Analysis of Squeezed Vacuum States of Light by Means of Wigner Functions

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1 Second-order Squeezing

$$
\begin{aligned}
i\hbar\partial_t \rho &= \hbar\omega[a^\dagger a, \rho] + \hbar\chi^{(2)}\left[\frac{\alpha^* e^{2i\omega t}a^2 - \alpha e^{-2i\omega t}(a^\dagger)^2}{2i}, \rho\right] \\
&= \hbar\omega[a^\dagger a, \rho] + \frac{\hbar\chi^{(2)}}{2i}([\alpha^* e^{2i\omega t}a^2, \rho] - [\alpha e^{-2i\omega t}(a^\dagger)^2, \rho])
\end{aligned}
$$

Let $\rho = \exp(-i\omega t a^\dagger a)\tilde{\rho}\exp(i\omega t a^\dagger a)$, then

$$
\partial_t \tilde{\rho} = -\frac{\chi^{(2)}}{2}(\alpha^*[a^2, \tilde{\rho}] - \alpha[(a^\dagger)^2, \tilde{\rho}])
$$

We want to look at the time evolution of the Wigner quasiprobability distribution. The Von Neumann equation above can be translated into a partial differential equation for the Wigner function by the following rules:

$$
\begin{aligned}
a\rho &\leftrightarrow \left(z + \frac{1}{2}\partial_{z^*}\right)W & z &\leftrightarrow q + ip \\
a^\dagger\rho &\leftrightarrow \left(z^* - \frac{1}{2}\partial_z\right)W & z^* &\leftrightarrow q - ip \\
\rho a &\leftrightarrow \left(z - \frac{1}{2}\partial_{z^*}\right)W & \partial_z &\leftrightarrow \frac{1}{2}(\partial_q - i\partial_p) \\
\rho a^\dagger &\leftrightarrow \left(z^* - \frac{1}{2}\partial_z\right)W & \partial_{z^*} &\leftrightarrow \frac{1}{2}(\partial_q + i\partial_p)
\end{aligned}
$$

Then the Von Neumann equation turns into

$$
\partial_t W = -\frac{\chi^{(2)}}{2}(2\alpha z^*\partial_z W + 2\alpha^* z\partial_{z^*}W)
$$

Now we want to evaluate how the Wigner function evolves in time given the distribution at time 0. That is, we want to calculate the trajectory $(q(t), p(t))$ that satisfies

$$
W(q(t), p(t); t) = W(q(0), p(0); 0)
$$

for all time t.

Differentiating both sides by t, we get, by the chain rule,

$$
\frac{dz}{dt}\partial_z W + \frac{dz^*}{dt}\partial_{z^*}W + \partial_t W = 0
$$

if we view W as a function of $z = q + ip$ and $z^* = q - ip$.

By the time evolution equation, this equation can be summarized into

$$
\left(\frac{dz}{dt} - \chi^{(2)}\alpha z^*\right)\partial_z W + \left(\frac{dz^*}{dt} - \chi^{(2)}\alpha^* z\right)\partial_{z^*}W = 0
$$

Therefore, if the trajectory $z(t) = q(t) + ip(t)$ satisfies

$$
\frac{dz}{dt} = \chi^{(2)}\alpha z^*
$$

then for any initial condition we can determine the Wigner function at time t.

Separating into real and imaginary parts, we have

$$\frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix} = \chi^{(2)} \begin{bmatrix} \Re(\alpha) & \Im(\alpha) \\ \Im(\alpha) & -\Re(\alpha) \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$$

and solving this linear ode results in

$$\begin{bmatrix} q \\ p \end{bmatrix} = R_{\theta/2} \begin{bmatrix} e^{\chi^{(2)}rt} & 0 \\ 0 & e^{-\chi^{(2)}rt} \end{bmatrix} R_{-\theta/2} \begin{bmatrix} q(0) \\ p(0) \end{bmatrix}$$

when $\alpha = re^{i\theta}$ and $R_{\theta/2}$ is the rotation matrix by $\theta/2$. Thus, we have shown that for *any* initial distribution the distribution is squeezed by $e^{\chi^2 r}$.

2 Third-order Squeezing

$$\begin{aligned} i\hbar\partial_t\rho &= \hbar\omega[a^\dagger a, \rho] + \hbar\chi^{(3)}\left[\frac{\alpha^* e^{3i\omega t}a^3 - \alpha e^{-3i\omega t}(a^\dagger)^3}{2i}, \rho\right] \\ &= \hbar\omega[a^\dagger a, \rho] + \frac{\hbar\chi^{(3)}}{2i}([\alpha^* e^{3i\omega t}a^3, \rho] - [\alpha e^{-3i\omega t}(a^\dagger)^3, \rho]) \end{aligned}$$

Let $\rho = \exp(-i\omega t a^\dagger a)\tilde{\rho}\exp(i\omega t a^\dagger a)$, then

$$\partial_t\tilde{\rho} = -\frac{\chi^{(3)}}{2}(\alpha^*[a^3, \tilde{\rho}] - \alpha[(a^\dagger)^3, \tilde{\rho}])$$

Now, the time evolution of the Wigner function is given by

$$\partial_t W = -\frac{\chi^{(3)}}{2}\left(\alpha^*\left(3z^2\partial_{z^*}W + \frac{1}{4}\partial_{z^*}^3 W\right) + \alpha\left(3(z^*)^2\partial_z W + \frac{1}{4}\partial_z^3 W\right)\right)$$

We consider the rotated distribution $\tilde{W}(z, z^*; t) := W(\omega z, \omega^{-1}z^*; t)$ when $\omega = \exp(i2\pi/3)$ is the third root of unity. We will show that $\tilde{W}$ also satisfies the above equation. Note that

$$\begin{aligned} \partial_t\tilde{W}(z, z^*; t) &= \partial_t W(\omega z, \omega^{-1}z^*; t) \\ \partial_z\tilde{W}(z, z^*; t) &= \omega\partial_z W(\omega z, \omega^{-1}z^*; t) \\ \partial_z^2\tilde{W}(z, z^*; t) &= \omega^2\partial_z^2 W(\omega z, \omega^{-1}z^*; t) \\ \partial_z^3\tilde{W}(z, z^*; t) &= \partial_3^3 W(\omega z, \omega^{-1}z^*; t) \end{aligned}$$

by the chain rule.

Then we have:

$$\begin{aligned} \partial_t\tilde{W}(z, z^*; t) &= \partial_t W(\omega z, \omega^{-1}z^*; t) \\ &= -\frac{\chi^{(3)}}{2}\left(\alpha^*\left(3(\omega z)^2\partial_{z^*}W(\omega z, \omega^{-1}z^*; t) + \frac{1}{4}\partial_{z^*}^3 W(\omega z, \omega^{-1}z^*; t)\right)\right. \\ &\left. + \alpha\left(3(\omega^{-1}z^*)^2\partial_z W(\omega z, \omega^{-1}z^*; t) + \frac{1}{4}\partial_z^3 W(\omega z, \omega^{-1}z^*; t)\right)\right) \\ &= -\frac{\chi^{(3)}}{2}\left(\alpha^*\left(3z^2\partial_{z^*}\tilde{W}(z, z^*; t) + \frac{1}{2}\partial_{z^*}^3\tilde{W}(z, z^*; t)\right)\right. \\ &\left. + \alpha\left(3(z^*)^2\partial_z\tilde{W}(z, z^*; t) + \frac{1}{4}\partial_{z^*}^3\tilde{W}(z, z^*; t)\right)\right) \end{aligned}$$

Thus, we know that

$$\partial_t(W - \tilde{W}) = -\frac{\chi^{(3)}}{2}\left(\alpha^*\left(3z^2\partial_{z^*} + \frac{1}{4}\partial_{z^*}^3\right) + \alpha\left(3(z^*)^2\partial_z + \frac{1}{4}\partial_z^3\right)\right)(W - \tilde{W})$$

If we have that for time 0, $(W - \tilde{W})(q, p; 0) = 0$, that is, if the probability distribution is symmetric with respect to 120° rotations, then for the rest of the time, $W - \tilde{W}$ stays 0. That is, the symmetry is preserved in time. This can explain why the squeezed vacuum subject to the third order generation process displays a Wigner quasiprobability distribution that is shaped like an equilateral triangle.