General Relativity Lecture1
Introduction and The Geometric Viewpoint on Physics

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Definition: spacetime A spacetime is a manifold of events that is endowed with a metric. There're some concepts that need to be explained further:

- manifold: A set of points with well-understood connectedness properties.
- event: when and where something happens. It can be labeled with coordinates, but event itself exists independent of these labels.
- metric: A notion of distance between events in manifold. Without this, a manifold has no notion of distance encoded in it.

Begin with Special Relativity

It's the simplest theory of spacetime, and it corresponds to GR in no-gravity limit.

Definition: Inertial Reference Frame Visualize lattice of clocks & measuring rods that allows us to label/assign coordinates to any event that happens in spacetime. It has some properties:

1. Lattice moves freely through spacetime. That means if no forces act on it, it does not rotate.
2. Measuring rods are orthogonal to each other. Tick marks are uniformly spaced.
3. Clocks tick uniformly.
4. Clocks synchronized using "Einstein synchronization procedure". This procedure takes advantage of the fact that the speed of light is the same to allobservers.

something about Einstein sync procedure Clock 1 & 2 are still in the frame. $t_{1,\mathrm{e}}$ is when clock1 emits a pulse of light. This light will follow a little trajectory through the spacetime, and it goes out and strikes clock2 at one point. Then it will be bounced back at time $t_{2,\mathrm{b}}$. Finally it will be received by clock1 at time $t_{1,\mathrm{r}}$. We require that

$$t_{2,\mathrm{b}} = \frac{1}{2}\left(t_{1,\mathrm{e}} + t_{1,\mathrm{r}}\right) \tag{1}$$

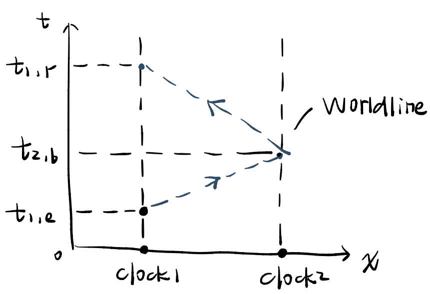

Figure 1: Einstein sync procedure

Now we need to add something about the **units**. We choose basic unit of length to be the distance light travels in basic unit of time. For example, if time unit is 1 second, then length unit is 1 light second. The speed of light c actually means:

$$c = \frac{1\mathrm{light} - \mathrm{time} - \mathrm{unit}}{\mathrm{time} - \mathrm{unit}} = 1 \tag{2}$$

When we set c as 1, all velocities we measure are going to be dimensionless. Actually we measure them as fractions of the speed of light.

Let's talk about a geometric object. Let O is an observer in the IRF defined a few moments ago, and O observes two events P and Q. We can difine displacement from P to Q as:

$$\Delta\vec{x} \underset{O}{\overset{\bullet}{=}} (t_Q - t_P, x_Q - x_P, y_Q - y_P, z_Q - z_P) \tag{3}$$

So position vector in 4-dimensional spacetime has 4 components, and we need to make a requirement that is seems a bit weird: notate 4 components as x^0, x^1, x^2, x^3, and let x^0 be the time t.

Specially, Greek indices tend to be used to label spacetime indices. And Latin indices are often used to pick out spatial components at a moment.

Now there is a different inertial observer $\bar{O}$. If he observe two events P and Q, he will get the displacement $\Delta x^{\bar{\mu}}$ by his own observation. Now the transformation between $\Delta x^{\bar{\mu}}$ and Δx^{μ} is called **Lorenz transformation**.

$$\begin{aligned}
\Delta x^{\bar{0}} &= \gamma\Delta x^0 - \gamma v \Delta x^1 \\
\Delta x^{\bar{1}} &= -\gamma v\Delta x^0 + \gamma \Delta x^1 \\
\Delta x^{\bar{2}} &= \Delta x^2 \\
\Delta x^{\bar{3}} &= \Delta x^3
\end{aligned} \tag{4}$$

The transformation above describes the condition that $\bar{O}$ moves with v along axis 1 with speed v as seen by O. And γ is defined by(set c as 1):

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \tag{5}$$

With **Einstein summation convention**, the transformation can be written in a more simple way:

$$\Delta x^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\nu} \Delta x^{\nu} \tag{6}$$

And the transformation matrix has a more general form:

$$\Lambda^{\bar{\mu}}{}_{\nu} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\nu}} \tag{7}$$

In the end, we can define a new notion. A spacetime vector is any quartet of numbers (component) which transforms between IRFs like displacement vector:

$$A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\alpha} A^{\alpha} \tag{8}$$

General Relativity Lecture2
Introduction to Tensors

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In frame O, we can write down 4 special vectors. They are called **basis vector**. We define them as:

$$\begin{aligned}\vec{e}_0 &= (1,0,0,0)\\ \vec{e}_1 &= (0,1,0,0)\\ \vec{e}_2 &= (0,0,1,0)\\ \vec{e}_3 &= (0,0,0,1)\end{aligned} \tag{1}$$

We can write this in a compact way:

$$(\vec{e}_\alpha)^\beta = \delta_\alpha{}^\beta \tag{2}$$

With this, any vector can be expressed as a combination of basis vectors:

$$\vec{A} = A^\alpha \vec{e}_\alpha \tag{3}$$

Now we wonder how basis vectors transform between two frames. The vector $\vec{A}$ has different representation in frame O and frame $\bar{O}$, but always remember that vector itself is the same:

$$\vec{A} = A^\alpha \vec{e}_\alpha = A^{\bar{\mu}} \vec{e}_{\bar{\mu}} \tag{4}$$

Now use the Lorenz transformation matrix:

$$\vec{A} = A^\alpha \vec{e}_\alpha = \left(\Lambda^{\bar{\mu}}{}_\beta A^\beta\right) \vec{e}_{\bar{\mu}} = A^\beta \Lambda^{\bar{\mu}}{}_\beta \vec{e}_{\bar{\mu}} = A^\alpha \Lambda^{\bar{\mu}}{}_\alpha \vec{e}_{\bar{\mu}} \tag{5}$$

Thus we get:

$$\vec{e}_\alpha = \Lambda^{\bar{\mu}}{}_\alpha \vec{e}_{\bar{\mu}} \tag{6}$$

But if we want to get $\vec{e}_{\bar{\mu}}$ from $\vec{e}_\alpha$, what we need to do is to calculate the inverse Lorentz transformation. In physics, it means reverse velocity vector. Actually the matrix $\Lambda^{\bar{\mu}}{}_\alpha$ can be seen as a function of velocity vector $\underset{\sim}{v}$ (add a tilde under the letter to represent a 3-vector), thus it can be written as $\Lambda^{\bar{\mu}}{}_\alpha\left(\underset{\sim}{v}\right)$. Now the inverse Lorentz transformation is:

$$\vec{e}_\alpha = \Lambda^{\bar{\mu}}{}_\alpha\left(\underset{\sim}{v}\right) \vec{e}_{\bar{\mu}} \longleftrightarrow \vec{e}_{\bar{\mu}} = \Lambda^{\nu}{}_{\bar{\mu}}\left(-\underset{\sim}{v}\right) \vec{e}_\nu \tag{7}$$

Next we will introduce the scalar product between two 4-vectors. But before this, let's recall an invariant that is so important in SR. That is the **spacetime interval**:

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{8}$$

Now let spacetime interval be the scalar product between displacement vector and itself:

$$\Delta s^2 \equiv \Delta\vec{x} \cdot \Delta\vec{x} \tag{9}$$

Since 4-vectors have the same transformation properties as $\Delta\vec{x}$, we similarly define $\vec{A}\cdot\vec{A}$:

$$\vec{A}\cdot\vec{A} = -\left(A^0\right)^2 + \left(A^1\right)^2 + \left(A^2\right)^2 + \left(A^3\right)^2 \tag{10}$$

And this must be a Lorentz invariant!

In 3-dimensional space we know that $\vec{A}\cdot\vec{A}$ must be positive, but not necessarily in 4-dimensional spacetime. We can make the following classification:

- if $\vec{A} \cdot \vec{A} < 0$, say $\vec{A}$ is **timelike**
- if $\vec{A} \cdot \vec{A} > 0$, say $\vec{A}$ is **spacelike**
- if $\vec{A} \cdot \vec{A} = 0$, say $\vec{A}$ is **lightlike** or **null**

More generally, the scalar product between $\vec{A}$ and $\vec{B}$ is:

$$\vec{A} \cdot \vec{B} = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 \tag{11}$$

This is also a Lorentz invariant (can be easily proven).

Now let's write (11) with basis vectors and components:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= (A^\alpha \vec{e}_\alpha) \cdot \left(B^\beta \vec{e}_\beta\right) \\ &= A^\alpha B^\beta \vec{e}_\alpha \cdot \vec{e}_\beta \\ &\equiv A^\alpha B^\beta \eta_{\alpha\beta} \end{aligned} \tag{12}$$

We difine tensor $\eta_{\alpha\beta}$ above. Its components is easy and clear:

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

Actually this is the so-called **the matric tensor** we mentioned at the very first beginning. Note that so far we have not given a strict definition of a tensor.

Let's consider two events that are extremly near to each other, then the distance between them can be written as:

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \tag{14}$$

The fact that this equation holds means that the $\mathrm{d}\vec{x}$ can be written as:

$$\mathrm{d}\vec{x} = \mathrm{d}x^\alpha \vec{e}_\alpha \tag{15}$$

When this is true, we say that $\vec{e}_\alpha$ is a **coordinate basis vector**.

The notion of coordinate basis vector is not interesting in Cartesian coordinates. But how about curvilinear coordinates? Take the spherical coordinate as an example, the $\mathrm{d}\vec{x}$ can be written as:

$$\mathrm{d}\vec{x} = \mathrm{d}x^i \vec{e}_i = \mathrm{d}r\vec{e}_r + \mathrm{d}\theta\vec{e}_\theta + \mathrm{d}\phi\vec{e}_\phi \tag{16}$$

The equation above might be a little confusing. Because to make it works, $\vec{e}_r$ must be dimensionless, $\vec{e}_\theta$ and $\vec{e}_\phi$ must have the dimension of length. Now let's go back to a more familiar form:

$$\mathrm{d}\vec{x} = \mathrm{d}r\vec{e}_{\hat{r}} + r\mathrm{d}\theta\vec{e}_{\hat{\theta}} + r\sin\theta\mathrm{d}\phi\vec{e}_{\hat{\phi}} \tag{17}$$

This is easier for us to understand because the basis vector in it is **orthonormal**, and that is the property we want the basis vector to have. But the basis vectors in (16) have:

$$\vec{e}_r \cdot \vec{e}_r = 1, \vec{e}_\theta \cdot \vec{e}_\theta = r^2, \vec{e}_\phi \cdot \vec{e}_\phi = r^2\sin^2\theta \tag{18}$$

From the displacement vector, we can get another important 4-vector called **4-velocity**:

$$\vec{u} \equiv \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} \tag{19}$$

In the equation, $\mathrm{d}\tau$ means time interval as measured along the trajectory of observer with 4-velocity $\vec{u}$. In other word, it's the interval of **proper time**. The components of 4-velocity is:

$$\vec{u} = \left(\gamma, \gamma \underset{\sim}{v}\right) \tag{20}$$

The **4-momentum** is defined as:

$$\vec{p} = m\vec{u} = \left(\gamma m, \gamma m \underset{\sim}{v}\right) = \left(E, \underset{\sim}{p}\right) \tag{21}$$

m is the rest mass of object.

Now let's couple these two new 4-vectors to scalar product. First is $\vec{u}$:

$$\vec{u} \cdot \vec{u} = -\gamma^2 + \gamma^2 v^2 \equiv -1 \tag{22}$$

Now someone might wonder why the right hand side must be -1. We can consider this in a rest frame. In the rest frame, v is 0 and γ is 1, so the result is -1. And this is an invariant. What about $\vec{p}$? We can write down:

$$\vec{p} \cdot \vec{p} = m^2 \vec{u} \cdot \vec{u} = -m^2 = -E^2 + \left| \underset{\sim}{p} \right|^2 \tag{23}$$

thus we have:

$$E^2 - \left| \underset{\sim}{p} \right|^2 = m^2 \tag{24}$$

If you think you're unfamiliar with this, now let's put some c factor to it:

$$E^2 - p^2 c^2 = m^2 c^4 \tag{25}$$

So conservation of 4-momentum puts both conservation of energy and conservation of momentum into one mathematical object. Futhermore, if we have N particles interacting, then the total 4-momentum is conserved in the interaction:

$$\vec{p}_{\text{tot}} = \sum_{i=1}^{N} \vec{p}_i = \text{const} \tag{26}$$

Usually when we want to simplify our algebra, we can choose a special frame called "**center of momentum** frame". In this frame, total momentum of whole system equals 0:

$$\vec{p}_{\text{tot}} \overset{\text{COM}}{=\!=\!=} \left(E, \underset{\sim}{0} \right) \tag{27}$$

And this is very useful when study particles collison.

Next we will deduce a very useful result follows from invariance of scalar product. Let $\vec{p}$ be 4-momentum of particle A. Let $\vec{u}$ be the 4-velocity of observer O. The question is: what does O measure as the energy of particle A? So in O's IRF, 4-momentum of particle can be written as:

$$\vec{p} \underset{O}{\overset{\bullet}{=\!=}} \left(E_O, \underset{\sim}{p_O} \right) \tag{28}$$

and $\vec{u}$ is :

$$\vec{u} \underset{O}{\overset{\bullet}{=\!=}} \left(1, \underset{\sim}{0} \right) \tag{29}$$

Their scalar product is:

$$\vec{p} \cdot \vec{u} = -E_O \tag{30}$$

Because of the invariance of scalar product, you just need to measure $\vec{p}$ and $\vec{u}$, then take the scalar product between them. Then the answer you want pops out. No nonsense of Lorentz transformation, just do the scalar product!

At last let's talk about **4-velocity of acceleration**:

$$\vec{a} = \frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau} \tag{31}$$

There is an important property of $\vec{a}$. It is always the case that $\vec{a}$ dotted into $\vec{u}$ equals 0:

$$\vec{a} \cdot \vec{u} = 0 \tag{32}$$

It's pretty easy to prove this. Always remember $\vec{u} \cdot \vec{u} = -1$, then

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \vec{u} \cdot \vec{u} = 2\vec{a} \cdot \vec{u} = 0 \tag{33}$$

General Relativity Lecture3
Tensors(continued)

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Definition Tensor of type $\begin{pmatrix} 0 \\ N \end{pmatrix}$ is a function or mapping of N vectors into Lorentz invariant scalars which is linear in its N arguments.

If we want to understand this in a more specific way, we can look into the scalar product we've learned. We know that the scalar product has some linear properties:

$$\vec{A} \cdot \vec{B} = \eta_{\alpha\beta} A^{\alpha} B^{\beta} = a$$
$$\left(\gamma \vec{A}\right) \cdot \vec{B} = \eta_{\alpha\beta} \left(\gamma A^{\alpha}\right) B^{\beta} = \gamma a$$
$$\vec{A} \cdot \left(\vec{B} + \vec{C}\right) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} = \eta_{\alpha\beta} A^{\alpha} B^{\beta} + \eta_{\alpha\beta} A^{\alpha} C^{\beta}$$

We can abstractly define tensor as a two-slot mathematical machine:

$$\boldsymbol{\eta}\left(\vec{A}, \vec{B}\right) \equiv \vec{A} \cdot \vec{B} = \eta_{\alpha\beta} A^{\alpha} B^{\beta} = a \tag{1}$$

Because $\vec{A}, \vec{B}$ and a are frame-independent geometric object, so the tensor must be frame-independent as well. But always remember that different representations of the tensor are used by different observers.

To get the components used by a particular observer, just plug basis vectors into its slots. We can take the metric tensor $\boldsymbol{\eta}$ as example:

$$\boldsymbol{\eta}\left(\vec{e}_{\alpha}, \vec{e}_{\beta}\right) \equiv \eta_{\alpha\beta}, \quad \boldsymbol{\eta}\left(\vec{e}_{\bar{\alpha}}, \vec{e}_{\bar{\beta}}\right) \equiv \eta_{\bar{\alpha}\bar{\beta}} \tag{2}$$

Now relate this equation to transformation of basis vector:

$$\eta_{\bar{\alpha}\bar{\beta}} = \boldsymbol{\eta}\left(\Lambda^{\mu}{}_{\bar{\alpha}} \vec{e}_{\mu}, \Lambda^{\nu}{}_{\bar{\beta}} \vec{e}_{\nu}\right) = \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} \boldsymbol{\eta}\left(\vec{e}_{\mu}, \vec{e}_{\nu}\right) = \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} \eta_{\mu\nu} \tag{3}$$

Next let's consider a special subset of tensors. That is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensors. In some textbooks, it is called "1-forms". And sometimes, it is also called "dual vectors". According to the definition of tensors, 1-form is mapping from a single vector to Lorentz invariant scalars. It means if we put a vector into its slot, we will get a scalar. If we want to know the components of 1-form, let's put basis vector in it:

$$\tilde{p}\left(\vec{e}_{\alpha}\right) = p_{\alpha} \tag{4}$$

With this, we can get the result that putting any vector in it:

$$\tilde{p}\left(\vec{A}\right) = \tilde{p}\left(A^{\alpha} \vec{e}_{\alpha}\right) = A^{\alpha} \tilde{p}\left(\vec{e}_{\alpha}\right) = A^{\alpha} p_{\alpha} \tag{5}$$

The operation above is also called contraction. Certainly the transformation of 1-form between different frame is pretty easy:

$$p_{\bar{\alpha}} = \Lambda^{\mu}{}_{\bar{\alpha}} p_{\mu} \tag{6}$$

Now we want a set of geometric objects, which are called basis 1-form. With this, we can write every 1-form as:

$$\tilde{p} = p_{\alpha} \tilde{\omega}^{\alpha} \tag{7}$$

To get the certain form of this geometric object, we can use equation (5):

$$\tilde{p}\left(\vec{A}\right) = p_\beta \tilde{\omega}^\beta \left(A^\alpha \vec{e}_\alpha\right) = p_\beta A^\alpha \tilde{\omega}^\beta \left(\vec{e}_\alpha\right) \tag{8}$$

Thus we get:

$$\tilde{\omega}^\beta \left(\vec{e}_\alpha\right) = \delta_\alpha{}^\beta \tag{9}$$

So via the pointview of linear algebra, the basis 1-form is really like basis vectors, but enter in a "dual" way. It's akin to row vectors versus column vectors. And this also reminds us that the contraction always happens between two objects with dual natures. For example, we can definitely write something like $\sum_{\mu=0}^{3} A^\mu B^\mu$, but it plays no role in our physics!

Another example in Quantum Mechanics Supposed that we have two wave functions $\psi\left(\vec{x}\right)$ and $\phi\left(\vec{x}\right)$. The operation between two functions we always do is like:

$$\int \psi^*\left(\vec{x}\right)\phi\left(\vec{x}\right)\mathrm{d}^3\vec{x} = \langle\psi|\phi\rangle \tag{10}$$

instead of

$$\int \psi\left(\vec{x}\right)\phi\left(\vec{x}\right)\mathrm{d}^3\vec{x} \tag{11}$$

Now let's consider an important example. Imagine there is a trajectory of the observer in spacetime. We can define the 4-velocity of this observer as:

$$\vec{u} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} \tag{12}$$

Suppose spacetime is filled with some field $\phi\left(t,x,y,z\right)$, so what's the rate of change of ϕ along trajectory? In 3-space, it is pretty easy to solve:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\phi}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial\phi}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial\phi}{\partial z} = \vec{v}\cdot\vec{\nabla}\phi \tag{13}$$

In spacetime, it can be similarly generalized as:

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\partial\phi}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\partial\phi}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\partial\phi}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}\tau}\frac{\partial\phi}{\partial z} \\ &= u^t\frac{\partial\phi}{\partial t} + u^x\frac{\partial\phi}{\partial x} + u^y\frac{\partial\phi}{\partial y} + u^z\frac{\partial\phi}{\partial z}\end{aligned} \tag{14}$$

To simplify this, we can introduce some mathematical notation:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = u^\alpha\frac{\partial\phi}{\partial x^\alpha} \equiv u^\alpha\partial_\alpha\phi \tag{15}$$

So the generalization of gradient is an example of 1-form.

$$\tilde{d}\phi = \left\{\partial_\alpha\phi\right\} \tag{16}$$

*a new notation:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = u^\alpha\partial_\alpha\phi \equiv \nabla_{\vec{u}}\phi \tag{17}$$

Actually the notion of gradient as 1-form gives us a nice way to think about basis 1-forms:

$$\tilde{\omega}^\alpha\left(\vec{e}_\beta\right) = \delta^\alpha{}_\beta \quad \text{vs} \quad \partial_\beta x^\alpha = \delta^\alpha{}_\beta \tag{18}$$

So we have:

$$\tilde{d}x^\alpha \equiv \tilde{\omega}^\alpha \tag{19}$$

In the end let's go back to metric tensor. We already know that metric with both slots filled with vectors is a Lorentz invariant scalar. But what about only one slot is filled? Actually you can easily figure out that it's 1-form!

$$\tilde{A}\left(\bullet\right) = \boldsymbol{\eta}\left(\vec{A},\bullet\right) \tag{20}$$

And we can write its components:

$$A_\alpha = \boldsymbol{\eta}\left(\vec{A}, \vec{e}_\alpha\right) = \boldsymbol{\eta}\left(A^\beta \vec{e}_\beta, \vec{e}_\alpha\right) = \eta_{\alpha\beta} A^\beta \tag{21}$$

The equation below tells us that matric converts vectors into 1-forms by ”lowering” indices. Of course this process is invertible. But we need to define inverse metric:

$$\eta^{\alpha\beta}\eta_{\beta\gamma} \equiv \delta^\alpha{}_\gamma \tag{22}$$

Use this, we can turn 1-form into vector:

$$\eta^{\alpha\beta} p_\beta = p^\alpha \tag{23}$$

New definition of tensor A tensor of type $\begin{pmatrix} M \\ N \end{pmatrix}$ is a linear mapping of M 1-forms and N vectors to Lorentz invariant scalars.

And metric allows us convert nature of slots on a tensor.

$$\begin{pmatrix} M \\ N \end{pmatrix} \xrightarrow{\text{lower}} \begin{pmatrix} M-1 \\ N+1 \end{pmatrix} \tag{24}$$

Just like:

$$\eta_{\alpha\mu} R^\mu{}_{\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} \tag{25}$$

Or

$$\begin{pmatrix} M \\ N \end{pmatrix} \xrightarrow{\text{raise}} \begin{pmatrix} M+1 \\ N-1 \end{pmatrix} \tag{26}$$

$$\eta^{\alpha\mu} S_{\mu\beta\gamma} = S^\alpha{}_{\beta\gamma} \tag{27}$$

The last question is : do we need give the notion of basis to every type of tensor? The answer is NO. Because with the operation of tensor product, the basis of any type of tensor can be written as the tensor product of basis vectors and basis 1-form.

$$R = R^\alpha{}_{\beta\gamma\delta} \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \otimes \tilde{\omega}^\delta \tag{28}$$

General Relativity Lecture4
Volume and Volume Elements; Conservation Laws

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Now let's comb through the physical quantities that we have introduced. The first is 4-velocity :

$$\vec{u} = \left(\gamma, \gamma \underset{\sim}{v}\right), \vec{u} \cdot \vec{u} = -1 \tag{1}$$

The definition of it means it's timelike and normalized. But if we want to describe the motion of a photon, obviously the 4-velocity is not a good choice.

And the second is 4-momentum. The definition and properties are writing below:

$$\vec{p} = m\vec{u} = \left(E, \underset{\sim}{p}\right), \vec{p} \cdot \vec{p} = -m^2, E^2 - \left|\underset{\sim}{p}\right|^2 = m^2 \tag{2}$$

In this case $m = 0$ is permitted. What's more, when the object we focus on is a photon, we always use the formula below:

$$\vec{p} = \hbar\omega\left(1, \hat{u}\right) \tag{3}$$

and $\hat{u}$ is the unit vector in direction of motion.

Next we're going to talk about something more interesting. Thus far, the discussions in physics have primarily centered on individual particles. So it is necessary for us to expand our research objects into a continuum. Let's begin with a simplest continuum: dust. The dust is a cluster of particles with mass and energy, but there is no interaction between two particles. When there is a cloud of dust, each element has its own rest frame. But different element has different rest frame.

To analysize a cloud of dust, the first step we can take is to figure out how many bits of dust are in element per volume, in other word the number density. Let n_0 be the number density in rest frame of element. But in most cases we are not in rest frame of element. So naturally we want to know what will happen when we move out of rest frame.

Actually 2 things are going to happen:

1. Number in volume stays same while volume Lorentz contracts, which means number density is no longer n_0. New number density is:
$$n = \gamma n_0 \tag{4}$$
2. If dust is flowing through space, we can define a 3-vector to describe the property of flux. Let $\underset{\sim}{n}$ be the number of particles crossing unit area in unit time, and we can get:
$$\underset{\sim}{n} = n\underset{\sim}{v} = \gamma n_0 \underset{\sim}{v} \tag{5}$$

With the fact above, it's natural to define a new 4-vector:

$$\vec{N} = \left(n, n\underset{\sim}{v}\right) = n_0\left(\gamma, \gamma\underset{\sim}{v}\right) = n_0\vec{u} \tag{6}$$

From this, it's pretty easy to get:

$$\vec{N} \cdot \vec{N} = -n_0^2 \longrightarrow n_0 = \sqrt{-\vec{N} \cdot \vec{N}} \tag{7}$$

Now we got $\vec{N}$. How cna we utilize it to find a systematic way to pick out flux across a surface? We can use a conclusion that we introduced in last lecture:$\tilde{dx}^\alpha = \tilde{\omega}^\alpha$. Then the flux of $\vec{N}$ in the x^α direction is:

$$\left(\tilde{dx}^\alpha\right)_\beta N^\beta = \text{this flux} \tag{8}$$

For time component, from formula above we can get:

$$\left(\tilde{d t}^{\alpha}\right)_{\beta} N^{\beta}=N^{0}=n \tag{9}$$

This is really an interesting result. Because we're tend to think about: what the time component of flux means? It means the thing is sitting there, just move in time and doesn't do anything. So we described the flux of a thing in time as just being its density.

More generally, we can define surface as solution of some scalar function in spacetime:

$$\psi(t, x, y, z)=\text{const} \tag{10}$$

Then the flux through this surface is easy to calculate:

$$\left(\tilde{d \psi}\right)_{\alpha} N^{\alpha}=\text{flux through surface} \tag{11}$$

When it comes to flux, there is always a conservation laws: the flux out of the sides must come at the expense of the density of dust there. Write this in mathematical way:

$$\frac{\partial n}{\partial t}=-\underset{\sim}{\nabla} \cdot \underset{\sim}{n} \tag{12}$$

or in a more compact way:

$$\partial_{\alpha} N^{\alpha}=0 \tag{13}$$

As we all know, the differential form is equivalent to the integral form:

$$\frac{\partial}{\partial t} \int_{V^{3}} n \mathrm{d} V=-\int_{\partial V^{3}} \underset{\sim}{n} \cdot \mathrm{d} \underset{\sim}{a} \tag{14}$$

We're going to spend some time on volume and volume integral. As we know, three vectors in 3-dimentional space can define a volume:

$$\underset{\sim}{A} \cdot\left(\underset{\sim}{B} \times \underset{\sim}{C}\right) \tag{15}$$

This formula is cyclic which means we can move three vectors in a cyclic way. But it can be written in a simpler way with Levi-Civita symbol:

$$\epsilon_{i j k} A^{i} B^{j} C^{k} \tag{16}$$

Actually we can consider Levi-Civita symbol as components of a $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ tensor. Put 3 vectors in it, and it will produce volume of figure bounded by these vectors.

$$V^{3}=\boldsymbol{\epsilon}\left(\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}\right) \tag{17}$$

But if only put 2 vectors, then we will get

$$\boldsymbol{\epsilon}\left(\bullet, \underset{\sim}{B}, \underset{\sim}{C}\right) \longrightarrow \Sigma_{i}=\epsilon_{i j k} B^{j} C^{k} \tag{18}$$

Here Σ is a 1-form whose magnitude is the area of the side spanned by $\underset{\sim}{B}$ and $\underset{\sim}{C}$.

We can use things above to write Gauss's theorem in geometric language:

$$\int_{V^{3}}\left(\underset{\sim}{\nabla} \cdot \underset{\sim}{A}\right) \mathrm{d} V=\int_{\partial V^{3}} \underset{\sim}{A} \cdot \mathrm{d} \underset{\sim}{\Sigma} \tag{19}$$

And we can define a differential triple: $\mathrm{d}\vec{x}_1, \mathrm{d}\vec{x}_2, \mathrm{d}\vec{x}_3$, which fulfills:

$$\mathrm{d} V=\epsilon_{i j k} \mathrm{d} x_{1}^{i} \mathrm{d} x_{2}^{j} \mathrm{d} x_{3}^{k} \tag{20}$$

To generalize such things to spacetime, we need introduce something more. Imagine a parallelepiped with sides $\vec{A}, \vec{B}, \vec{C}, \vec{D}$. We can define 4-volume of such things:

$$4-\text{volume}=\epsilon_{\alpha \beta \gamma \sigma} A^{\alpha} B^{\beta} C^{\gamma} D^{\sigma} \tag{21}$$

And the definiton of 4-D Levi-Civita cymbol is:

$$\epsilon_{\alpha\beta\gamma\sigma} = \begin{cases} +1, \alpha\beta\gamma\sigma \text{ in even permutations} \\ -1, \alpha\beta\gamma\sigma \text{ in odd permutations} \\ 0, \text{have repeated index} \end{cases} \tag{22}$$

The area of each "face" of this figure is a 3-volume:

$$\Sigma_\alpha = \epsilon_{\alpha\beta\gamma\delta} B^\beta C^\gamma D^\delta \tag{23}$$

With this, we can get the generalization of Gauss's theorem:

$$\int_{V^4} (\partial_\alpha V^\alpha)\, \mathrm{d}V^4 = \int_{\partial V^4} V^\alpha \mathrm{d}\Sigma_\alpha \tag{24}$$

Now put $\vec{N}$ to this formula. Because $\partial_\alpha N^\alpha = 0$, so we get:

$$\int_{V^4} (\partial_\alpha N^\alpha)\, \mathrm{d}V^4 = 0, \quad \int_{\partial V^4} N^\alpha \mathrm{d}\Sigma_\alpha = 0 \tag{25}$$

Now consider the 4-vol that be drawn in a 2-dimensional plane, which is shown by fig 1.

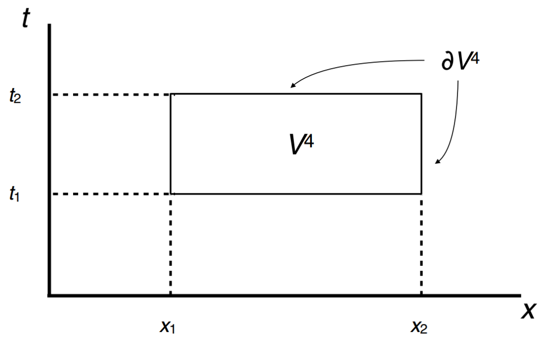

Figure 1: the 4-volume we consider

Now just do the integral:

$$\begin{aligned} \int_{\partial V^4} N^\alpha \mathrm{d}\Sigma_\alpha = & \int_{t=t_2} N^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{t=t_1} N^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ & + \int_{x=x_2} N^1 \mathrm{d}t\mathrm{d}y\mathrm{d}z - \int_{x=x_1} N^1 \mathrm{d}t\mathrm{d}y\mathrm{d}z \\ & + \cdots\cdots = 0 \end{aligned} \tag{26}$$

Let $t_2 \to t_1 + \mathrm{d}t$, rearrange:

$$\int_{t_1+\mathrm{d}t} N^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{t_1} N^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z = -\mathrm{d}t \left[\int_{x_2} N^1 \mathrm{d}y\mathrm{d}z - \int_{x_1} N^1 \mathrm{d}y\mathrm{d}z + \cdots \right] \tag{27}$$

When $t \to 0$, things will become

$$\frac{\int_{t_1+\mathrm{d}t} N^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{t_1} N^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z}{\mathrm{d}t} \longrightarrow \frac{\partial}{\partial t} \int N^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{28}$$

On the other side, get flux of $\underset{\sim}{N}$ through the six sides.

$$\frac{\partial}{\partial t} \int N^0 \mathrm{d}V = -\int \underset{\sim}{N} \cdot \mathrm{d}\underset{\sim}{a} \tag{29}$$

Next let's switch to another important example of matter: electric current. Define:

$$\vec{J} = \left(\rho, \underset{\sim}{J}\right) \tag{30}$$

There is conservation of charge:

$$\frac{\partial \rho}{\partial t} = -\underset{\sim}{\nabla} \cdot \underset{\sim}{J} \longleftrightarrow \partial_\alpha J^\alpha = 0 \tag{31}$$

If we want to represent the electric and magnetic field in the space, obviously we need an object with 6 indepent components. In this case 4-vector is not the perfect choice. Instead, an antisymmetric tensor maybe better. For an antisymmetric tensor, the components at diagnal must be zeros.

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \tag{32}$$

With this, Maxwell's equations can be written as:

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu \tag{33}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{34}$$

What's more, we can prove that the divergence of current is zero:

$$\begin{aligned} 4\pi\partial_\mu J^\mu &= \partial_\mu \partial_\nu F^{\mu\nu} \\ &= \partial_\nu \partial_\mu F^{\nu\mu} \\ &= -\partial_\nu \partial_\mu F^{\mu\nu} \\ &= -\partial_\mu \partial_\nu F^{\mu\nu} = 0 \end{aligned} \tag{35}$$

In the end, a trick about symmetry is very useful. If $A^{\alpha\beta}$ is antisymmetric and $S^{\alpha\beta}$ is antisymmetric, then

$$A^{\alpha\beta} S_{\alpha\beta} = 0 \tag{36}$$

General Relativity Lecture5

The Stress Energy Tensor & The Christoffel Symbol

Lin Fu

March 2025

In the last lecture we talked about dust. Specifically, we learned the number density of dust. But that's not all, because there are many other properties of dust that we haven't yet learned about. So let's consider energy and momentum of dust. Suppose each dust particle has a rest mass m. In the rest frame of the dust element, its rest energy density is:

$$\rho_0 = mn_0 \tag{1}$$

Now go into frame moving with $\underset{\sim}{v}$ relative to this frame. The energy density becomes:

$$\rho = (\gamma m)(\gamma n_0) = \gamma^2 \rho_0 \tag{2}$$

This is not the transformation law of a 4-vector component, nor a scalar. But if ρ is a component of a tensor, then the thing can be permitted. So assemble ρ by combining energy (timelike component of $\vec{p}$) with number density (timelike component of $\vec{N}$):

$$\rho = p^t N^t \equiv T^{tt} \tag{3}$$

We say $\boldsymbol{T}$ is the tensor product of $\vec{N}$ and $\vec{p}$:

$$\boldsymbol{T} = \vec{N} \otimes \vec{p} = n_0 m \vec{u} \otimes \vec{u} = \rho_0 \vec{u} \otimes \vec{u} \tag{4}$$

The component of $\boldsymbol{T}$ is:

$$T^{\alpha\beta} = \rho_0 u^\alpha u^\beta \tag{5}$$

In other way, we can put basis 1-form in its two slots:

$$T^{\alpha\beta} = \boldsymbol{T}\left(\tilde{d}x^\alpha, \tilde{d}x^\beta\right) \tag{6}$$

The physical meaning of this formula is flux of 4-momentum p^α in the x^β direction. With this, $T^{00} = \rho_0 u^t u^t \equiv \rho$ means flux of p^t in t direction. That is the energy density.

Let's look at other components:

- T^{0i}: means flux of p^t in x^i direction. That is the energy flux.
- T^{i0}: means flux of p^i in t direction. That is the momentum density.
- T^{ij}: means flux of p^i in x^j direction. That's just momentum flux.

Go back to dust itself, we can write down:

$$\begin{gathered} T^{0i} = \rho_0 u^0 u^i = \gamma^2 \rho_0 v^i \\ T^{i0} = \rho_0 u^i u^0 = \gamma^2 \rho_0 v^i \\ T^{ij} = \rho_0 u^i u^j = \gamma^2 \rho_0 v^i v^j \end{gathered}$$

We find that energy flux and momentum density are exactly the same. And, the tensor $\boldsymbol{T}$ is symmetric.

Next we're going to learn about an very important example: perfect fluid. The definiton of perfect fluid has three keypoints:

1. No energy flow in "rest" frame.

2. No lateral stresses. Lateral stresses refer to T^{ij} when i and j are not equal to one another. This means fluid has no viscosity.

Things above tells us: perfect fluid is totally characterized by energy density ρ and its pressure P (pressure means isotropic spatial stress).

So in this frame, the stress energy tensor of perfect fluid can be written as:

$$T^{\alpha\beta} \doteq \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} = \mathrm{diag}(\rho, P, P, P) \tag{7}$$

But this result is frame dependent. We can write it in a more covariant way:

$$\boldsymbol{T} = \rho \vec{u} \otimes \vec{u} + (\boldsymbol{\eta} + \vec{u} \otimes \vec{u})\, P \tag{8}$$

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + P(\eta^{\alpha\beta} + u^\alpha u^\beta) = (\rho + P) u^\alpha u^\beta + P\eta^{\alpha\beta} \tag{9}$$

We might heard about Newtonian field equation before:

$$\nabla^2 \Phi_g = 4\pi G \rho \tag{10}$$

We called Φ_g the Newtonian gravitational interaction, and ρ is the mass density. But actually this equation should make us suspicious immediately because ρ is not a scalar, which means this equation only describe one specific component of a tensor. This cannot be a healthy theory. So Einstein's effort is to make this equation to describe tensor. In this case, the ρ should becomes $T^{\alpha\beta}$.

Next we want to prove the spatial component of $\boldsymbol{T}$ is symmetric, that is $T^{ij} = T^{ji}$. Consider a cube immersed in $T^{\alpha\beta}$, look at momentum flux into and out of box. We give every face except top and bottom a number, just shown in Figure.

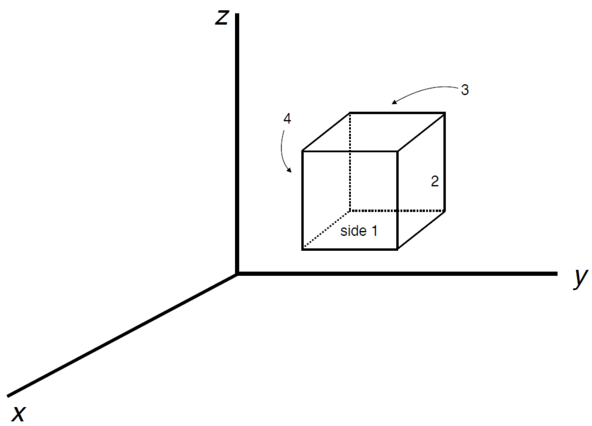

Figure 1: a cube immersed in $T^{\alpha\beta}$

So the force on 4 faces are:

$$\underset{\sim}{F_1} = T^{ix} l^2, \underset{\sim}{F_2} = T^{iy} l^2, \underset{\sim}{F_3} = -\underset{\sim}{F_1}, \underset{\sim}{F_4} = -\underset{\sim}{F_2} \tag{11}$$

Let us next examine the torques that these forces exert about an axis through the center of the cube, parallel to the z axis. So if we do this, we'll find that:

$$\tau^z = l^3 \left(T^{xy} - T^{yx}\right) \tag{12}$$

And the momentum of inertia of the cube is

$$I = \alpha \left(\rho l^3\right) l^2 \propto l^5 \tag{13}$$

The angular acceleration is:

$$\ddot{\theta} = \tau^2 / I \propto (T^{xy} - T^{yx}) / l^2 \tag{14}$$

When l is very small, it will unstablely rotate. This is absolutely unphysical. So we must have $T^{xy} = T^{yx}$.

With the stress energy tensor, conservation of energy and momentum can be written in a very simple and beautiful way:

$$\partial_\alpha T^{\alpha\beta} = 0 \tag{15}$$

Once we pick a frame, then

$$\text{energy conservation} \quad \partial_\alpha T^{\alpha 0} = 0 \quad \text{or} \quad \frac{\partial T^{00}}{\partial t} = -\frac{\partial T^{0i}}{\partial x^i} \tag{16}$$

$$\text{momentum conservation} \quad \partial_\alpha T^{i\alpha} = 0 \quad \text{or} \quad \frac{\partial T^{i0}}{\partial t} = -\frac{\partial T^{ij}}{\partial x^j} \tag{17}$$

All these can be written as integral equation

$$\frac{\partial}{\partial t}\int_{V^3} T^{00}\mathrm{d}^3x = -\int_{\partial V^3} T^{0i}\mathrm{d}\Sigma_i \tag{18}$$

$$\frac{\partial}{\partial t}\int_{V^3} T^{0j}\mathrm{d}^3x = -\int_{\partial V^3} T^{ij}\mathrm{d}\Sigma_i \tag{19}$$

Another example Let's consider a point particle of rest mass m_0, moving in worldline $\vec{z}(\tau)$. The stress energy tensor of it is:

$$T^{\mu\nu} = m_0 \int u^\mu u^\nu \delta^{(4)}\left[\vec{x} - \vec{z}(\tau)\right]\mathrm{d}\tau \tag{20}$$

The definiton of 4-Dirac function is

$$\delta^{(4)}\left[\vec{x} - \vec{z}(\tau)\right] = \delta\left[t - z^0(\tau)\right]\delta\left[x - z^1(\tau)\right]\delta\left[y - z^2(\tau)\right]\delta\left[z - z^3(\tau)\right] \tag{21}$$

And there is a rule:

$$\int f(x)\delta[g(x)]\mathrm{d}x = \frac{f(x_0)}{|g'|_{x=x_0}} \tag{22}$$

x_0 is the zero point of $g(x)$. With the rule, (20) can be written as:

$$T^{\mu\nu} = \frac{mu^\mu u^\nu}{u^0}\delta^{(3)}\left[\underset{\sim}{x} - \underset{\sim}{z}(\tau)\right] \tag{23}$$

Electromagnetic field If a region is filled with an electromagnetic field with field tensor $F^{\mu\nu}$, then

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right] \tag{24}$$

It's might a little bit hard to understand. But we can look into some specific components of it:

$$T^{00} = \frac{\underset{\sim}{E}\cdot\underset{\sim}{E} + \underset{\sim}{B}\cdot\underset{\sim}{B}}{8\pi} \tag{25}$$

$$T^{0i} = \frac{(\underset{\sim}{E}\times\underset{\sim}{B})^i}{4\pi} = T^{i0} \tag{26}$$

$$T^{ij} = \frac{1}{8\pi}[(\underset{\sim}{E}\cdot\underset{\sim}{E} + \underset{\sim}{B}\cdot\underset{\sim}{B})\delta^{ij} - 2(E^iE^j + B^iB^j)] \tag{27}$$

If we consider a simple condition $\boldsymbol{E} = E^x\boldsymbol{e}_x$, then

$$T^{\mu\nu} \doteq \frac{(E^x)^2}{8\pi}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{28}$$

The term T^{xx} indicates that this electromagnetic field generates an attractive stress along the x direction, but also a pressure in the y and z directions.

Prelude to curvature: flat spacetime in curvilinear coordinates Let's consider the transformation from Cartesian coordinates to cylindrical coordinate. To transform (t, x, y, z) to (t, r, ϕ, z), we can use

$$x = r\cos\phi, y = r\sin\phi \tag{29}$$

And we continue to use a coordinate basis. This means:

$$\mathrm{d}\vec{x} = \mathrm{d}t\vec{e}_t + \mathrm{d}r\vec{e}_r + \mathrm{d}y\vec{e}_y + \mathrm{d}z\vec{e}_z \tag{30}$$

We need to note that the basis vectors above are not normal. For example, $\vec{e}_\phi \cdot \vec{e}_\phi \neq 1$.

Next we need to learn about transformation between representations. Introduce the notation:

$$L^{\alpha}{}_{\bar{\mu}} = \frac{\partial x^\alpha}{\partial x^{\bar{\mu}}} \tag{31}$$

And let barred be cylindrical and unbarred be Cartesian. And we can use a matrix to represent this relationship:

$$\frac{x^\alpha}{x^{\bar{\mu}}} \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -r\sin\phi & r\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = L^{\alpha}{}_{\bar{\mu}} \tag{32}$$

Of course we can write the inverse of it:

$$\frac{\partial x^{\bar{\mu}}}{\partial x^\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi/r & 0 \\ 0 & \sin\phi & \cos\phi/r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = L^{\bar{\mu}}{}_{\alpha} \tag{33}$$

General Relativity Lecture6

The Principle of Equivalence

Lin Fu

March 2025

Still working in cylindrical coordinate, let's condiser what basis vectors looks like:

$$\vec{e}_r = \cos\phi\vec{e}_x + \sin\phi\vec{e}_y = L^{\alpha}{}_{r}\vec{e}_\alpha \tag{1}$$
$$\vec{e}_\phi = -r\sin\phi\vec{e}_x + r\cos\phi\vec{e}_y = L^{\alpha}{}_{\phi}\vec{e}_\alpha \tag{2}$$

And we can still define metric as $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$. So in this example the metric is:

$$g_{\alpha\beta} = \mathrm{diag}(-1, 1, r^2, 1) \tag{3}$$

With the metric, we can always know the invariant displacement between two events:

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = -\mathrm{d}t^2 + \mathrm{d}x^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2 \tag{4}$$

What's more, we can write the basis 1-form in this frame:

$$\tilde{d}r = L^{r}{}_{\alpha}\tilde{d}x^\alpha = \cos\phi\tilde{d}x + \sin\phi\tilde{d}y \tag{5}$$
$$\tilde{d}\phi = -\frac{\sin\phi}{r}\tilde{d}x + \frac{\cos\phi}{r}\tilde{d}y \tag{6}$$

Next we're going to talk about derivatives. There is an important result that we need to know:

$$\frac{\partial\vec{e}_r}{\partial r} = 0, \frac{\partial\vec{e}_r}{\partial\phi} = \frac{\vec{e}_\phi}{r}, \frac{\partial\vec{e}_\phi}{\partial r} = \frac{\vec{e}_\phi}{r}, \frac{\partial\vec{e}_\phi}{\partial\phi} = -r\vec{e}_r \tag{7}$$

Let's think about this in the context of the derivative of a simple tensor object, a vector $\vec{V} = V^\alpha\vec{e}_\alpha$. We can write its gradient:

$$\nabla\vec{V} = \partial_\beta(V^\alpha\vec{e}_\alpha)\tilde{\omega}^\beta \tag{8}$$

The derivative we need to evaluate here is:

$$\frac{\partial\vec{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta}\right)\vec{e}_\alpha + V^\alpha\frac{\partial\vec{e}_\alpha}{\partial x^\beta} \tag{9}$$

We can see that the formula above involves the derivatives of the basis vectors themselves.

Now we can introduce a new symbol, and this is based on that we consider $\partial_\beta\vec{e}_\alpha$ as linear combination of basis vectors:

$$\partial_\beta\vec{e}_\alpha = \Gamma^{\mu}{}_{\beta\alpha}\vec{e}_\mu \tag{10}$$

The $\Gamma^{\mu}{}_{\beta\alpha}$ is called Christoffel symbol. So in cylindrical coordinate, there are three non-zero Christoffel symbols:

$$\Gamma^{\phi}{}_{r\phi} = \frac{1}{r} = \Gamma^{\phi}{}_{\phi r}, \;\; \Gamma^{r}{}_{\phi\phi} = -r \tag{11}$$

Now let's continue to talk about derivative of vector:

$$\begin{aligned}\partial_\beta\vec{V} &= \vec{e}_\alpha\partial_\beta V^\alpha + V^\alpha\Gamma^{\mu}{}_{\beta\alpha}\vec{e}_r \\ &= (\partial_\beta V^\alpha + V^\mu\Gamma^{\alpha}{}_{\beta\mu})\vec{e}_\alpha \\ &= (\nabla_\beta V^\alpha)\vec{e}_\alpha\end{aligned} \tag{12}$$

We call $\nabla_\beta V^\alpha$ the covariant derivative.

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha{}_{\beta\mu} V^\mu \tag{13}$$

So the gradient can be written as:

$$\nabla \vec{V} = (\nabla_\beta V^\alpha) \tilde{\omega}^\beta \otimes \vec{e}_\alpha \tag{14}$$

And with this, we can calculate the divergence:

$$\nabla_\alpha V^\alpha = \partial_\alpha V^\alpha + \Gamma^\alpha{}_{\alpha\mu} V^\mu = \partial_t V^t + \partial_r V^r + \partial_z V^z + \partial_\phi V^\phi + V^r/r \tag{15}$$

Let's think about the covariant derivatives of other objects. For a scalar, things are pretty easy:

$$\nabla_\alpha \Phi = \partial_\alpha \Phi \tag{16}$$

For 1-form, we need to use the fact that 1-form contracted on vector is a scalar:

$$\begin{aligned} \nabla_\beta (p_\alpha A^\alpha) &= \partial_\beta (p_\alpha A^\alpha) \\ &= A^\alpha \partial_\beta p_\alpha + p_\alpha \partial_\beta A^\alpha \end{aligned} \tag{17}$$

Use $\partial_\beta A^\alpha = \nabla_\beta A^\alpha - \Gamma^\alpha{}_{\beta\mu} A^\mu$, then we get:

$$\begin{aligned} \nabla_\beta (p_\alpha A^\alpha) &= A^\alpha \partial_\beta p_\alpha + p_\alpha \nabla_\beta A^\alpha - p_\alpha \Gamma^\alpha{}_{\beta\mu} A^\mu \\ &= p_\alpha \nabla_\beta A^\alpha + A^\alpha (\partial_\beta p_\alpha - p_\mu \Gamma^\mu{}_{\beta\alpha}) \end{aligned} \tag{18}$$

For any derivative operation, we should have $\nabla_\beta (p_\alpha A^\alpha) = p_\alpha \nabla_\beta A^\alpha + A^\alpha \nabla_\beta p_\alpha$. So

$$\nabla_\beta p_\alpha = \partial_\beta p_\alpha - \Gamma^\mu{}_{\beta\alpha} p_\mu \tag{19}$$

And we can similarly analysize the basis 1-form,then we will get:

$$\partial_\beta \tilde{\omega}^\alpha = -\Gamma^\alpha{}_{\beta\mu} \tilde{\omega}^\mu \tag{20}$$

The minus sign here enforces

$$< \tilde{\omega}^\alpha, \vec{e}_\beta >= \delta^\alpha{}_\beta \tag{21}$$

It's also easy to calculate the covariant derivative of any type tensor:

$$\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu{}_{\beta\alpha} T^{\alpha\nu} + \Gamma^\nu{}_{\beta\alpha} T^{\mu\alpha} \tag{22}$$

$$\nabla_\beta T_{\mu\nu} = \partial_\beta T_{\mu\nu} - \Gamma^\alpha{}_{\beta\mu} T_{\alpha\nu} - \Gamma^\alpha{}_{\beta\nu} T_{\mu\alpha} \tag{23}$$

$$\begin{aligned} \nabla_\beta T^{\mu\nu\cdots}{}_{\rho\sigma\cdots} = \partial_\beta T^{\mu\nu\cdots}{}_{\rho\sigma\cdots} & \\ + \Gamma^\mu{}_{\beta\alpha} T^{\alpha\nu\cdots}{}_{\rho\sigma\cdots} &+ \Gamma^\nu{}_{\beta\alpha} T^{\mu\alpha\cdots}{}_{\rho\sigma\cdots} + \cdots \\ - \Gamma^\alpha{}_{\beta\rho} T^{\mu\nu\cdots}{}_{\alpha\sigma\cdots} &- \Gamma^\alpha{}_{\beta\sigma} T^{\mu\nu\cdots}{}_{\rho\alpha\cdots} - \cdots \end{aligned} \tag{24}$$

Actually there is a better way to get Christoffel symbol: via the metric. The derivation relies on a key property of tensor relationship: a tensorial equation that holds in one representation must hold in all representation. Change the representation or coordinates cannot change equation!

So let's do a warm up exercise: double gradient of a scalar. In Cartesian representation, it is

$$\nabla\nabla\phi \equiv \partial_\alpha \partial_\beta \phi \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \tag{25}$$

And it's symmetric when exchange α and β. In general representation, it is:

$$\nabla\nabla\phi = \nabla_\alpha \nabla_\beta \phi \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \tag{26}$$

And this must be symmetric in general!

$$\begin{aligned} \nabla_\alpha \nabla_\beta \phi &= \nabla_\beta \nabla_\alpha \phi \\ \nabla_\alpha (\partial_\beta \phi) &= \nabla_\beta (\partial_\alpha \phi) \\ \partial_\alpha \partial_\beta \phi - \Gamma^\mu{}_{\alpha\beta} \partial_\mu \phi &= \partial_\beta \partial_\alpha \phi - \Gamma^\mu{}_{\beta\alpha} \partial_\mu \phi \end{aligned} \tag{27}$$

Thus we get:

$$(\Gamma^{\mu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\beta\alpha})\partial_{\mu}\phi = 0 \tag{28}$$

This means Christoffel symbol is symmetric to its two lower indices.

We need more notation. First is:

$$A_{(\alpha\beta)} = \frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha}) \tag{29}$$

$$A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha}) \tag{30}$$

For Christoffel symbol, we have:

$$\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{(\alpha\beta)}, \Gamma^{\mu}{}_{\alpha\beta}A^{\alpha\beta} = 0 \ \text{ (if A is antisymmetric)} \tag{31}$$

The way to get Christoffel symbol is to calculate the gradient of the metric.

$$\nabla \boldsymbol{g} = \nabla_{\gamma} g_{\alpha\beta} \tilde{\omega}^{\alpha} \otimes \tilde{\omega}^{\beta} \otimes \tilde{\omega}^{\gamma} \tag{32}$$

$$\overset{Cartesian}{=\!=\!=\!=} \nabla_{\gamma} \eta_{\alpha\beta} \tilde{\omega}^{\alpha} \otimes \tilde{\omega}^{\beta} \otimes \tilde{\omega}^{\gamma} \tag{33}$$

This must be zero. So we require $\nabla_{\gamma} g_{\alpha\beta} = 0$. Now permute the indices:

$$\nabla_{\gamma} g_{\alpha\beta} = \partial_{\gamma} g_{\alpha\beta} - \Gamma^{\mu}{}_{\gamma\alpha} g_{\mu\beta} - \Gamma^{\mu}{}_{\gamma\beta} g_{\alpha\mu} = 0 \tag{34}$$

$$\nabla_{\alpha} g_{\beta\gamma} = \partial_{\alpha} g_{\beta\gamma} - \Gamma^{\mu}{}_{\alpha\beta} g_{\mu\gamma} - \Gamma^{\mu}{}_{\alpha\gamma} g_{\beta\mu} = 0 \tag{35}$$

$$\nabla_{\beta} g_{\gamma\alpha} = \partial_{\beta} g_{\gamma\alpha} - \Gamma^{\mu}{}_{\beta\gamma} g_{\mu\alpha} - \Gamma^{\mu}{}_{\beta\alpha} g_{\gamma\mu} = 0 \tag{36}$$

Now (34) - (35) - (36), we will get:

$$\begin{aligned} &\partial_{\gamma} g_{\alpha\beta} - \partial_{\alpha} g_{\beta\gamma} - \partial_{\beta} g_{\gamma\alpha} \\ &- g_{\mu\beta}(\Gamma^{\mu}{}_{\gamma\alpha} - \Gamma^{\mu}{}_{\alpha\gamma}) \\ &+ g_{\mu\gamma}(\Gamma^{\mu}{}_{\alpha\beta} + \Gamma^{\mu}{}_{\beta\alpha}) \\ &+ g_{\mu\alpha}(\Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\mu}{}_{\gamma\beta}) = 0 \end{aligned} \tag{37}$$

Use the symmetry of the Christoffel symbol, we will get:

$$g_{\mu\gamma}\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}(\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta}) \equiv \Gamma_{\gamma\alpha\beta} \tag{38}$$

So

$$\Gamma^{\mu}{}_{\alpha\beta} = g^{\mu\gamma}\Gamma_{\gamma\alpha\beta} \tag{39}$$

Everything we have discussed so far has pertained to special relativity. In special relativity, the entire manifold of spacetime can be “covered” be a single inertial reference frame. What is “special” about special relativity is the existence of such global Lorentz frames.

Gravity breaks this. When we include gravity, global inertial frames cannot exist if gravity acts on light. We can do a thought experiment. Imagine you are standing at the top of a tower whose height is h. Now drop rock of rest mass m off top of tower. At bottom, photonulator converts rock into a single photon, conserving energy.

$$E_{\text{tol}} = m + mgh = \hbar\omega_B \tag{40}$$

At top, re-rockulator converts photon back into rock. If there is no perpetual machine, the energy at the top must be:

$$E = \hbar\omega_T = m \tag{41}$$

So

$$\frac{E_T}{E_B} = \frac{m}{m + mgh} = \frac{\omega_T}{\omega_B} \longrightarrow \omega_T = (1 - gh)\,\omega_B \tag{42}$$

This is called gravitational redshift.

General Relativity Lecture7

Principle of Equivalence; Parallel Transport

Lin Fu

March 2025

Now review the content that we talked about in last lecture. We said that when we have gravity, we cannot cover all of spacetime with an inertial frame. We use a simple thought experiment to clarify the existence of gravitational redshift, which is a highly tested experimental fact. Next we will take another example to clarify that there is no a global Lorentz frame.

Suppose we could cover a large region with a single Lorentz frame. Consider worldline of successive crests of wave. Because of the existence of gravity, the worldline of crest 1 is not a diagonal, which is shown in figure 1. For crest 2, if we can use a global Lorentz frame, spacetime is translation invariant. So crest 2 must be congruent with crest 1. There are some spacing between two crests at the bottom and the top, which are written as Δt_B and Δt_T. We must have $\Delta t_B = \Delta t_T$ if globally Lorentz. What's more, that is the period of the wave, which is

$$\Delta t_B = \Delta t_T = \frac{2\pi}{\omega} \tag{1}$$

But ω is not the same at bottom and top because of gravitational redshift, which induces $\Delta t_B \neq \Delta t_T$. So we cannot have a global Lorentz frame.

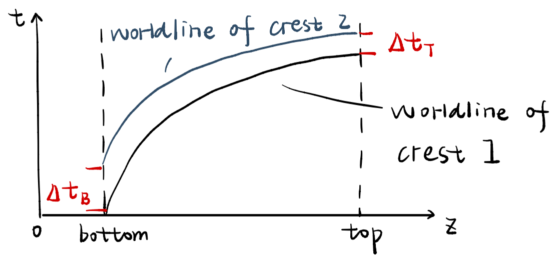

Figure 1: Worldline of two crests

Though there is no global Lorentz frame, a local Lorentz frame is permitted. We should note that Lorentz frame is often called inertial frame, which means no accelerations on observers at rest in that frame. Next thing we're going to talk about is **freely falling frame**(FFF). In that frame all objects experience same accelerations, in other word 0 relative acceleration (in absence of other forces). There is a key point: within this frame, objects maintain their relative velocities.

There is another thing that is worth noting. We cannot find a global freely falling frame in reality because gravity in life is not uniform. We call this variation in gravity **tides**. Tides break down the notion of uniform FFF. We can take an example. Just imagine that there is a very tall elevator which is freely falling, and there are three people respectively at top, middle and bottom of this elevator. They will see separation of free fall since gravity is not uniform.

We can represent this in a spacetime diagram. In the frame of people at middle, the worldlines of three people are shown in figure 2. We can see that tangents of worldlines do not remain parallel.

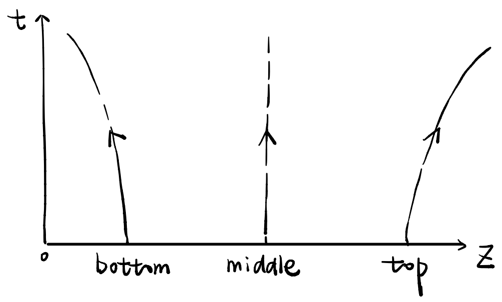

Figure 2: worldlines of three people

In geometry there is an axiom called Euclid's parallelism axiom: two lines that start parallel remain parallel. But that is only true on a flat manifold. Imagine that two people start from two different place at the equator of earth and move towards north together. Their trajectories are parallel at the beginning but they will cross at the north pole.

Weak Equivalence Principle Over sufficiently small regions, the motion of freely falling particles due to gravity cannot be distinguished from uniform acceleration. And there is another way to express this principle: in a local gravitational field, the inertial mass and gravitational mass of all point particles are equal.

There is a question: can we do physics by just applying special relativity in our new notion of inertial frame? The answer is no. Tides prevent this from working: different FFFs exist at different locations. So we have to link them up.

Reformulation of Equivalence Principle In sufficiently small regions of spacetime, we can find a representation such that the laws of physics reduce to those of special relativity. This is also called **Einstein Equivalence Principle**.

Strong Equivalence Principle Gravity falls in a gravitational field in a way that is indistinguishable from mass.

Next we will manifest that we can always find a local Lorentz frame, which means we can put metric in SP form over some finite region. Let $\{x^\alpha\}$ be our starting coordinates, and the metric on it is $g_{\alpha\beta}$. Let $\{x^{\bar{\alpha}}\}$ be coordinate in which spacetime is Lorentz in vicinity of event P. Assume there is a mapping between coordinates: $x^\alpha = x^\alpha(x^{\bar{\alpha}})$. This relationship can be expressed as a matrix:

$$L^\alpha{}_{\bar{\mu}} = \frac{\partial x^\alpha}{\partial x^{\bar{\mu}}} \tag{2}$$

The goal is to find a coordinate system such that

$$g_{\bar{\mu}\bar{\nu}} = L^\alpha{}_{\bar{\mu}} L^\beta{}_{\bar{\nu}} g_{\alpha\beta} = \eta_{\bar{\mu}\bar{\nu}} \tag{3}$$

We will do this by expanding all the relevant quantities in a Taylor series about this event. At each order in the expansion, we will compare the number of degrees of freedom offered by the coordinate transformation to the number of constraints imposed by the metric and its derivatives. The degrees of freedom are free for us to adjust in order to meet the constraints; the constraints are imposed by the original metric (and its derivatives).

Let us do the expansions in the barred-frame coordinates:

$$g_{\alpha\beta} = g_{\alpha\beta}|_P + (x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(\partial_{\bar{\gamma}} g_{\alpha\beta})\big|_P + \frac{1}{2}\,(x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(x^{\bar{\delta}} - x^{\bar{\delta}}_P)(\partial_{\bar{\gamma}}\partial_{\bar{\delta}} g_{\alpha\beta})\Big|_P + \cdots \tag{4}$$

$$L^\alpha{}_{\bar{\mu}} = L^\alpha{}_{\bar{\mu}}|_P + (x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(\partial_{\bar{\gamma}} L^\alpha{}_{\bar{\mu}})\big|_P + \frac{1}{2}\,(x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(x^{\bar{\delta}} - x^{\bar{\delta}}_P)(\partial_{\bar{\gamma}}\partial_{\bar{\delta}} L^\alpha{}_{\bar{\mu}})\Big|_P + \cdots \tag{5}$$

And before our calculation, there are something we should notice:

- $g_{\alpha\beta}|_P$, $\partial g_{\alpha\beta}|_P$, $\partial^2 g_{\alpha\beta}\big|_P$ are handed to us, thet are constraints.
- $L^\alpha{}_{\bar{\mu}}, \partial L^\alpha{}_{\bar{\mu}}, \partial^2 L^\alpha{}_{\bar{\mu}}$ are freely and specifiable. They are degrees of freedom.

The logic of the calculation is to make

$$\begin{aligned} L^{\alpha}{}_{\bar{\mu}} L^{\beta}{}_{\bar{\nu}} g_{\alpha\beta} = &(L^{\alpha}{}_{\bar{\mu}})_P (L^{\beta}{}_{\bar{\nu}})_P (g_{\alpha\beta})_P \\ &+ (x^{\bar{\gamma}} - x^{\bar{\gamma}}_P) \big(\text{Terms involving } (\partial_{\bar{\gamma}} g_{\alpha\beta})_P \text{ and } (\partial_{\bar{\gamma}} L^{\alpha}{}_{\bar{\mu}})_P\big) \\ &+ \frac{1}{2}(x^{\bar{\gamma}} - x^{\bar{\gamma}}_P)(x^{\bar{\delta}} - x^{\bar{\delta}}_P) \big(\text{Terms involving } (\partial_{\bar{\gamma}}\partial_{\bar{\delta}} g_{\alpha\beta})_P \text{ and } (\partial_{\bar{\gamma}}\partial_{\bar{\delta}} L^{\alpha}{}_{\bar{\mu}})_P\big) + \ldots \end{aligned}$$

as close to $\eta_{\bar{\mu}\bar{\nu}}$ as possible. At zeroth order we have 10 constraints imposed by the spacetime metric $g_{\alpha\beta}$ at P due to the fact that the metric is represented by a symmetric 4×4 matrix. But, we also have 16 degrees of freedom available to effect this transformation, since $L^{\alpha}{}_{\bar{\mu}}$ is a non-symmetric 4×4 matrix. . We in fact have 6 degrees of freedom left over. These extra degrees of freedom correspond to 3 boosts and 3 rotations at the event P.

Next is the first order $(\partial_{\bar{\gamma}} g_{\alpha\beta})_P$. This is a total of 40 constraints, arising from the 4 derivatives of the 10 metric components. Our degrees of freedom at first order come from $(\partial_{\bar{\gamma}} L^{\alpha}{}_{\bar{\mu}})_P$.

$$\partial_{\bar{\gamma}} L^{\alpha}{}_{\bar{\mu}} = \frac{\partial^2 x^{\alpha}}{\partial x^{\bar{\gamma}} x^{\bar{\mu}}} \tag{6}$$

We have 4 α components and $\bar{\gamma}$ and $\bar{\mu}$ should be symmetric. So we totally have 40 degrees of freedom.

At second order the constraints is given by $(\partial_{\bar{\gamma}}\partial_{\bar{\delta}} g_{\alpha\beta})_P$. Because $\bar{\gamma}, \bar{\delta}$ is symmetric and $g_{\alpha\beta}$ is symmetric, so we totally have 100 constraints. The degrees of freedom come from

$$\partial_{\bar{\gamma}}\partial_{\bar{\delta}} L^{\alpha}{}_{\bar{\mu}} = \frac{\partial^3 x^{\alpha}}{\partial x^{\bar{\mu}} \partial x^{\bar{\delta}} \partial x^{\bar{\gamma}}} \tag{7}$$

This provides 80 degrees of freedom.

Things above tell us that we can put any spacetime metric in the form

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} + \mathcal{O}[(\partial^2 g)(\delta x)^2] \tag{8}$$

This means that the metric can be put into the inertial, special relativity form at some event in spacetime, and the form accurately describes spacetime over a region with corrections scaling as the second derivative of the metric, times the spacetime separation squared. In Newtonian theory, the second derivative of the gravitational potential tells us about tides. The second derivative of any function describes the curvature of that function. We expect to develop a more rigorous notion of curvature that "looks like" two derivatives of the metric and with 20 independent components.

So naturally we want to define a curved manifold. A manifold with curvature is one on which initially parallel trajectories do not remain parallel. Interestingly, suface of a cylinder is not a curved manifold but it looks curved.

As we think about doing physics in curved manifolds, we need to develop tools for putting things like vectors and tensors into the manifold. There is an important concept called "tangent space". In flat manifolds all points gave the same tangent space. But for curved manifolds it's no.

Now consider a curve γ in a curved manifold, and there is a vector field in it, which is shown in figure 3. We want to differentiate the vector field $\vec{A}$.

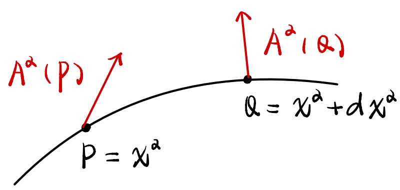

Figure 3: A vector field in a curved manifold

The first guess is

$$\frac{\partial A^{\alpha}}{\partial x^{\beta}} = \lim_{\mathrm{d}x^{\alpha} \to 0} \frac{A^{\alpha}(P) - A^{\alpha}(Q)}{\mathrm{d}x^{\alpha}} \tag{9}$$

This is mathematically meaning, but actually this is not a component of tensor. The events P and Q don't have the same tangent space, so this derivative misses how the basis vectors vary from P to Q. We hope to find

$$\partial_{\beta'} A^{\alpha'} \stackrel{?}{=} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \partial_{\beta} A^{\alpha} \tag{10}$$

But actually we get

$$\partial_{\beta'} A^{\alpha'} = \frac{\partial x^{\beta}}{\partial x^{\beta'}} \left(\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \partial_{\beta} A^{\alpha} + \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\alpha}} A^{\alpha} \right) \tag{11}$$

To address this, we need to introduce a notion of transporting the field $\vec{A}$ from P to Q. We thus compare the vector components at the same point, so that the same tangent space applies to them.

General Relativity Lecture8

Lie Transport, Killing Vectors, Tensor Densities

Lin Fu

March 2025

Recap In the last lecture, we've know that transformation of vector derivative is not tensorial:

$$\partial_{\beta'} A^{\alpha'} = \frac{\partial x^\beta}{\partial x^{\beta'}} \left(\frac{\partial x^{\alpha'}}{\partial x^\alpha} \partial_\beta A^\alpha + \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\alpha} A^\alpha \right) \tag{1}$$

It can be seen that the part marked in red in the above equation makes this equation not conform to the transformation law of tensors. So we must find some methods to fix it.

The thing we consider is how to differentiate the vector field $\vec{A}$ along the curve γ. In last lecture we took a very simple definition which was proven wrong. So our next plan is to define a notion of how to **transport** vector from P to Q.

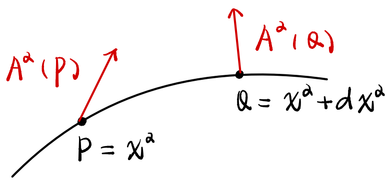

Figure 1: Two vectors at different points

Assume we can define an object $\Pi^\alpha{}_{\beta\mu}$ which does the following:

$$A^\alpha_{\mathrm{T}}(P \to Q) = A^\alpha(P) - \Pi^\alpha{}_{\beta\mu} \mathrm{d}x^\beta A^\mu \tag{2}$$

With this, we can define a derivative by comparing the transported vector field to the field at Q

$$\begin{aligned} D_\beta A^\alpha &\equiv \frac{A^\alpha(Q) - A^\alpha_{\mathrm{T}}(P \to Q)}{\mathrm{d}x^\beta} \\ &= \partial_\beta A^\alpha + \Pi^\alpha{}_{\beta\mu} A^\mu \end{aligned} \tag{3}$$

And we call $\Pi^\alpha{}_{\beta\mu}$ **connection**. Now we need to put some demands on the connection.

1. When we do the coordinates transformation, connection varies as

$$\Pi^{\alpha'}{}_{\beta'\mu'} A^{\mu'} = \left(\frac{\partial x^{\alpha'}}{\partial x^\alpha} \right) \left(\frac{\partial x^\beta}{\partial x^{\beta'}} \right) \Pi^\alpha{}_{\beta\mu} A^\mu - \left(\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\mu} \right) \left(\frac{\partial x^\beta}{\partial x^{\beta'}} \right) A^\mu \tag{4}$$

   You might think the second term of the equation above is annoying. Actullay its existence is just for canceling out the bad term in the partial derivative transformation.

2. If we demand that $D_\beta g_{\mu\nu} = 0$, we will get that the connection is actually the Christoffel symbol.

$$\Pi^\alpha{}_{\beta\mu} = \Gamma^\alpha{}_{\beta\mu} \tag{5}$$

   And the derivative we define is just the covariant derivative.

Imagine we put a tape measure along the curve γ, and there will be a lot of uniform tickmarks in the curve, as shown in the figure 2. We use λ to represent them. With this, tangent vector to this curve is

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \tag{6}$$

And we can define a new operation

$$u^\beta(\nabla_\beta A^\alpha) \equiv \frac{DA^\alpha}{\mathrm{d}\lambda} \tag{7}$$

This tells us how A^α changes as it is transported along curve.

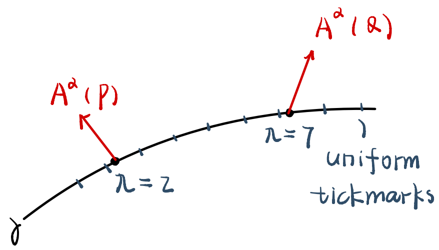

Figure 2: A curve with uniform tickmarks

Now we can give parallel transport a definition:

$$\frac{DA^\alpha}{\mathrm{d}\lambda} = 0 \tag{8}$$

To explain why, we can look at the initial definition:

$$u^\beta \nabla_\beta A^\alpha = u^\beta(\partial_\beta A^\alpha + \Gamma^\alpha{}_{\beta\mu} A^\mu) \tag{9}$$

When P and Q are close enough that they fit in the same LLF, no Christoffel symbol in that frame! So things become:

$$u^\beta \partial_\beta A^\alpha = \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\frac{\partial}{\partial x^\beta}A^\alpha = \frac{\mathrm{d}A^\alpha}{\mathrm{d}\lambda} = 0 \tag{10}$$

This means A^α holds components constant as we slide vector along curve γ.

There is an important thing: parallel transport is not the unique way to transport vector. Actually we can switch our perspective from transporting vector to transform coordinates.

$$x^\alpha + \mathrm{d}x^\alpha = x^\alpha + u^\alpha \mathrm{d}\lambda \equiv (x')^\alpha \tag{11}$$

Define a new transport as below:

$$\begin{aligned} A^\alpha_{\mathrm{LT}}(P \to Q) &= \frac{\partial (x')^\alpha}{\partial x^\beta} A^\beta(P) \\ &= (\delta^\alpha{}_\beta + (\partial_\beta u^\alpha)\mathrm{d}\lambda)\, A^\beta(P) \\ &= A^\alpha(P) + \mathrm{d}\lambda(\partial_\beta u^\alpha) A^\beta(P) \end{aligned} \tag{12}$$

And we can also express field at Q in forms of field at P using Taylor expansion:

$$\begin{aligned} A^\alpha(Q) &= A^\alpha(x^\beta + \mathrm{d}x^\beta) \\ &= A^\alpha(x^\beta) + \mathrm{d}x^\beta\, (\partial_\beta A^\alpha)|_P \\ &= A^\alpha(P) + (u^\beta \mathrm{d}\lambda)\, (\partial_\beta A^\alpha)|_P \end{aligned} \tag{13}$$

With two things above, we can define a new derivative

$$\mathscr{L}_{\vec{u}} A^\alpha \equiv \frac{A^\alpha(Q) - A^\alpha_{\mathrm{LT}}(P \to Q)}{\mathrm{d}\lambda} \tag{14}$$

This is called **Lie derivative**. And it can be proven that

$$\mathscr{L}_{\vec{u}} A^\alpha = u^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta u^\alpha \tag{15}$$

$$= u^\beta \nabla_\beta A^\alpha - A^\beta \nabla_\beta u^\alpha \tag{16}$$

What's more, we can write it using commutator

$$\mathscr{L}_{\vec{u}} \vec{A} = \left[\vec{u}, \vec{A}\right] \tag{17}$$

For other types of tensor, we have

$$\mathscr{L}_{\vec{u}} \Phi = u^\alpha \partial_\alpha \Phi = u^\alpha \nabla_\alpha \Phi \tag{18}$$

$$\mathscr{L}_{\vec{u}} p_\alpha = u^\beta \nabla_\beta p_\alpha + p_\beta \nabla_\alpha u^\beta \tag{19}$$

$$\mathscr{L}_{\vec{u}} T^\alpha{}_\beta = u^\mu \nabla_\mu T^\alpha{}_\beta - T^\mu{}_\beta \nabla_\mu u^\alpha + T^\alpha{}_\mu \nabla_\beta u^\mu \tag{20}$$

With Lie derivative, we can define a new transportation. If a tensor fulfills

$$\mathscr{L}_{\vec{u}}(\text{tensor}) = 0 \tag{21}$$

We say the tensor is **Lie transported**.

Suppose a tensor is Lie transported. If that's the case, define coordinates centered on curve for which $\vec{u}$ is tangent. This demands that $x^0 = \lambda$ on the curve, and $x^{1,2,3}$ is constant on the curve. Then

$$u^\alpha = \delta^\alpha{}_0 \longrightarrow \partial_\mu u^\alpha = 0 \tag{22}$$

Then

$$\mathscr{L}_{\vec{u}}(\text{tensor}) = \frac{\partial(\text{tensor})}{\partial x^0} = 0 \tag{23}$$

This means the tensor does not vary with this parameter along the curve.

Suppose "tensor" is the metric. Let us say a vector $\vec{\xi}$ exists such that $\mathscr{L}_{\vec{\xi}} g_{\mu\nu} = 0$. From the discussion above, this tells us that there exists a coordinate x^0 such that $\partial g_{\alpha\beta}/\partial x^0 = 0$. The converse is also true: if the metric is constant with respect to some coordinate, then a vector $\vec{\xi}$ exists such that the metric is Lie transported along $\vec{\xi}$.

And, we can expand the Lie derivative $\mathscr{L}_{\vec{\xi}} g_{\alpha\beta} = 0$:

$$\xi^\gamma \nabla_\gamma g_{\alpha\beta} + g_{\alpha\gamma} \nabla_\beta \xi^\gamma + g_{\gamma\beta} \nabla_\alpha \xi^\gamma = 0 \tag{24}$$

But remember that the covariant derivative of the metric is zero. The first term on the left-hand side is zero; on the other two terms, we can move the metric inside the derivatives and lower the indices on the components of ξ:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \tag{25}$$

This result is called **Killing's equation**, and $\vec{\xi}$ is known as a **Killing vector**.

Tensor densities Tensor "densities" are a further category of mathematical objects which will prove useful in our future work. These objects are best regarded as things that transform between different representations almost, but not quite, in the way that tensors transform. When we change representation, we find that they "miss" the correct transformation law by a factor that looks like the determinant of the matrix which effects the transformation.

One of the most important tensor densities is the Levi-Civita symbol.

$$\begin{aligned} \tilde{\epsilon}_{\alpha\beta\gamma\delta} &= +1 \quad \text{for indices 0123 and even permutations} \\ &= -1 \quad \text{for odd permutations of 0123} \\ &= 0 \quad \text{for any index repeated} \end{aligned} \tag{26}$$

Remember that we write this with a tilde to indicate that the Levi-Civita symbol is not a tensor. To show that this is not a tensor, we quote a theorem: Given any 4×4 matrix $\boldsymbol{M}$ whose components are $M^\alpha{}_\mu$,

$$\tilde{\epsilon}_{\alpha\beta\gamma\delta} M^\alpha{}_\mu M^\beta{}_\nu M^\gamma{}_\rho M^\delta{}_\sigma = \tilde{\epsilon}_{\mu\nu\rho\sigma} |\boldsymbol{M}| \tag{27}$$

where $|\boldsymbol{M}|$ is the determinant of the matrix $\boldsymbol{M}$.

Let's take in particular $M^{\alpha}{}_{\bar{\alpha}} = \partial x^{\alpha}/\partial x^{\bar{\alpha}}$. Then,

$$\tilde{\epsilon}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} = \left|\frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}}\right| \tilde{\epsilon}_{\alpha\beta\gamma\delta} \left(\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\right) \left(\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\right) \left(\frac{\partial x^{\gamma}}{\partial x^{\bar{\gamma}}}\right) \left(\frac{\partial x^{\delta}}{\partial x^{\bar{\delta}}}\right) \tag{28}$$

The extra factor pushes us away from a tensor relationship. If the determinant of the transformation matrix were not present, then this would be tensorial. Instead, we call this a "tensor density of weight 1".

General Relativity Lecture9

Geodesics

Lin Fu

March 2025

In last lecture we introduced the notion of tensor densities. One example is Levi-Civita symbol:

$$\tilde{\epsilon}_{\alpha'\beta'\gamma'\delta'} = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \tilde{\epsilon}_{\alpha\beta\gamma\delta} \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \frac{\partial x^{\delta}}{\partial x^{\delta'}} \tag{1}$$

we call this a "tensor density of weight 1". And another example is metric.

$$g_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} g_{\alpha\beta} \tag{2}$$

Now take determinant of both sides:

$$\boldsymbol{g}' = \left|\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\right|^{2} \boldsymbol{g} = \left|\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right|^{-2} \boldsymbol{g} \tag{3}$$

So determinant of $g_{\mu\nu}$ is tensor density of weight -2.

We can convert tensor density of w into a proper tensor by multiplying by $|\boldsymbol{g}|^{w/2}$. For example, we can make Levi-Civita symbol into

$$\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{|\boldsymbol{g}|}\tilde{\epsilon}_{\alpha\beta\gamma\delta} \tag{4}$$

And if we higher all indicies, we will get

$$\epsilon^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{|\boldsymbol{g}|}}\tilde{\epsilon}^{\alpha\beta\gamma\delta} \tag{5}$$

We can use this to form covariant volume operators:

$$\mathrm{d}V^4 = \sqrt{|\boldsymbol{g}|}\tilde{\epsilon}_{\alpha\beta\gamma\delta}\mathrm{d}x_0^{\alpha}\mathrm{d}x_1^{\beta}\mathrm{d}x_2^{\gamma}\mathrm{d}x_3^{\delta} \tag{6}$$

If working in an orthogonal basis, it will be like:

$$\mathrm{d}V^4 = \sqrt{|\boldsymbol{g}|}\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 \tag{7}$$

To explain this point more straightforwardly, we can write this in 3D spherical coordinates.

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \quad g_{ij} = \mathrm{diag}(1, r^2, r^2\sin^2\theta) \tag{8}$$

So the volume element is:

$$\mathrm{d}V^3 = \sqrt{|\boldsymbol{g}|}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{9}$$

Which is familiar to us.

There is a party trick: using determinant of $g_{\mu\nu}$ to compute some christoffel symbols.

$$\begin{aligned}\Gamma^{\mu}{}_{\mu\alpha} &= g^{\mu\beta}\Gamma_{\beta\mu\alpha} \\ &= \frac{1}{2}g^{\mu\beta}\left(\partial_{\mu}g_{\beta\alpha} + \partial_{\alpha}g_{\beta\mu} - \partial_{\beta}g_{\mu\alpha}\right)\end{aligned} \tag{10}$$

We should notice that the first and third term of the equation above should be antisymmetric when exchange μ and β. So they will disappear when contracting with $g^{\mu\beta}$ which is symmetric. Finally we get

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\mu\beta} \tag{11}$$

We can prove that

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{\sqrt{|\boldsymbol{g}|}} \partial_\alpha \left(\sqrt{|\boldsymbol{g}|} \right) = \partial_\alpha \left(\ln \sqrt{|\boldsymbol{g}|} \right) \tag{12}$$

The proof of this relies on some results of linear algebra.

$$\begin{aligned} \delta \ln (\det \boldsymbol{M}) &= \ln [\det (\boldsymbol{M} + \delta \boldsymbol{M})] - \ln (\det \boldsymbol{M}) \\ &= \ln \left[\frac{\det (\boldsymbol{M} + \delta \boldsymbol{M})}{\det \boldsymbol{M}} \right] \\ &= \ln \left[\det \left(\boldsymbol{I} + \boldsymbol{M}^{-1} \cdot \delta \boldsymbol{M} \right) \right] \end{aligned} \tag{13}$$

And there is an important identity: if $\boldsymbol{\epsilon}$ is a "small" matrix, then

$$\det(\boldsymbol{I} + \boldsymbol{\epsilon}) \simeq 1 + \mathrm{Tr}(\boldsymbol{\epsilon}) \tag{14}$$

Let's consider $\boldsymbol{M}^{-1} \cdot \delta \boldsymbol{M}$ as our $\boldsymbol{\epsilon}$, then we get

$$\delta \ln(\det \boldsymbol{M}) = \ln[1 + \mathrm{Tr}(\boldsymbol{M}^{-1} \cdot \delta \boldsymbol{M})] = \mathrm{Tr}(\boldsymbol{M}^{-1} \cdot \delta \boldsymbol{M}) \tag{15}$$

Now set the matrix $\boldsymbol{M}$ to the metric components $g_{\alpha\beta}$, then we will get

$$\delta \ln |\boldsymbol{g}| = \mathrm{Tr} \left(g^{\mu\beta} \delta g_{\beta\gamma} \right) \tag{16}$$

We should point out that the trace of a matrix is

$$\mathrm{Tr}(\boldsymbol{\epsilon}) = g^{\alpha\beta} \epsilon_{\alpha\beta} \equiv \epsilon^{\beta}{}_{\beta} \tag{17}$$

So Eq.(16) becomes

$$\delta \ln |\boldsymbol{g}| = g^{\mu\beta} \delta g_{\beta\mu} \tag{18}$$

Imagine that this variation arises from a small variation in the coordinates, δx^α; divide by this variation, take the limit, and find

$$\partial_\alpha \ln |\boldsymbol{g}| = g^{\mu\beta} \partial_\alpha g_{\beta\mu} \tag{19}$$

Combining this with Eq.(11), then we get

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\beta\mu} = \frac{1}{2} \partial_\alpha \ln |\boldsymbol{g}| = \partial_\alpha \left(\ln \sqrt{|\boldsymbol{g}|} \right) \tag{20}$$

A particularly useful application of this party trick is to the divergence of a vector:

$$\begin{aligned} \nabla_\alpha A^\alpha &= \partial_\alpha A^\alpha + \Gamma^{\alpha}{}_{\alpha\beta} A^\beta \\ &= \partial_\alpha A^\alpha + \Gamma^{\beta}{}_{\beta\alpha} A^\alpha \\ &= \partial_\alpha A^\alpha + A^\alpha \frac{1}{\sqrt{|\boldsymbol{g}|}} \partial_\alpha \left(\sqrt{|\boldsymbol{g}|} \right) \\ &= \frac{1}{\sqrt{|\boldsymbol{g}|}} \partial_\alpha \left(\sqrt{|\boldsymbol{g}|} A^\alpha \right) \end{aligned} \tag{21}$$

This form enables a really lovely way to write Gauss's theorem that works very nicely even in arbitrary curved geometries:

$$\int_{V^4} (\nabla_\alpha A^\alpha) \sqrt{|\boldsymbol{g}|} \mathrm{d}^4 x = \int_{V^4} \partial_\alpha \left(\sqrt{|\boldsymbol{g}|} A^\alpha \right) \mathrm{d}^4 x = \oint_{\partial V^4} A^\alpha \sqrt{|\boldsymbol{g}|} \mathrm{d}\Sigma_\alpha \tag{22}$$

We need to think: can we use this party trick when dealing with tensors? The answer is no, except a spacial example.

$$\nabla_\alpha A^{\alpha\beta} = \partial_\alpha A^{\alpha\beta} + \Gamma^{\alpha}{}_{\alpha\gamma} A^{\gamma\beta} + \Gamma^{\beta}{}_{\alpha\gamma} A^{\alpha\gamma} \tag{23}$$

For the second term we can use the trick, but for the third term we cannot. If $A^{\alpha\gamma}$ is antisymmetric, then the third term will disappear. Even if the divergence can be simplified, we cannot integrate it up to get something useful.

Next we're going to talk about some physics. How do we formulate kinematics of bodies in curved spacetime? Now go into a FFF, use locally Lorentz representation, consider a "test body": no charge, no spatial extent, no spin, nothing but mass. In this frame, body moves on a purely inertial trajectory: $x^\alpha = x_0^\alpha + u^\alpha\tau$. This is the "straight line" in this representation.

We can think of it more geometrically: **the trajectory parallel transports its tangent vector.** As usual, imagine trajectories that are parameterized by some λ which grows along their worldlines. We define $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\lambda$. If the tangent vector is itself parallel transported along the worldline, then it satisfies

$$u^\alpha \nabla_\alpha u^\beta = 0 \longrightarrow \frac{Du^\alpha}{\mathrm{d}\lambda} = 0 \tag{24}$$

We can expand this and write it in some equivalent ways:

$$u^\alpha \partial_\alpha u^\beta + \Gamma^\beta{}_{\alpha\mu} u^\alpha u^\mu = 0 \tag{25}$$

$$\frac{\mathrm{d}u^\beta}{\mathrm{d}\lambda} + \Gamma^\beta{}_{\alpha\mu} u^\alpha u^\mu = 0 \tag{26}$$

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\beta{}_{\alpha\mu} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{27}$$

This result is called **the geodesic equation**, and the curves $x^\alpha(\lambda)$ which solve it are geodesics.

As a brief but occasionally useful aside, a more general form of the geodesic equation is to allow its normalization to change as it is transported:

$$\frac{Du^\alpha}{\mathrm{d}\lambda^*} = \kappa(\lambda^*) u^\alpha \tag{28}$$

It can be proven that we can always change our parameterization from λ^* to λ in such a way that RHS is zero. Define $v^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\lambda$. If

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda^*} = \exp\left[\int_0^\lambda \kappa(\lambda^*)\mathrm{d}\lambda^*\right] \tag{29}$$

then we will get $v^\alpha \nabla_\alpha v^\beta = 0$.

A choice of λ which maintains normalization is known as an **affine parameterization**. An affine parameterization corresponds to one which has uniformly spaced "tickmarks" in local Lorentz frames along the worldline. At least for timelike trajectories, a very natural choice for this is the proper time τ. As such, it is very common to see Eq.(26)-(27) written using τ rather than λ. And there is an important result: if λ is an affine parameterization, then so is

$$\lambda' = a\lambda + b \tag{30}$$

with a and b both constants.

There is second way to get geodesic equation. This is based on intuition that shortest path between two points is a straight line. The way we're going to talk about is akin to the least action principle in classical mechanics.

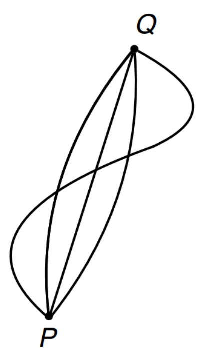

Figure 1: All possible trajectories from P to Q

Consider every possible path that connects P to Q, which is shown in Fig. 1. The accumulated proper time for an observer who follows one of these paths is

$$\Delta\tau = \int_P^Q \mathrm{d}\lambda \sqrt{-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}} \tag{31}$$

We can define an action:

$$I = \frac{1}{2}\int_P^Q \left(g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{32}$$

We require that $\delta I = 0$. Then we will get

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta = 0 \tag{33}$$

General Relativity Lecture10

Spacetime Curvature

Lin Fu

April 2025

Recap In last lecture we've talked about the geodesics: the trajectory parallel transports its tangent vector. It corresponds to the free-fall.

Can we write this in terms of momentum? For a particle with rest mass m, moving along a timelike trajactory, we have:

$$p^\alpha \nabla_\alpha p^\beta = 0 \longrightarrow m\frac{\mathrm{d}p^\beta}{\mathrm{d}\tau} + \Gamma^\beta{}_{\mu\nu} p^\mu p^\nu = 0 \tag{1}$$

Now change the affine parameter $\Delta\lambda = \Delta\tau/m$, then

$$p^\alpha = m\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \tag{2}$$

And we have

$$\frac{\mathrm{d}p^\beta}{\mathrm{d}\lambda} + \Gamma^\beta{}_{\mu\nu} p^\mu p^\nu = 0 \tag{3}$$

We can take the limit $m \to 0$. Massless particles move on light-like trajectories in relativistic kinematics; the very notion of proper time does not make sense along such trajectories. However, if we imagine a limit of a massive body approaching the lightlike trajectory, we can consider $m \to 0$ while $\Delta tau \to 0$, but doing so in such a way that $\Delta\tau/m$ is always constant. Equation (3) is a form of the geodesic equation that is perfectly suited for studying light-like trajectories.

Things that are conserved along geodesics Now ask ourselves a question: can we lower the indice of $p^\alpha \nabla_\alpha p^\beta = 0$ and turn it into $p^\alpha \nabla_\alpha p_\beta = 0$? This is right because the covariant derivative of metric $g_{\alpha\beta}$ is zero. So

$$p^\alpha \nabla_\alpha p_\beta = p^\alpha \nabla_\alpha g_{\beta\mu} p^\mu = g_{\beta\mu} p^\alpha \nabla_\alpha p^\mu = 0 \tag{4}$$

Now expand $p^\alpha \nabla_\alpha p_\beta = 0$:

$$m\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} - \Gamma^\gamma{}_{\beta\alpha} p^\alpha p_\gamma = 0 \tag{5}$$

In other form:

$$m\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} = \Gamma_{\gamma\beta\alpha} p^\alpha p^\gamma = \frac{1}{2}\left(\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}\right) p^\alpha p^\gamma \tag{6}$$

Look carefully at the last two terms in (6): the derivatives of the metric are antisymmetric on exchange of α and γ, but they are contracted with the symmetric combination $p^\alpha p^\gamma$. So things becomes:

$$m\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} = \frac{1}{2}(\partial_\beta g_{\alpha\gamma}) p^\alpha p^\gamma \tag{7}$$

If the metric is independent of a particular x^β, then we will have

$$\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} = 0 \longrightarrow p_\beta \text{ is constant on the geodesic} \tag{8}$$

This reminds us of a result from Lagrangian mechanics: if a Lagrangian does not depend on a coordinate, then the momentum conjugate to that coordinate is constant.

As we talked in lectures before, if the metric was independent of a particular coordinate, then there exists a Killing vector ξ^β associated with this symmetry. Next look at how $p^\beta \xi_\beta$ evolves on geodesic worldline. We are going to examine that

$$m\frac{D}{\mathrm{d}\tau}(p^\beta \xi_\beta) = p^\alpha \nabla_\alpha (p^\beta \xi_\beta) = 0 \tag{9}$$

To do this, we can expand this with Leibniz's rule:

$$m\frac{D}{\mathrm{d}\tau}(p^\beta \xi_\beta) = \xi_\beta (p^\alpha \nabla_\alpha p^\beta) + p^\alpha p^\beta \nabla_\alpha \xi_\beta \tag{10}$$

Because p^α solves the geodesic equation so the first term disappear. For the second term, we can use the fact that for any two indices tensor we have:

$$M_{\alpha\beta} = M_{(\alpha\beta)} + M_{[\alpha\beta]} \tag{11}$$

So

$$\begin{aligned} p^\alpha p^\beta \nabla_\alpha \xi_\beta &= p^\alpha p^\beta \left[\nabla_{(\alpha}\xi_{\beta)} + \nabla_{[\alpha}\xi_{\beta]}\right] \\ &= \frac{1}{2} p^\alpha p^\beta (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) = 0 \end{aligned} \tag{12}$$

We use the Killing's equation above:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \tag{13}$$

We thus see that $p^\beta \xi_\beta$ is constant along any geodesic trajectory.

Important examples If $\partial_t g_{\alpha\beta} = 0$, we know that there exists a "timelike" Killing vector $\vec{\xi}^{\mathrm{T}}$, and p_t is conserved along all trajectories. To tie this to special relativity intuition, we define the conserved energy along the geodesic:

$$p_t = -E \tag{14}$$

This is useful for spacetime that is **asymptotically flat**.

Consider a spacetime

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{15}$$

with $\Phi \ll 1$ and $\Phi = \Phi(x,y,z)$. Now we want to study the slow motion in this spacetime. The geodesic equation is

$$m\frac{\mathrm{d}p^\beta}{\mathrm{d}\tau} + \Gamma^\beta{}_{\mu\nu} p^\mu p^\nu = 0 \tag{16}$$

Because we are considering the slow motion limit, the second term in the geodesic equation is dominated by the terms with $\mu = \nu = 0$. So the equation becomes:

$$m\frac{\mathrm{d}p^\beta}{\mathrm{d}\tau} \simeq -\Gamma^\beta{}_{00} p^0 p^0 \simeq -m^2 \Gamma^\beta{}_{00} \tag{17}$$

For $\Gamma^\beta{}_{00}$, now we just consider the spatial components $\beta = i$, then

$$\Gamma^i{}_{00} = g^{i\alpha}\Gamma_{\alpha 00} = \frac{1}{2} g^{i\alpha}(\partial_0 g_{0\alpha} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}) \tag{18}$$

The first two terms of equation above equals zero. Using the fact that $g^{i\alpha} = (1-2\Phi)^{-1}\delta^{i\alpha}$, we will get

$$\Gamma^i{}_{00} = -\frac{1}{2}(1-2\Phi)^{-1}\delta^{ij}\partial_j(-2\Phi) = \delta^{ij}\partial_j\Phi + \mathcal{O}(\Phi^2) \tag{19}$$

Now go back to equation of motion (17), we will get

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = -m\delta^{ij}\partial_j\Phi \tag{20}$$

This is exactly the law of motion that describes a body moving under the influence of a Newtonian gravitational potential Φ.

Quantifying curvature Our concept of curvature centers on the idea that initially parallel trajectories do not remain parallel. Consider a vector parallel transported around a closed figure on a curved manifold. If the manifold is flat, then the vector transported back should be parallel to the original one.

But if we do this on surface of a ball, it will hit different, which shown in Fig 1. It's easy to see that the one that be transported back is not parallel to the original one. This is because, on a curved manifold, the sum of the angles of a triangle is not equal to 180 degrees, and there will be a difference between the angle of the transported vector and that of the initial vector.

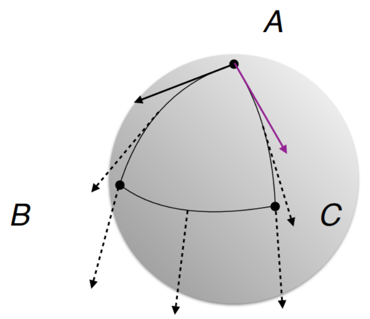

Figure 1: Transport a vector on a sphere

So let's do it more precisely. In Figure 2 there are some coordinate lines, and 4 intersection point called A, B, C, D. For our analysis, consider parallel transport of a vector with components V^α around the following "diamond", from $A \to B \to C \to D \to A$. When move from A to B, x^λ keeps constant but x^σ changes. So we can say that the vector is transported along $\vec{e}_\sigma$. This requires that

$$\begin{aligned} &\nabla_{\vec{e}_\sigma} V^\alpha = 0 \\ &\to \partial_\sigma V^\alpha + \Gamma^\alpha{}_{\sigma\mu} V^\mu = 0 \\ &\to \frac{\partial V^\alpha}{\partial x^\sigma} = -\Gamma^\alpha{}_{\sigma\mu} V^\mu \end{aligned} \tag{21}$$

Treating this as a differential equation for V^α, we find

$$V^\alpha(B) = V^\alpha_{\text{init}} - \int_1 \Gamma^\alpha{}_{\sigma\mu} V^\mu \mathrm{d}x^\sigma \tag{22}$$

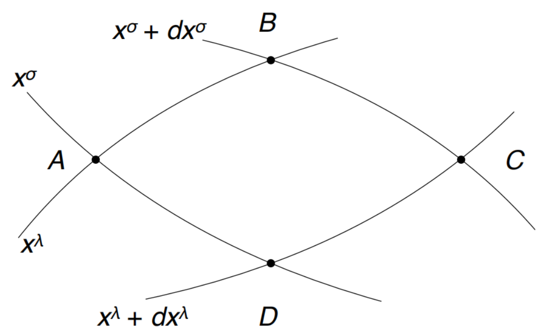

Figure 2: Several coordinate lines

Similarly, we can write down:

$$V^\alpha(C) = V^\alpha(B) - \int_2 \Gamma^\alpha{}_{\lambda\mu} V^\mu \mathrm{d}x^\lambda \tag{23}$$

$$V^\alpha(D) = V^\alpha(C) + \int_3 \Gamma^\alpha{}_{\sigma\mu} V^\mu \mathrm{d}x^\sigma \tag{24}$$

$$V^\alpha_{\text{final}} = V^\alpha(D) + \int_4 \Gamma^\alpha{}_{\lambda\mu} V^\mu \mathrm{d}x^\lambda \tag{25}$$

We are interested in seeing how the vector changes after going around this loop:

$$\begin{aligned}\delta V^{\alpha} &= V^{\alpha}_{\text{final}} - V^{\alpha}_{\text{init}} \\ &= \int_4 \Gamma^{\alpha}{}_{\lambda\mu} V^{\mu} \mathrm{d}x^{\lambda} - \int_2 \Gamma^{\alpha}{}_{\lambda\mu} V^{\mu} \mathrm{d}x^{\lambda} \\ &\quad + \int_3 \Gamma^{\alpha}{}_{\sigma\mu} V^{\mu} \mathrm{d}x^{\sigma} - \int_1 \Gamma^{\alpha}{}_{\sigma\mu} V^{\mu} \mathrm{d}x^{\sigma}\end{aligned} \tag{26}$$

We should notice that path 2 is evaluated along constant $x^{\sigma} + \mathrm{d}x^{\sigma}$, but path 4 is evaluated along constant x^{σ}. So we can combine them into one integral:

$$\int_4 \Gamma^{\alpha}{}_{\lambda\mu} V^{\mu} \mathrm{d}x^{\lambda} - \int_2 \Gamma^{\alpha}{}_{\lambda\mu} V^{\mu} \mathrm{d}x^{\lambda} = \int_2 \left(-\delta x^{\sigma}\right) \frac{\partial}{\partial x^{\sigma}} \left(\Gamma^{\alpha}{}_{\lambda\mu} V^{\mu}\right) \mathrm{d}x^{\lambda} \tag{27}$$

Now we can rewrite Eq.(26) as:

$$\delta V^{\alpha} = \int_{x^{\sigma}}^{x^{\sigma}+\mathrm{d}x^{\sigma}} \delta x^{\lambda} \frac{\partial}{\partial x^{\lambda}} \left(\Gamma^{\alpha}{}_{\sigma\mu} V^{\mu}\right) \mathrm{d}x^{\sigma} - \int_{x^{\lambda}}^{x^{\lambda}+\mathrm{d}x^{\lambda}} \delta x^{\sigma} \frac{\partial}{\partial x^{\sigma}} \left(\Gamma^{\alpha}{}_{\lambda\mu} V^{\mu}\right) \mathrm{d}x^{\lambda} \tag{28}$$

Expanding the derivatives, this becomes

$$\delta V^{\alpha} = \delta x^{\lambda} \delta x^{\sigma} \left[\partial_{\lambda} \Gamma^{\alpha}{}_{\sigma\mu} V^{\mu} - \partial_{\sigma} \Gamma^{\alpha}{}_{\lambda\mu} V^{\mu} + \Gamma^{\alpha}{}_{\sigma\mu} \partial_{\lambda} V^{\mu} - \Gamma^{\alpha}{}_{\lambda\mu} \partial_{\sigma} V^{\mu}\right] \tag{29}$$

Remember that the vector is parallel transported, and so we can write

$$\partial_{\lambda} V^{\mu} = -\Gamma^{\mu}{}_{\lambda\nu} V^{\nu}, \;\; \partial_{\sigma} V^{\mu} = -\Gamma^{\mu}{}_{\sigma\nu} V^{\nu} \tag{30}$$

Inserting this, we find

$$\delta V^{\alpha} = \delta x^{\lambda} \delta x^{\sigma} \left[\left(\partial_{\lambda} \Gamma^{\alpha}{}_{\sigma\mu} - \partial_{\sigma} \Gamma^{\alpha}{}_{\lambda\mu}\right) V^{\mu} + \left(\Gamma^{\alpha}{}_{\lambda\mu} \Gamma^{\mu}{}_{\sigma\nu} - \Gamma^{\alpha}{}_{\sigma\mu} \Gamma^{\mu}{}_{\lambda_{\nu}}\right) V^{\nu}\right] \tag{31}$$

Relabel dummy indices on the last two terms to exchanging μ and ν, finally we get

$$\delta V^{\alpha} = R^{\alpha}{}_{\mu\lambda\sigma} \delta x^{\lambda} \delta x^{\sigma} V^{\mu} \tag{32}$$

where we have introduced

$$R^{\alpha}{}_{\mu\lambda\sigma} = \partial_{\lambda} \Gamma^{\alpha}{}_{\sigma\mu} - \partial_{\sigma} \Gamma^{\alpha}{}_{\lambda\mu} + \Gamma^{\alpha}{}_{\lambda\nu} \Gamma^{\nu}{}_{\sigma\mu} - \Gamma^{\alpha}{}_{\sigma\nu} \Gamma^{\nu}{}_{\lambda\mu} \tag{33}$$

This is called **Riemann curvature tensor**.

There is an equivalent definition of Riemann curvature tensor:

$$\left[\nabla_{\lambda}, \nabla_{\sigma}\right] V^{\alpha} = R^{\alpha}{}_{\mu\lambda\sigma} V^{\mu} \tag{34}$$

$$\left[\nabla_{\lambda}, \nabla_{\sigma}\right] p_{\alpha} = -R^{\mu}{}_{\alpha\lambda\sigma} p_{\mu} \tag{35}$$

General Relativity Lecture11

More on Spacetime Curvature

Lin Fu

April 2025

Recap In last lecture we got a very important mathematical object called Riemann curvature tensor.

$$R^{\alpha}{}_{\mu\lambda\sigma} = \partial_\lambda \Gamma^{\alpha}{}_{\sigma\mu} - \partial_\sigma \Gamma^{\alpha}{}_{\lambda\mu} + \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\nu}{}_{\sigma\mu} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\nu}{}_{\lambda\mu} \tag{1}$$

What we're going to do is to research its symmetry.

First just look at the definition (1), it's easy to see that

$$R^{\alpha}{}_{\mu\lambda\sigma} = -R^{\alpha}{}_{\mu\sigma\lambda} \tag{2}$$

This corresponds to reversing direction of transport.

To go further, it is useful to lower the first index:

$$R_{\alpha\mu\lambda\sigma} = g_{\alpha\nu} R^{\nu}{}_{\mu\lambda\sigma} \tag{3}$$

If we go into a LLF, then all Christoffel symbols will equal zero but their derivatives do not. So we get

$$R_{\alpha\mu\lambda\sigma} = \partial_\lambda \Gamma_{\alpha\sigma\mu} - \partial_\sigma \Gamma_{\alpha\lambda\mu} \tag{4}$$

Inserting the definition of the Christoffel symbols, we get:

$$R_{\alpha\mu\lambda\sigma} = \frac{1}{2}\left(\partial_\lambda\partial_\mu g_{\alpha\sigma} - \partial_\lambda\partial_\alpha g_{\sigma\mu} - \partial_\sigma\partial_\mu g_{\alpha\lambda} + \partial_\sigma\partial_\alpha g_{\lambda\mu}\right) \tag{5}$$

Just stare at this, there are some symmetries we can get:

- Antisymmetry on the last two indices:$R_{\alpha\mu\lambda\sigma} = -R_{\alpha\mu\sigma\lambda}$.
- Antisymmetry on the first two indices:$R_{\alpha\mu\lambda\sigma} = -R_{\mu\alpha\lambda\sigma}$.
- Block symmetry of the first two with the last two:$R_{\alpha\mu\lambda\sigma} = R_{\lambda\sigma\alpha\mu}$.
- Cyclic permutation of last three indices sums to zero:

$$R_{\alpha\mu\lambda\sigma} + R_{\alpha\lambda\sigma\mu} + R_{\alpha\sigma\mu\lambda} = 0 \tag{6}$$

This can be rewritten as $R_{\alpha[\mu\lambda\sigma]} = 0$.

$$R_{\alpha[\mu\lambda\sigma]} = \frac{1}{3!}\left(R_{\alpha\mu\lambda\sigma} - R_{\alpha\lambda\mu\sigma} + R_{\alpha\lambda\sigma\mu} - R_{\alpha\mu\sigma\lambda} + R_{\alpha\sigma\mu\lambda} - R_{\alpha\sigma\lambda\mu}\right) \tag{7}$$

And there is an important result: the number of independent components of Riemann curvature tensor is

$$N_{\text{Riemann}}(n) = \frac{n^4 - 2n^3 + 3n^2 - 2n}{8} - \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} = \frac{n^2(n^2-1)}{12} \tag{8}$$

With the Riemann tensor in hand, it is possible to actually write down the metric in a freely falling representation and show how the 20 components of Riemann allow us to build the metric that is locally flat, with quadratic corrections.

$$g_{tt} = -1 - R_{tjtk}x^j x^k + \mathcal{O}\left(x^3\right) \tag{9}$$

$$g_{tj} = -\frac{2}{3}R_{tijk}x^i x^k + \mathcal{O}\left(x^3\right) \tag{10}$$

$$g_{jk} = \delta_{jk} - \frac{1}{3}R_{jikl}x^i x^l + \mathcal{O}\left(x^3\right) \tag{11}$$

This type of coordinates is called **Riemann normal coordinates**.

Variants of the curvature tensor Next we are going to study the **trace** of Riemann tensor. To make this make sense, we can only contract the first and third (or second and fourth) index because of the antisymmetry:

$$R^{\alpha}{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} \equiv R_{\mu\nu} \tag{12}$$

This is called **Ricci curvature tensor**. It's symmetric on μ and ν (not proven here). In 4-dimensional spacetime, the Ricci tensor has 10 independent components.

The trace of Ricci tensor

$$R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \equiv R \tag{13}$$

is called the **Ricci scalar**. And there is another variant of curvature, which is defined as below:

$$C_{\alpha\mu\lambda\sigma} = R_{\alpha\mu\lambda\sigma} - \frac{2}{n-2}\left(g_{\alpha[\lambda}R_{\sigma]\mu} - g_{\mu[\lambda}R_{\sigma]\alpha}\right) + \frac{2}{(n-2)(n-1)} g_{\alpha[\lambda}g_{\sigma]\mu}R \tag{14}$$

This quantity is known as the **Weyl curvature tensor**. The Weyl tensor has the same symmetries as the Riemann tensor, but has no trace. Because of this, it only has 10 independent components. To some extent, we can say that Weyl tensor plus Ricci tensor have the same information about curvature as Riemann tensor has.

Breakdown of parallelism Our original motivating idea to lay out the notion of "curvature" was that initially parallel trajectories do not remain parallel. Here we make this notion precise.

Consider two nearby geodesics $\gamma_{\vec{u}}$ and $\gamma_{\vec{v}}$, each parameterized by λ. Let $\vec{u}$ be tangent vector to $\gamma_{\vec{u}}$ and $\vec{v}$ be tangent vector to $\gamma_{\vec{v}}$. A is at λ_0 on $\gamma_{\vec{u}}$ and A' is at λ_0 on $\gamma_{\vec{v}}$. And let $\vec{\xi}$ points from event at λ on $\gamma_{\vec{u}}$ to the event at λ on $\gamma_{\vec{v}}$.

$$\vec{\xi} = \vec{x}\left(\gamma_{\vec{v}}, \lambda\right) - \vec{x}\left(\gamma_{\vec{u}}, \lambda\right) \tag{15}$$

What's more, we assume that two geodesics are initially parallel:

$$\vec{u}(\lambda_0) = \vec{v}(\lambda_0) \tag{16}$$

So we can use

$$\left(u^{\alpha}\nabla_{\alpha}\xi^{\beta}\right)\Big|_{\lambda=\lambda_0} = 0 \tag{17}$$

as boundary condition.

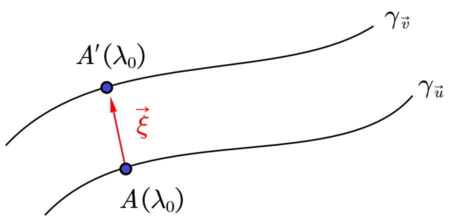

Figure 1: Two nearby geodesics

Now develop some intuition for what is happening by working in LLF centered on A:

$$g_{\mu\nu}\big|_A = \eta_{\mu\nu}, \quad \Gamma^{\mu}{}_{\alpha\beta}\big|_A = 0 \tag{18}$$

$$g_{\mu\nu}\big|_{A'} = \eta_{\mu\nu}, \quad \Gamma^{\mu}{}_{\alpha\beta}\big|_{A'} = \partial_{\gamma}\Gamma^{\mu}{}_{\alpha\beta}\xi^{\gamma} \tag{19}$$

And think about the geodesic equation:

$$\left.\frac{\mathrm{d}^2 x^{\alpha}}{\mathrm{d}\lambda^2}\right|_A = 0 \tag{20}$$

$$\left.\frac{\mathrm{d}^2 x^{\alpha}}{\mathrm{d}\lambda^2}\right|_{A'} + \left[\Gamma^{\alpha}{}_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda}\right]_{A'} = 0 \tag{21}$$

Let's rewrite two equations above:

$$\left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right|_{A'} = v^\mu|_{A'} = u^\mu \tag{22}$$

$$\left.\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2}\right|_{A'} = -\partial_\beta \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu \xi^\beta \tag{23}$$

Differentiate Eq. (15), we will get:

$$\left.\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2}\right|_{A'} - \left.\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2}\right|_{A} \equiv \frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\lambda^2} = -\partial_\beta \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu \xi^\beta \tag{24}$$

Actually Eq.(24) is the key of problem. But that's not enough because it is not tensorial. So our next goal is to make it tensorial. First we can introduce a new notation.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = u^\alpha \partial_\alpha \longrightarrow \text{rewrite using } \frac{D}{\mathrm{d}\lambda} = u^\alpha \nabla_\alpha \tag{25}$$

Now operate this on ξ^α:

$$\frac{D\xi^\alpha}{\mathrm{d}\lambda} = u^\beta \nabla_\beta \xi^\alpha = u^\beta \partial_\beta \xi^\alpha + u^\beta \Gamma^\alpha{}_{\beta\mu}\xi^\mu = \frac{\mathrm{d}\xi^\alpha}{\mathrm{d}\lambda} + u^\beta \Gamma^\alpha{}_{\beta\mu}\xi^\mu \tag{26}$$

And do it again:

$$\begin{aligned}
\frac{D^2\xi^\alpha}{\mathrm{d}\lambda^2} &= u^\gamma \nabla_\gamma \left(\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha{}_{\beta\mu}\xi^\mu u^\beta\right) \\
&= \frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\lambda^2} + u^\gamma \Gamma^\alpha{}_{\gamma\mu}\frac{\mathrm{d}\xi^\mu}{\mathrm{d}\lambda} + \left(u^\gamma \nabla_\gamma \Gamma^\alpha{}_{\beta\mu}\right) u^\beta \xi^\mu + \Gamma^\alpha{}_{\beta\mu}\left(u^\gamma \nabla_\gamma u^\beta\right)\xi^\mu + \Gamma^\alpha{}_{\beta\mu} u^\beta \left(u^\gamma \nabla_\gamma \xi^\mu\right) \\
&= \frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\lambda^2} + \partial_\gamma \Gamma^\alpha{}_{\beta\mu} u^\beta u^\gamma \xi^\mu \\
&= \partial_\gamma \Gamma^\alpha{}_{\beta\mu} u^\beta u^\gamma \xi^\mu - \partial_\beta \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu \xi^\beta
\end{aligned} \tag{27}$$

Relabel dummy indices, finally we get:

$$\begin{aligned}
\frac{D^2\xi^\alpha}{\mathrm{d}\lambda^2} &= \left(\partial_\gamma \Gamma^\alpha{}_{\beta\mu} - \partial_\mu \Gamma^\alpha{}_{\gamma\beta}\right) u^\beta u^\gamma \xi^\mu \\
&= R^\alpha{}_{\beta\gamma\mu} u^\beta u^\gamma \xi^\mu
\end{aligned} \tag{28}$$

This is called **equation of geodesic deviation**.

Bianchi identity There are two equivalent forms of this identity:

$$\nabla_{[\alpha} R_{\beta\gamma]\mu\delta} = 0 \tag{29}$$

$$\nabla_\alpha R_{\beta\gamma\mu\delta} + \nabla_\beta R_{\gamma\alpha\mu\delta} + \nabla_\gamma R_{\alpha\beta\mu\delta} = 0 \tag{30}$$

And we can define **Einstein curvature tensor**:

$$G_{\alpha\mu} = R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R \tag{31}$$

Then it will fulfills:

$$\nabla^\mu G_{\alpha\mu} = 0 \tag{32}$$

General Relativity Lecture12

The Einstein Field Equation

Lin Fu

April 2025

1 A little bit more math

In last lecture, we've learned an important indentity: **Bianchi indentity**

$$\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\gamma R_{\alpha\beta\mu\nu} = 0 \tag{1}$$

Now we're going to contract this with metric $g^{\beta\mu}$:

$$\nabla_\alpha R_{\gamma\nu} + \nabla^\mu R_{\gamma\alpha\mu\nu} - \nabla_\gamma R_{\alpha\nu} = 0 \tag{2}$$

Let's now contract once more using metric $g^{\gamma\nu}$:

$$\nabla_\alpha R - \nabla^\mu R_{\alpha\mu} - \nabla^\nu R_{\alpha\nu} = 0 \tag{3}$$

Notice that the second and third term are both the divergence, so we can rewrite the equation above as:

$$\nabla_\alpha R - 2\nabla^\mu R_{\alpha\mu} = 0 \longrightarrow \nabla^\mu \left(R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R \right) = 0 \tag{4}$$

We define **Einstein tensor**

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{5}$$

And, if we calculate the trace of Einstein tensor we will get:

$$G^\mu{}_\mu = g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = R - \frac{1}{2} g^\mu{}_\mu R \equiv G \tag{6}$$

G is called **Einstein scalar**. With the fact that the trace of metric equals the dimention of spacetime, we will get:

$$g^\mu{}_\mu = 4 \longrightarrow G = -R \tag{7}$$

This result tells us: Einstein tensor is the trace-reversed Ricci tensor.

2 Making a theory of gravity: Guiding principles

Now we've had all mathematical tools for developing the gravity theory, and there are two main ingredients go into this.

First, using the **principle of equivalence** (in the form of the Einstein equivalence principle) to express laws of physics (particularly laws of motion) in a form appropriate to physics in a curved spacetime. We have already had some practice with using the equivalence principle. The key idea, sometimes called the "minimal coupling" principle, is to begin with a law of physics that is valid in inertial coordinates in flat spacetime; to then use that form of the law in a local Lorentz frame in which, by the Einstein equivalence principle, it should hold as well; and then to rewrite the law in a covariant, tensorial form. We finally assert that this law holds in curved spacetime.

For example, in a freely falling frame, bodies that move unaffected by non-gravitational forces follow a trajectory describe by

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0 \tag{8}$$

But this is not tensorial. A more general statement of such motion is that the tangent vector is parallel transported along the trajectory, so the fully covariant formulation of becomes

$$u^\alpha \nabla_\alpha u^\beta = 0 \tag{9}$$

Second, we need some sort of field equation that connects spacetime to sources of matter and energy. We require that whatever emerges must recover Newtonian gravity in limit.

$$\nabla^2 \Phi = \delta^{ij} \partial_i \partial_j \Phi = 4\pi G \rho \tag{10}$$

In last lecture we've known that the equation of motion under the influence of a Newtonian gravitational potential Φ is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\delta^{ij} \partial_j \Phi \tag{11}$$

This is very much not a covariant formulation. So we'd better begin with geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \tag{12}$$

If we set slow motion limit

$$\frac{\mathrm{d}x^0}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau} \gg \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \tag{13}$$

Then Eq.(12) can be simplified into:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{00} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 = 0 \tag{14}$$

We can expand the Christoffel symbol $\Gamma^\mu{}_{00}$:

$$\Gamma^\mu{}_{00} = \frac{1}{2} g^{\mu\nu} \left(\partial_0 g_{\nu 0} + \partial_0 g_{0\nu} - \partial_\nu g_{00} \right) \tag{15}$$

In Newtonian limit, most things we talk about is static. So just neglect time derivatives to recover this limit:

$$\Gamma^\mu{}_{00} = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{00} \tag{16}$$

We also imagine that the spacetime metric is almost that of special relativity, writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{17}$$

where all of the components of $h_{\mu\nu}$ are "small":$\|h_{\mu\nu}\| \ll 1$. It's not hard to show that the inverse tensor corresponding to (17) is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}\left(h^2\right) \tag{18}$$

With this, the Christoffel becomes:

$$\Gamma^\mu{}_{00} = -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00} + \mathcal{O}\left(h^2\right) \tag{19}$$

Because we've said that we should neglect all time derivatives, so $\Gamma^0{}_{00} = 0$.

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0 \tag{20}$$

And spatial motion is:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = \frac{1}{2} \eta^{ij} \partial_j h_{00} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 \longrightarrow \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2} \delta^{ij} \partial_j h_{00} \tag{21}$$

If we take

$$h_{00} = -2\Phi \quad \text{or} \quad g_{00} = \eta_{00} + h_{00} = -\left(1 + 2\Phi\right) \tag{22}$$

this will corresponds to (11).

3 A field equation for relativistic gravity

Rewrite Newton field equation (10):

$$\eta^{ij}\partial_i\partial_j\Phi = 4\pi G\rho \tag{23}$$

The key point is: ρ is not a scalar but a component of a tensor. And we've know that $\rho = T_{00}$. Now we want to promote the whole equation to something tensorial.

Now we turn the right hand side of the equation into $T_{\mu\nu}$. On the left hand side, we've know that the metric has the meaning of potential, so just change Φ into the metric. In that case, two derivatives of the spacetime metric emerges. This corresponds to curvature.

$$(\text{curvature tensor}) = T_{\mu\nu} \tag{24}$$

Now there are some candidates for this curvature tensor. But we should notice that $T_{\mu\nu}$ has a special property: $\nabla_\mu T^{\mu\nu} = 0$ (conservation of energy and momentum). So the curvature tensor should also has zero divergence. This reminds us that the best choice is Einstein curvature tensor $G_{\mu\nu}$.

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{25}$$

Always remember that Einstein tensor is the trace-reversed Ricci tensor. So if we trace-reverse Eq.(25), we will get:

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{26}$$

Let's take our source to be a static perfect fluid:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu} \tag{27}$$

In Newtonian limit, $\rho \gg P$. And we assume that the fluid is static:

$$u^\mu = (u^0, 0, 0, 0) \tag{28}$$

4-velocity must has $g_{\mu\nu}u^\mu u^\nu = -1$, so we have

$$g_{00}\left(u^0\right)^2 = -1, g_{00} = -1 + h_{00} \longrightarrow u^0 = 1 + \frac{1}{2}h_{00} + \mathcal{O}\left(h^2\right) \tag{29}$$

Lowering the index so that we have the component we need in our stress-energy tensor yields

$$u_0 = g_{0\mu}u^\mu = (\eta_{00} + h_{00})u^0 = -1 + \frac{1}{2}h_{00} + \mathcal{O}(h^2) \tag{30}$$

We now have enough pieces to generate at least one component of the proposed field equation:

$$R_{00} = \kappa\left(T_{00} - \frac{1}{2}g_{00}T\right) \tag{31}$$

On the right hand side we have

$$T_{00} = \rho u_0 u_0 = \rho(1 - h_{00}) \tag{32}$$
$$T = g^{\mu\nu}T_{\mu\nu} = \rho u^\mu u_\mu = -\rho \tag{33}$$

So we find

$$T_{00} - \frac{1}{2}g_{00}T = \rho(1 - h_{00}) - \frac{1}{2}\left(-1 + h_{00}\right)(-\rho) = \frac{1}{2}\rho + \mathcal{O}\left(h\right) \tag{34}$$

Let's now build the left-hand side:

$$\begin{aligned}
R_{00} &= R^\alpha{}_{0\alpha 0} = R^i{}_{0i0} \\
&= \partial_i\Gamma^i{}_{00} - \partial_0\Gamma^i{}_{0i} + \mathcal{O}\left(\Gamma^2\right) \\
&= \frac{1}{2}\partial_i\left[g^{i\mu}\left(\partial_0 g_{\mu 0} + \partial_0 g_{0\mu} - \partial_\mu g_{00}\right)\right] \\
&= -\frac{1}{2}\partial_i\left[\eta^{i\mu}\partial_\mu h_{00}\right] \\
&= -\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} \\
&= -\frac{1}{2}\nabla^2 h_{00}
\end{aligned} \tag{35}$$

Finally Eq.(31) becomes

$$\nabla^2 h_{00} = -\kappa\rho \tag{36}$$

Recovering Newtonian free-fall motion in the limit requires that $h_{00} = -2\Phi$. Making this substitution, we see that this limit of our proposed field equation duplicates the Newtonian field equation if we put $\kappa = 8\pi G$. Finally we get

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{37}$$

This is called **the Einstein field equation**.

4 A few remarks

Actually we can add any divergence free tensor onto LHS and still have a good field equation. Take metric for example:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{38}$$

Λ is called **the cosmological constant**.

And we can define

$$T^{\lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \tag{39}$$

This can be thought of as a rather bizarre perfect fluid, one that has

$$\rho = \frac{\Lambda}{8\pi g}, \quad P = -\rho \tag{40}$$

in a freely-falling frame. Such a stress-energy tensor arises in quantum field theory, and represents a form of vacuum energy, as it is isotropic and invariant to Lorentz transformations in the local Lorentz frame.

What's more, people often set $G = 1, c = 1$. When we do that, mass, length, and time are all measured in the same units. A useful way to keep terms straight in your head is to note that factors of G and c can be used to convert between different kinds of units (See it at Scott A. Hughes's lecture notes).

General Relativity Lecture13

The Einstein Field Equation (Variant Derivation)

Lin Fu

April 2025

In this lecture, we will introduce another way to get Einstein field equation — via the **Einstein-Hillbert action**.

The basic idea is to imagine that there is some function $\mathcal{L}$ which characterizes the fields you wish to study, and that an action can be found by integrating this over all of spacetime:

$$S = \int \mathrm{d}^4 x \mathcal{L} \tag{1}$$

Since the action must be a Lorentz scalar, is not uncommon to write this

$$S = \int \mathrm{d}^4 x \sqrt{-g} \hat{\mathcal{L}} \tag{2}$$

including the factor $\sqrt{-g}$ needed to make a proper volume element in a spacetime $g_{\mu\nu}$. Use the least action principle, then we'll get:

$$\delta S = \int \mathrm{d}^4 x \left[\frac{\partial \hat{\mathcal{L}}}{\partial (\text{fields})} \right] \delta (\text{fields}) = 0 \tag{3}$$

Because δ(fields) is arbitrary, so we have

$$\frac{\partial \hat{\mathcal{L}}}{\partial (\text{fields})} = 0 \tag{4}$$

This will lead to Euler-Lagrange equations for fields.

Example Consider a Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$. When the field varies:

$$\phi \to \phi + \delta\phi, \;\; \partial_\mu \phi \to \partial_\mu \phi + \partial_\mu (\delta \phi) \tag{5}$$

The variation of the action is

$$\begin{aligned} \delta S &= \int \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu (\delta \phi) \right] \\ &= \int \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \delta \phi \end{aligned} \tag{6}$$

Then we get the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0 \tag{7}$$

A common form of Lagrangian is

$$\mathcal{L} = -\frac{1}{2} \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} m^2 \phi^2 \tag{8}$$

Take derivatives of this, we will get:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \;\; \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = -\eta^{\mu\nu} \partial_\nu \phi \tag{9}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = -\eta^{\mu\nu} \partial_\mu \partial_\nu \phi = -\Box \phi \tag{10}$$

Finally we get

$$\Box\phi - m^2\phi = 0 \tag{11}$$

This is simply the **Klein-Gordon equation** for a massive scalar field.

Next we would like to develop a similar Lagrangian that is appropriate for a relativistic theory of gravity. Two main points guide our thinking:

- The action must be a scalar, so the Lagrangian must also be a scalar.
- The Lagrangian must be built out of curvature scalars so that it is not something that can be eliminated at any point simply by changing representation.

Based on that, the simplest Lagrangian we can choose is $\hat{\mathcal{L}} = R$. Then we write the action S as:

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta} \tag{12}$$

The least action principle tells us:

$$\delta S = \frac{1}{16\pi G}\int \mathrm{d}^4x\frac{\delta}{\delta g^{\alpha\beta}}\left[\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}\right]\delta g^{\alpha\beta} = 0 \tag{13}$$

Now start calculating! First we can expand $\delta\left[\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}\right]$:

$$\delta\left[\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}\right] = \left(\delta\sqrt{-g}\right)R + \sqrt{-g}\left(\delta g^{\alpha\beta}\right)R_{\alpha\beta} + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} \tag{14}$$

The first term in (14) can be found using some of the tricks associated with the determinant of the metric we discussed in an earlier lecture; we find:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} \tag{15}$$

And for Ricci tensor we have:

$$\delta R_{\alpha\beta} = \nabla_\mu\delta\Gamma^\mu{}_{\beta\alpha} - \nabla_\beta\delta\Gamma^\mu{}_{\mu\alpha} \tag{16}$$

We are not done: we still need to examine the variation in the Christoffel symbol. With some effort, one can show that:

$$\delta\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\left[\nabla_\gamma\left(g_{\alpha\nu}g_{\beta\lambda}g^{\mu\gamma}\delta g^{\nu\lambda}\right) - \nabla_\alpha\left(g_{\beta\gamma}\delta g^{\mu\gamma}\right) - \nabla_\beta\left(g_{\alpha\gamma}\delta g^{\mu\gamma}\right)\right] \tag{17}$$

Using this, one can finally assemble:

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\alpha\nabla_\beta\left(g^{\alpha\beta}g_{\mu\nu}\delta g^{\mu\nu} - \delta g^{\alpha\beta}\right) \equiv \nabla_\alpha v^\alpha \tag{18}$$

This final form recognizes that the expression we at last landed on can be thought of as the covariant divergence of a vector field v^α whose form can be read out of Eq.(18).

Putting these three terms together, the variational principle we seek to enforce takes the form

$$\delta S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)\delta g^{\alpha\beta} + \nabla_\alpha v^\alpha\right] \tag{19}$$

$$= \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[G_{\alpha\beta}\delta g^{\alpha\beta} + \nabla_\alpha v^\alpha\right] \tag{20}$$

The first term inside the square brackets looks pretty much just like what we want. The second term does not. Because this term is in the form of a divergence, one can imagine invoking the divergence theorem and converting it into a boundary term. We presumably could then discard this term by requiring the variation to vanish on the boundary. Enforcing $\delta S = 0$, we get:

$$G_{\alpha\beta} = 0 \tag{21}$$

This is called **vacuum Einstein equation**.

More generally, we can write our action for everything — matter, fields, spacetime — in the form

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{16\pi G} + \mathcal{L}_M\right) \tag{22}$$

Let $\delta S = 0$, then we get:

$$\frac{\sqrt{-g}}{16\pi G} G_{\alpha\beta} + \frac{\partial\left(\sqrt{-g}\mathcal{L}_M\right)}{\partial g^{\alpha\beta}} = 0 \tag{23}$$

We define:

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\mathcal{L}_M\right)}{\partial g^{\alpha\beta}} \tag{24}$$

Then this will give us the same Einstein field equation.

Example Choose $\mathcal{L}_M$ as:

$$\mathcal{L}_M = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{25}$$

Going through a bunch of complex calculation, finally we will get

$$\delta S = -\frac{1}{4} \int d^4x \sqrt{-g} \left[F_{\mu\alpha} F^{\mu}{}_{\beta} + F_{\alpha\mu} F_{\beta}{}^{\mu} - \frac{1}{2} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right] \delta g^{\alpha\beta} \tag{26}$$

One can find out that the stuff inside the square brackets is the stress-energy tensor of electromagnetic field.

General Relativity Lecture14

Linearized Gravity 1: Principles and Static Limit

Lin Fu

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Let's begin with the Einstein equation $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$. Actually we can regard this as a set of differential equations for the spacetime given a source. Just like through Maxwell's equations we can calculate the electromagnetic field that arises from the given distribution of charge and current, we can compute the spacetime that arises from the given distribution of mass and energy.

On the left-hand side, we can regard the Einstein tensor as a differential operator acting on the metric of spacetime:

$$G_{\alpha\beta} = \mathcal{D}^2_{\alpha\beta}\left(g_{\mu\nu}\right) \tag{1}$$

In this schematic form, $\mathcal{D}^2_{\alpha\beta}$ is a 2nd-order, nonlinear partial differential operator. On the right-hand side, we assume that the source is perfect fluid:

$$T_{\alpha\beta} = (\rho + P)\, u_\alpha u_\beta + P g_{\alpha\beta} \tag{2}$$

And there is a constraint: $g_{\mu\nu}u^\mu u^\nu = -1$. So the whole thing can be seen as a differential equation of the spacetime metric.

In general, we have three way to solve this awful equation:

1. Solve for weak gravity. Consider $g_{\alpha\beta}$ is close to $\eta_{\alpha\beta}$.
2. Consider symmetric solutions, with the technique of perturbation.
3. Numerical solutions of the whole monster, with no simplifications.

Let's dive into the first way. Now we're going to consider the weak gravity condition (also called linearized GR). In that case, we can choose coordinates such that $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $\|h_{\alpha\beta}\| \ll 1$. We called such kind of coordinates "nearly Lorentz coordinates".

Now consider coordinates transformation of nearly flat spacetime:

$$g_{\bar\mu\bar\nu} = L^\alpha{}_{\bar\mu} L^\beta{}_{\bar\nu} g_{\alpha\beta}, \quad L^\alpha{}_{\bar\mu} = \frac{\partial x^\alpha}{\partial x^{\bar\mu}} \tag{3}$$

In flat spacetime, the transformation is just Lorentz transformation. What will happen if we apply this to nearly flat spacetime?

$$\begin{aligned} g_{\bar\mu\bar\nu} &= \Lambda^\alpha{}_{\bar\mu}\Lambda^\beta{}_{\bar\nu}\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right) \\ &= \Lambda^\alpha{}_{\bar\mu}\Lambda^\beta{}_{\bar\nu}\eta_{\alpha\beta} + \Lambda^\alpha{}_{\bar\mu}\Lambda^\beta{}_{\bar\nu} h_{\alpha\beta} \\ &\equiv \eta_{\bar\mu\bar\nu} + h_{\bar\mu\bar\nu} \end{aligned} \tag{4}$$

This is interesting because we see that $h_{\alpha\beta}$ transforms just like the ordinary tensor in flat spacetime. On this basis we regard $h_{\alpha\beta}$ as an ordinary tensor field living in $\eta_{\alpha\beta}$ metric. And we can lower and raise the indices of h using $\eta_{\alpha\beta}$

$$h^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}h_{\mu\nu} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu} + \mathcal{O}\left(h^2\right) \tag{5}$$

Notice that $g^{\alpha\beta}$ is actually the inverse metric and we haven't considered this yet. Use the definition:

$$g^{\alpha\beta}g_{\beta\gamma} = \left(\eta^{\alpha\beta} + m^{\alpha\beta}\right)\left(\eta_{\beta\gamma} + h_{\beta\gamma}\right) = \delta^\alpha{}_\gamma \tag{6}$$

Expand this:

$$\delta^\alpha{}_\gamma + m^\alpha{}_\gamma + h^\alpha{}_\gamma + \mathcal{O}\left(h^2\right) = \delta^\alpha{}_\gamma \longrightarrow m^\alpha{}_\gamma = -h^\alpha{}_\gamma \tag{7}$$

Finally we get:

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \tag{8}$$

Let's consider another kind of coordinates transformation —— coordinates shift, which is defined by

$$x^{\alpha'} = x^{\alpha} + \xi^{\alpha}\left(x^{\beta}\right) \tag{9}$$

Then the transformation matrix is

$$L^{\alpha'}{}_{\beta} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} = \delta^{\alpha}{}_{\beta} + \partial_{\beta}\xi^{\alpha} \tag{10}$$

Because we work in the nearly Lorentz coordinates, so $\partial_{\beta}\xi^{\alpha}$ must be very small. We call this kind of transformation "infinitesimal transformation". With $L^{\alpha'}{}_{\beta}L^{\beta}{}_{\gamma'} = \delta^{\alpha'}{}_{\gamma'}$, it's easy to prove that

$$L^{\alpha}{}_{\beta'} = \delta^{\alpha}{}_{\beta} - \partial_{\beta}\xi^{\alpha} + \mathcal{O}\left[(\partial\xi)^2\right] \tag{11}$$

Now we want to see how does metric change under this transformation.

$$\begin{aligned} g_{\mu'\nu\prime} &= L^{\alpha}{}_{\mu'}L^{\beta}{}_{\nu'}g_{\alpha\beta} \\ &= \left(\delta^{\alpha}{}_{\mu} - \partial_{\mu}\xi^{\alpha}\right)\left(\delta^{\beta}{}_{\nu} - \partial_{\nu}\xi^{\beta}\right)\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right) \\ &= \eta_{\mu\nu} + h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + \mathcal{O}\left[h\left(\partial\xi\right)\right] \end{aligned} \tag{12}$$

The effect of this coordinate transformation is to change the perturbation around flat spacetime:

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} \tag{13}$$

This makes us recall that in electromagnetism we can add a gradient to the potential and leaves the fields unchanged $A_{\mu} \to A_{\mu} + \partial_{\mu}\Lambda$. Similarly, if we change the perturbation like (13), the Riemann curvature tensor stays the same! Because of this, the transformation (9) is also called **gauge transformation** in linearized gravity. It is simple to show that the Riemann tensor is given by:

$$R_{\mu\alpha\nu\beta} = \frac{1}{2}\left(\partial_{\alpha}\partial_{\nu}h_{\mu\beta} + \partial_{\mu}\partial_{\beta}h_{\alpha\nu} - \partial_{\alpha}\partial_{\beta}h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h_{\alpha\beta}\right) \tag{14}$$

When we change the metric using (13), the change of Riemann tensor is

$$\begin{aligned} \delta R_{\mu\alpha\nu\beta} =& \frac{1}{2}\left(-\partial_{\alpha}\partial_{\nu}\partial_{\mu}\xi_{\beta} - \partial_{\alpha}\partial_{\nu}\partial_{\beta}\xi_{\mu} - \partial_{\mu}\partial_{\beta}\partial_{\alpha}\xi_{\nu} - \partial_{\mu}\partial_{\beta}\partial_{\nu}\xi_{\alpha}\right. \\ & \left.+\partial_{\mu}\partial_{\nu}\partial_{\alpha}\xi_{\beta} + \partial_{\alpha}\partial_{\beta}\partial_{\nu}\xi_{\mu} + \partial_{\alpha}\partial_{\beta}\partial_{\mu}\xi_{\nu} + \partial_{\mu}\partial_{\nu}\partial_{\eta}\xi_{\alpha}\right) \end{aligned} \tag{15}$$

Because partial derivatives commute, one can see that $\delta R_{\mu\alpha\nu\beta} = 0$.

We can build up Ricci tensor using Riemann tensor:

$$\begin{aligned} R_{\alpha\beta} &= \eta^{\mu\nu}R_{\mu\alpha\nu\beta} \\ &= \frac{1}{2}\left(\partial_{\alpha}\partial^{\mu}h_{\mu\beta} + \partial_{\beta}\partial^{\mu}h_{\mu\alpha} - \partial_{\alpha}\partial_{\beta}h - \Box h_{\alpha\beta}\right) \end{aligned} \tag{16}$$

In the above equation $\Box = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$, $\;h = \eta^{\mu\nu}h_{\mu\nu} = h^{\mu}{}_{\mu}$. And Ricci scalar is

$$R = \eta^{\alpha\beta}R_{\alpha\beta} = \partial^{\alpha}\partial^{\mu}h_{\mu\alpha} - \Box h \tag{17}$$

The Einstein tensor is:

$$\begin{aligned} G_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R \\ &= \frac{1}{2}\left(\partial_{\alpha}\partial^{\mu}h_{\mu\beta} + \partial_{\beta}\partial^{\mu}h_{\mu\alpha} - \partial_{\alpha}\partial_{\beta}h - \Box h_{\alpha\beta} + \eta_{\alpha\beta}\Box h - \eta_{\alpha\beta}\partial^{\mu}\partial^{\nu}h_{\mu\nu}\right) \end{aligned} \tag{18}$$

To simplify this, define a new tensor:

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h \tag{19}$$

The trace of this is

$$\bar{h} = \eta^{\alpha\beta}\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h\right) = h - 2h = -h \tag{20}$$

Because of this, we call $\bar{h}_{\alpha\beta}$ the trace reversed metric perturbation. The reason we introduce it is that the Einstein tensor itself is the trace reverse of Ricci tensor. So insert $h_{\alpha\beta} = \bar{h}_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}h$ to Eq.(18), then we'll get

$$G_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha\partial^\mu\bar{h}_{\mu\beta} + \partial_\beta\partial^\mu\bar{h}_{\mu\alpha} - \eta_{\alpha\beta}\partial^\mu\partial^\nu\bar{h}_{\mu\nu} - \Box\bar{h}_{\alpha\beta}\right) \tag{21}$$

Three of these terms look like divergence of $\bar{h}_{\mu\nu}$.

Just like in electrodynamics we often set $\nabla\cdot\boldsymbol{A} = 0$, can we similarly set $\partial^\mu\bar{h}_{\mu\nu} = 0$? Actually these are four conditions, and gauge generators ξ_ν are 4 four free functions. We have enough degrees of freedom to do this! Writing

$$h_{\mu\nu}^{\text{new}} = h_{\mu\nu}^{\text{old}} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \tag{22}$$

we trace reverse and find that

$$\bar{h}_{\mu\nu}^{\text{new}} = \bar{h}_{\mu\nu}^{\text{old}} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \eta_{\mu\nu}\partial^\alpha\xi_\alpha \tag{23}$$

Take the divergence of this:

$$\begin{aligned}\partial^\mu\bar{h}_{\mu\nu}^{\text{new}} &= \partial^\mu\bar{h}_{\mu\nu}^{\text{old}} - \Box\xi_\nu - \partial_\nu\partial^\mu\xi_\mu + \eta_{\mu\nu}\partial^\mu\partial^\alpha\xi_\alpha \\ &= \partial^\mu\bar{h}_{\mu\nu}^{\text{old}} - \Box\xi_\nu\end{aligned} \tag{24}$$

If we choose $\Box\xi_\nu = \partial^\mu\bar{h}_{\mu\nu}^{\text{old}}$, then $\partial^\mu\bar{h}_{\mu\nu}^{\text{new}} = 0$! Then

$$G_{\alpha\beta} = -\frac{1}{2}\Box\bar{h}_{\alpha\beta} \tag{25}$$

We call this Lorentz gauge in linearized gravity. And we find the linearized Einstein field equation takes the form

$$\Box\bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta} \tag{26}$$

Static solutions Let us begin by considering a perfect fluid source that is static ($u^t = 1 + \mathcal{O}(h), u^i = 0$) and non-relativistic ($\rho \gg P$). We can write the stress energy tensor

$$T_{\alpha\beta} \simeq \rho u_\alpha u_\beta \tag{27}$$

Thanks to the static condition, the only non-negligible component is $T_{00} = \rho + \mathcal{O}(h)$. The only non-trivial field equation component is then

$$\Box\bar{h}_{00} = -16\pi G\rho \tag{28}$$

Since the source is static, the differential operator simplifies to

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu \to \nabla^2 \tag{29}$$

Then the equation becomes:

$$\nabla^2\bar{h}_{00} = -16\pi G\rho \tag{30}$$

We assume the solution of Eq.(30) is

$$\bar{h}_{00} = -4\Phi, \quad \text{all other } \bar{h}_{\mu\nu} = 0 \tag{31}$$

To get the metric perturbation, we trace reverse again:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{32}$$

where $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = 4\Phi$. Then we get:

$$h_{00} = h_{11} = h_{22} = h_{33} = -2\Phi \tag{33}$$

So the metric is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} = \operatorname{diag}(-1-2\Phi, 1-2\Phi, 1-2\Phi, 1-2\Phi) \tag{34}$$

Then the line element is

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + (1-2\Phi)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{35}$$

General Relativity Lecture15

Linearized Gravity 2: Dynamic Sources

Lin Fu

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When sources are not dynamical, the linearized Einstein field equation is:

$$\Box \bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta} \tag{1}$$

Next we'll talk about how to solve this equation when sources are dynamical. Our tool for beginning this investigation takes advantage of the fact that Eq.(1) is a linear equation. And we can write this into a simpler way:

$$\mathcal{D} f(t, \boldsymbol{x}) = s(t, \boldsymbol{x}) \tag{2}$$

In this $\mathcal{D}$ is a linear operator, $f(t, \boldsymbol{x})$ is the field and $s(t, \boldsymbol{x})$ is the source. We can use **Green's function** to solve this.

The first thing is to relapce source with delta function:

$$s(t, \boldsymbol{x}) \longrightarrow \delta(t - t') \, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \tag{3}$$

And we assert solution exists, and call resulting solution $G(t, \boldsymbol{x}; t', \boldsymbol{x}')$. Then we have

$$\mathcal{D} G(t, \boldsymbol{x}; t', \boldsymbol{x}') = \delta(t - t') \, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \tag{4}$$

What's more, we call $(t, \boldsymbol{x})$ field point, and $(t', \boldsymbol{x}')$ source point. Suppose we have constructed the Green's function corresponding to our operator $\mathcal{D}$, then the solution for our field is then given by

$$f(t, \boldsymbol{x}) = \int \mathrm{d}t' \int \mathrm{d}^3\boldsymbol{x}' G(t, \boldsymbol{x}; t', \boldsymbol{x}') \, s(t', \boldsymbol{x}') \tag{5}$$

To prove Eq.(5) is right, we can let $\mathcal{D}$ act on it:

$$\begin{aligned}
\mathcal{D} f(t, \boldsymbol{x}) &= \int \mathrm{d}t' \int \mathrm{d}^3\boldsymbol{x}' s(t', \boldsymbol{x}') \, \mathcal{D} G(t, \boldsymbol{x}; t', \boldsymbol{x}') \\
&= \int \mathrm{d}t' \int \mathrm{d}^3\boldsymbol{x}' s(t', \boldsymbol{x}') \, \delta(t - t') \, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \\
&= s(t, \boldsymbol{x})
\end{aligned} \tag{6}$$

So the task becomes: find the Green's function.

Now go back to our problem. If $\mathcal{D} \to \Box$, then Green's function is called **radiative Green's function**.

$$G(t, \boldsymbol{x}; t', \boldsymbol{x}') = -\frac{\delta\left[t' - (t - |\boldsymbol{x} - \boldsymbol{x}'|)\right]}{4\pi |\boldsymbol{x} - \boldsymbol{x}'|} \tag{7}$$

Notice the form

$$t - |\boldsymbol{x} - \boldsymbol{x}'| \equiv t_{\mathrm{ret}} \tag{8}$$

that enters the time dependence. The name t_{ret} we assign means that this is **retarded time**. It reflects that things happened at source point needs time to get field point.

Apply Green's function to our linearized Einstein field equation, we'll get:

$$\begin{aligned}
\bar{h}_{\alpha\beta}(t, \boldsymbol{x}) &= -16\pi G \int \mathrm{d}t' \int \mathrm{d}^3\boldsymbol{x}' T_{\alpha\beta}(t', \boldsymbol{x}') \left(-\frac{\delta\left[t' - (t - |\boldsymbol{x} - \boldsymbol{x}'|)\right]}{4\pi |\boldsymbol{x} - \boldsymbol{x}'|} \right) \\
&= 4G \int \mathrm{d}^3\boldsymbol{x}' \frac{T_{\alpha\beta}(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}
\end{aligned} \tag{9}$$

This solution is exact within the confines of the linearized approximation to general relativity. We made a specific gauge choice in order to write the field equation in a form which is amenable to our use of the radiative Green's function. But the gauge we chose is unfortunately masking some of the character of the spacetime.

For example, consider an electromagnetic potential whose components in some frame are given by

$$
\begin{aligned}
A^0 &= \frac{q}{r} + \frac{q\omega R \sin(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}{r} \\
A^i &= \frac{qk^i R \sin(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}{r} + \frac{qx^i R \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}{r^3}
\end{aligned} \tag{10}
$$

Using this to compute components of the field tensor, we'll find:

$$
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{pmatrix} \times \frac{q}{r^3} \tag{11}
$$

This potential describes an electric field $\boldsymbol{E} = q\boldsymbol{x}/r^3$ and a magnetic field $\boldsymbol{B} = 0$. It is simply a Coulomb point charge at the origin. There is no radiation in this problem at all.

This motivates a way of understanding which aspects of a spacetime are truly radiative in all gauges, and which are not (at least when we linearize around a flat background spacetime). The end result is that we will find that the degrees of freedom represented by the 10 independent components of the spacetime perturbation $h_{\alpha\beta}$ describe 4 degrees of freedom governed by Poisson-like equations (and thus are not radiative) and 2 degrees of freedom governed by a wave equation. These represent 2 polarizations of gravitational radiation. The remaining 4 degrees of freedom are purely gauge in nature.

Now consider $h_{\mu\nu}$ as a tensor field in a flat background, and choose time and space coordinates. We're going to examine how components of $h_{\mu\nu}$ break up into subgroups with respect to spatial-only coordinates transformation.

$$
\begin{aligned}
h_{\mu\nu} \mapsto & h_{tt} \equiv -2\phi \,(\text{scalar}) \\
& h_{ti} \,(\text{a } \ 3-\text{vector}) \\
& h_{ij} \,(\text{a } \ 3\times 3 \ \text{ symmetric } \ \text{tensor})
\end{aligned} \tag{12}
$$

First let's look at the 3-vector h_{ti}. Using the fact that any 3-vector can be written as a divergence-free function plus the gradient of a scalar, we can write it as:

$$
h_{ti} = \beta_i + \partial_i\gamma, \quad \partial_i\beta_i = 0 \tag{13}
$$

γ has 1 degrees of freedom, and β_i has 2 degrees of freedom. Extending this logic to h_{ti}, we find that the most general form of this 3×3 symmetric tensor is

$$
h_{ij} = h_{ij}^{\mathrm{TT}} + \frac{1}{3}H\delta_{ij} + \partial_{(i}\epsilon_{j)} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\lambda \tag{14}
$$

Let's go through the quantities we have introduced here and examine their properties:

- The function H is a scalar under rotations, and represents 1 degree of freedom. We put $H = \delta^{ij}h_{ij}$, so that it is the trace of this 3×3 tensor. This means there is no contribution to the trace from any other terms in Eq.(14).

- The function λ is also a scalar under rotations, and represents 1 degree of freedom. It represents a trace-free double gradient of a scalar. The operator which acts upon it is defined to have zero trace since the trace is already bound up in the function H.

- The functions ϵ_j behave as a 3-vector under rotations. In order that it have no trace, we require that $\partial_i\epsilon_i = 0$. This means that this quantity represents 3 free functions plus 1 constraint, for a total of 2 degrees of freedom.

- The functions h_{ij}^{TT} describe the remaining divergence-free, trace-free degrees of freedom in h_{ij}. The 6 functions of h_{ij}^{TT} are subject to 3 constraints to make this quantity divergence-free, $\partial_i h_{ij}^{\mathrm{TT}} = 0$, plus 1 constraint to make it trace free, $\delta^{ij} h_{ij}^{\mathrm{TT}} = 0$. These functions thus represent 2 degrees of freedom. The label "TT" indicates that these represent the *transverse* and *trace-free* parts of h_{ij}. "Trace-free" is hopefully clear; the reason why divergenceless corresponds to transverse will be made clear in the next lecture.

These choices sum to a total of 6 degrees of freedom. Including the 3 degrees of freedom associated with γ and β_i and the 1 associated with ϕ, we see that all 10 components of $h_{\mu\nu}$ are accounted for in the degrees of freedom provided by this decomposition.

Our goal is: develop Einstein field equation in terms of $(\phi, \gamma, \beta_i, H, \lambda, \epsilon_i, h_{ij}^{\mathrm{TT}})$. Next write the gauge generator as:

$$\xi_\alpha = (\xi_t, \xi_i) = (A, B_i + \partial_i C) \quad \text{with } \partial_i B_i = 0 \tag{15}$$

When we use this form of the generator to change gauge, we find that the functions we introduced to decompose the metric change as follows:

$$\begin{aligned}
\phi &\to \phi + \partial_t A, \\
\beta_i &\to \beta_i - \partial_t B_i, \\
\gamma &\to \gamma - A - \partial_t C, \\
H &\to H - 2\nabla^2 C, \\
\lambda &\to \lambda - 2C, \\
\epsilon_i &\to \epsilon_i - 2B_i, \\
h_{ij}^{\mathrm{TT}} &\to h_{ij}^{\mathrm{TT}}.
\end{aligned} \tag{16}$$

If we stare at these functions for a little while, we notice that the following combinations of metric functions are totally gauge invariant:

$$\begin{aligned}
\Phi &= \phi + \partial_t \gamma - \frac{1}{2}\partial_t^2 \lambda \\
\Theta &= \frac{1}{3}\left(H - \nabla^2 \lambda\right) \\
\Psi_i &= \beta_i - \frac{1}{2}\partial_t \epsilon_i, \;\; \partial_i \Psi_i = 0
\end{aligned} \tag{17}$$

These quantities, plus h_{ij}^{TT}, represent the gauge-invariant, physical degrees of freedom in the metric $h_{\mu\nu}$. Notice that there are only 6 such degrees of freedom; in the 10 independent components of $h_{\mu\nu}$, 4 are purely gauge degrees of freedom.

We would now like to see what this decomposition of spacetime implies for the Einstein field equations. Before doing so, we first introduce a decomposition of the stress-energy tensor similar to what we used in the metric:

$$\begin{aligned}
T_{tt} &= \rho \\
T_{ti} &= S_i + \partial_i S \\
T_{ij} &= P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i \partial_j - \frac{1}{3}\nabla^2\right)\sigma
\end{aligned} \tag{18}$$

These quantities are subject to the constraints:

$$\begin{aligned}
\partial_i S_i = 0, &\;\; \partial_i \sigma_i = 0 \\
\partial_i \sigma_{ij} = 0, &\;\; \delta^{ij}\sigma_{ij} = 0
\end{aligned} \tag{19}$$

When we enforce $\nabla^\alpha T_{\alpha\beta} = 0$, we find that

$$\begin{aligned}
\nabla^2 S &= \partial_t \rho \\
\nabla^2 \sigma &= -\frac{3}{2}P + \frac{3}{2}\partial_t S \\
\nabla^2 \sigma_i &= 2\partial_t S_i
\end{aligned} \tag{20}$$

This amplifies the message that some of the quantities we introduced in the decomposition of the stress - energy tensor are not independent. We can freely specify ρ, P, S_i, and σ_{ij}, a total of 6 functions describing the source; the remaining 4 functions S, σ, and σ_i which complete the source are determined by the behavior of the 6 we can freely specify.

With some labor, we develop the ten components of the Einstein tensor and write the results using the gauge-invariant metric functions we worked out. The result of this exercise is

$$\begin{aligned}
G_{tt} &= -\nabla^2\Theta \\
G_{ti} &= -\frac{1}{2}\nabla^2\Psi_i - \partial_i\partial_t\Theta \\
G_{ij} &= -\frac{1}{2}\Box h_{ij}^{\mathrm{TT}} - \partial_{(i}\Psi_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi + \Theta\right) + \delta_{ij}\left(\frac{1}{2}\nabla^2\left(2\Phi + \Theta\right) - \partial_t^2\Theta\right)
\end{aligned} \tag{21}$$

Enforcing $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$ yields the following 6 differential equations governing the gauge-invariant degrees of freedom in spacetime:

$$\nabla^2\Phi = 4\pi G\left(\rho + 3P - \partial_t S\right) \tag{22}$$

$$\nabla^2\Theta = -8\pi G\rho \tag{23}$$

$$\nabla^2\Psi_i = -16\pi G S_i \tag{24}$$

$$\Box h_{ij}^{\mathrm{TT}} = -16\pi G\sigma_{ij} \tag{25}$$

This exercise confirms that, although the solution we derived earlier appears to be one in which the metric perturbation is entirely radiative, this is quite misleading. No more than two components of the spacetime — those described by the transverse, traceless piece of the metric, h_{ij}^{TT} — satisfy a wave equation in all gauges. Therefore, only those two components truly characterize radiative components of the spacetime. Four of the remaining components of the spacetime are governed by Poisson-type equations, and are often called the “longitudinal degrees of freedom” associated with spacetime. The last four components represent gauge degrees of freedom.

General Relativity Lecture16

Gravitational Radiation 1

Lin Fu

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In previous lecture, we've showed that radiative components of the spacetime can be characterized by

$$\Box h_{ij}^{\mathrm{TT}} = -16\pi G\sigma_{ij} \tag{1}$$

Today's goal is to understand h_{ij}^{TT} in terms of observables, and understand how to compute h_{ij}^{TT} given a source.

To begin, we simply write down a metric that has $\Phi = \Theta = \Psi_i = 0$, and choose $h_{ij}^{\mathrm{TT}} = h_{ij}^{\mathrm{TT}}(t-z)$ — radiation propagating in the z direction. Our solution takes the form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx}^{\mathrm{TT}} & h_{xy}^{\mathrm{TT}} & 0 \\ 0 & h_{yx}^{\mathrm{TT}} & h_{yy}^{\mathrm{TT}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

Because of the symmetry, we have $h_{xy}^{\mathrm{TT}} = h_{yx}^{\mathrm{TT}}$. Because of the property of trace-free, we have $h_{yy}^{\mathrm{TT}} = -h_{xx}^{\mathrm{TT}}$.

Consider a freely falling body in a spacetime $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We've known that the trajectory is a geodesic:

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu}u^\mu u^\nu = 0 \tag{3}$$

Imagine $h_{\mu\nu} = 0$ initially, then $h_{\mu\nu} \neq 0$ comes along and passes over bodies. So $u^\mu = (1, 0, 0, 0)$ initially. When $t = 0$, non-zero wave comes along, then

$$\begin{aligned} \left.\frac{du^\alpha}{d\tau}\right|_{t=0} &= -\left.\left(\Gamma^\alpha{}_{\mu\nu}u^\mu u^\nu\right)\right|_{t=0} \\ &= -\Gamma^\alpha{}_{00} = -\eta^{\alpha\beta}\Gamma_{\beta 00} \\ &= -\frac{1}{2}\eta^{\alpha\beta}\left(\partial_t h_{\beta 0}^{\mathrm{TT}} + \partial_t h_{0\beta}^{\mathrm{TT}} - \partial_\beta h_{00}^{\mathrm{TT}}\right) \\ &= 0 \end{aligned} \tag{4}$$

This means the body does not appear to move. Can we say that the gravitational wave doesn't work? Always bear in mind that geodesics describe motion in some specifield coordinates. Free fall in these coordinates means the body remains fixed in these coordinates! These coordinates follow the body: wiggle right along with body.

So let's switch the thought. Consider two nearby bodies that each follow geodesics.

Figure 1: Two nearby bodies

Let's compute the time takes for a light pulse to travel from A to B. The 4-momentum of the light is

$$p^\mu = \left(\frac{dt}{d\lambda}, \frac{dx}{d\lambda}, 0, 0\right) \tag{5}$$

and it has $g_{\mu\nu}p^\mu p^\nu = 0$:

$$-\left(\frac{dt}{d\lambda}\right)^2 + (1+h^{\rm TT}_{xx})\left(\frac{dx}{d\lambda}\right)^2 = 0 \tag{6}$$

So we have:

$$dt = \sqrt{1+h^{\rm TT}_{xx}}dx \simeq \left(1+\frac{h^{\rm TT}_{xx}}{2}\right)dx \tag{7}$$

The total time is:

$$\begin{aligned} T_{A\to B\to A} &= \int_0^\epsilon \left(1+\frac{h^{\rm TT}_{xx}}{2}\right)dx + \int_\epsilon^0 \left(1+\frac{h^{\rm TT}_{xx}}{2}\right)(-dx) \\ &= 2\epsilon + \frac{1}{2}\int_0^\epsilon h^{\rm TT}_{xx}dx - \frac{1}{2}\int_\epsilon^0 h^{\rm TT}_{xx}dx \end{aligned} \tag{8}$$

We can see that time of arrival of pulses depends on $h^{\rm TT}_{xx}$.

Next let's consider geodesics deviation of freely falling worldlines of these bodies. Initially, both have

$$\vec{u} = (1,0,0,0) + \mathcal{O}(h) \tag{9}$$

The equation of geodesic deviation is:

$$\frac{D^2\xi^\alpha}{d\tau^2} = R^\alpha{}_{\mu\nu\beta}u^\mu u^\nu \xi^\beta \tag{10}$$

When linearized, $R \sim \mathcal{O}(h)$, so the equation becomes:

$$\frac{\partial^2\xi^i}{\partial t^2} = R^i{}_{00j}\xi^j + \mathcal{O}(h^2) \tag{11}$$

For our radiative perturbation, the only unique non-zero Riemann components are:

$$R_{x0x0} = R^x{}_{0x0} = -\frac{1}{2}\partial_t^2 h^{\rm TT}_{xx} \tag{12}$$

$$R_{y0y0} = R^y{}_{0y0} = -\frac{1}{2}\partial_t^2 h^{\rm TT}_{yy} = \frac{1}{2}\partial_t^2 h^{\rm TT}_{xx} \tag{13}$$

$$R_{y0x0} = R^y{}_{0x0} = -\frac{1}{2}\partial_t^2 h^{\rm TT}_{xy} \tag{14}$$

Insert them into Eq.(11), we will get:

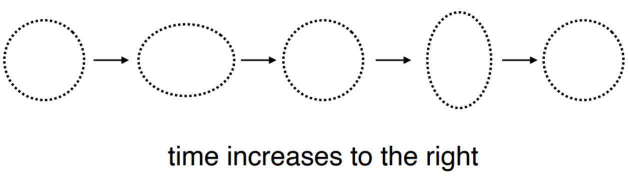
$$\partial_t^2\xi^x = \frac{1}{2}\partial_t^2 h^{\rm TT}_{xx}\xi^x + \frac{1}{2}\partial_t^2 h^{\rm TT}_{xy}\xi^y \tag{15}$$

$$\partial_t^2\xi^y = \frac{1}{2}\partial_t^2 h^{\rm TT}_{xy}\xi^x - \frac{1}{2}\partial_t^2 h^{\rm TT}_{xx}\xi^y \tag{16}$$

$$\partial_t^2\xi^z = 0 \tag{17}$$

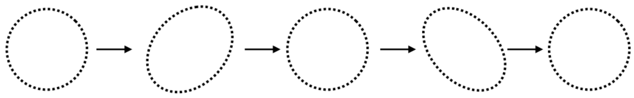
Assume that $h^{\rm TT}_{ij} \ll 1, \xi^i = \xi^i_0 + \delta\xi^i, \delta\xi^i \sim h$. Consider 2 limits:

- First, imagine that $h^{\rm TT}_{xx}$ is a sinusoid, and $h^{\rm TT}_{xy} = 0$. Imagine a ring of test masses in the (x, y) plane. As this gravitational wave passes by, the separation vector components $(\delta Y^x, \delta Y^y)$ oscillate with the following pattern:



We call this the "plus" polarization: it tidally stretches and squeezes along orthogonal axes that are aligned with the x and y axes of our coordinates. We denote this polarization h_+.

- Next, the opposite limit: h_{xy}^{TT} is a sinusoid, $h_{xx}^{\mathrm{TT}} = 0$. The separation vector now undergoes the following pattern as time goes on:



time increases to the right

  This pattern is the "cross" polarization, since it tidally stretches and squeezes along orthogonal axes aligned at 45° to our coordinates' x and y axes. We denote this polarization $h_\times$.

Recall that h_{ij}^{TT} has only two independent degrees of freedom: the 6 components of the symmetric tensor are constrained by 3 conditions to make it divergence free, plus 1 trace-free condition. The polarizations h_+ and $h_\times$ completely account for these two independent degrees of freedom.

Compute h_{ij}^{TT} given a source