined on the z-axis where $\theta = 0$ or π. The 1-form β is not well-defined on the entire z-axis, but the potential 1-form β' vanishes on the positive z-axis and has a singularity along the negative

z-axis. It is sometimes called a **Dirac string** – a term commonly reserved for solutions representing magnetic monopoles.

The impossibility of a global potential β can be seen by integrating $*\varphi$ over the unit 2-sphere

$$\int_{S^2} *\varphi = \int_0^{\pi}\int_0^{2\pi} -2q \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = -8\pi q,$$

and using Stokes' theorem (note that S^2 has no boundary)

$$\int_{S^2} *\varphi = \int_{S^2} \mathrm{d}\beta = \int_{\partial S^2} \beta = 0.$$

Problems

Problem 17.13 Let

$$\alpha = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}.$$

Show that α is a closed 1-form on $\mathbb{R}^2 - \{\mathbf{0}\}$. Compute its integral over the unit circle S^1 and show that it is not exact. What does this tell us of the de Rham cohomology of $\mathbb{R}^2 - \{\mathbf{0}\}$ and S^1?

Problem 17.14 Prove that every closed 1-form on S^2 is exact. Show that this statement does not extend to 2-forms by showing that the 2-form

$$\alpha = r^{-3/2}(x \, \mathrm{d}y \wedge \mathrm{d}z + y \, \mathrm{d}z \wedge \mathrm{d}x + z \, \mathrm{d}x \wedge \mathrm{d}y)$$

is closed, but has non-vanishing integral on S^2.

Problem 17.15 Show that the Maxwell 2-form satisfies the identities

$$\varphi \wedge *\varphi = *\varphi \wedge \varphi = 4(\mathbf{B}^2 - \mathbf{E}^2)\Omega$$
$$\varphi \wedge \varphi = - * \varphi \wedge *\varphi = 8\mathbf{B} \cdot \mathbf{E}\Omega$$

where $\Omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$.

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