10 Topology

Up till now we have focused almost entirely on the role of algebraic structures in mathematical physics. Occasionally, as in the previous chapter, it has been necessary to use some differential calculus, but this has not been done in any systematic way. Concepts such as *continuity* and *differentiability*, central to the area of mathematics known as *analysis*, are essentially geometrical in nature and require the use of *topology* for their rigorous definition. In broad terms, a *topology* is a structure imposed on a set to allow for the definition of *convergence* and *limits* of sequences or subsets. A space with a topology defined on it will be called a *topological space*, and a *continuous map* between topological spaces is one that essentially preserves limit points of subsets. The most general approach to this subject turns out to be through the concept of *open sets*.

Consider a two-dimensional surface S embedded in Euclidean three-dimensional space $\mathbb{E}^3$. In this case we have an intuitive understanding of a 'continuous deformation' of the surface as being a transformation of the surface that does not involve any tearing or pasting. Topology deals basically with those properties that are invariant under continuous deformations of the surface. Metric properties are not essential to the concept of continuity, and since operations such as 'stretching' are permissible, topology is sometimes called 'rubber sheet geometry'. In this chapter we will also define the concept of a *metric space*. Such a space always has a naturally defined topology associated with it, but the converse is not true in general – it is quite possible to define topology on a space without having a concept of distance defined on the space.

10.1 Euclidean topology

The archetypal model for a topological space is the real line and the Euclidean plane $\mathbb{R}^2$. On the real line $\mathbb{R}$, an **open interval** is any set $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. A set $U \subseteq \mathbb{R}$ is called a **neighbourhood** of $x \in \mathbb{R}$ if there exists $\epsilon > 0$ such that the open interval $(x - \epsilon,\ x + \epsilon)$ is a subset of U. We say a sequence of real numbers $\{x_n\}$ **converges to** $x \in \mathbb{R}$, written $x_n \to x$, if for every $\epsilon > 0$ there exists an integer $N > 0$ such that $|x - x_n| < \epsilon$ for all $n > N$; that is, for sufficiently large n the sequence x_n enters and stays in every neighbourhood U of x. The point x is then said to be the **limit** of the sequence $\{x_n\}$.

Exercise: Show that the limit of a sequence is unique: if $x_n \to x$ and $x_n \to x'$ then $x = x'$.

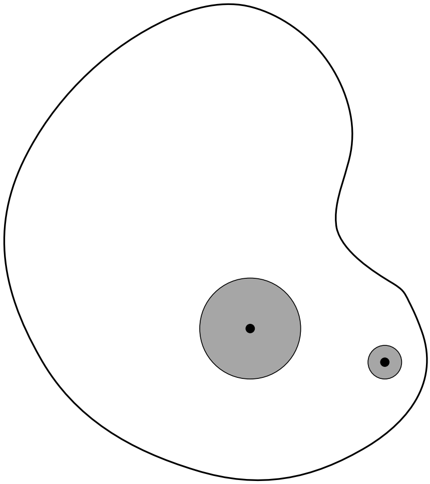

Figure 10.1 Points in an open set can be 'thickened' to an open ball within the set

Similar definitions apply to the Euclidean plane $\mathbb{R}^2$, where we set $|\mathbf{y} - \mathbf{x}| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$. In this case, open intervals are replaced by **open balls**

$$B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 \mid |\mathbf{y} - \mathbf{x}| < r\}$$

and a set $U \subseteq \mathbb{R}^2$ is said to be a **neighbourhood** of $\mathbf{x} \in \mathbb{R}^2$ if there exists a real number $\epsilon > 0$ such that the open ball $B_\epsilon(\mathbf{x}) \subseteq U$. A sequence of points $\{\mathbf{x}_n\}$ **converges to** $\mathbf{x} \in \mathbb{R}^2$, or $\mathbf{x}$ is the **limit** of the sequence $\{\mathbf{x}_n\}$, if for every $\epsilon > 0$ there exists an integer $N > 0$ such that

$$\mathbf{x}_n \in B_\epsilon(\mathbf{x}) \qquad \text{for all} \quad n > N.$$

Again we write $\mathbf{x}_n \to \mathbf{x}$, and the definition is equivalent to the statement that for every neighbourhood U of $\mathbf{x}$ there exists $N > 0$ such that $\mathbf{x}_n \in U$ for all $n > N$.

An **open set** U in $\mathbb{R}$ or $\mathbb{R}^2$ is a set that is a neighbourhood of every point in it. Intuitively, U is open in $\mathbb{R}$ (resp. $\mathbb{R}^2$) if every point in U can be 'thickened out' to an open interval (resp. open ball) within U (see Fig. 10.1). For example, the unit ball $B_1(\mathbf{O}) = \{\mathbf{y} \mid |\mathbf{y}|^2 < 1\}$ is an open set since, for every point $\mathbf{x} \in B_1(\mathbf{O})$ the open ball $B_\epsilon(\mathbf{x}) \subseteq B_1(\mathbf{O})$ where $\epsilon = 1 - |\mathbf{x}| > 0$.

On the real line it may be shown that the most general open set consists of a union of non-intersecting open intervals,

$$\dots, (a_{-1}, a_0), \quad (a_1, a_2), \quad (a_3, a_4), \quad (a_5, a_6), \dots$$

where $\dots a_{-1} < a_0 \le a_1 < a_2 \le a_3 < a_4 \le a_5 < a_6 \le \dots$ In $\mathbb{R}^2$ open sets cannot be so simply categorized, for while every open set is a union of open balls, the union need not be disjoint.

In standard analyis, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous at* x if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|y - x| < \delta \implies |f(y) - f(x)| < \epsilon.$$

Hence, for every $\epsilon > 0$, the inverse image set $f^{-1}(f(x) - \epsilon,\ f(x) + \epsilon)$ is a neighbourhood of x, since it includes an open interval $(x - \delta, x + \delta)$ centred on x. As every neighbourhood of $f(x)$ contains an interval of the form $(f(x) - \epsilon,\ f(x) + \epsilon)$ the function f is continuous at x if and only if the inverse image of every neighbourhood of $f(x)$ is a neighbourhood of x. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **continuous on** $\mathbb{R}$ if it is continuous at every point $x \in \mathbb{R}$.

Theorem 10.1 *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ if and only if the inverse image $V = f^{-1}(U)$ of every open set $U \subseteq \mathbb{R}$ is an open subset of $\mathbb{R}$.*

Proof: Let f be continuous on $\mathbb{R}$. Since an open set U is a neighbourhood of every point $y \in U$, its inverse image $V = f^{-1}(U)$ must be a neighbourhood of every point $x \in V$. Hence V is an open set.

Conversely let $f : \mathbb{R} \to \mathbb{R}$ be any function having the property that $V = f^{-1}(U)$ is open for every open set $U \subseteq \mathbb{R}$. Then for any $x \in \mathbb{R}$ and every $\epsilon > 0$ the inverse image under f of the open interval $(f(x) - \epsilon,\ f(x) + \epsilon)$ is an open set including x. It therefore contains an open interval of the form $(x - \delta,\ x + \delta)$, so that f is continuous at x. Since x is an arbitrary point, the function f is continuous on $\mathbb{R}$. ■

In general topology this will be used as the defining characteristic of a continuous map. In $\mathbb{R}^2$ the treatment is almost identical. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be *continuous at* $\mathbf{x}$ if for every $\epsilon > 0$ there exists a real number $\delta > 0$ such that

$$|\mathbf{y} - \mathbf{x}| < \delta \implies |f(\mathbf{y}) - f(\mathbf{x})| < \epsilon.$$

An essentially identical proof to that given in Theorem 10.1 shows that a function f is *continuous on* $\mathbb{R}^2$ if and only if the inverse image $f^{-1}(U)$ of every open set $U \subseteq \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$. The same applies to real-valued functions $f : \mathbb{R}^2 \to \mathbb{R}$. Thus continuity of functions can be described entirely by their inverse action on open sets. For this reason, open sets are regarded as the key ingredients of a topological space. Experience from Euclidean spaces and surfaces embedded in them has taught mathematicians that the most important properties of open sets can be summarized in a few simple rules, which are set out in the next section (see also [1–8]).

10.2 General topological spaces

Given a set X, a **topology** on X consists of a family of subsets $\mathcal{O}$, called **open sets**, which satisfy the following conditions:

(Top1) The empty set $\emptyset$ is open and the entire space X is open, $\{\emptyset, X\} \subset \mathcal{O}$.
(Top2) If U and V are open sets then so is their intersection $U \cap V$,

$$U \in \mathcal{O} \text{ and } V \in \mathcal{O} \implies U \cap V \in \mathcal{O}.$$

(Top3) If $\{V_i \mid i \in I\}$ is any family of open sets then their union $\bigcup_{i \in I} V_i$ is open.

Successive application of (Top2) implies that the intersection of any finite number of open sets is open, but $\mathcal{O}$ is not in general closed with respect to infinite intersections of open sets. On the other hand, $\mathcal{O}$ is closed with respect to arbitrary unions of open sets. The pair $(X, \mathcal{O})$, where $\mathcal{O}$ is a topology on X, is called a **topological space**. We often refer simply to a topological space X when the topology $\mathcal{O}$ is understood. The elements of the underlying space X are normally referred to as **points**.

Example 10.1 Define $\mathcal{O}$ to be the collection of subsets U of the real line $\mathbb{R}$ having the property that for every $x \in U$ there exists an open interval $(x - \epsilon,\ x + \epsilon) \subseteq U$ for some $\epsilon > 0$. These sets agree with the definition of open sets given in Section 10.1. The empty set is assumed to belong to $\mathcal{O}$ by default, while the whole line $\mathbb{R}$ is evidently open since every point lies in an open interval. Thus (Top1) holds for the family $\mathcal{O}$. To prove (Top2) let U and V be open sets such that $U \cap V \neq \emptyset$, the case where $U \cap V = \emptyset$ being trivial. For any $x \in U \cap V$ there exist positive numbers ϵ_1 and ϵ_2 such that

$$(x - \epsilon_1,\ x + \epsilon_1) \subseteq U \quad \text{and} \quad (x - \epsilon_2,\ x + \epsilon_2) \subseteq V.$$

If $\epsilon = \min(\epsilon_1, \epsilon_2)$ then $(x - \epsilon,\ x + \epsilon) \subseteq U \cap V$, hence $U \cap V$ is an open set.

For (Top3), let U be the union of an arbitrary collection of open sets $\{U_i \mid i \in I\}$. If $x \in U$, then $x \in U_j$ for some $j \in I$ and there exists $\epsilon > 0$ such that $(x - \epsilon,\ x + \epsilon) \subseteq U_j \subseteq U$. Hence U is open and the family $\mathcal{O}$ forms a topology for $\mathbb{R}$. It is often referred to as the **standard topology** on $\mathbb{R}$. Any open interval $(a,\ b)$ where $a < b$ is an open set, for if $x \in (a,\ b)$ then $(x - \epsilon,\ x + \epsilon) \subset (a,\ b)$ for $\epsilon = \frac{1}{2}\min(x - a, b - x)$. A similar argument shows that the intervals may also be of semi-infinite extent, such as $(-\infty,\ a)$ or $(b,\ \infty)$. Notice that infinite intersections of open sets do not generally result in an open set. For example, an isolated point $\{a\}$ is not an open set since it contains no finite open interval, yet it is the intersection of an infinite sequence of open intervals such as

$$(a - 1,\ a + 1),\ \ (a - \tfrac{1}{2},\ a + \tfrac{1}{2}),\ \ (a - \tfrac{1}{3}, a + \tfrac{1}{3}),\ \ (a - \tfrac{1}{4}, a + \tfrac{1}{4}), \dots$$

Similar arguments can be used to show that the open sets defined on $\mathbb{R}^2$ in Section 10.1 form a topology. Similarly, in $\mathbb{R}^n$ we define a topology where a set U is said to be open if for every point $\mathbf{x} \in U$ there exists an open ball

$$B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 \mid |\mathbf{y} - \mathbf{x}| < r\} \subset U,$$

where

$$|\mathbf{y} - \mathbf{x}| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}.$$

This topology will again be termed the **standard topology** on $\mathbb{R}^n$.

Example 10.2 Consider the family $\mathcal{O}'$ of all open intervals on $\mathbb{R}$ of the form $(-a, b)$ where $a, b > 0$, together with the empty set. All these intervals contain the origin 0. It is not hard to show that (Top1)–(Top3) hold for this family and that $(X, \mathcal{O}')$ is a topological space. This space is not very ‘nice’ in some of its properties. For example no two points $x, y \in \mathbb{R}$ lie in non-intersecting neighbourhoods. In a sense all points of the line are ‘arbitrarily close’ to each other in this topology.

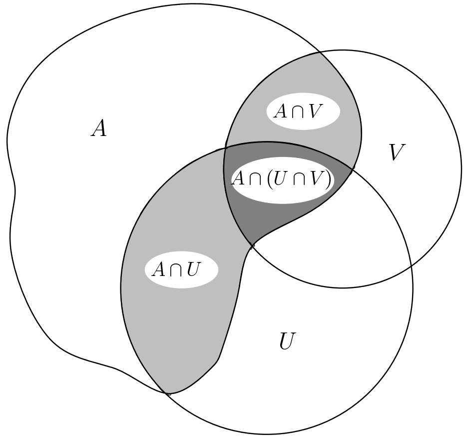

Figure 10.2 Relative topology induced on a subset of a topological space

A subset V is called **closed** if its complement $X - V$ is open. The empty set and the whole space are clearly closed sets, since they are both open sets and are the complements of each other. The intersection of an arbitrary family of closed sets is closed, as it is the complement of a union of open sets. However, only finite unions of closed sets are closed in general.

Example 10.3 Every *closed interval* $[a, b] = \{x \mid a \le x \le b\}$ where $-\infty < a \le b < \infty$ is a closed set, as it is the complement of the open set $(-\infty, a) \cup (b, \infty)$. Every singleton set consisting of an isolated point $\{a\} \equiv [a, a]$ is closed. Closed intervals $[a, b]$ are not open sets since the end points a or b do not belong to any open interval included in $[a, b]$.

If A is any subset of X the **relative topology** on A, or **topology induced** on A, is the topology whose open sets are

$$\mathcal{O}_A = \{A \cap U \mid U \in \mathcal{O}\}.$$

Thus a set is open in the relative topology on A iff it is the intersection of A and an open set U in X (see Fig. 10.2). That these sets form a topology on A follows from the following three facts:

1. $\emptyset \cap A = \emptyset, \quad X \cap A = A.$
2. $(U \cap A) \cap (V \cap A) = (U \cap V) \cap A.$
3. $\displaystyle\bigcup_{i\in I}(U_i \cap A) = \Big(\bigcup_{i\in I} U_i\Big) \cap A.$

A subset A of X together with the relative topology $\mathcal{O}_A$ induced on it is called a **subspace** of $(X, \mathcal{O})$.

Example 10.4 The relative topology on the half-open interval $A = [0, 1) \subset \mathbb{R}$, induced on A by the standard topology on $\mathbb{R}$, is the union of half-open intervals of the form $[0, a)$

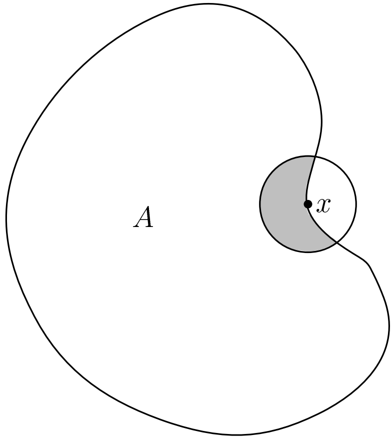

Figure 10.3 Accumulation point of a set

where $0 < a < 1$, and all intervals of the form (a, b) where $0 < a < b \leq 1$. Evidently some of the open sets in this topology are not open in $\mathbb{R}$.

Exercise: Show that if $A \subseteq X$ is an open set then all open sets in the relative topology on A are open in X.

Exercise: If A is a closed set, show that every closed set in the induced topology on A is closed in X.

A point x is said to be an **accumulation point** of a set A if every open neighbourhood U of x contains points of A other than x itself, as shown in Fig. 10.3. What this means is that x may or may not lie in A, but points of A 'cluster' arbitrarily close to it (sometimes it is also called a *cluster point* of A). A related concept is commonly applied to sequences of points $x_n \in X$. We say that the sequence $x_n \in X$ **converges to** $x \in X$ or that x is a **limit point** of $\{x_n\}$, denoted $x_n \to x$, if for every open neighbourhood U of x there is an integer N such that $x_n \in U$ for all $n \geq N$. This differs from an accumulation point in that we could have $x_n = x$ for all $n > n_0$ for some n_0.

The **closure** of any set A, denoted $\overline{A}$, is the union of the set A and all its accumulation points. The **interior** of A is the union of all open sets $U \subseteq A$, denoted A^o. The difference of these two sets, $b(A) = \overline{A} - A^o$, is called the **boundary** of A.

Theorem 10.2 *The closure of any set A is a closed set. The interior A^o is the largest open set included in A. The boundary $b(A)$ is a closed set.*

Proof: Let x be any point not in $\overline{A}$. Since x is not in A and is not an accumulation point of A, it has an open neighbourhood U_x not intersecting A. Furthermore, U_x cannot contain any other accumulation point of A else it would be an open neighbourhood of that point not intersecting A. Hence the complement $X - \overline{A}$ of the closure of A is the union of the open sets U_x. It is therefore itself an open set and its complement $\overline{A}$ is a closed set.

Since the interior A^o is a union of open sets, it is an open set by (Top3). If U is any open set such that $U \subseteq A$ then, by definition, $U \subseteq A^o$. Thus A^o is the largest open subset of A. Its complement is closed and the boundary $b(A) = \overline{A} \cap (X - A^o)$ is necessarily a closed set. ■

Exercise: Show that a set A is closed if and only if it contains its boundary, $A \supseteq b(A)$.

Exercise: A set A is open if and only if $A \cap b(A) = \emptyset$.

Exercise: Show that all accumulation points of A lie in the boundary $b(A)$.

Exercise: Show that a point x lies in the boundary of A iff every neighbourhood of x contains points both in A and not in A.

Example 10.5 The closure of the open ball $B_a(\mathbf{x}) \subset \mathbb{R}^n$ (see Example 10.1) is the *closed ball*

$$\overline{B_a}(\mathbf{x}) = \{\mathbf{y} \mid |\mathbf{y} - \mathbf{x}| \leq a\}.$$

Since every open ball is an open set, it is its own interior, $B_a^o(\mathbf{x}) = B_a(\mathbf{x})$ and its boundary is the $(n-1)$-sphere of radius a, centre $\mathbf{x}$,

$$b(B_a(\mathbf{x})) = S_a^{n-1}(\mathbf{x}) = \{y \mid |\mathbf{y} - \mathbf{x}| = a\}.$$

Example 10.6 A set whose closure is the entire space X is said to be **dense** in X. For example, since every real number has rational numbers arbitrarily close to it, the rational numbers $\mathbb{Q}$ are a countable set that is dense in the set of real numbers. In higher dimensions the situation is similar. The set of points with rational coordinates $\mathbb{Q}^n$ is a countable set that is dense in $\mathbb{R}^n$.

Exercise: Show that $\mathbb{Q}$ is neither an open or closed set in $\mathbb{R}$.

Exercise: Show that $\mathbb{Q}^0 = \emptyset$ and $b(\mathbb{Q}) = \mathbb{R}$.

It is sometimes possible to compare different topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ on a set X. We say $\mathcal{O}_1$ is **finer** or **stronger** than $\mathcal{O}_2$ if $\mathcal{O}_1 \supseteq \mathcal{O}_2$. Essentially, $\mathcal{O}_1$ has more open sets than $\mathcal{O}_2$. In this case we also say that $\mathcal{O}_2$ is **coarser** or **weaker** than $\mathcal{O}_1$.

Example 10.7 All topologies on a set X lie somewhere between two extremes, the discrete and indiscrete topologies. The **indiscrete** or **trivial** topology consists simply of the empty set and the whole space itself, $\mathcal{O}_1 = \{\emptyset, X\}$. It is the coarsest possible topology on X – if $\mathcal{O}$ is any other topology then $\mathcal{O}_1 \subseteq \mathcal{O}$ by (Top1). The **discrete topology** consists of all subsets of X, $\mathcal{O}_2 = 2^X$. This topology is the finest possible topology on X, since it includes all other topologies $\mathcal{O}_2 \supseteq \mathcal{O}$. For both topologies (Top1)–(Top3) are trivial to verify.

Given a set X, and an arbitrary collection of subsets $\mathcal{U}$, we can ask for the weakest topology $\mathcal{O}(\mathcal{U})$ containing $\mathcal{U}$. This topology is the intersection of all topologies that contain

$\mathcal{U}$ and is called the **topology generated** by $\mathcal{U}$. It is analogous to the concept of the vector subspace $L(M)$ generated by an arbitrary subset M of a vector space V (see Section 3.5).

A contructive way of defining $\mathcal{O}(\mathcal{U})$ is the following. Firstly, adjoin the empty set $\emptyset$ and the entire space X to $\mathcal{U}$ if they are not already in it. Next, extend $\mathcal{U}$ to a family $\hat{\mathcal{U}}$ consisting of all finite intersections $U_1 \cap U_2 \cap \dots \cap U_n$ of sets $U_i \in \mathcal{U} \cup \{\emptyset, X\}$. Finally, the set $\mathcal{O}(\mathcal{U})$ consisting of arbitrary unions of sets from $\hat{\mathcal{U}}$ forms a topology. To prove (Top2),

$$\bigcup_{i\in I}\Big(\bigcap_{a=1}^{n_i} U_{ia}\Big) \cap \bigcup_{j\in J}\Big(\bigcap_{b=1}^{n_j} V_{jb}\Big) = \bigcup_{i\in I}\bigcup_{j\in J}\Big(U_{i1}\cap\dots\cap U_{in_i}\cap V_{j1}\cap\dots\cap V_{jn_j}\Big).$$

Property (Top3) follows immediately from the contruction.

Example 10.8 On the real line $\mathbb{R}$, the family $\mathcal{U}$ of all open intervals generates the standard topology since every open set is a union of open sets of the form $(x-\epsilon,\ x+\epsilon)$. Similarly, the standard topology on $\mathbb{R}^2$ is generated by the set of open balls

$$\mathcal{U} = \{B_a(\mathbf{r}) \mid\ > 0,\ \mathbf{r} = (x, y) \in \mathbb{R}^2\}.$$

To prove this statement we must show that every set that is an intersection of two open balls $B_a(\mathbf{r})$ and $B_b(\mathbf{r}')$ is a union of open balls from $\mathcal{U}$. If $\mathbf{x} \in B_a(\mathbf{r})$, let $\epsilon < a$ be such that $B_\epsilon(\mathbf{x}) \subset B_a(\mathbf{r})$. Similarly if $\mathbf{x} \in B_b(\mathbf{r}')$, let $\epsilon' < a$ be such that $B_{\epsilon'}(\mathbf{x}) \subset B_b(\mathbf{r}')$. Hence, if $\mathbf{x} \in B_a(\mathbf{r}) \cap B_b(\mathbf{r}')$ then $B_{\epsilon''}(\mathbf{x}) \subset B_a(\mathbf{r}) \cap B_b(\mathbf{r}')$ where $\epsilon'' = \min(\epsilon, \epsilon')$. The proof easily generalizes to intersections of any finite number of open balls. Hence the standard topology of $\mathbb{R}^2$ is generated by the set of all open balls. The extension to $\mathbb{R}^n$ is straightforward.

Exercise: Show that the discrete topology on X is generated by the family of all singleton sets $\{x\}$ where $x \in X$.

A set A is said to be a **neighbourhood** of $x \in X$ if there exists an open set U such that $x \in U \subset A$. If A itself is open it is called an **open neighbourhood** of x. A topological space X is said to be **first countable** if every point $x \in X$ has a countable collection $U_1(x), U_2(x), \dots$ of open neighbourhoods of x such that every open neighbourhood U of x includes one of these neighbourhoods $U \supset U_n(x)$. A stronger condition is the following: a topological space $(X, \mathcal{O})$ is said to be **second countable** or **separable** if there exists a countable set $U_1,\ U_2,\ U_3, \dots$ that generates the topology of X.

Example 10.9 The standard topology of the Euclidean plane $\mathbb{R}^2$ is separable, since it is generated by the set of all rational open balls,

$$\mathcal{B}_{\text{rat}} = \{B_a(\mathbf{r}) \mid a > 0 \in \mathbb{Q}, \mathbf{r} = (x, y) \text{ s.t. } x, y \in \mathbb{Q}\}.$$

The set $\mathcal{B}_{\text{rat}}$ is countable as it can be put in one-to-one correspondence with a subset of $\mathbb{Q}^3$. Since the rational numbers are dense in the real numbers, every point $\mathbf{x}$ of an open set U lies in a rational open ball. Thus every open set is a union of rational open balls. By a similar argument to that used in Example 10.8 it is straightforward to prove that the intersection of two sets from $\mathcal{B}_{\text{rat}}$ is a union of rational open balls. Hence $\mathbb{R}^2$ is separable. Similarly, all spaces $\mathbb{R}^n$ where $n \geq 1$ are separable.

Let X and Y be two topological spaces. Theorem 10.1 motivates the following definition: a function $f : X \to Y$ is said to be **continuous** if the inverse image $f^{-1}(U)$ of every open set U in Y is open in X. If f is one-to-one and its inverse $f^{-1} : Y \to X$ is continuous, the function is called a **homeomorphism** and the topological spaces X and Y are said to be **homeomorphic** or **topologically equivalent**, written $X \cong Y$. The main task of *topology* is to find **topological invariants** – properties that are preserved under homeomorphisms. They may be real numbers, algebraic structures such as groups or vector spaces constructed from the topological space, or specific properties such as *compactness* and *connectedness*. The ultimate goal is to find a set of topological invariants that characterize a topological space. In the language of category theory, Section 1.7, continuous functions are the morphisms of the category whose objects are topological spaces, and homeomorphisms are the isomorphism of this category.

Example 10.10 Let $f : X \to Y$ be a continuous function between topological spaces. If the topology on X is discrete then every function f is continuous, for no matter what the topology on Y, every inverse image set $f^{-1}(U)$ is open in X. Similarly if the topology on Y is indiscrete than the function f is always continuous since the only inverse images in X of open sets are $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$, which are always open sets by (Top1).

Problems

Problem 10.1 Give an example in $\mathbb{R}^2$ of each of the following:

(a) A family of open sets whose intersection is a closed set that is not open.
(b) A family of closed sets whose union is an open set that is not closed.
(c) A set that is neither open nor closed.
(d) A countable dense set.
(e) A sequence of continuous functions $f_n : \mathbb{R}^2 \to \mathbb{R}$ whose limit is a discontinuous function.

Problem 10.2 If $\mathcal{U}$ generates the topology on X show that $\{A \cap U \mid U \in \mathcal{U}\}$ generates the relative topology on A.

Problem 10.3 Let X be a topological space and $A \subset B \subset X$. If B is given the relative topology, show that the relative topology induced on A by B is identical to the relative topology induced on it by X.

Problem 10.4 Show that for any subsets U, V of a topological space $\overline{U \cup V} = \overline{U} \cup \overline{V}$. Is it true that $\overline{U \cap V} = \overline{U} \cap \overline{V}$? What corresponding statements hold for the interior and boundaries of unions and intersections of sets?

Problem 10.5 If A is a dense set in a topological space X and $U \subseteq X$ is open, show that $U \subseteq \overline{A \cap U}$.

Problem 10.6 Show that a map $f : X \to Y$ between two topological spaces X and Y is continuous if and only if $f(\overline{U}) \subseteq \overline{f(U)}$ for all sets $U \subseteq X$. Show that f is a homeomorphism only if $f(\overline{U}) = \overline{f(U)}$ for all sets $U \subseteq X$.

Problem 10.7 Show the following:

(a) In the trivial topology, every sequence x_n converges to every point of the space $x \in X$.

(b) In R^2 the family of open sets consisting of all open balls centred on the origin $B_r(0)$ is a topology. Any sequence $\mathbf{x}_n \to \mathbf{x}$ converges to all points on the circle of radius $|\mathbf{x}|$ centred on the origin.
(c) If C is a closed set of a topological space X it contains all limit points of sequences $x_n \in C$.
(d) Let $f : X \to Y$ be a continuous function between topological spaces X and Y. If $x_n \to x$ is any convergent sequence in X then $f(x_n) \to f(x)$ in Y.

Problem 10.8 If W, X and Y are topological spaces and the functions $f : W \to X$, $g : X \to Y$ are both continuous, show that the function $h = g \circ f : W \to Y$ is continuous.

10.3 Metric spaces

To generalize the idea of 'distance' as it appears in $\mathbb{R}$ and $\mathbb{R}^2$, we define a **metric space** [9] to be a set M with a **distance function** or **metric** $d : M \times M \to \mathbb{R}$ such that

(Met1) $d(x, y) \geq 0$ for all $x, y \in M$.
(Met2) $d(x, y) = 0$ if and only if $x = y$.
(Met3) $d(x, y) = d(y, x)$.
(Met4) $d(x, y) + d(y, z) \geq d(x, z)$.

Condition (Met4) is called the **triangle inequality** – the length of any side of a triangle xyz is less than the sum of the other two sides. For every x in a metric space (M, d) and positive real number $a > 0$ we define the **open ball** $B_a(x) = \{y \mid d(x, y) < a\}$.

In n-dimensional Euclidean space $\mathbb{R}^n$ the distance function is given by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2},$$

but the following could also serve as acceptable metrics:

$$d_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|,$$
$$d_2(\mathbf{x}, \mathbf{y}) = \max\big(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\big).$$

Exercise: Show that $d(\mathbf{x}, \mathbf{y})$, $d_1(\mathbf{x}, \mathbf{y})$ and $d_2(\mathbf{x}, \mathbf{y})$ satisfy the metric axioms (Met1)–(Met4).

Exercise: In $\mathbb{R}^2$ sketch the open balls $B_1((0, 0))$ for the metrics d, d_1 and d_2.

If (M, d) is a metric space, then a subset $U \subset M$ is said to be open if and only if for every $x \in U$ there exists an open ball $B_\epsilon(x) \subseteq U$. Just as for $\mathbb{R}^2$, this defines a natural topology on M, called the **metric topology**. This topology is generated by the set of all open balls $B_a(x) \subset M$. The proof closely follows the argument in Example 10.8.

In a metric space (M, d), a sequence x_n converges to a point x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. Equivalently, $x_n \to x$ if and only if for every $\epsilon > 0$ the sequence eventually enters and stays in the open ball $B_\epsilon(x)$. In a metric space the limit point x of a sequence x_n is unique, for if $x_n \to x$ and $x_n \to y$ then $d(x, y) \leq d(x, x_n) + d(x_n, y)$ by the triangle inequality. By choosing n large enough we have $d(x, y) < \epsilon$ for any $\epsilon > 0$. Hence $d(x, y) = 0$, and $x = y$ by (Met2). For this reason, the concept of convergent sequences is more useful in metric spaces than in general topological spaces (see Problem 10.7).

In a metric space (M, d) let x_n be a sequence that converges to some point $x \in M$. Then for every $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$. For, let N be an integer such that $d(x_k, x) < \frac{1}{2}\epsilon$ for all $k > N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon \qquad \text{for all } n,\ m > N.$$

A sequence having this property, $d(x_n, x_m) \to 0$ as $n, m \to \infty$, is termed a **Cauchy sequence**.

Example 10.11 Not every Cauchy sequence need converge to a point of M. For example, in the open interval $(0, 1)$ with the usual metric topology, the sequence $x_n = 2^{-n}$ is a Cauchy sequence yet it does not converge to any point in the open interval. A metric space (M, d) is said to be **complete** if every Cauchy sequence $x_1, x_2, \dots$ converges to a point $x \in M$. Completeness is not a topological property. For example the real line $\mathbb{R}$ is a complete metric space, and the Cauchy sequence 2^{-n} has the limit 0 in $\mathbb{R}$. The topological spaces $\mathbb{R}$ and $(0, 1)$ are homeomorphic, using the map $\varphi : x \mapsto \tan \frac{1}{2}\pi(2x - 1)$. However one space is complete while the other is not with respect to the metrics generating their topologies.

Problems

Problem 10.9 Show that every metric space is first countable. Hence show that every subset of a metric space can be written as the intersection of a countable collection of open sets.

Problem 10.10 If $\mathcal{U}_1$ and $\mathcal{U}_2$ are two families of subsets of a set X, show that the topologies generated by these families are homeomorphic if every member of $\mathcal{U}_2$ is a union of sets from $\mathcal{U}_1$ and vice versa. Use this property to show that the metric topologies on $\mathbb{R}^n$ defined by the metrics d, d_1 and d_2 are all homeomorphic.

Problem 10.11 A topological space X is called **normal** if for every pair of disjoint closed subsets A and B there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Show that every metric space is normal.

10.4 Induced topologies

Induced topologies and topological products

Given a topological space $(X, \mathcal{O})$ and a map $f : Y \to X$ from an arbitrary set Y into X, we can ask for the weakest topology on Y for which this map is continuous – it is useless to ask for the finest such topology since, as shown in Example 10.10, the discrete topology on Y always achieves this end. This is known as the topology **induced** on Y by the map f. Let $\mathcal{O}_f$ be the family of all inverse images of open sets of X,

$$\mathcal{O}_f = \{f^{-1}(U) \mid U \in \mathcal{O}\}.$$

Since f is required to be continuous, all members of this collection must be open in the induced topology. Furthermore, $\mathcal{O}_f$ is a topology on Y since (i) property (Top1) is trivial, as $\emptyset = f^{-1}(\emptyset)$ and $Y = f^{-1}(X)$; (ii) the axioms (Top2) and (Top3) follow from the set-theoretical identities

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \quad \text{and} \quad \bigcup_{i \in I} f^{-1}(U_i) = f^{-1}\Big(\bigcup_{i \in I} U_i\Big).$$

Hence $\mathcal{O}_f$ is a topology on Y and is included in any other topology such that the map f is continuous. It must be the topology induced on Y by the map f since it is the coarsest possible such topology.

Example 10.12 Let $(X, \mathcal{O})$ be any topological space and A any subset of X. In the topology induced on A by the natural inclusion map $i_A : A \to X$ defined by $i_A(x) = x$ for all $x \in A$, a subset B of A is open iff it is the intersection of A with an open set of X; that is, $B = A \cap U$ where U is open in X. This is precisely the relative topology on A defined in Section 10.2. The relative topology is thus the coarsest topology on A for which the inclusion map is continuous.

More generally, for a collection of maps $\{f_i : Y \to X_i \mid i \in I\}$ where X_i are topological spaces, the weakest topology on Y such that all these maps are continuous is said to be the topology **induced** by these maps. To create this topology it is necessary to consider the set of all inverse images of open sets $\mathcal{U} = \{f_i^{-1}(U_i) \mid U_i \in \mathcal{O}_i\}$. This collection of sets is not itself a topology in general, the topology generated by these sets *will* be the coarsest topology on Y such that each function f_i is continuous.

Given two topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, let $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ be the natural projection maps defined by

$$\mathrm{pr}_1(x, y) = x \quad \text{and} \quad \mathrm{pr}_2(x, y) = y.$$

The **product topology** on the set $X \times Y$ is defined as the topology induced by these two maps. The space $X \times Y$ together with the product topology is called the **topological product** of X and Y. It is the coarsest topology such that the projection maps are continuous. The inverse image under pr_1 of an open set U in X is a 'vertical strip' $U \times Y$, while the inverse image of an open set V in Y under pr_2 is a 'horizontal strip' $X \times V$. The intersection of any pair of these strips is a set of the form $U \times V$ where U and V are open sets from X and Y respectively (see Fig. 10.4). Since the topology generated by the vertical and horizontal strips consists of all possible unions of such intersections, it follows that in the product topology a subset $A \subset X \times Y$ is open if for every point $(x, y) \in A$ there exist open sets $U \in \mathcal{O}_X$ and $V \in \mathcal{O}_Y$ such that $(x, y) \in U \times V \subseteq A$.

Given an arbitrary collection of sets $\{X_i \mid i \in I\}$, their **cartesian product** $P = \prod_{i \in I} X_i$ is defined as the set of maps $f : I \to \bigcup_i X_i$ such that $f(i) \in X_i$ for each $i \in I$. For a finite number of sets, taking $I = \{1, 2, \dots, n\}$, this concept is identical with the set of n-tuples from $X_1 \times X_2 \times \dots \times X_n$. The product topology on P is the topology induced by the projection maps $\mathrm{pr}_i : P \to X_i$ defined by $\mathrm{pr}_i(f) = f(i)$. This topology is coarser than the topology generated by all sets of the form $\prod_{i \in I} U_i$ where U_i is an open subset of X_i.

Example 10.13 Let S^1 be the unit circle in $\mathbb{R}^2$ defined by $x^2 + y^2 = 1$, with the relative topology. The product space $S^1 \times S^1$ is homeomorphic to the torus T^2 or 'donut', with topology induced from its embedding as a subset of $\mathbb{R}^3$. This can be seen by embedding S^1 in the $z = 0$ plane of $\mathbb{R}^3$, and attaching a vertical unit circle facing outwards from each

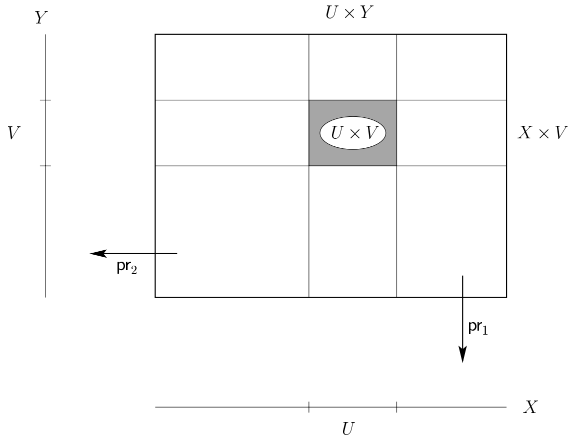

Figure 10.4 Product of two topological spaces

point on S^1. As the vertical circles 'sweep around' the horizontal cirle the resulting circle is clearly a torus.

The following is an occasionally useful theorem.

Theorem 10.3 *If X and Y are topological spaces then for each point $x \in X$ the* **injection map** $\iota_x : Y \to X \times Y$ *defined by $\iota_x(y) = (x, y)$ is continuous. Similarly the map $\iota'_y : X \to X \times Y$ defined by $\iota'_y(x) = (x, y)$ is continuous.*

Proof: Let U and V be open subsets of X and Y respectively. Then

$$(\iota_x)^{-1}(U \times V) = \begin{cases} V & \text{if } x \in U, \\ \emptyset & \text{if } x \notin U. \end{cases}$$

Since every open subset of $U \times V$ in the product topology is a union of sets of type $U \times V$, it follows that the inverse image under ι_x of every open set in $X \times Y$ is an open subset of Y. Hence the map ι_x is continuous. Similarly for the map ι'_y. ■

Topology by identification

We may also reverse the above situation. Let $(X, \mathcal{O})$ be a topological space, and $f : X \to Y$ a map from X onto an arbitrary set Y. In this case the topology on Y **induced** by f is defined to be the *finest* topology such that f is continuous. This topology consists of all subsets $U \subseteq Y$ such that $f^{-1}(U)$ is open in X; that is, $\mathcal{O}_Y = \{f^{-1}(U) \mid U \in \mathcal{O}\}$.

Exercise: Show that $\mathcal{O}_Y$ is a topology on Y.

Exercise: Show that $\mathcal{O}_Y$ is the strongest topology such that f is continuous.

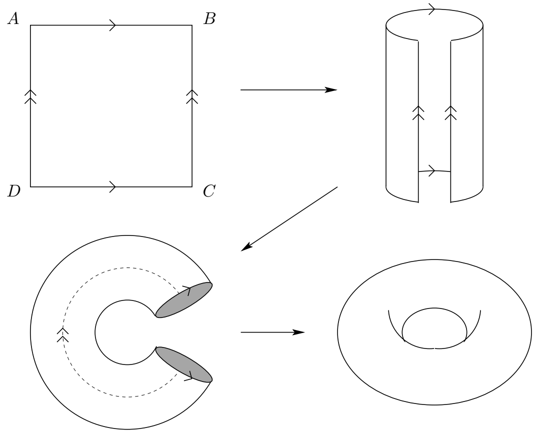

Figure 10.5 Construction of a torus by identification of opposite sides of a square

A common instance of this type of induced topology occurs when there is an equivalence relation E defined on a topological space X. Let $[x] = \{y \mid yEx\}$ be the equivalence class containing the point $x \in X$. In the factor space $X/E = \{[x] \mid x \in X\}$ define the topology obtained **by identification** from X to be the topology induced by the natural map $i_E : X \to X/E$ associating each point $x \in X$ with the equivalence class to which it belongs, $i_E(x) = [x]$. In this topology a subset A is open iff its inverse image $i_E^{-1}(A) = \{x \in X \mid [x] \in A\} = \bigcup_{[x]\in A}[x]$ is an open subset of X. That is, a subset of equivalence classes $A = \{[x]\}$ is open in the identification topology on X/E iff the union of the sets $[x]$ that belong to A is an open subset of X.

Exercise: Verify directly from the last statement that the axioms (Top1)–(Top3) are satisfied by this topology on X/E.

Example 10.14 As in Example 1.4, we say two points (x, y) and (x', y') in the plane $\mathbb{R}^2$ are equivalent if their coordinates differ by integral amounts,

$$(x, y) \equiv (x'y') \quad \text{iff} \quad x - x' = n,\ \ y - y' = m \quad (n, m \in \mathbb{Z}).$$

The topology on the space $\mathbb{R}^2/\equiv$ obtained by identification can be pictured as the unit square with opposite sides identified (see Fig. 10.5). To understand that this is a representation of the torus T^2, consider a square rubber sheet. Identifying sides AD and BC is equivalent to joining these two sides together to form a cylinder. The identification of AB and CD is now equivalent to identifying the circular edges at the top and bottom of the cylinder. In three-dimensions this involves bending the cylinder until top and bottom join up to form the inner tube of a tyre – remember, distances or metric properties need not be preserved for a topological transformation. The n-torus T^n can similarly be defined as the topological space obtained by identification from the corresponding equivalence relation on R^n, whereby points are equivalent if their coordinates differ by integers.

Example 10.15 Let $\dot{\mathbb{R}}^3 = \mathbb{R}^3 - \{\mathbf{0}\}$ be the set of non-zero 3-triples of real numbers given the relative topology in $\mathbb{R}^3$. Define an equivalence relation on $\dot{\mathbb{R}}^3$ whereby $(x, y, z) \equiv (x', y', z')$ iff there exists a real number $\lambda \neq 0$ such that $x = \lambda x'$, $y = \lambda y'$ and $z = \lambda z'$. The factor space $P^2 = \dot{\mathbb{R}}^3/\equiv$ is known as the **real projective plane**.

Each equivalence class $[(x, y, z)]$ is a straight line through the origin that meets the unit 2-sphere S^2 in two diametrically opposite points. Define an equivalence relation on S^2 by identifying diametrically opposite points, $(x, y, z) \sim (-x, -y, -z)$ where $x^2 + y^2 + z^2 = 1$. The topology on P^2 obtained by identification from $\dot{\mathbb{R}}^3$ is thus identical with that of the 2-sphere S^2 with diametrically opposite points identified.

Generalizing, we define **real projective *n*-space** P^n to be $\dot{\mathbb{R}}^{n+1}/\equiv$ where $(x_1, x_2, \dots, x_{n+1}) \equiv (x'_1, x'_2, \dots, x'_{n+1})$ if and only if there exists $\lambda \neq 0$ such that $x_1 = \lambda x'_1$, $x_2 = \lambda x'_2, \dots, x_{n+1} = \lambda x'_{n+1}$. This space can be thought of as the set of all straight lines through the origin in $\mathbb{R}^{n+1}$. The topology of P^n is homeomorphic with that of the n-sphere S^n with opposite points identified.

Problems

Problem 10.12 If $f : X \to Y$ is a continuous map between topological spaces, we define its **graph** to be the set $G = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$. Show that if G is given the relative topology induced by the topological product $X \times Y$ then it is homeomorphic to the topological space X.

Problem 10.13 Let X and Y be topological spaces and $f : X \times Y \to X$ a continuous map. For each fixed $a \in X$ show that the map $f_a : Y \to X$ defined by $f_a(y) = f(a, y)$ is continuous.

10.5 Hausdorff spaces

In some topologies, for example the indiscrete topology, there are so few open sets that different points cannot be separated by non-intersecting neighbourhoods. To remedy this situation, conditions known as *separation axioms* are sometimes imposed on topological spaces. One of the most common of these is the **Hausdorff condition**: for every pair of points $x, y \in X$ there exist open neighbourhoods U of x and V of y such that $U \cap V = \emptyset$. A topological space satisfying this property is known as a **Hausdorff space**. In an intuitive sense, no pair of distinct points of a Hausdorff space are 'arbitrarily close' to each other.

A typical 'nice' property of Hausdorff spaces is the fact that the limit of any convergent sequence $x_n \to x$, defined in Problem 10.7, is unique. Suppose, for example, that $x_n \to x$ and $x_n \to x'$ in a Hausdorff space X. If $x \neq x'$ let U and U' be disjoint open neighbourhoods such that $x \in U$ and $x' \in U'$, and N an integer such that $x_n \in U$ for all $n > N$. Since $x_n \notin U'$ for all $n > N$ the sequence x_n cannot converge to x'. Hence $x = x'$.

In a Hausdorff space every singleton set $\{x\}$ is a closed set, for let $Y = X - \{x\}$ be its complement. Every point $y \in Y$ has an open neighbourhood U_y that does not intersect some open neighbourhood of x. In particular $x \notin U_y$. By (Top3) the union of all these open neighbourhoods, $Y = \bigcup_{y \in Y} U_y = X - \{x\}$, is open. Hence $\{x\} = X - Y$ is closed since it is the complement of an open set.

Exercise: Show that on a finite set X, the only Hausdorff topology is the discrete topology. For this reason, finite topologies are of limited interest.

Theorem 10.4 *Every metric space (X, d) is a Hausdorff space.*

Proof: Let $x, y \in X$ be any pair of unequal points and let $\epsilon = \frac{1}{4}d(x, y)$. The open balls $U = B_\epsilon(x)$ and $V = B_\epsilon(y)$ are open neighbourhoods of x and y respectively. Their intersection is empty, for if $z \in U \cap V$ then $d(x, z) < \epsilon$ and $d(y, z) < \epsilon$, which contradicts the triangle inequality (Met4),

$$d(x, y) \leq d(x, z) + d(z, y) \leq 2\epsilon < \tfrac{1}{2}d(x, y).$$

■

An immediate consequence of this theorem is that the standard topology on $\mathbb{R}^n$ is Hausdorff for all $n > 0$.

Theorem 10.5 *If X and Y are topological spaces and $f : X \to Y$ is a one-to-one continuous mapping, then X is Hausdorff if Y is Hausdorff.*

Proof: Let x and x' be any pair of distinct points in X and set $y = f(x)$, $y' = f(x')$. Since f is one-to-one these are distinct points of Y. If Y is Hausdorff there exist non-intersecting open neighbourhoods U_y and $U_{y'}$ in Y of y and y' respectively. The inverse images of these sets under f are open neighbourhoods of x and x' respectively that are non-intersecting, since $f^{-1}(U_y) \cap f^{-1}(U_{y'}) = f^{-1}(U_y \cap U_{y'}) = f^{-1}(\emptyset) = \emptyset$. ■

This shows that the Hausdorff condition is a genuine topological property, invariant under topological transformations, for if $f : X \to Y$ is a homeomorphism then $f^{-1} : Y \to X$ is continuous and one-to-one.

Corollary 10.6 *Any subspace of a Hausdorff space is Hausdorff in the relative topology.*

Proof: Let A be any subset of a topological space X. In the relative topology the inclusion map $i_A : A \to X$ is continuous. Since it is one-to-one, Theorem 10.5 implies that A is Hausdorff. ■

Theorem 10.7 *If X and Y are Hausdorff topological spaces then their topological product $X \times Y$ is Hausdorff.*

Proof: Let (x, y) and (x', y') be any distinct pair of points in $X \times Y$, so that either $x \neq x'$ or $y \neq y'$. Suppose that $x \neq x'$. There then exist open sets U and U' in X such that $x \in U$, $x' \in U'$ and $U \cap U' = \emptyset$. The sets $U \times Y$ and $U' \times Y$ are disjoint open neighbourhoods of (x, y) and (x', y') respectively. Similarly, if $y \neq y'$ a pair of disjoint neighbourhoods of the form $X \times V$ and $X \times V'$ can be found that separate the two points. ■

Problems

Problem 10.14 If Y is a Hausdorff topological space show that every continuous map $f : X \to Y$ from a topological space X with indiscrete topology into Y is a *constant map*; that is, a map of the form $f(x) = y_0$ where y_0 is a fixed element of Y.

Problem 10.15 Show that if $f : X \to Y$ and $g : X \to Y$ are continuous maps from a topological space X into a Hausdorff space Y then the set of points A on which these maps agree, $A = \{x \in X \mid f(x) = g(x)\}$, is closed. If A is a dense subset of X show that $f = g$.

10.6 Compact spaces

A collection of sets $\mathcal{U} = \{U_i \mid i \in I\}$ is said to be a **covering** of a subset A of a topological space X if every point $x \in A$ belongs to some member of the collection. If every member $\mathcal{U}$ is an open set it is called an **open covering**. A subset of the covering, $\mathcal{U}' \subseteq \mathcal{U}$, which covers A is referred to as a **subcovering**. If $\mathcal{U}'$ consists of finitely many sets $\{U_1, U_2, \dots, U_n\}$ it is called a **finite subcovering**.

A topological space $(X, \mathcal{O})$ is said to be **compact** if every open covering of X contains a finite subcovering. The motivation for this definition lies in the following theorem, the proof of which can be found in standard books on analysis [10–12].

Theorem 10.8 (Heine–Borel) *A subset A of $\mathbb{R}^n$ is closed and bounded (included in a central ball, $A \subset B_a(\mathbf{0})$ for some $a > 0$) if and only if every open covering $\mathcal{U}$ of A has a finite subcovering.*

Theorem 10.9 *Every closed subspace A of a compact space X is compact in the relative topology.*

Proof: Let $\mathcal{U}$ be any covering of A by sets that are open in the relative topology. Each member of this covering must be of the form $U \cap A$, where U is open in Y. The sets $\{U\}$ together with the open set $X - A$ form an open covering of X that, by compactness of X, must have a finite subcovering $\{U_1, U_2, \dots, U_n, X - A\}$. The sets $\{U_1 \cap A, \dots, U_n \cap A\}$ are thus a finite subfamily of the original open covering $\mathcal{U}$ of A. Hence A is compact in the relative topology. ■

Theorem 10.10 *If $f : X \to Y$ is a continuous map from a compact topological space X into a topological space Y, then the image set $f(X) \subseteq Y$ is compact in the relative topology.*

Proof: Let $\mathcal{U}$ be any covering of $f(X)$ consisting entirely of open sets in the relative topology. Each member of this covering is of the form $U \cap f(X)$, where U is open in Y. Since f is continuous, the sets $f^{-1}(U)$ form an open covering of X. By compactness of X, a finite subfamily $\{f^{-1}(U_i) \mid i = 1, \dots, n\}$ serves to cover X, and the corresponding sets $U_i \cap f(X)$ evidently form a finite subcovering of $f(X)$. ■

Compactness is therefore a topological property, invariant under homeomorphisms.

Example 10.16 If E is an equivalence relation on a compact topological space X, the map $i_E : X \to X/E$ is continuous in the topology on X/E obtained by identification from X. By

Theorem 10.10 the topological space X/E is compact. For example, the torus T^2 formed by identifying opposite sides of the closed and compact unit square in $\mathbb{R}^2$ is a compact space.

Theorem 10.11 *The topological product $X \times Y$ is compact if and only if both X and Y are compact.*

Proof: If $X \times Y$ is compact then X and Y are compact by Theorem 10.10 since both the projection maps $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ are continuous in the product topology.

Conversely, suppose X and Y are compact. Let $\mathcal{W} = \{W_i \mid i \in I\}$ be an open covering of $X \times Y$. Since each set W_i is a union of such sets of the form $U \times V$ where U and V are open sets of X and Y respectively, the family of all such sets $U_j \times V_j$ $(j \in J)$ that are subsets of W_i for some $i \in I$ is an open cover of $X \times Y$. Given any point $y \in Y$, the set of all U_j such that $y \in V_j$ is an open cover of X, and since X is compact there exists a finite subcover $\{U_{j_1}, U_{j_2}, \dots, U_{j_n}\}$. The set $A_y = V_{j_1} \cap V_{j_2} \cap \dots \cap V_{j_n}$ is an open set in Y by condition (Top2), and $y \in A_y$ since $y \in V_{j_k}$ for each $k = 1, \dots, n$. Thus the family of sets $\{A_y \mid y \in Y\}$ forms an open cover of Y. As Y is compact, there is a finite subcovering $A_{y_1}, A_{y_2}, \dots, A_{y_m}$. The totality of all the sets $U_{j_k} \times V_{j_k}$ associated with these sets A_{y_a} forms a finite open covering of $X \times Y$. For each such set select a corresponding member W_i of the original covering $\mathcal{W}$ of which it is a subset. The result is a finite subcovering of $X \times Y$, proving that $X \times Y$ is compact. ■

Somewhat surprisingly, this statement extends to arbitrary infinite products (*Tychonoff's theorem*). The interested reader is referred to [8] or [2] for a proof of this more difficult result.

Theorem 10.12 *Every infinite subset of a compact topological space has an accumulation point.*

Proof: Suppose X is a compact topological space and $A \subset X$ has no accumulation point. The aim is to show that A is a finite set. Since every point in $x \in A - X$ has an open neighbourhood U_x such that $U_x \cap A = \emptyset$ it follows that $A \subseteq X$ is closed since its complement $A - X = \bigcup_{x \in X - A} U_x$ is open. Hence, by Theorem 10.9, A is compact. Since each point $a \in A$ is not an accumulation point, there exists an open neighbourhood U_a of a such that $U_a \cap A = \{a\}$. Hence each singleton $\{a\}$ is an open set in the relative topology induced on A, and the relative topology on A is therefore the discrete topology. The singleton sets $\{a \mid a \in A\}$ therefore form an open covering of A, and since A is compact there must be a finite subcovering $\{a_1\}, \{a_2\}, \dots, \{a_n\}$. Thus $A = \{a_1, a_2, \dots, a_n\}$ is a finite set. ■

Theorem 10.13 *Every compact subspace of a Hausdorff space is closed.*

Proof: Let X be a Hausdorff space and A a compact subspace in the relative topology. If $a \in A$ and $x \in X - A$ then there exist disjoint open sets U_a and V_a such that $a \in U_a$ and $x \in V_a$. The family of open sets $U_a \cap A$ is an open covering of A in the relative topology. Since A is compact there is a finite subcovering $\{U_{a_1} \cap A, \dots, U_{a_n} \cap A\}$. The intersection of the corresponding neighbourhoods $W = V_{a_1} \cap \dots \cap V_{a_n}$ is an open set that contains x. As all its points lie outside every $U_{a_i} \cap A$ we have $W \cap A = \emptyset$. Thus every point $x \in X - A$ has an open neighbourhood with no points in A. Hence A includes all its accumulation points and must be a closed set. ∎

In a metric space (M, d) we will say a subset A is **bounded** if $\sup\{d(x, y) \mid x, y \in A\} < \infty$.

Theorem 10.14 *Every compact subspace of a metric space is closed and bounded.*

Proof: Let A be a compact subspace of a metric space (M, d). Since M is a Hausdorff space by Theorem 10.4 it follows by the previous theorem that A is closed. Let $\mathcal{U} = \{B_1(a) \cap A \mid a \in A\}$ be the open covering of A consisting of intersections of A with unit open balls centred on points of A. Since A is compact, a finite number of these open balls $\{B_1(a_1),\ B_1(a_2), \dots,\ B_1(a_n)\}$ can be selected to cover A. Let the greatest distance between any pair of these points be $D = \max d(a_i, a_j)$. For any pair of points $a, b \in A$, if $a \in B_1(a_k)$ and $b \in B_1(a_l)$ then by the triangle inequality

$$d(a, b) \leq d(a, a_i) + d(a_i, a_j) + \cdots + d(a_j, b) \leq D + 2.$$

Thus A is a bounded set. ∎

Problems

Problem 10.16 Show that every compact Hausdorff space is normal (see Problem 10.11).

Problem 10.17 Show that every one-to-one continuous map $f : X \to Y$ from a compact space X onto a Hausdorff space Y is a homeomorphism.

10.7 Connected spaces

Intuitively, we can think of a topological space X as being 'disconnected' if it can be decomposed into two disjoint subsets $X = A \cup B$ without these sets having any boundary points in common. Since the boundary of a set is at the same time the boundary of the complement of that set, the only way such a decomposition can occur is if there exists a set A other than the empty set or the whole space X that has no boundary points at all. Since $b(A) = \overline{A} - A^o$, the only way $b(A) = \emptyset$ can occur is if $\overline{A} = A^o$. As $\overline{A} \supseteq A \supseteq A^o$, the set A must equal both its closure and interior; in particular, it would need to be both open and closed at the same time. This motivates the following definition: a topological space X is said to be **connected** if the only subsets that are both open and closed are the empty set and the space X itself. A space is said to be **disconnected** if it is not connected. In other words, X is disconnected if $X = A \cup B$ where A and B are disjoint sets that are both open and closed. A subset $A \subset X$ is said to be **connected** if it is connected in the relative topology.

Example 10.17 The indiscrete topology on any set X is connected, since the only open sets are $\emptyset$ or X. The discrete topology on any set X having more than one point is disconnected since every non-empty subset is both open and closed.

Example 10.18 The real numbers $\mathbb{R}$ are connected in the standard topology. To show this, let $A \subset \mathbb{R}$ be both open and closed. If $x \in A$ set y to be the least upper bound of those real numbers such that $[x, y) \subset A$. If $y < \infty$ then y is an accumulation point of A, and therefore $y \in A$ since A is closed. However, since A is an open set there exists an interval $(y - a, y + a) \subset A$. Thus $[x, y + a) \subset A$, contradicting the stipulation that y is the least upper bound. Hence $y = \infty$. Similarly $(-\infty, x] \subset A$ and the only possibility is that $A = \emptyset$ or $A = \mathbb{R}$.

Theorem 10.15 *The closure of a connected set is connected.*

Proof: Let $A \subset X$ be a connected set. Suppose U is a subset of the closure $\overline{A}$ of A, which is both open and closed in $\overline{A}$, and let $V = \overline{A} - U$ be the complement of U in $\overline{A}$. Since A is connected and the sets $U \cap A$ and $V \cap A$ are both open and closed in A, one of them must be the empty set, while the other is the whole set A. If, for example, $V \cap A = \emptyset$ then $U \cap A = A$, so that $A \subset U \subseteq \overline{A}$. Since U is closed in $\overline{A}$ we must have that $U = \overline{A}$ and $V = \overline{A} - U = \emptyset$. If $U \cap A = \emptyset$ then $U = \emptyset$ by an identical argument. Hence $\overline{A}$ is connected. ■

The following theorem is used in many arguments to do with connectedness of topological spaces or their subspaces. Intuitively, it says that connectedness is retained if any number of connected sets are 'attached' to a given connected set.

Theorem 10.16 *Let A_0 be any connected subset of a topological space X and $\{A_i \mid i \in I\}$ any family of connected subsets of X such that $A_0 \cap A_i \neq \emptyset$ for each member of the family. Then the set $A = A_0 \cup \big(\bigcup_{i \in I} A_i\big)$ is a connected subset of X.*

Proof: Suppose $A = U \cup V$ where U and V are disjoint open sets in the relative topology on A. For all $i \in I$ the sets $U \cap A_i$ and $V \cap A_i$ are disjoint open sets of A_i whose union is A_i. Since A_i is connected, either $U \cap A_i = \emptyset$ or $V \cap A_i = \emptyset$. This also holds for A_0: either $U \cap A_0 = \emptyset$ or $V \cap A_0 = \emptyset$, say the latter. Then $U \cap A_0 = A_0$, so that $A_0 \subseteq U$. Since $A_0 \cap A_i \neq \emptyset$ we have $U \cap A_i \neq \emptyset$ for all $i \in I$. Hence $V \cap A_i = \emptyset$ and $U \cap A_i = A_i$; that is, $A_i \subseteq U$ for all $i \in I$. Hence $U = A$ and $V = \emptyset$, showing that A is a connected subset of X. ■

A theorem similar to Theorem 10.10 is available for connectedness: the image of a connected space under a continuous map is connected. This also shows that connectedness is a topological property, invariant under homeomorphisms.

Theorem 10.17 *If $f : X \to Y$ is a continuous map from a connected topological space X into a topological space Y, its image set $f(X)$ is a connected subset of Y.*

Proof: Let B be any non-empty subset of $f(X)$ that is both open and closed in the relative topology. This means there exists an open set $U \subseteq Y$ and a closed set $C \subseteq Y$ such that $B = U \cap f(X) = C \cap f(X)$. Since f is a continuous map, the inverse image set $f^{-1}(B) = f^{-1}(U) = f^{-1}(C)$ is both open and closed in X. As X is connected it follows that $B = f(X)$; hence $f(X)$ is connected. ■

A useful application of these theorems is to show the topological product of two connected spaces is connected.

Theorem 10.18 *The topological product $X \times Y$ of two topological spaces is connected if and only if both X and Y are connected spaces.*

Proof: By Theorem 10.3, the maps $\iota_x : Y \to X \times Y$ and $\iota'_y : X \to X \times Y$ defined by $\iota_x(y) = \iota'_y(x) = (x, y)$ are both continuous. Suppose that both X and Y are connected topological spaces. Select a fixed point $y_0 \in Y$. By Theorem 10.17 the set of points $X \times y_0 = \{(x, y_0) \mid x \in X\} = \iota'_{y_0}(X)$ is a connected subset of $X \times Y$. Similarly, the sets $\{x \times Y = \iota_x(Y) \mid x \in X\}$ are connected subsets of $X \times Y$, each of which intersects $X \times y_0$ in the point (x, y_0). The union of these sets is clearly $X \times Y$, which by Theorem 10.16 must be connected.

Conversely, suppose $X \times Y$ is connected. The spaces X and Y are both connected, by Theorem 10.17, since they are the images of the continuous projection maps $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$, respectively. ■

Example 10.19 The spaces $\mathbb{R}^n$ are connected by Example 10.18 and Theorem 10.18. To show that the 2-sphere S^2 is connected consider the 'punctured' spheres $S' = S^2 - \{N = (0, 0, 1)\}$ and $S'' = S^2 - \{S = (0, 0, -1)\}$ by removing the north and south poles, respectively. The set S' is connected since it is homeomorphic to the plane $\mathbb{R}^2$ under stereographic projection (Fig. 10.6),

$$x' = \frac{x}{1 - z}, \quad y' = \frac{y}{1 - z} \quad \text{where} \quad z = \pm\sqrt{1 - x^2 - y^2}, \tag{10.1}$$

which has continuous inverse

$$x = \frac{2x'}{r'^2 + 1}, \quad y = \frac{2y'}{r'^2 + 1}, \quad z = \frac{r'^2 - 1}{r'^2 + 1} \quad \left(r'^2 = x'^2 + y'^2\right). \tag{10.2}$$

Similarly S'' is connected since it is homeomorphic to $\mathbb{R}^2$. As $S' \cap S'' \neq \emptyset$ and $S^2 = S' \cup S''$ it follows from Theorem 10.16 that S^2 is a connected subset of $\mathbb{R}^3$. A similar argument can be used to show that the n-sphere S^n is a connected topological space for all $n \geq 1$.

A **connected component** C of a topological space X is a maximal connected set; that is, C is a connected subset of X such that if $C' \supseteq C$ is any connected superset of C then $C' = C$. A **connected component of a subset** $A \subset X$ is a connected component with respect to the relative topology on A. By Theorem 10.15 it is immediate that any connected component is a closed set, since it implies that $C = \overline{C}$. A topological space X is connected if and only if the whole space X is its only connected component. In the discrete topology the connected components consist of all singleton sets $\{x\}$.

Exercise: Show that any two distinct components A and B are *separated*, in the sense that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Theorem 10.19 *Each connected subset A of a topological space lies in a unique connected component.*

Proof: Let C be the union of all connected subsets of X that contain the set A. Since these sets all intersect the connected subset A it follows from Theorem 10.16 that C is a connected set. It is clearly maximal, for if there exists a connected set C_1 such that $C \subseteq C_1$, then C_1 is in the family of sets of which C is the union, so that $C \supset C_1$. Hence $C = C_1$.

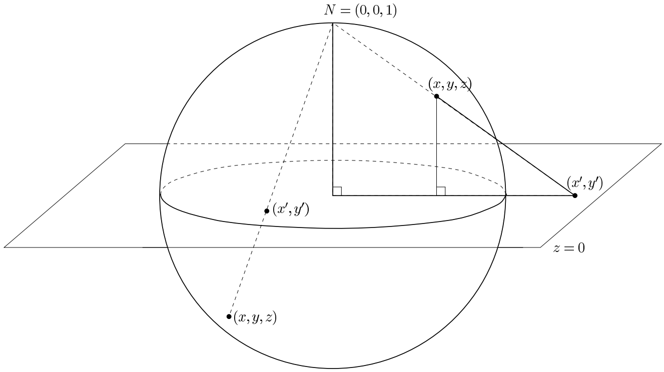

Figure 10.6 Stereographic projection from the north pole of a sphere

To prove uniqueness, suppose C' were another connected component such that $C' \supset A$. By Theorem 10.16, $C \cup C'$ is a connected set and by maximality of C and C' we have $C \cup C' = C = C'$. ■

Problems

Problem 10.18 Show that a topological space X is connected if and only if every continuous map $f : X \to Y$ of X into a discrete topological space Y consisting of at least two points is a constant map (see Problem 10.14).

Problem 10.19 From Theorem 10.16 show that the unit circle S^1 is connected, and that the punctured n-space $\dot{\mathbb{R}}^n = \mathbb{R}^n - \{\mathbf{0}\}$ is connected for all $n > 1$. Why is this not true for $n = 1$?

Problem 10.20 Show that the real projective space defined in Example 10.15 is connected, Hausdorff and compact.

Problem 10.21 Show that the rational numbers $\mathbb{Q}$ are a disconnected subset of the real numbers. Are the irrational points a disconnected subset of $\mathbb{R}$? Show that the connected components of the rational numbers $\mathbb{Q}$ consist of singleton sets $\{x\}$.

10.8 Topological groups

There are a number of useful ways in which topological and algebraic structure can be combined. The principal requirement connecting the two types of structure is that the functions representing the algebraic laws of composition be continuous with respect to the topology imposed on the underlying set. In this section we combine group theory with topology.

A **topological group** is a set G that is both a group and a Hausdorff topological space such that the map $\psi : G \times G \to G$ defined by $\psi(g, h) = gh^{-1}$ is continuous. The topological group G is called **discrete** if the underlying topology is discrete.

The maps $\phi : G \to G$ and $\tau : G \to G$ defined by $\phi(g, h) = gh$ and $\tau(g) = g^{-1}$ are both continuous. For, by Theorem 10.3 the injection map $i : G \to G \times G$ defined by $i(h) = \iota_e(h) = (e, h)$ is continuous. The map τ is therefore continuous since it is a composition of continuous maps, $\tau : \psi \circ i$. Since $\phi(g, h) = \psi(g, \tau(h))$ it follows immediately that ϕ is also a continuous map.

Exercise: Show that τ is a homeomorphism of G.

Exercise: If ϕ and τ are continuous maps, show that ψ is continuous.

Example 10.20 The additive group $\mathbb{R}^n$, where the 'product' is vector addition

$$\phi(\mathbf{x}, \mathbf{y}) = (x^1, \dots, x^n) + (y^1, \dots, y^n) = (x^1 + y^1, \dots, x^n + y^n)$$

and the inverse map is

$$\tau(\mathbf{x}) = -\mathbf{x} = (-x^1, \dots, -x^n),$$

is an abelian topological group with respect to the Euclidean topology on $\mathbb{R}^n$. The n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is also an abelian topological group, where group composition is addition modulo 1.

Example 10.21 The set $M_n(\mathbb{R})$ of $n \times n$ real matrices has a topology homeomorphic to the Euclidean topology on $\mathbb{R}^{n^2}$. The determinant map $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is clearly continuous since $\det \mathsf{A}$ is a polynomial function of the components of A. Hence the general linear group $GL(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R})$ since it is the inverse image of the open set $\dot{\mathbb{R}} = \mathbb{R} - \{0\}$ under the determinant map. If $GL(n, \mathbb{R})$ is given the induced relative topology in $M_n(\mathbb{R})$ then the map ψ reads in components,

$$(\psi(\mathsf{A}, \mathsf{B}))_{ij} = \sum_{k=1}^{n} A_{ik}\big(B^{-1}\big)_{kj}.$$

These are continuous functions since $\big(B^{-1}\big)_{ij}$ are rational polynomial functions of the components B_{ij} with non-vanishing denominator $\det \mathsf{B}$.

A subgroup H of G together with its relative topology is called a **topological subgroup** of G. To show that any subgroup H becomes a topological subgroup with respect to the relative topology, let $U' = H \cap U$ where U is an arbitrary open subset of G. By continuity of the map ϕ, for any pair of points $g, h \in H$ such that $gh \in U' \subset U$ there exist open sets A and B of G such that $A \times B \subset \phi^{-1}(U)$. It follows that $\phi(A' \times B') \subset H \cap U$, where $A' = A \cap H$, $B' = B \cap H$, and the continuity of $\phi\big|_H$ is immediate. Similarly the inverse map τ is continuous when restricted to H. If H is a closed set in G, it is called a **closed subgroup** of G.

For each $g \in G$ let the **left translation** $L_g : G \to G$ be the map

$$L_g(h) \equiv L_g h = gh,$$

as defined in Example 2.25. The map L_g is continuous since it is the composition of two continuous maps, $L_g = \phi \circ \iota_g$, where $\iota_g : G \to G \times G$ is the injection map $\iota_g(h) = (g, h)$

(see Theorem 10.3). It is clearly one-to-one, for $gh = gh' \Longrightarrow h = g^{-1}gh' = h'$, and its inverse is the continuous map $L_{g^{-1}}$. Hence L_g is a homeomorphism. Similarly, every **right translation** $R_g : G \to G$ defined by $R_g h = hg$ is a homeomorphism of G, as is the **inner automorphism** $C_g : G \to G$ defined by $C_g h = ghg^{-1} = L_h \circ R_{h^{-1}}(g)$.

Connected component of the identity

If G is a topological group we will denote by G_0 the connected component containing the identity element e, simply referred to as the **component of the identity**.

Theorem 10.20 *Let G be a topological group, and G_0 the component of the identity. Then G_0 is a closed normal subgroup of G.*

Proof: By Theorem 10.17 the set $G_0 g^{-1}$ is connected, since it is a continuous image under right translation by g^{-1} of a connected set. If $g \in G_0$ then $e = gg^{-1} \in G_0 g^{-1}$. Hence $G_0 g^{-1}$ is a closed connected subset containing the identity e, and must therefore be a subset of G_0. We have therefore $G_0 G_0^{-1} \subseteq G_0$, showing that G_0 is a subgroup of G. Since it is a connected component of G it is a closed set. Thus, G_0 is a closed subgroup of G.

For any $g \in G$, the set $gG_0 g^{-1}$ is connected as it is the image of G_0 under the inner automorphism $h \mapsto C_g(h)$. Since this set contains the identity e, we have $gG_0 g^{-1} \subseteq G_0$, and G_0 is a normal subgroup. ■

A topological space X is said to be **locally connected** if every neighbourhood of every point of X contains a connected open neighbourhood. A topological group G is locally connected if it is locally connected at the identity e, for if V is a connected open neighbourhood of e then $gV = L_g V$ is a connected open neighbourhood of any selected point $g \in G$. If K is any subset of a group G, we call the smallest subgroup of G that contains K the **subgroup generated by** K. It is the intersection of all subgroups of G that contain K.

Theorem 10.21 *In any locally connected group G the component of the identity G_0 is generated by any connected neighbourhood of the identity e.*

Proof: Let V be any connected neighbourhood of e, and H the subgroup generated by V. For any $g \in H$, the left coset $gV = L_g V \subset H$ is a neighbourhood of g since L_g is a homeomorphism. Hence H is an open subset of G. On the other hand, if H is an open subgroup of G it is also closed since it is the complement in G of the union of all cosets of H that differ from H itself. Thus H is both open and closed. It is therefore the connected component of the identity, G_0. ■

Let H be a closed subgroup of a topological group G, we can give the factor space G/H the natural topology induced by the canonical projection map $\pi : g \mapsto gH$. This is the finest topology on G/H such that π is a continuous map. In this topology a collection of cosets $U \subseteq G/H$ is open if and only if their union is an open subset of G. Clearly π is an **open map** with respect to this topology, meaning that $\pi(V)$ is open for all open sets $V \subseteq G$.

Theorem 10.22 *If G is a topological group and H a closed connected subgroup such that the factor space G/H is connected, then G is connected.*

Proof: Suppose G is not connected. There then exist open sets U and V such that $G = U \cup V$, with $U \cap V = \emptyset$. Since π is an open map the sets $\pi(U)$ and $\pi(V)$ are open in G/H and $G/H = \pi(U) \cup \pi(V)$. But G/H is connected, $\pi(U) \cap \pi(V) \neq \emptyset$, so there exists a coset $gH \in \pi(U) \cap \pi(V)$. As a subset of G this coset clearly meets both U and V, and $gH = (gH \cap U) \cup (gH \cap V)$, contradicting the fact that gH is connected (since it is the image under the continuous map L_g of a connected set H). Hence G is connected. ■

Example 10.22 The general linear group $GL(n, \mathbb{R})$ is not connected since the determinant map $\det : GL(n, \mathbb{R}) \to \mathbb{R}$ has image $\dot{\mathbb{R}} = \mathbb{R} - \{0\}$, which is a disconnected set. The component G_0 of the identity I is the set of $n \times n$ matrices with determinant > 0, and the group of components is discrete

$$GL(n, \mathbb{R})/G_0 \cong \{1, -1\} = Z_2.$$

Note, however, that the complex general linear group $GL(n, \mathbb{C})$ is connected, as may be surmised from the fact that the Jordan canonical form of any non-singular complex matrix can be continuously deformed to the identity matrix I.

The special orthogonal groups $SO(n)$ are all connected. This can be shown by induction on the dimension n. Evidently $SO(1) = \{1\}$ is connected. Assume that $SO(n)$ is connected. It will be shown in Chapter 19, Example 19.10, that $SO(n+1)/SO(n)$ is homeomorphic to the n-sphere S^n. As this is a connected set (see Example 10.19) it follows from Theorem 10.22 that $SO(n+1)$ is connected. By induction, $SO(n)$ is a connected group for all $n = 1, 2, \dots$ However the orthogonal groups $O(n)$ are not connected, the component of the identity being $SO(n)$ while the remaining orthogonal matrices have determinant -1.

Similarly, $SU(1) = \{1\}$ and $SU(n+1)/SU(n) \cong S^{2n-1}$, from which it follows that all special unitary groups $SU(n)$ are connected. By Theorem 10.22 the unitary groups $U(n)$ are also all connected, since $U(n)/SU(n) = S^1$ is connected.

Problem

Problem 10.22 If G_0 is the component of the identity of a locally connected topological group G, the factor group G/G_0 is called the **group of components of** G. Show that the group of components is a discrete topological group with respect to the topology induced by the natural projection map $\pi : g \mapsto gG_0$.

10.9 Topological vector spaces

A **topological vector space** is a vector space V that has a Hausdorff topology defined on it, such that the operations of vector addition and scalar multiplication are continuous functions on their respective domains with respect to this topology,

$$\psi : V \times V \to V \quad \text{defined by} \quad \psi(u, v) = u + v,$$
$$\tau : \mathbb{K} \times V \to V \quad \text{defined by} \quad \tau(\lambda, v) = \lambda v.$$

We will always assume that the field of scalars is either the real or complex numbers, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$; in the latter case the topology is the standard topology in $\mathbb{R}^2$.

Recall from Section 10.3 that a sequence of vectors $v_n \in V$ is called **convergent** if there exists a vector $v \in V$, called its **limit**, such that for every open neighbourhood U of v there is an integer N such that $v_n \in U$ for all $n \geq N$. We also say the sequence **converges to** v, denoted

$$v_n \to v \quad \text{or} \quad \lim_{n\to\infty} v_n = v.$$

The following properties of convergent sequences are easily proved:

$$v_n \to v \quad \text{and} \quad v_n \to v' \implies v = v', \tag{10.3}$$

$$v_n = v \text{ for all } n \implies v_n \to v, \tag{10.4}$$

$$\text{if } \{v'_n\} \text{ is a subsequence of } v_n \to v \quad \text{then} \quad v'_n \to v, \tag{10.5}$$

$$u_n \to u, \quad v_n \to v \implies u_n + \lambda v_n \to u + \lambda v, \tag{10.6}$$

where $\lambda \in \mathbb{K}$ is any scalar. Also, if λ_n is a convergent sequence of scalars in $\mathbb{K}$ then

$$\lambda_n \to \lambda \implies \lambda_n u \to \lambda u. \tag{10.7}$$

Example 10.23 The vector spaces $\mathbb{R}^n$ are topological vector spaces with respect to the Euclidean topology. It is worth giving the full proof of this statement, as it sets the pattern for a number of other examples. A set $U \subset \mathbb{R}^n$ is open if and only if for every $\mathbf{x} \in U$ there exists $\epsilon > 0$ such that

$$I_\epsilon(\mathbf{x}) = \{\mathbf{y} \mid |y_i - x_i| < \epsilon, \text{ for all } i = 1, \dots, n\} \subseteq U.$$

To show that vector addition ψ is continuous, it is necessary to show that $N = \psi^{-1}(I_\epsilon(\mathbf{x}))$ is an open subset of $\mathbb{R}^n \times \mathbb{R}^n$ for all $\mathbf{x} \in \mathbb{R}^n$, $\epsilon > 0$. If $\psi(\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v} = \mathbf{x}$ then for any $(\mathbf{u}', \mathbf{v}') \in I_{\epsilon/2}(\mathbf{u}) \times I_{\epsilon/2}(\mathbf{v})$, we have for all $i = 1, \dots, n$

$$\begin{aligned} \big|\big(x_i - (u'_i + v'_i)\big)\big| &= \big|\big((u_i - u'_i) + (v_i - v'_i)\big)\big| \\ &\leq \big|\big((u_i - u'_i)\big)\big| + \big|\big((v_i - v'_i)\big)\big| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $I_{\epsilon/2}(\mathbf{u}) \times I_{\epsilon/2}(\mathbf{v}) \subset N$, and continuity of ψ is proved.

For continuity of the scalar multiplication function τ, let $M = \tau^{-1}(I_\epsilon(\mathbf{x})) \subset \mathbb{R} \times \mathbb{R}^n$. If $\mathbf{x} = a\mathbf{u}$, let $\mathbf{v} \in I_\delta(\mathbf{u})$ and $b \in I_{\delta'}(a)$. Then, setting $A = \max_i |u_i|$, we have

$$\begin{aligned} |bv_i - au_i| &= |bv_i - bu_i + bu_i - au_i| \\ &\leq |b||v_i - u_i| + |b - a||u_i| \\ &\leq (|a| + \delta')\delta + \delta' A \\ &\leq \epsilon \quad \text{if } \delta' = \frac{\epsilon}{2A} \text{ and } \delta = \frac{\epsilon}{2|a|A + \epsilon}. \end{aligned}$$

A similar proof may be used to show that the complex vector space $\mathbb{C}^n$ is a topological vector space.

Example 10.24 The vector space $\mathbb{R}^\infty$ consisting of all infinite sequences $\mathbf{x} = (x_1, x_2, \dots)$ is an infinite dimensional vector space. We give it the product topology as described, whereby a set U is open if for every point $\mathbf{x} \in U$ there is a finite sequence of integers $\mathbf{i} = (i_1, i_2, \dots, i_n)$ such that

$$I_{\mathbf{i},\epsilon}(\mathbf{x}) = \{\mathbf{y} \mid \left|y_{i_k} - x_{i_k}\right| < \epsilon \text{ for } k = 1, \dots, n\} \subset U.$$

This neighbourhood of $\mathbf{x}$ is an infinite product of intervals of which all but a finite number consist of all of $\mathbb{R}$. To prove that ψ and τ are continuous functions, we again need only show that $\psi^{-1}\big(I_{\mathbf{i},\epsilon}(\mathbf{x})\big)$ and $\tau^{-1}\big(I_{\mathbf{i},\epsilon}(\mathbf{x})\big)$ are open sets. The argument follows along essentially identical lines to that in Example 10.23. To prove continuity of the scalar product τ, we set $A = \max_{i \in \mathbf{i}} |u_i|$, where $\mathbf{x} = a\mathbf{u}$ and continue as in the previous example.

Example 10.25 Let S be any set, and set $\mathcal{F}(S)$ to be the set of bounded real-valued functions on S. This is obviously a vector space, with vector addition defined by $(f + g)(x) = f(x) + g(x)$ and scalar multiplication by $(af)(x) = af(x)$. A metric can be defined on this space by setting $d(f, g)$ to be the least upper bound of $|f(x) - g(x)|$ on S. Conditions (Met1)–(Met4) are easy to verify. The vector space $\mathcal{F}(S)$ is a topological vector space with respect to the metric topology generated by this distance function. For example, let $f(x) = u(x) + v(x)$, then if $|u'(x) - u(x)| < \epsilon/2$ and $|v'(x) - v(x)| < \epsilon/2$ it follows at once that $|f(x) - (u'(x) + v'(x))| < \epsilon$. To prove continuity of scalar addition we again proceed as in Example 10.23. If $f(x) = au(x)$, let $|b - a| < \epsilon/2A$ where A is an upper bound of $u(x)$ in S and $|v(x) - u(x)| < \epsilon/(2|a|A + \epsilon)$ for all $x \in S$; then

$$|bv(x) - f(x)| < \epsilon \qquad \text{for all } x \in S.$$

Banach spaces

A **norm** on a vector space V is a map $\|\cdot\| : V \to \mathbb{R}$, associating a real number $\|v\|$ with every vector $v \in V$, such that

(Norm1) $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$.
(Norm2) $\|\lambda v\| = |\lambda| \, \|v\|$.
(Norm3) $\|u + v\| \leq \|u\| + \|v\|$.

In most cases the field of scalars is taken to be the complex numbers, $\mathbb{K} = \mathbb{C}$, although much of what we say also applies to real normed spaces. We have met this concept earlier, in the context of a complex inner product space (see Section 5.2).

A norm defines a distance function $d : V \times V \to \mathbb{R}$ by

$$d(u, v) = \|u - v\|.$$

The properties (Met1)–(Met3) are trivial to verify, while the triangle inequality

$$d(u, v) \le d(u, w) + d(w, v) \tag{10.8}$$

is an immediate consequence of (Norm3),

$$\|u - v\| = \|u - w + w - v\| \le \|u - w\| + \|w - v\|.$$

We give V the standard metric topology generated by open balls $B_a(v) = \{u \mid d(u, v) < a\}$ as in Section 10.3. This makes it into a topological vector space. To show that the function $(u, v) \mapsto u + v$ is continuous with respect to this topology,

$$\|u' - u\| < \frac{\epsilon}{2} \text{ and } \|v' - v\| < \frac{\epsilon}{2} \implies \|u' + v' - (u + v)\| < \epsilon$$

on using the triangle inequality. The proof that $(\lambda, v) \mapsto \lambda v$ is continuous follows the lines of Example 10.23.

Exercise: Show that the 'norm' is a continuous function $\|\cdot\| : V \to \mathbb{R}$.

Example 10.26 The vector space $\mathcal{F}(S)$ of bounded real-valued functions on a set S defined in Example 10.25 has a norm

$$\|f\| = \sup_{x \in S} |f(x)|,$$

giving rise to the distance function $d(f, g)$ of Example 10.25. This is called the *supremum norm*.

Convergence of sequences is defined on V by

$$u_n \to u \text{ if } d(u_n, u) = \|u_n - u\| \to 0.$$

As in Section 10.3, every convergent sequence $u_i \to u$ is a Cauchy sequence

$$\|u_i - u_j\| \le \|u - u_i\| + \|u - u_j\| \ \to\ 0 \text{ as } i, j \to \infty,$$

but the converse need not always hold. We say a normed vector space $(V, \|\cdot\|)$ is **complete**, or is a **Banach space**, if every Cauchy sequence converges,

$$\|u_i - u_j\| \to 0 \text{ as } i, j \to \infty \implies u_i \to u \text{ for some } u \in V.$$

Exercise: Give an example of a vector subspace of $\mathbb{C}^\infty$ that is an incomplete normed vector space.

Example 10.27 On the vector space $\mathbb{C}^n$ define the standard norm

$$\|\mathbf{x}\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}.$$

Conditions (Norm1) and (Norm2) are trivial, while (Norm3) follows from Theorem 5.6, since this norm is precisely that defined in Eq. (5.11) from the inner product $\langle x|y\rangle = \sum_{i=1}^{n} \overline{x_i} y_i$. If $\|\mathbf{x}_n - \mathbf{x}_m\| \to 0$ for a sequence of vectors $\mathbf{x}_n$, then each component is a Cauchy sequence $|x_{ni} - x_{mi}| \to 0$, and therefore has a limit $x_n i \to x_i$. It is straightforward to show that $\|\mathbf{x}_n - \mathbf{x}\| = 0$ where $\mathbf{x} = (x_1, x_2, \dots, x_n)$. Hence this normed vector space is complete.

Example 10.28 Let $\mathcal{D}([-1, 1])$ be the vector space of bounded differentiable complex-valued functions on the closed interval $[-1, 1]$. As in Example 10.26 we adopt the supremum norm $\|f\| = \sup_{x\in[-1,1]} |f(x)|$. This normed vector space is not complete, for consider the sequence of functions

$$f_n(x) = \big|x\big|^{1+1/n} \quad (n = 1, 2, \dots).$$

These functions are all differentiable on $[-1, 1]$ and have zero derivative at $x = 0$ from both the left and the right. Since they approach the bounded function $|x|$ as $n \to \infty$ this is necessarily a Cauchy sequence. However, the limit function $|x|$ is not differentiable at $x = 0$, and the norm is incomplete.

By a **linear functional** on a Banach space V we always mean a *continuous* linear map $\varphi : V \to \mathbb{C}$. The vector space of all linear functionals on V is called the **dual space** V' of V. If V is finite dimensional then V' and V^* coincide, since all linear functionals are continuous with respect to the norm

$$\|\mathbf{u}\| = \sqrt{|u_1|^2 + \dots + |u_n|^2},$$

but for infinite dimensional spaces it is important to stipulate the continuity requirement. A linear map φ on a Banach space is said to be **bounded** if there exists $M > 0$ such that

$$|\varphi(x)| \le M\|x\| \qquad \text{for all } x \in V.$$

The following theorem shows that the words 'bounded' and 'continuous' are interchangeable for linear functionals on a Banach space.

Theorem 10.23 *A linear functional $\varphi : V \to \mathbb{R}$ on a Banach space V is continuous if and only if it is bounded.*

Proof: If φ is bounded, let $M > 0$ be such that $|\varphi(x)| \le M\|x\|$ for all $x \in V$. Then for any pair of vectors $x, y \in V$

$$|\varphi(x - y)| \le M\|x - y\|,$$

and for any $\epsilon > 0$ we have

$$\|x - y\| < \frac{\epsilon}{M} \Longrightarrow |\varphi(x) - \varphi(y)| = |\varphi(x - y)| \le \epsilon.$$

Hence φ is continuous.

Conversely, suppose φ is continuous. In particular, it is continuous at the origin $x = 0$ and there exists $\delta > 0$ such that

$$\|x\| < \delta \Longrightarrow |\varphi(x)| < 1.$$

For any vector $y \in V$ we have

$$\left\| \frac{\delta y}{\|y\|} \right\| < \delta$$

whence

$$\left| \varphi\Big(\frac{\delta y}{\|y\|} \Big) \right| < 1.$$

Thus

$$|\varphi(y)| < \frac{\|y\|}{\delta},$$

showing that φ is bounded. ■

Example 10.29 Let ℓ^1 be the vector space of all complex infinite sequences $\mathbf{x} = (x_1, x_2, \dots)$ that are absolutely convergent,

$$\|\mathbf{x}\| = \sum_{i=1}^{\infty} |x_i| \ < \ \infty.$$

If $c = (c_1, c_2, \dots)$ is a bounded infinite sequence of complex numbers, $|c_i| \le C$ for all $i = 1, 2, \dots$, then

$$\varphi_c(\mathbf{x}) = c_1 x_1 + c_2 x_2 + \dots$$

is a continuous linear functional on ℓ^1. Linearity is obvious as long as the series converges, and convergence and boundedness are proved in one step,

$$\|\varphi_c(\mathbf{x})\| = \sum_{i=1}^{\infty} |c_i x_i| \le |C| \sum_{i=1}^{\infty} |x_i| = |C| \, \|\mathbf{x}\| < \infty.$$

Hence φ_c is a continuous linear operator by Theorem 10.23.

Problems

Problem 10.23 Prove the properties (10.3)–(10.7).

Problem 10.24 Show that a linear map $T : V \to W$ between topological vector spaces is continuous everywhere on V if and only if it is continuous at the origin $0 \in V$.

Problem 10.25 Give an example of a linear map $T : V \to W$ between topological vector spaces V and W that is not continuous.

Problem 10.26 Complete the proof that a normed vector space is a topological vector space with respect to the metric topology induced by the norm.

Problem 10.27 Show that a real vector space V of dimension ≥ 1 is not a topological vector space with respect to either the discrete or indiscrete topology.

Problem 10.28 Show that the following are all norms in the vector space $\mathbb{R}^2$:

$$\|\mathbf{u}\|_1 = \sqrt{(u_1)^2 + (u_2)^2},$$
$$\|\mathbf{u}\|_2 = \max\{|u_1|, |u_2|\},$$
$$\|\mathbf{u}\|_3 = |u_1| + |u_2|.$$

What are the shapes of the open balls $B_a(\mathbf{u})$? Show that the topologies generated by these norms are the same.

Problem 10.29 Show that if $x_n \to x$ in a normed vector space then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \to x.$$

Problem 10.30 Show that if x_n is a sequence in a normed vector space V such that every subsequence has a subsequence convergent to x, then $x_n \to x$.

Problem 10.31 Let V be a Banach space and W be a vector subspace of V. Define its *closure* $\overline{W}$ to be the union of W and all limits of Cauchy sequences of elements of W. Show that $\overline{W}$ is a closed vector subspace of V in the sense that the limit points of all Cauchy sequences in $\overline{W}$ lie in $\overline{W}$ (note that the Cauchy sequences may include the newly added limit points of W).

Problem 10.32 Show that every space $\mathcal{F}(S)$ is complete with respect to the supremum norm of Example 10.26. Hence show that the vector space ℓ_∞ of bounded infinite complex sequences is a Banach space with respect to the norm $\|\mathbf{x}\| = \sup(x_i)$.

Problem 10.33 Show that the set V' consisting of bounded linear functionals on a Banach space V is a normed vector space with respect to the norm

$$\|\varphi\| = \sup\{M \mid |\varphi(x)| \le M\|x\| \text{ for all } x \in V\}.$$

Show that this norm is complete on V'.

Problem 10.34 We say two norms $\|u\|_1$ and $\|u\|_2$ on a vector space V are *equivalent* if there exist constants A and B such that

$$\|u\|_1 \le A\|u\|_2 \quad \text{and} \quad \|u\|_2 \le B\|u\|_1$$

for all $u \in V$. If two norms are equivalent then show the following:

(a) If $u_n \to u$ with respect to one norm then this is also true for the other norm.
(b) Every linear functional that is continuous with respect to one norm is continuous with respect to the other norm.
(c) Let $V = \mathcal{C}[0, 1]$ be the vector space of continuous complex functions on the interval $[0, 1]$. By considering the sequence of functions

$$f_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$

show that the norms

$$\|f\|_1 = \sqrt{\int_0^1 |f|^2 \, dx} \quad \text{and} \quad \|f\|_2 = \max\{|f(x)| \mid 0 \le x \le 1\}$$

are not equivalent.

(d) Show that the linear functional defined by $F(f) = f(1)$ is continuous with respect to $\|\cdot\|_2$ but not with respect to $\|\cdot\|_1$.

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