MATH-2790 Differential Equations
HW assignment 4 – Solution

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1. Using a change of variables rewrite the following equation as separable

   a. $(7x+2y)(5x+9y)\,dx + (10x+5y)(7x+4y)\,dy = 0$

   Solution:
   Opening the parentheses we obtain

   $$\left(35x^2+73xy+18y^2\right)dx + \left(70x^2+75xy+20y^2\right)dy = 0.$$

   Switching to $u := \frac{y}{x}, x$ we obtain

   $$x^2\left(\left(35+73u+18u^2\right)dx + \left(70+75u+20u^2\right)(udx+xdu)\right) = 0.$$

   Dividing by x^2 and simplifying we have

   $$\left(35+143u+93u^2+20u^3\right)dx + x\left(70+75u+20u^2\right)du = 0,$$

   and after separating the variables

   $$\frac{1}{x}dx + \frac{70+75u+20u^2}{35+143u+93u^2+20u^3}du = 0.$$

   b. $(2x+11y)(2x+5y)\,dx + (7x+3y)(7x+4y)\,dy = 0$

   Solution:
   Opening the parentheses we obtain

   $$\left(4x^2+32xy+55y^2\right)dx + \left(49x^2+49xy+12y^2\right)dy = 0.$$

   Switching to $u := \frac{y}{x}, x$ we obtain

   $$x^2\left(\left(4+32u+55u^2\right)dx + \left(49+49u+12u^2\right)(udx+xdu)\right) = 0.$$

   Dividing by x^2 and simplifying we have

   $$\left(4+81u+104u^2+12u^3\right)dx + x\left(49+49u+12u^2\right)du = 0,$$

   and after separating the variables

   $$\frac{1}{x}dx + \frac{49+49u+12u^2}{4+81u+104u^2+12u^3}du = 0.$$

2. Solve the following initial value problem

$$x^2 + 2y^2 = xyy', \quad y(-1) = 1.$$

Solution:
The coefficients in front of dx and dy are

$$x^2 + 2y^2, \quad -xy.$$

These functions are both homogeneous of degree 2. Therefore we introduce a new parameter

$$u := \frac{y}{x},$$

and calculate

$$dy = xdu + udx.$$

Substituting u instead of $\frac{y}{x}$ and $xdu + udx$ instead of dy in the original equation we obtain

$$x^2(1 + 2u^2)dx - x^2u(xdu + udx) = 0.$$

Cancelling x^2 and rearranging we have

$$\frac{dx}{x} = \frac{u}{1+u^2}du,$$

which is a separable equation. Taking indefinite integrals on both sides we obtain

$$\ln|x| = \frac{1}{2}\ln\left(1 + u^2\right) + C,$$

where C is the constant of integration. Taking the exponential of both sides we have

$$x = D\sqrt{1+u^2},$$

where we have denoted $D := -e^C$, and used the initial condition $x = -1$ to assume that $x < 0$. Then

$$u = \pm\sqrt{\frac{x^2}{D} - 1},$$

and going back to y we have

$$y = xu = -x\sqrt{\frac{x^2}{D} - 1}.$$

The $-$ sign is due to the fact that $y(-1) = 1 > 0$ according to the initial conditions. To find the value of D we substitute the values of x, y and obtain

$$1 = -(-1)\sqrt{\frac{(-1)^2}{D} - 1} \Rightarrow D = \frac{1}{2}.$$

Therefore a solution to the initial value problem is

$$y = -x\sqrt{2x^2-1}, \quad x < -\sqrt{\frac{1}{2}}.$$

3. Compute the Wronskian

a. $W\left\{7x^2+8x, 13x^2+10x+2, 10x^2+18x+2\right\}$

Solution:
First we compute derivatives of the given functions

$$\begin{aligned}
&\left(7x^2+8x\right)' = 14x+8, \quad \left(7x^2+8x\right)'' = 14,\\
&\left(13x^2+10x+2\right)' = 26x+10, \quad \left(13x^2+10x+2\right)'' = 26,\\
&\left(10x^2+18x+2\right)' = 20x+18, \quad \left(10x^2+18x+2\right)'' = 20.
\end{aligned}$$

Then we compute the Wronskian

$$W = \begin{vmatrix} 7x^2+8x & 13x^2+10x+2 & 10x^2+18x+2 \\ 14x+8 & 26x+10 & 20x+18 \\ 14 & 26 & 20 \end{vmatrix} = 320.$$

b. $W\left\{6x^2+11x, 8x^2+6x+9, 8x^2+17x+9\right\}$

Solution:
First we compute derivatives of the given functions

$$\begin{aligned}
&\left(6x^2+11x\right)' = 12x+11, \quad \left(6x^2+11x\right)'' = 12,\\
&\left(8x^2+6x+9\right)' = 16x+6, \quad \left(8x^2+6x+9\right)'' = 16,\\
&\left(8x^2+17x+9\right)' = 16x+17, \quad \left(8x^2+17x+9\right)'' = 16.
\end{aligned}$$

Then we compute the Wronskian

$$W = \begin{vmatrix} 6x^2+11x & 8x^2+6x+9 & 8x^2+17x+9 \\ 12x+11 & 16x+6 & 16x+17 \\ 12 & 16 & 16 \end{vmatrix} = 1188.$$

4. Find a fundamental set of solutions for the following equation

a. $11y'' - 231y' + 1144y = 0$

Solution:
The differential operator corresponding to this equation is

$$\mathcal{D} = 11\frac{d^2}{dx^2} - 231\frac{d}{dx} + 1144,$$

and the polynomial corresponding to this operator is

$$P_{\mathcal{D}}(\lambda) = 11\lambda^2 - 231\lambda + 1144.$$

Breaking this polynomial into linear factors we obtain

$$P_{\mathcal{D}}(\lambda) = 11\left(\lambda - 8\right)\left(\lambda - 13\right).$$

Hence a fundamental set of solutions of the ODE is given by

$$\left\{e^{13x}, \quad e^{8x}\right\}.$$

b. $5y'' - 100y' + 480y = 0$

Solution:
The differential operator corresponding to this equation is

$$\mathcal{D} = 5\frac{d^2}{dx^2} - 100\frac{d}{dx} + 480,$$

and the polynomial corresponding to this operator is

$$P_{\mathcal{D}}(\lambda) = 5\lambda^2 - 100\lambda + 480.$$

Breaking this polynomial into linear factors we obtain

$$P_{\mathcal{D}}(\lambda) = 5\left(\lambda - 8\right)\left(\lambda - 12\right).$$

Hence a fundamental set of solutions of the ODE is given by

$$\left\{e^{12x}, \quad e^{8x}\right\}.$$