Chapter I

LINEAR EQUATIONS IN LINEAR ALGEBRA

1 Definitions and Properties

A linear equation in the variables $x_, x_2, ..., x_n$ is an equation that can be written as

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b,$$

where a_i are the coefficients, $a_i, b \in \mathbf{K}, \mathbf{K} = \mathbb{R}, \mathbb{C}, \text{or } \mathbb{Z}_2$ (for instance), $\mathrm{i} = \overline{1, \mathrm{n}}, \mathrm{n} \geq 2, \mathrm{n} \in \mathbb{N}$.

A system of linear equations or **a linear system** is a collection of one or more linear equations involving the same variables $x_1, x_2, ..., x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m. \end{cases}$$

A solution of the system is a list $s_1, s_2, ..., s_n$ of numbers that makes every statement true. The set of all possible solutions is called **the solution** of the linear system.

Two linear systems are called **equivalent** if they have the same solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions. A system is said to be **inconsistent** if it has no solution.

Let $A \in M_{m,n}(\mathbf{K})$ be the **coefficient matrix** of a linear system. Then, the system can be put into the equivalent form $Ax = b$, where $x = (x_1, x_2, ..., x_n)^t$ and $b = (b_1, b_2, ..., b_m)^t$.

We denote by $\bar{A} = [A \mid b]$ **the augmented matrix** of the linear system:

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ \dots & \dots & \dots & \dots & | & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m \end{pmatrix}$$

Row Reduction and Echelon Form

A **nonzero row or column** in a matrix means a row or a column that contains at least one nonzero entry.

A leading entry or **a pivot** of a row is the leftmost nonzero entry in a nonzero row.

Definition 1.1. *A rectangular matrix is in* **echelon form** *(EF) if it has the following three properties:*

- *All nonzero rows are above any rows of all zeros.*
- *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*
- *All entries in a column below a leading entry are zeros.*

Examples: 1. The matrix A_1 satisfies all the above conditions, so it is in EF:

$$A_1 = \begin{pmatrix} \mathbf{9} & -4 & 11 & 2 \\ 0 & \mathbf{3} & 5 & 0 \\ 0 & 0 & 0 & \mathbf{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero entries $a_{11} = 9, a_{21} = 3, a_{34} = 2$ are the pivots.
2. The matrix A_2 satisfies all the above conditions, so it is in EF:

$$A_2 = \begin{pmatrix} \mathbf{1} & 0 & 0 & 2 & 2 \\ 0 & \mathbf{3} & 0 & -2 & 2 \\ 0 & 0 & \mathbf{3} & 2 & 4 \end{pmatrix}$$

The nonzero entries $a_{11} = 1, a_{21} = 3, a_{33} = 3$ are the pivots.

Definition 1.2. *If a matrix in echelon form satisfies the following additional conditions, then it is in* **reduced echelon form** *(REF):*

- *The leading entry in each row is* 1.

- *Each leading* 1 *is the only nonzero entry in its column.*

Examples: 1. The matrix C satisfies all the above conditions, so it is in REF:

$$C = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Elementary row operations:

- **Replacement** Replace one row by the sum of itself and a multiple of another row.
- **Interchange** Interchange two rows.
- **Scaling** Multiply all entries in a row by a nonzero constant.

Row Reduction Algorithm

Step 1 Begin with the leftmost nonzero column. This is a pivot column. The pivot is at the top. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Step 2 Use row replacement operations to create zeros in all positions below the pivot.

Step 3 Ignore the row containing the pivot position and all rows, if any, above it. Apply Step 1 and Step 2 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

If we want the REF, we perform one more step:

Step 4 Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by scaling operation.

Remark The Steps 1-3 are called **the forward phase** of the row reduction algorithm. Step 4 is called the backward phase. A computer program usually selects as a pivot the entry in a column having the largest absolute value. This method, named **partial pivoting**, reduce roundoff errors in the calculations.

Definition 1.3. *Two matrices are called* **row equivalent** *if there is a sequence of elementary row operations that transforms one matrix into the other.*

Uniqueness of the REF

Each matrix is row equivalent to one and only one REF matrix.

Remark Let $A \in M_{m,n}(\mathbb{K}), \mathbb{K} = \mathbb{R}, \mathbb{C}$ be a matrix and $C \in M_{m,n}(\mathbb{K})$ its corresponding REF. Then:

1. Rank(A) = Rank(C) and is equal to the number of the pivots.
2. If $m = n$ (i.e. A is a square matrix) and I_n is the identity matrix, considering the augmented matrix: $[A \mid I_n] \sim [C \mid D]$, then:

- If $C = I_n$ it follows that $D = A^{-1}$.
- If $C \neq I_n$ it follows that A is singular.

Examples Looking back to the previous examples we can conclude that $\text{Rank A}_1 = 3, \text{Rank A}_2 = 3, \text{and Rank C} = 3$.

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system, when the algorithm is applied to the augmented matrix of the system.

The variables corresponding to the pivot columns in the matrix are called **basic variables** or **leading variables**. The other are called **free variables**.

Theorem 1.1. *The linear system $Ax = b$ is consistent iff the rank of the matrix A is the same with the rank of the augmented matrix $\bar{A} = [A \mid b]$ (Kronecker-Capelli Theorem) or, iff the right most column of the augmented matrix is not a pivot column.*

Theorem 1.2. *A nonhomogeneous system $Ax = b$ of linear equations in n unknowns has a unique solution if and only if rank $[A \mid b] = rank(A) = n$.*

Remark: A nonhomogeneous system of m equations in n unknowns cannot have a unique solution if $m < n$.

If a linear system is consistent, then the solution set contains either a unique solution (when there is no free variables) or infinitely many solutions (when there is at least one free variable).

A system of linear equations is said to be **homogeneous** if it can be written as $Ax = 0, A \in M_{m,n}(\mathbb{K}), \mathbb{K} = \mathbb{R}, \mathbb{C}, \text{or } \mathbb{Z}_2$.

Proposition 1.1. *An homogeneous linear system has a nontrivial solution iff the system has at least one free variable.*

2 Solved Problems

1. Find the EF and the REF of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -3 & 0 \end{pmatrix}$.

   Solution: We start with the leftmost nonzero entry and we create zeros below it:

$$A = \begin{pmatrix} \mathbf{1} & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -3 & 0 \end{pmatrix} \overset{\substack{-2L1+L2\to L2 \\ L1+L3\to L3}}{\approx} \begin{pmatrix} 1 & 2 & 1 \\ 0 & \mathbf{-2} & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

   The next pivot is $a_{22} = -2$, so we repeat the procedure creating a zero entry below it:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & \mathbf{-2} & 1 \\ 0 & -1 & 1 \end{pmatrix} \overset{-\frac{1}{2}L2+L3\to L3}{\approx} \begin{pmatrix} \mathbf{1} & 2 & 1 \\ 0 & \mathbf{-2} & 1 \\ 0 & 0 & \mathbf{\frac{1}{2}} \end{pmatrix}.$$

   In order to obtain the REF, first we scale the last row, then we create zeros above the last pivot:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & \mathbf{\frac{1}{2}} \end{pmatrix} \overset{2L3\to L3}{\approx} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \overset{\substack{-L3+L1\to L1 \\ -L3+L2\to L2}}{\approx} \begin{pmatrix} 1 & 2 & 0 \\ 0 & \mathbf{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   We repeat the procedure for the second pivot $a_{22} = -2$, first scaling, then create zeros above it:

$$\overset{-\frac{1}{2}L2\to L2}{\approx} \begin{pmatrix} 1 & 2 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{-2L2+L1\to L1}{\approx} \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}.$$

3 Exercises

1. Which one of the following matrices corresponds to a reduced echelon form of a matrix?

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$D = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix};\ F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Find the EF and the REF for the matrix $A = \begin{pmatrix} 0 & 2 & -3 & 1 \\ 0 & 5 & 4 & -2 \\ 0 & 0 & 7 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$.

3. Find the inverse for each of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 4 \\ 1 & 2 & 3 & 5 \\ 2 & 4 & 0 & 4 \end{pmatrix};\ A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix};\ A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

4. Using EF, solve the systems $Ax = b$ if

a) $A = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}$, $b = \begin{pmatrix} 5 \\ 6 \\ 9 \\ 9 \end{pmatrix}$.

b) $A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & 2 & 1 \\ -2 & 1 & 3 \end{pmatrix}$, $b = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$.

c) $A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 \\ 1 & 1 & 3 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

5. Study if the system whose augmented matrix is

a) $\bar{A} = \left(\begin{array}{cccc|c} -2 & 4 & 2 & -1 & -11 \\ 1 & -2 & 0 & 1 & 3 \\ 4 & -8 & 6 & 7 & -5 \end{array}\right)$;

b) $\bar{A} = \left(\begin{array}{ccccc|c} 2 & -1 & 0 & 5 & 8 & 0 \\ -5 & 3 & 3 & 1 & -1 & 2 \\ 0 & -4 & 1 & -3 & 3 & -4 \\ 1 & 2 & -2 & 0 & 2 & 5 \\ 3 & 6 & 7 & 2 & 0 & 1 \end{array}\right)$,

is consistent and if so, solve it.

6. Find the rank of the matrices whose EF is:

$$\text{a) } C = \begin{pmatrix} 2 & 3 & 5 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{b) } C = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 5 & 8 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{c) } C = \begin{pmatrix} -3 & 2 & 8 & 1 & 5 \\ 0 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 9 & 5 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

7. The matrix $A \in M_{3,4}(\mathbb{R})$ has the EF given by $\begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Is the system $Ax = b$ consistent and if so, find the set of solutions, where

$$b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

8. Using the EF (or REF), solve the following systems $Ax = b$:

a)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & -1 & 5 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.$$

b)

$$A = \begin{pmatrix} 1 & 3 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ 2 & -1 & -2 & 0 \\ 3 & 2 & -1 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 5 \end{pmatrix}.$$