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Constraint Programming | Due Date: 24 April 2017

Simple linear inequality

Propagator Definition

Propagator $p_{lin} \in constraint \equiv a{\cdot}x{+}b{\cdot}y \leq c$ where x, y are variables and a, b, c are integer constants

$$p_{lin}(s) = \begin{cases} a > 0 \implies x \mapsto \{n \in s(x) | n \leq \left\lceil \frac{\max\{c-(b\cdot m)|m \in s(y)\}}{a} \right\rceil\} \\ a < 0 \implies x \mapsto \{n \in s(x) | n \geq \left\lfloor \frac{\max\{c-(b\cdot m)|m \in s(y)\}}{a} \right\rfloor\} \\ b > 0 \implies y \mapsto \{n \in s(y) | n \leq \left\lceil \frac{\max\{c-a\cdot m|m \in s(x)\}}{b} \right\rceil\} \\ b < 0 \implies y \mapsto \{n \in s(y) | n \geq \left\lfloor \frac{\max\{c-a\cdot m|m \in s(x)\}}{b} \right\rfloor\} \end{cases}$$

Idempotence

Theorem 0.1. *The propagator p_{lin} is idempotent*

Proof sketch:

Proof. Assume by contradiction that p_{lin} is not idempotent, then there exists two stores $p_{lin}(s) = s'$ and $p_{lin}(s') = s''$ such that $s' \neq s''$. If s' is failed then s'' must also be failed by the definition of the propagator. From now on we therefore consider the case where s' is not failed.

p_{lin} is by definition contracting which means that if $s' \neq s''$ the following must hold: $s'' < s'$, which implies that $s''(x) \subset s'(x) \vee s''(y) \subset s'(y)$.

There are four cases to consider here:

- $a > 0 \wedge b > 0 \implies$ for it to be possible that $s'' < s'$ it must be that $\min s'(x)$ is less than $\min s(x)$ or $\min s'(y)$ is less than $\min s(y)$. This is a contradiction of the contracting-property of p_{lin} since the minimum value of x or y can only be increased from store s to s', never decreased. To decrease the minimum value of x or y it is necessary to add a value to the store, which contradicts the contracting property.
- $a < 0 \wedge b < 0 \implies$ for it to be possible that $s'' < s'$ it must be that $\max s'(x) \neq \max s(x)$ since the constraint is for greater-than-or-equal ($\geq$), if max value is pruned than the store is failed, i.e $a < 0 \wedge b < 0 \wedge \max s'(x) \neq \max s(x) \implies \max s'(x) = \emptyset$. The same holds for y.

- $a < 0 \wedge b > 0 \implies$ for it to be possible that $s'' < s'$ it must be that $\max s'(x) \neq \max s(x)$ or that $\min s'(y) < \min s(y)$ in both cases idempotence still holds by the same reasoning as above.
- $a > 0 \wedge b < 0 \implies$ same argument as point above.

$\square$

Subsumption

Detect subsumption early:

$$\begin{gathered} p_{lin}(s) : \textbf{let } s' = p_{lin}(s) \\ \textbf{if } \max(a \cdot s(x)) + \max(b \cdot s(y)) \leq c \\ \textbf{then } \langle subsumed, s' \rangle \textbf{ else } \langle fix, s' \rangle \end{gathered}$$

Changing Propagation Order

No, it is not true. Assuming a propagator exhibits the properties described and proved in the course notes [1], we know that when executing a set of propagators on a store s until all propagators are at fixpoint, the fixpoint will be the weakest simultaneous fixpoint, no matter in which order the propagators were applied. I.e the order of propagation does not matter when running until simultaneous fixpoint. However, this does not imply that $p_1(p_2(s)) \neq p_2(p_1(s))$ if the propagators are not at fixpoint. Intermediate stores in the propagation are affected by the order in which propagators are applied.

Theorem 0.2.

$$\neg(p_1(p_2(s)) = p_2(p_1(s)) \quad \forall p, s)$$

Proof. By construction of counterexample.

Let s_1 be a constraint store and p_1, p_2 be two correct propagators according to the definition in [1], i.e monotonic, contracting and solution preserving propagators.

$$s_1 = \{x \mapsto \{1, \ldots, 10\}, y \mapsto \{1, \ldots, 10\}\}$$

p_1 implementing the constraint $x \leq 3$ and p_2 implementing the constraint $x + y \geq 10$.

$$p_1(s) = \begin{cases} x \mapsto \{n \in s(x) | n \leq 3\} \\ y \mapsto s(y) \end{cases}$$

$$p_2(s) = \begin{cases} x \mapsto \{n \in s(x) | n \geq 10 - \max s(y)\} \\ y \mapsto \{n \in s(y) | n \geq 10 - \max s(x)\} \end{cases}$$

$$p_1(s_1) = s_2 = \{x \mapsto \{1, \ldots, 3\}, y \mapsto \{1, \ldots, 10\}\}$$

$$p_2(s_1) = s_1 = \{x \mapsto \{1, \dots, 10\}, y \mapsto \{1, \dots, 10\}\}$$

$$p_2(s_2) = s_3 = \{x \mapsto \{1, \dots, 3\}, y \mapsto \{7, \dots, 10\}\}$$

$$p_1(p_2(s_1)) = p_1(s_1) = s_2$$

$$p_2(p_1(s_1)) = p_2(s_2) = s_3$$

$$s_2 \neq s_3$$

□

Idempotent Propagators

Yes it is true. This follows from the fact that propagators are contracting and that constraint stores have finite sets of variables and values. For any correct propagator p and a arbitrary store s with finite variables and values, it is possible to iterate the propagator to idempotency through n iterations, where $n \in \mathbb{N}$.

Theorem 0.3.

$$\exists n \in \mathbb{N} \textit{ such that } p^n \textit{ is idempotent} \quad \forall p, s$$

Proof. By definition a constraint store that is input to constraint propagation has a finite set of values and variables [1]. This implies that the set of variables $var(p)$ is a finite set where $|var(p)| \in \mathbb{N}$ and that the set of values $s(x)$ for all variables $x \in var(p)$ is a finite set where $|s(x)| \in \mathbb{N}$.

A correct propagator is contracting, this implies that if $p^n(s)$ is not idempotent then $p^{n+1}(s) < p^n(s)$. I.e $p^{n+1}(s)$ is strictly stronger than $p^n(s)$, which means that $\exists x \in var(p)$ where $p^{n+1}(s)(x) \subset p^n(s)(x)$.

A propagator p must by definition be idempotent on a store s if $s(x) = \emptyset \quad \forall x \in var(p)$, i.e $p(s) = s$.

Now we can see that $p^n(s)$ for arbitrary natural number n and constraint store s computes a sequence of stores $s > s_1 > s_2 > s_3 > \cdots > s_{n-1}$ that is a well founded order (this is a property of the $>$ relation on stores[1]).

$$\implies \exists n \in \mathbb{N} \text{ such that } p^n \text{ is idempotent for any store } s \text{ and correct propagator } p$$

□

The idempotence property is not true for arbitrary functions on arbitrary sets. A counterexample is a function f on sets that is not contracting.

Theorem 0.4. *Arbitrary functions on arbitrary sets are not always idempotent*

Proof. By construction of a counterexample.

let $f \in X \to X$ be a function taking a set X of a natural number x as argument.

$$f(\{x\}) = \{x + 1\}$$

Example:

$$f^3(\{1\}) = f(f(f(\{1\}))) = f(f(\{2\})) = f(\{3\}) = \{4\}$$
$$\lim_{n\to\infty} f^n = \{\infty\}$$

$\square$

References

[1] Christian Schulte. Course notes constraint programming (id2204) vt 2012, 2012. [Online; accessed 19-April-2017].