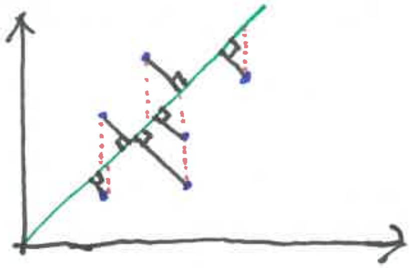
Exercise session 2

1.)



$$\bar{p} = \frac{1}{N}\sum p_i = (-1, 0)$$

$$C = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}, \quad x_i' = x_i - \underbrace{\mu}_{= \bar{p}}$$

$$C_{xx} = \frac{1}{N}\sum x_i'^2 = \frac{1}{5}\cdot 50 = 10$$

$$C_{xy} = \frac{1}{5}\cdot 22, \quad C_{yy} = \frac{1}{5}\cdot 10$$

$$Cv = \lambda v$$

$$|\lambda I - 5C| = \begin{vmatrix} \lambda - 50 & -22 \\ -22 & \lambda - 10 \end{vmatrix} = (\lambda-50)(\lambda-10) - (-22)\cdot(-22)$$

$$= \lambda^2 - 60 + 16 = 0$$

$$\lambda = 30 \pm \sqrt{884}$$

$$\bar{n} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$a(x - x_0) + b(y - y_0) = 0$$

$$(2v - \sqrt{884})x + 22y + 20 - \sqrt{884} = 0$$

$$V = (20 - \sqrt{884}, 22)$$

2.) $X = \begin{pmatrix} -6 & 1 \\ -3 & 1 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}, \beta = \begin{pmatrix} K \\ L \end{pmatrix}, Y = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

$e = Y - X\beta \quad , \quad S(\beta) = \sum e^2 = e^T e = (Y - X\beta)^T (Y - X\beta)$

$= Y^T Y - 2 Y^T \beta + (X\beta)^T (X\beta)$

$\frac{S(\beta)}{\delta \beta} = -2X^T Y + 2X^T X \beta = 0$

$\beta = (X^T X)^T (X^T Y) = \begin{pmatrix} 55 & -5 \\ -5 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 22 \\ 0 \end{pmatrix}$

$= \frac{1}{250} \begin{pmatrix} 5 & 5 \\ 5 & 55 \end{pmatrix} \begin{pmatrix} 22 \\ 0 \end{pmatrix} = \begin{pmatrix} 0{,}44 \\ 0{,}44 \end{pmatrix} = \begin{pmatrix} K \\ L \end{pmatrix}$

3.) $[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \; \ldots \;]$

$[1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1]$

$1\cdot 1 + 1\cdot 5 + 1\cdot 10 + 1\cdot 10 + 1\cdot 5 + 1\cdot 1 = 32$

So $k = \frac{1}{32}$

$$g * \left[\tfrac{1}{2} \; \tfrac{1}{2}\right] = \left[\tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot\tfrac{1}{2},\ \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2},\ \tfrac{1}{2}\cdot\tfrac{1}{2} + 0\right]$$

$$= \left[\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}\right] = 4\cdot[1 \quad 2 \quad 1] = g * g$$

$$g * g * g * g = \frac{1}{16}[1 \quad 4 \quad 6 \quad 4 \quad 1]$$

$$g^{5*} = \frac{1}{32}[1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1]$$ so yes!

4.) $(f * g) = \sum_{m=-M}^{M} f(n-m)\, g[m]$
$\uparrow$ index

$d_x^1 * f(x) = \frac{1}{2}\left(f(x+1)\cdot(1) + f(x-0)\cdot 0 + f(x-1)\cdot(-1)\right)$

$-1 \quad 0 \quad 1$

$= \frac{1}{2}\left(f(x+1) - f(x-1)\right)$

Taylor expansion:

$f(x+h) \approx f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + R_3(x)$

$R_3 = \frac{1}{3!} f'''(c) h^3, \ \exists c \in [x, x+h]$

$d_x^1 * f(x) = \frac{1}{2}\Big(f(x) + f'(x)\cdot 1 + \cancel{\frac{f''(x)}{2}\cdot 1^2}$

$- f(x) - f'(x)\cdot(-1) - \cancel{\frac{f''}{2!}(-1)^2}\Big) + R_3(x)$

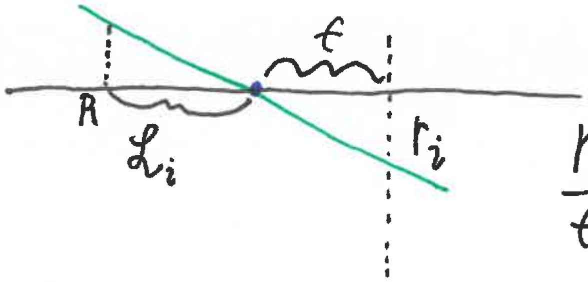
$= 2f'(x) + R_3(x)$

$d_x^2 * f(x) = f(x+1) - f(x-0) =$

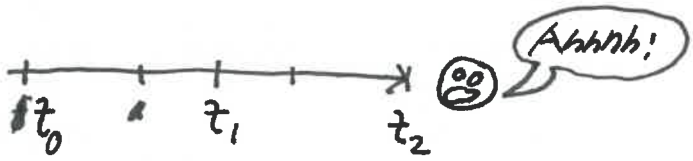
$= \cancel{f(x)} + f'(x) + \frac{f''(x)}{2!}\cdot(1)^2 + R_3(x) \cancel{- f(x)}$

$= f'(x) + \frac{f''(x)}{2!} + R_3(x)$

5.)



$$\frac{r_i}{f} = \frac{R}{L_i} \Leftrightarrow r_i L_i = R f$$



$$r_0 L_0 = r_1 L_1 \Rightarrow L_1 = \frac{L_0}{\sqrt{1,5}}$$

$r_0 = \sqrt{500\pi}$ $\quad r_2 = \sqrt{2}\, r_0 \quad L_2 = \frac{L_0}{\sqrt{2}}$

$t_1 = 3$

$$V = \frac{\sqrt{1,5}\, z_0 - z_0}{3\sqrt{1,5}}$$

$r_1 = \sqrt{1,5}\, r_0$

$$t_2 = \frac{L_0 - L_2}{V} = 3\, \frac{\sqrt{3} - \sqrt{1,5}}{\sqrt{3} - \sqrt{2}}$$

6.)

y \ x	1	2	3	4	5	6	
1	1	1	1	0	0	0	3
2	1	1	0	0	0	1	3
3	1	1	0	0	0	1	3
4	1	1	0	0	1	0	3
5	1	0	0	0	1	0	2
6	0	0	0	1	1	1	3
	5	4	1	1	3	3	

$$m_{ij} = \sum_y \sum_x I(x,y)\, x^i y^j$$

$m_{00} = 17$

$m_{01} = 5 \cdot 1 + 2 \cdot 4 + 3 \cdot 1 + 4 \cdot 1 + 5 \cdot 3 + 6 \cdot 3 = 53$

$m_{10} = 58$

$m_{20} = 248$

$$x_0 = \frac{m_{10}}{m_{00}} \approx 3,41$$

$$y_0 = \frac{m_{01}}{m_{00}} \approx 3,12$$