Exercise Session 3

1.) $P(C_1)$, $P(x|C_1)$, $C_{opt} = \arg\max_{C_i} P(C_i|x)$

$$P(x, c_i) = P(x|c_i)\,P(c_i) = P(c_i|x)\,P(x)$$

$$\Rightarrow P(c_i|x) = \frac{P(x|c_i)\,P(c_i)}{P(x)}$$

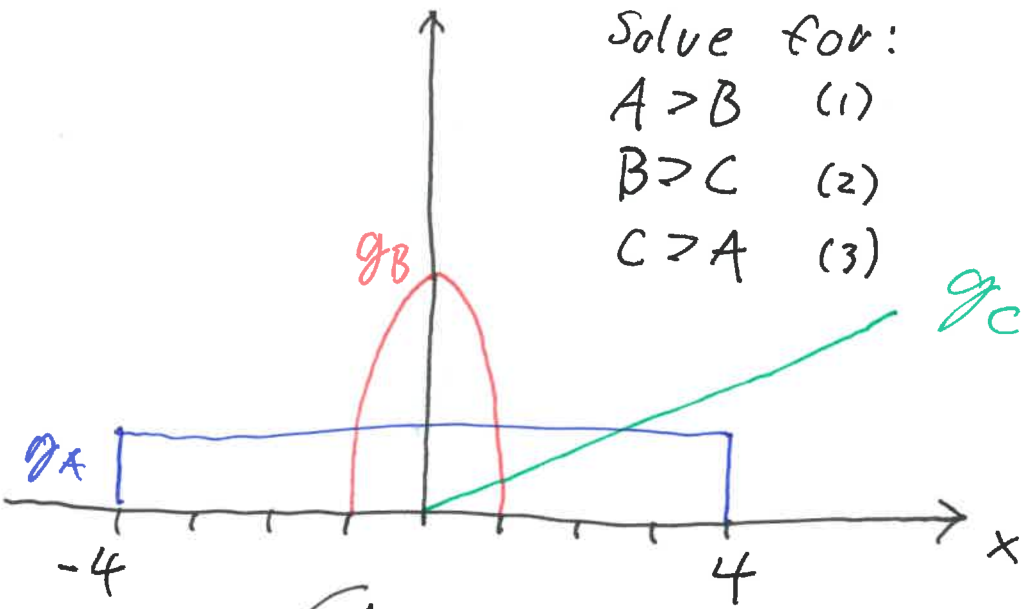
$$C_{opt} = \arg\max_{c_i} \frac{P(x|c_i)\,P(c_i)}{P(x)}$$

$$\propto \arg\max_{c_i} P(x|c_i)\,P(c_i)$$

$$g_A(x) = P(A)\cdot P(x|A) = \frac{1}{2}\cdot\frac{1}{8} = \frac{1}{16} \quad \text{for } x \in [-4, 4]$$

$$g_B(x) = P(B)\,P(x|B) = \frac{1}{3}\cdot 3\cdot\frac{1-x^2}{4} \quad \text{for } x \in [-1, 1]$$

$$g_C(x) = P(C)\,P(x|C) = \frac{1}{6}\cdot\frac{x}{8} = \frac{x}{48} \quad \text{for } x \in [0, 4]$$



Solve for:

$A > B$ (1)

$B > C$ (2)

$C > A$ (3)

$$C_{opt} = \begin{cases} A, & g_A > g_B,\ g_A > g_C \\ B, & g_B > g_A,\ g_B > g_C \\ C, & g_C > g_A,\ g_C > g_B \end{cases}$$

ineq (1)

$$g_A > g_B \iff \frac{1}{16} > \frac{1-x^2}{4} \iff x^2 > 1 - \frac{1}{4} = \frac{3}{4}$$

$$\rightarrow x < -\frac{\sqrt{3}}{2} \text{ and } x > \frac{\sqrt{3}}{2}$$

~~ineq (2)~~

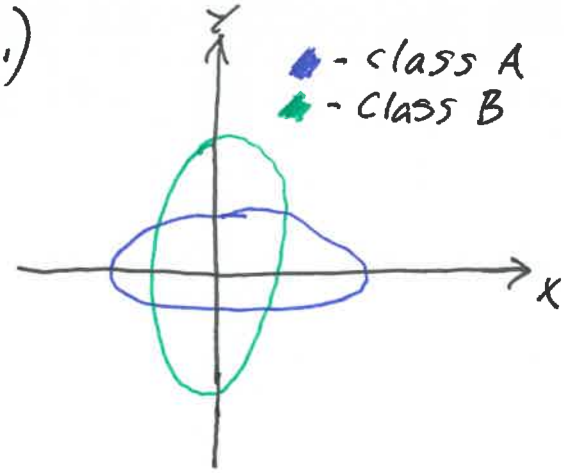
$$\cancel{g_B > g_C \iff \frac{1-x^2}{4} > \frac{x}{48} \iff 12 - 12x^2}$$

ineq (3)

$$g_C > g_A \iff \frac{x}{48} > \frac{1}{16} \Rightarrow x > 3$$

$$C_{opt} = \begin{cases} A, & \left(x < -\frac{\sqrt{3}}{2} \vee x > \frac{\sqrt{3}}{2}\right) \wedge x < 3 \\ B, & -\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2} \\ C, & 3 < x < 4 \end{cases}$$

2.)



$$P(C|X) = \frac{P(X|C)\,P(C)}{P(X)}$$

Want to find:

$$P(A)P(Z|A) > P(B)P(Z|B)$$

$$\frac{3}{\cancel{4}} \cdot \frac{1}{\cancel{2\pi |\Sigma_A|^{\frac{1}{2}}}} e^{-\frac{z^T \Sigma_A^{-1} z}{2}} > \frac{1}{\cancel{4}} \cdot \frac{1}{\cancel{2\pi |\Sigma_B|^{\frac{1}{2}}}} e^{-\frac{z^T \Sigma_B z}{2}}$$

$$\ln(3) + \frac{-(x\ y)\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}{2} > \frac{-(x\ y)\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}{2}$$

$$\ln(3) - \left(\frac{x^2}{8} + \frac{y^2}{2}\right) > -\left(\frac{x^2}{2} + \frac{y^2}{8}\right)$$

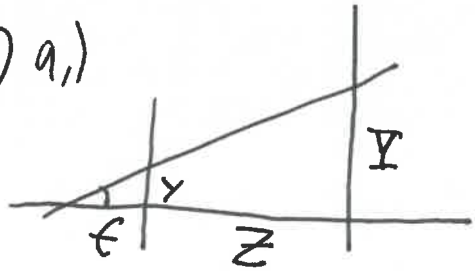
$$\frac{3}{8}x^2 - \frac{3}{8}y^2 + \ln(3) > 0$$

$$x^2 - y^2 + \frac{8}{3}\ln(3) > 0$$

answer:

$$\begin{cases} \text{Class A} & \text{if } x^2 - y^2 + \frac{8}{3}\ln(3) > 0 \\ \text{class B} & \text{, otherwise} \end{cases}$$

3.) a.)

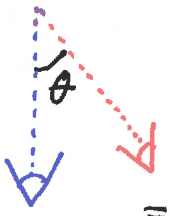


$$\frac{y}{f} = \frac{Y}{Z} \Rightarrow y = \frac{Y}{Z} \cdot f = \frac{Y}{Z}$$

Similarly.... $x = \frac{X}{Z}$

$$p = (x, y, z) \rightarrow p = \left(-\frac{X}{z}, -\frac{Y}{z}\right)$$

b.)



$$p = R(p-t) + t = Rp - Rt + t$$
$$= Rp - R(t - R^{-1}t) = R(p-e)$$

$$\Rightarrow e = t - R^{-1}t = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ 0 \\ \cos\theta - 1 \end{pmatrix}$$

c.) $p'^T E p = 0$, $E = R T_e$

$$T_e = \begin{pmatrix} 0 & 1-\cos\theta & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} \qquad (e \times X = T_e \cdot x)$$

$$E = R \cdot T_e = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1-\cos\theta & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cos\theta(1-\cos\theta) & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta(1-\cancel{\cos\theta)+} \; \cancel{+\cos\theta\sin\theta} & 0 \end{pmatrix}$$

$$(p'_x \;\; p'_y \;\; -1) \, E \begin{pmatrix} p_x \\ p_y \\ -1 \end{pmatrix} = 0$$

d.)

$$\begin{pmatrix} p_x' & p_y' & -1 \end{pmatrix} \begin{pmatrix} 0 & \cos\theta(\cos\theta+1) & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ -1 \end{pmatrix} = 0$$

$$p_y' p_x (\cos\theta - 1) + p_x' p_y (\cos\theta - 1) - p_y' \sin\theta + p_y \sin\theta = 0$$

4.)

$$\frac{y}{f} = \frac{Y}{Z}, \quad \frac{x}{f} = \frac{X}{Z}$$

$$\frac{d}{dx}\left(\frac{x}{f}\right) = \frac{\dot{x}}{f} = \left(\frac{\dot{X}}{f}\right) = \frac{\dot{X}Z - \dot{Z}X}{Z^2} = \frac{\dot{X}}{Z} - \frac{\dot{Z}X}{Z^2}$$

$$\frac{\dot{y}}{f} = \left(\frac{\dot{Y}}{Z}\right) = \frac{\dot{Y}}{Z} - \frac{\dot{Z}Y}{Z^2}$$

$$(\dot{X}, \dot{Y}, \dot{Z}) = -(T_x, T_y, T_z)$$

$$\left.\begin{matrix} \frac{\dot{X}}{Z} - \frac{\dot{Z}X}{Z^2} \\ \frac{\dot{Y}}{Z} - \frac{\dot{Z}Y}{Z^2} \end{matrix}\right\} \Rightarrow \begin{matrix} \frac{\dot{x}}{f} = \frac{-T_x}{Z} - \frac{-T_z X}{Z^2} = \frac{-T_x}{Z} + \frac{T_z x}{fZ} \\ \frac{\dot{y}}{f} = -\frac{T_y}{Z} + \frac{T_z y}{fZ} \end{matrix}$$

{multiply by fZ}

$$\Rightarrow \left.\begin{matrix} \dot{x}Z = T_z \cdot x - T_x \cdot f \\ \dot{y}Z = T_z \cdot y - T_y \cdot f \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \dot{x}Z + fT_x = T_z x \\ \dot{y}Z + fT_y = T_z y \end{matrix}\right.$$

$$\Rightarrow \begin{pmatrix} f & 0 & \dot{x}_a & 0 \\ 0 & f & \dot{y}_a & 0 \\ f & 0 & 0 & \dot{x}_b \\ 0 & f & 0 & \dot{y}_b \end{pmatrix} \begin{pmatrix} T_x \\ T_y \\ Z_a \\ Z_b \end{pmatrix} = T_z \begin{pmatrix} x_a \\ y_a \\ x_b \\ y_b \end{pmatrix}$$

4.) ... $\begin{pmatrix} \frac{T_x}{T_z} \\ \frac{T_y}{T_z} \\ \frac{z_a}{T_z} \\ \frac{z_b}{T_z} \end{pmatrix}$ after something.....

$$\left(\frac{T_x}{T_z}, \frac{T_y}{T_z}, 1\right) \sim (T_x, T_y, T_z)$$