The Fourier Transform

DD2423 Image Analysis and Computer Vision

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Relation continuous/discrete Fourier transform

- Continuous

$$\hat{f}(\omega) = \int_{x \in \mathbb{R}^n} f(x) e^{-i\omega^T x} dx$$

- Discrete

$$\hat{f}(u) = \frac{1}{\sqrt{M}^n} \sum_{x \in I^n} f(x) e^{-\frac{2\pi i u^T x}{M}}$$

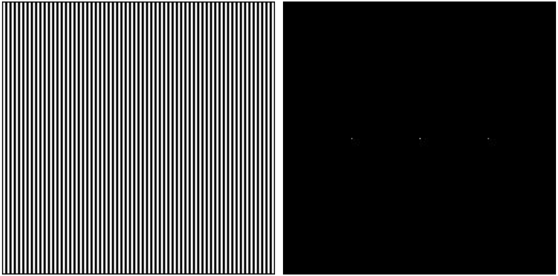
- Frequency variables are related (in 1D) by

$$\omega = \frac{2\pi u}{M}$$

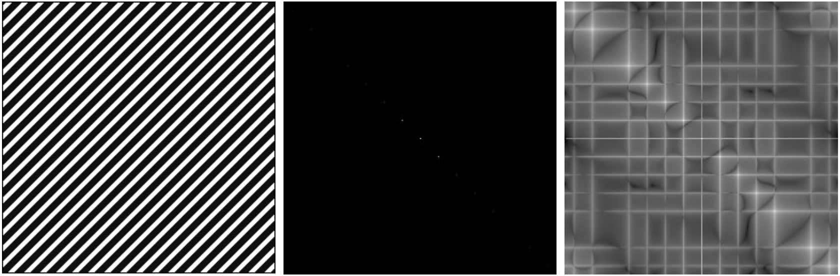
- Note: u assumes values $0 \ldots M-1 \Rightarrow \omega \in [0, 2\pi)$.
- By periodic extension, we can map this integral to $[-\pi, \pi)$.

More insight (2 pixel wide stripes)

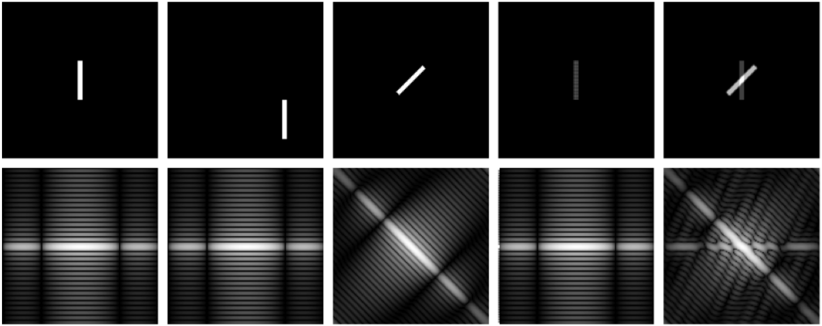
- The maximum frequency which can be represented in the spatial domain are one pixel wide stripes (period=2): $\omega_{max} = 2\pi\frac{1}{2} = \pi$
- So, 2 pixel wide stripes (period=4) give $\omega = 2\pi\frac{1}{4} = \frac{1}{2}\omega_{max}$
- Plotting magnitude of the Fourier transform:
  - Two points are halfway between the center and the edge of the image, i.e. the represented frequency is half of the maximum.
  - One point in the middle shows the DC-value (image mean).



- If you take the logarithm of the Fourier transform, you see many minor frequencies. Since an image can only be represented by square pixels, discretization noise appear along diagonals.



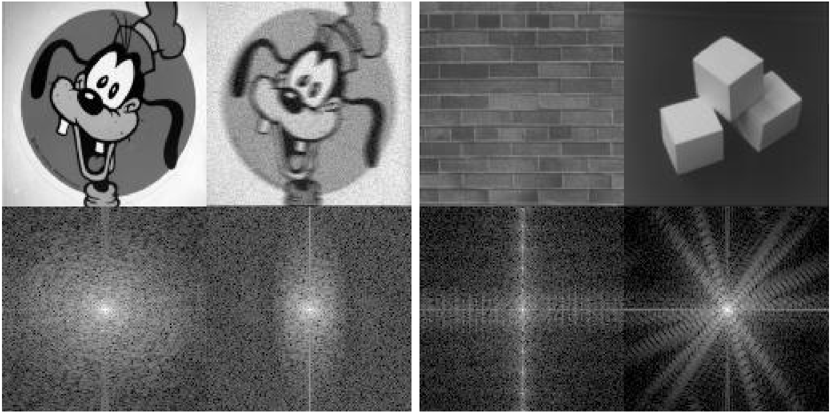
Example images and Fourier transforms



Observations:

- Translation only affects phase (not shown), not magnitude (shown)
- A rotation in one domain becomes a rotation in the other domain
- The Fourier transform is a linear operation

Example images and Fourier transforms



Exercise

Given a simple 4-pixel "image" $f(m) = [3, 2, 2, 1]$, what is its Fourier Transform $\hat{f}(u)$?

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$$\hat{\mathrm{f}}(u) = \frac{1}{\sqrt{M}} \sum_m \mathrm{f}(m) e^{-\frac{2\pi i u m}{M}}$$

$$\hat{\mathrm{f}}(0) = \frac{1}{2} \sum_m \mathrm{f}(m)(1)^m = \frac{1}{2}(3 + 2 + 2 + 1) = 4$$

$$\hat{\mathrm{f}}(1) = \frac{1}{2} \sum_m \mathrm{f}(m)(-i)^m = \frac{1}{2}(3 + 2(-i) - 2 + 1(i)) = \frac{1}{2}(1 - i)$$

$$\hat{\mathrm{f}}(2) = \frac{1}{2} \sum_m \mathrm{f}(m)(-1)^m = \frac{1}{2}(3 - 2 + 2 - 1) = 1$$

$$\hat{\mathrm{f}}(3) = \frac{1}{2} \sum_m \mathrm{f}(m)(i)^m = \frac{1}{2}(3 + 2(i) - 2 + 1(-i)) = \frac{1}{2}(1 + i)$$

Property I - Separability

$$\hat{f}(u,v) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) e^{-2\pi i (\frac{mu}{M} + \frac{nv}{N})} \tag{1}$$

$$\hat{f}(u,v) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(m,n) e^{-2\pi i \frac{nv}{N}} \right) e^{-2\pi i \frac{mu}{M}}$$

$$f(m,n) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{f}(u,v) e^{2\pi i \frac{nv}{N}} \right) e^{2\pi i \frac{mu}{M}}$$

2D DFT can be implemented as a series of 1D DFTs along each column, followed by 1D DFTs along each row.

Property II - Linearity

$$\mathcal{F}\left[a\,\mathrm{f}_1(m,n) \;+\; b\,\mathrm{f}_2(m,n))\right] = a\,\hat{f}_1(u,v) \;+\; b\,\hat{f}_2(u,v)$$

$$a\,\mathrm{f}_1(m,n) \;+\; b\,\mathrm{f}_2(m,n)) = \mathcal{F}^{-1}\left[a\,\hat{f}_1(u,v) \;+\; b\,\hat{f}_2(u,v)\right]$$

You can add two functions (images) or rescale a function, either before or after computing the Fourier transform. It leads to the same result.

Property III - Modulation

- If the original function is multiplied with an exponential like the one below and transformed, it will result in a shift of the origin of the frequency plane to point (u_0, v_0).

$$\mathcal{F}\left[f(m,n)e^{2\pi i(\frac{mu_0}{M}+\frac{nv_0}{N})}\right] = \hat{f}(u-u_0, v-v_0)$$

- For $(u_0, v_0) = (M/2, N/2)$

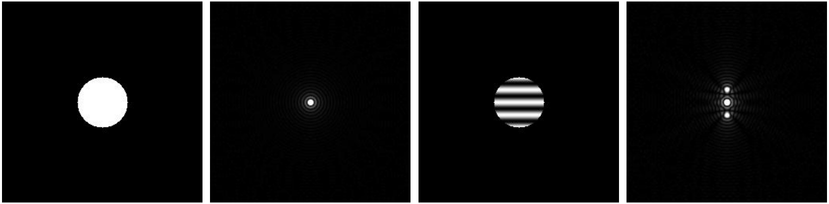
$$e^{2\pi i(\frac{mu_0}{M}+\frac{nv_0}{N})} = e^{\pi i(m+n)} = (-1)^{m+n}$$

  and

$$f(m,n)(-1)^{m+n} \Longleftrightarrow \hat{f}(u-M/2, v-N/2)$$

- Conclusion: the origin of the Fourier transform can be moved to the center by multiplying the original function by $(-1)^{m+n}$.

From left: Original image, magnitude of the Fourier spectrum, original multiplied by $1 + \cos \omega y$ at a relative frequency of 16, magnitude of the Fourier spectrum.



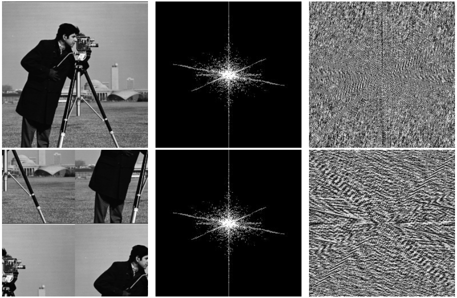
Note: $1 + \cos \omega y = 1 + \frac{1}{2} e^{i\omega y} + \frac{1}{2} e^{-i\omega y}$.

Property IV - Translation

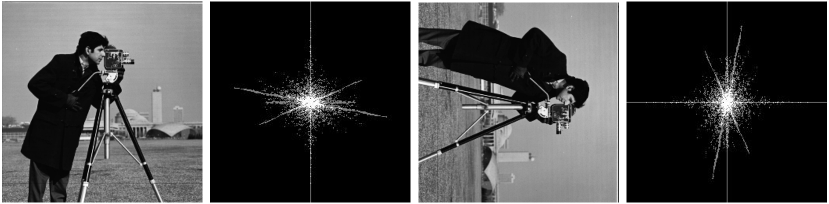
- If the image is moved, the Fourier spectrum undergoes a phase shift, but magnitude of the spectrum remains the same.

$$\mathcal{F}\left[f(m-m_0, n-n_0)\right] = \hat{f}(u,v)e^{-2\pi i(\frac{m_0 u}{M}+\frac{n_0 v}{N})}$$

$$|\hat{f}(u,v)e^{-2\pi i(\frac{m_0 u}{M}+\frac{n_0 v}{N})}| = |\hat{f}(u,v)|$$



Rotation of the original image rotates $\hat{f}$ by the same angle.



Exercise: Introduce polar coordinates and perform direct substitution.

Property VI - Scaling

$$A = \begin{pmatrix} S_1 & & \\ & \ddots & \\ & & S_n \end{pmatrix} \text{ (diagonal)}$$

$$g(x) = f\left(S_1 x_1, \ldots, S_n x_n\right)$$

$$\hat{g}(\omega) = \frac{1}{|S_1 \ldots S_n|} \hat{f}\left(\frac{\omega_1}{S_1}, \ldots, \frac{\omega_n}{S_n}\right)$$

Conclusion: compression in spatial domain is same as expansion in Fourier domain (and vise versa).

Property VII - Periodicity

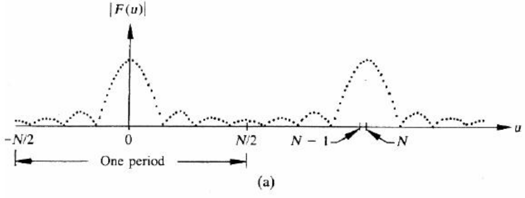
- The DFT and its inverse are periodic with period N, for an N x N image. This means:

$$\hat{f}(u,v) = \hat{f}(u+N,v) = \hat{f}(u,v+N) = \hat{f}(u+N,v+N)$$

- This property defines the implied symmetry in the Fourier spectrum as well as that $\hat{f}$ repeats itself infinitely.
- However, only one period is enough to reconstruct the original function f.

Property VIII - Conjugate Symmetry

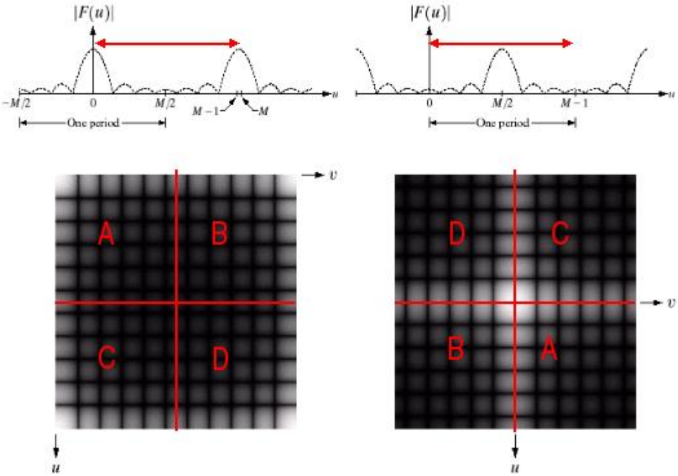
- The Fourier transform satisfies $\hat{f}(u,v) = \hat{f}^*(-u,-v)$ and $|\hat{f}(u,v)| = |\hat{f}(-u,-v)|$.
- With periodicity and above, we have that $\hat{f}$ has period N and is (conjugate) symmetric around the origin.



- Thus we don't need $2N^2$ (N^2 real and N^2 imaginary) values to represent a $N \times N$ image in Fourier domain, but N^2 if we exploit the symmetry.

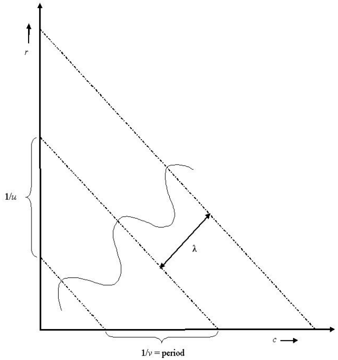
a b
c d

FIGURE 4.34
(a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$.
(b) Shifted spectrum showing a full period in the same interval.
(c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions.
(d) Centered Fourier spectrum.



Wavelength in two dimensions

- The wavelength of the sinusoid is: $\lambda = \frac{1}{\sqrt{u^2+v^2}}$, where (u, v) are the frequencies along (r, c) and the periods are $1/u$ and $1/v$.



Theorem 3

Multiplication in the spatial domain is same as convolution in the Fourier domain.

$$\mathcal{F}(\mathrm{hf}) = \mathcal{F}(\mathrm{h}) * \mathcal{F}(\mathrm{f})$$

Exercise: Prove it!

Transfer functions

- A linear, shift invariant system (such as a filter) is completely specified by it's response to an impulse, which is called the impulse response.
- The transfer function H is the Fourier transform of the impulse response.
- Using the convolution theorem to describe the effects of the system:

$$g(x,y) = h(x,y) * f(x,y)$$

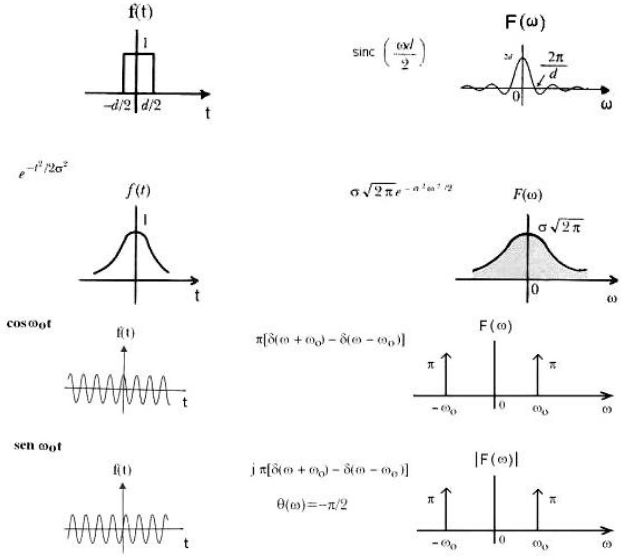
$$G(u,v) = H(u,v) \cdot F(u,v)$$

- Convolution with an impulse function

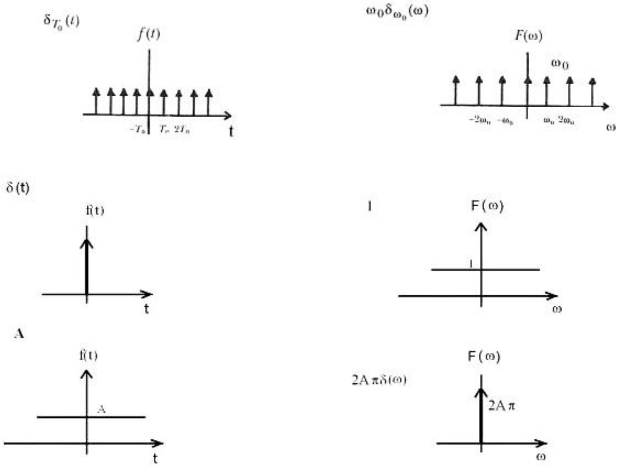
$$h(x,y) * \delta(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x_0,y_0)\delta(x-x_0,y-y_0)dx_0 dy_0 = h(x,y)$$

results in a "copy" of $h(x,y)$ to the location of the impulse

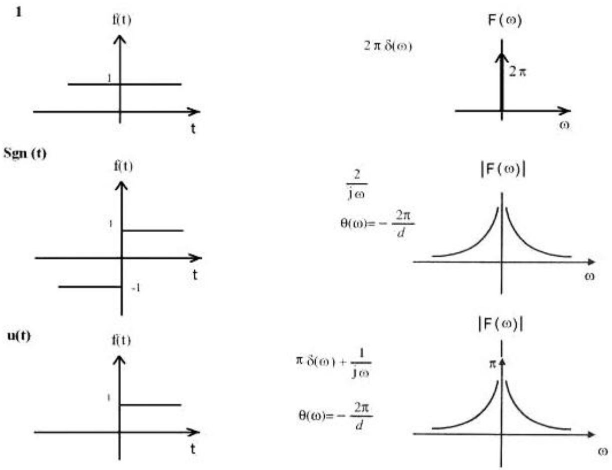
Transfer function examples



Transfer function examples



Transfer function examples

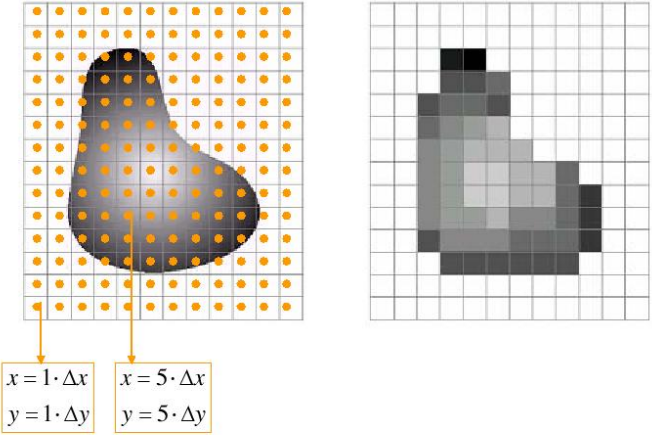


Sampling

Our signals (images) are not in a continuous domain, but in a discrete.

- A continuous function $f(x, y)$ (an image) can be sampled using a discrete grid of sampling points.
- The image is sampled at points $(j\Delta x, k\Delta y)$, with $j = 1, ..., M$ and $k = 1, ..., N$, where is (M, N) is the size of the image in pixels.
- Here Δx and Δy are called the sampling interval.

Sampling



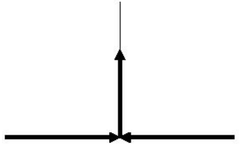
Sampling

Dirac (continuous domain) and Kronecker (discrete) delta functions.

Ideal impulse defined using ***Dirac distribution***

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)\,dxdy = 1$$

and $\delta(x,y)=0$ *for all* $x, y \neq 0$



- The 'sifting property' of the dirac function provides a value of the function $f(x,y)$ at point (a,b)

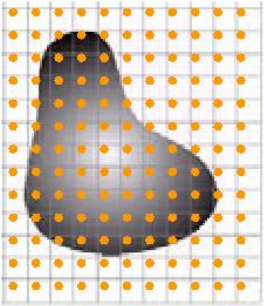
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-a,y-b)dxdy = f(a,b)$$

- The sifting property can be used to describe the sampling process of a continuous function $f(x,y)$.

Sampling

- The ideal sampling $s(x,y)$ in the regular grid can be represented using a collection of Dirac functions δ.

$$s(x,y) = \sum_{j=1}^{M} \sum_{k=1}^{N} \delta(x - j\Delta x, y - k\Delta y)$$



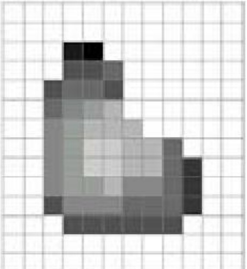
Sampling

- The sampled image $f_s(x,y)$ is the product of the continuous image $f(x,y)$ and the sampling function $s(x,y)$.

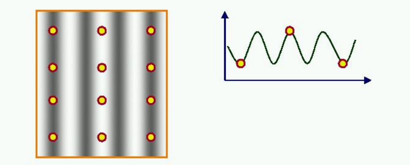
$$f_s(x,y) = f(x,y)s(x,y) = f(x,y)\sum_{j=1}^{M}\sum_{k=1}^{N}\delta(x-j\Delta x, y-k\Delta y) =$$

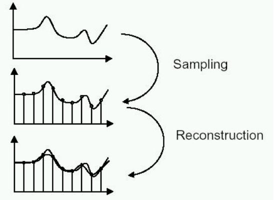
$$= \sum_{j=1}^{M}\sum_{k=1}^{N} f(j\Delta x, k\Delta y)\delta(x-j\Delta x, y-k\Delta y)$$

- Note: Sampling is not a convolution, but a product $f(x,y)s(x,y)$.



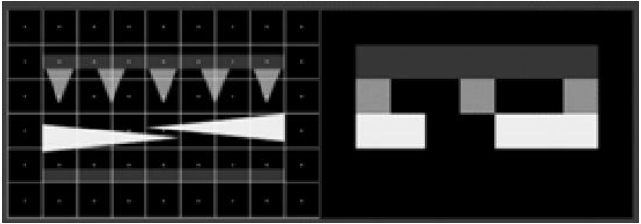
- Sources of error during sampling:
  - Intensity quantization (not enough intensity resolution).
  - Spatial aliasing (not enough spatial resolution).
  - Temporal aliasing (not enough temporal resolution).
- Sampling Theorem answers (more later):
  - How many samples are required to describe the given signal without loss of information?
  - What signal can be reconstructed given the current sampling rate?



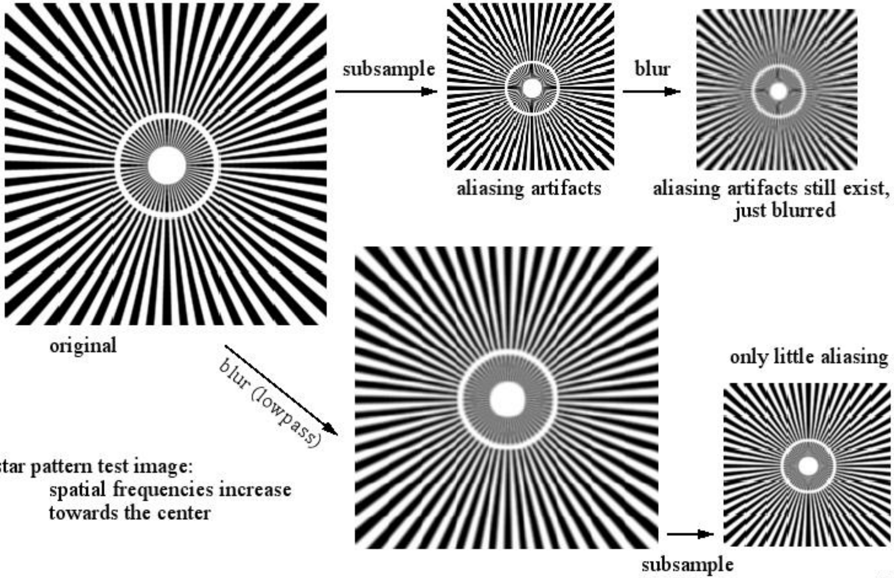


Aliasing and anti-aliasing

- Artifacts produced by under-sampling or poor reconstruction. Fine structures disappear and distort coarser structure.
- Spatial and temporal aliasing.
- Anti-aliasing: sample at higher rate or prefiltering. Tools: Fourier transform, convolution and sampling theory.



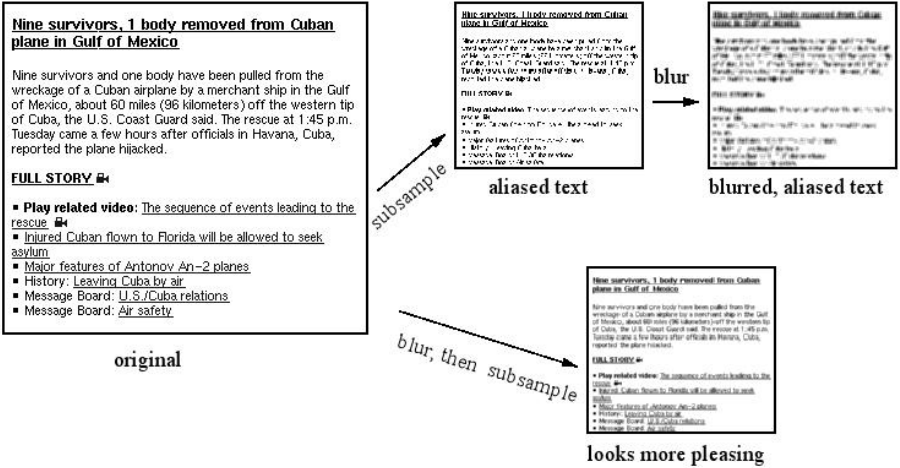
Example: Aliasing



star pattern test image:
spatial frequencies increase
towards the center

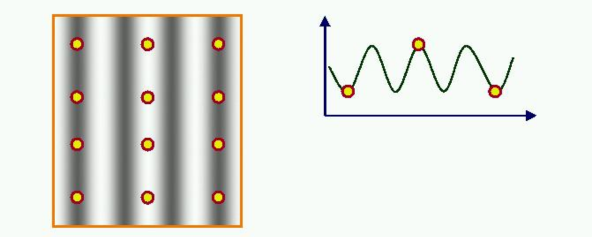
Example: Aliasing

Low pass filtering (blurring) important!



Sampling Theorem

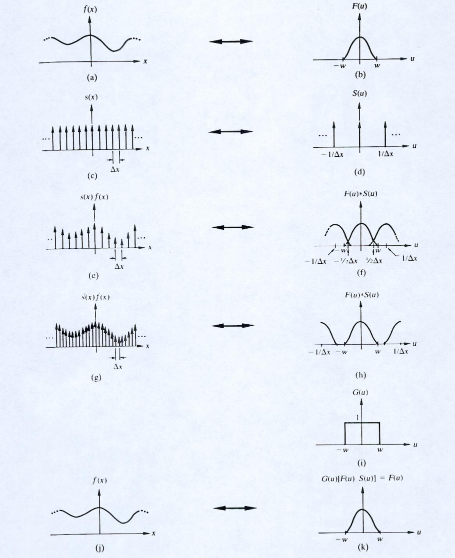
- How many samples are required to describe the given signal without loss of information?
- What signal can be reconstructed given the current sampling rate?



Sampling Theorem

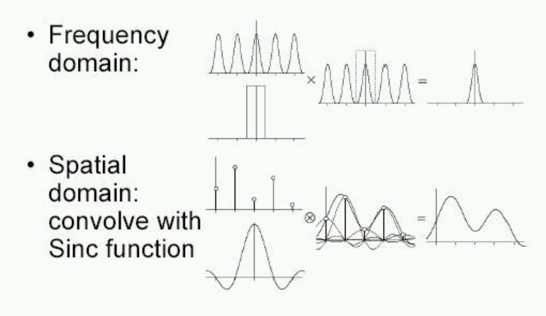
- A signal is band limited if its highest frequency is bounded. This frequency is called the bandwidth.
- The sine/cosine component of the highest frequency determines the highest “frequency content” of the signal.
- If the signal is sampled at a rate equal or greater to than twice its highest frequency, the original signal can be completely recovered from it samples (Shannon).
- The minimum sampling rate for band limited function is called Nyquist rate.

Reconstruction



Reconstruction

- For reconstruction, we need to convolve with a sinc function.
  - It is the Fourier transform of the box function.
  - It has infinite support.
- May be approximated by a Gaussian, cubic or even “tent” function.



Aliasing

- If the signal is undersampled, aliasing occurs.
- Prevent aliasing by:

  increasing sampling rate, or [infeasible]

  decreasing highest frequency before sampling. [blurring]

Summary of good questions

- How do you interpret a point in the Fourier domain in the spatial domain?
- How do you apply a discrete Fourier transform?
- What happens to the Fourier transform, if you translate an image?
- What happens to the Fourier transform, if you rotate an image?
- In what sense is the Fourier transform symmetric?
- How can a Dirac function be used to model sampling?
- What does the Sampling Theorem mean in practice?
- What can you do to get rid of aliasing in the sampling process?

Readings

- Gonzalez and Woods: Chapter 4.3-4.7
- Szeliski: Chapter 3.4