

Homework1

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Exercise 1:

Open loop control

1.1. Find the equilibrium points corresponding to the constant input u^* . You don't need to find explicit expressions for the equilibrium points; it is enough that you find an expression that relates the equilibrium points and the corresponding constant input.

Solution:

By setting

$$\begin{cases} f_1 = x_2 = 0 \\ f_2 = -x_1 - 0.2x_2 + x_3^2 = 0 \\ f_3 = -1.6(1 - x_1)x_3 + u^* = 0 \end{cases}$$

we can get

$$\begin{cases} x_2 = 0 \\ x_1 = x_3^2 \\ x_3 - x_3^3 = \frac{u^*}{2.6} \end{cases}$$

1.2. Plot u^* as a function of x_1^* . Remember that $0 \leq x_1 \leq 1$. Consider $u^* = 0.8$. What equilibrium points x_1^* does that correspond to? Are they stable or unstable?

Solution: Plot u^* as a function of x_1^*

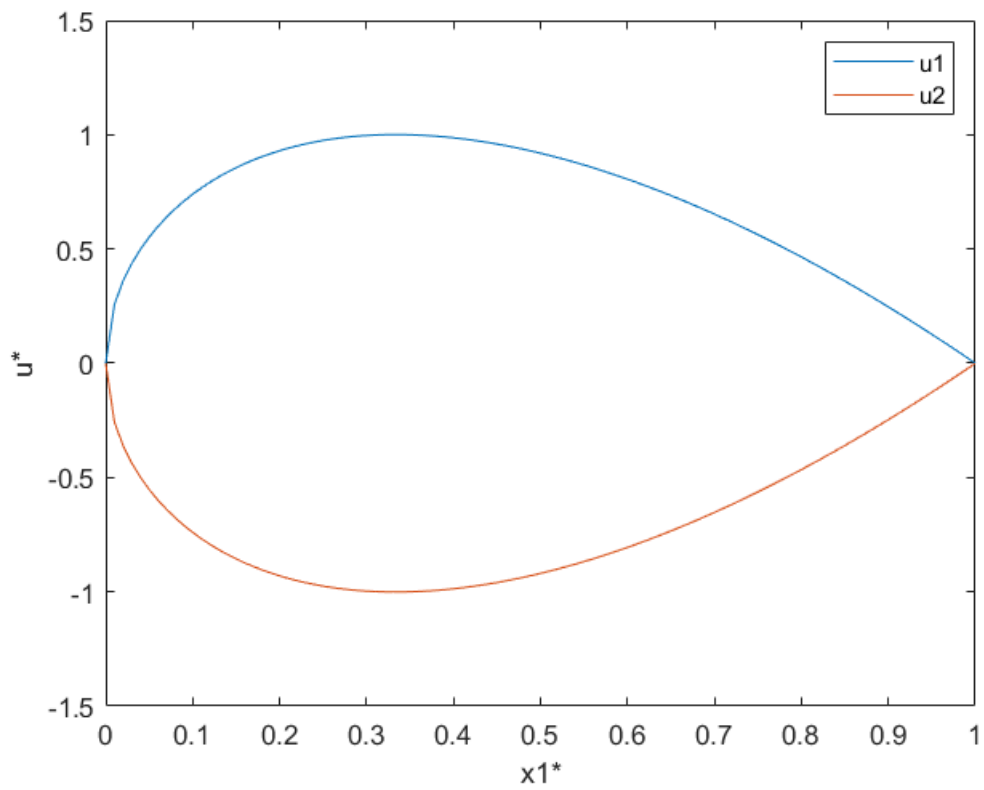
when $x_3 > 0$

$$u^* = 2.6 \cdot \sqrt{x_1} (1 - x_1)$$

else if $x_3 \leq 0$

$$u^* = 2.6 \cdot \sqrt{x_1} (1 - x_1)$$

which is shown as the following figure.



when $u^*=0.8$

$$x_3^*(1 - x_3^{*2}) = \frac{0.8}{2.6}$$

$$x_3^*(1 - x_1^*) = \frac{0.8}{2.6}$$

$$x_1^* = 1 - \frac{0.8}{2.6x_3^*}$$

so x_1^* correspond to the value of x_3^* .

when $u^*=0.8$

$$x_3^{*3} - x_3^* + \frac{0.8}{2.6} = 0$$

$$\begin{aligned} x_3 &= 3 \times 1 \\ &-1.1282 \\ &0.7773 \\ &0.3509 \end{aligned}$$

$$x_1^* = x_3^{*2}$$

so x_1^* does correspond to the value of x_3^* .

$$x_1 = 3 \times 1$$

1.2727
0.6041
0.1231

Because x_1 belong to $[0,1]$, so

$x_1 = 0.6041$ and 0.1231

Stable Analysis: Jacobian matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -0.2 & 2 * x_3 \\ 2.6 * x_3 & 0 & -2.6 + 2.6 * x_1 \end{bmatrix}$$

The Jacobian matrix are M1 and M2 separately.

M1 = 3x3

0	1.0000	0
-1.0000	-0.2000	1.5546
2.0210	0	-1.0293

M2 = 3x3

0	1.0000	0
-1.0000	-0.2000	0.7018
0.9123	0	-2.2799

The equilibrium point(0.6041, 0, 0.7773) and (0.1231, 0, 0.3509)

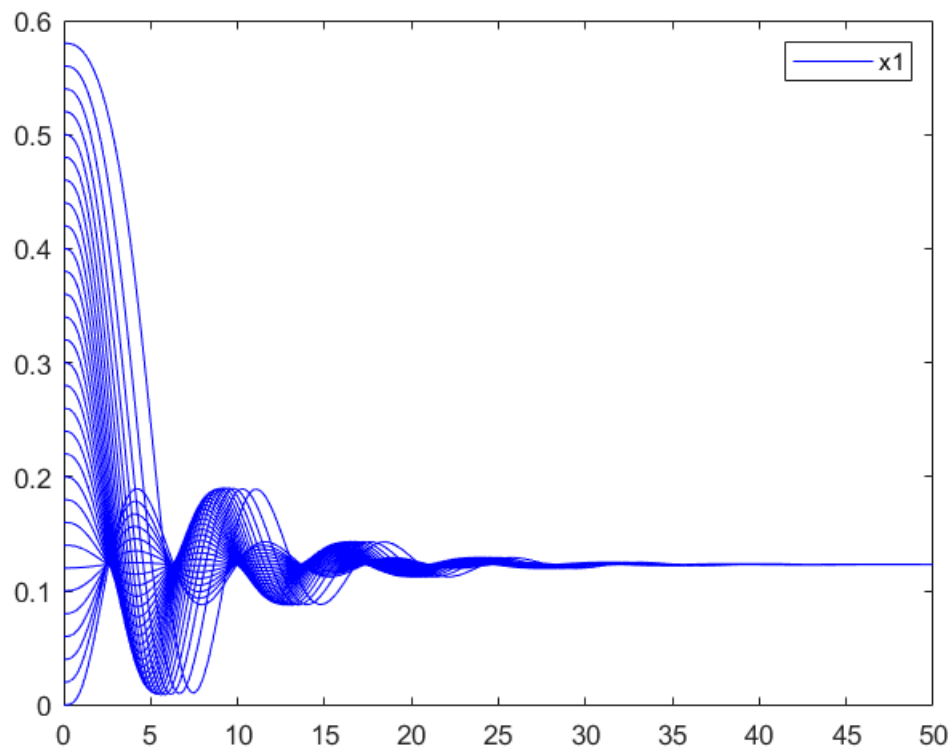
The first eigenvalue are $[-0.9995 + 1.3212i, -0.9995 - 1.3212i, 0.7697]$, because $0.7697 > 0$, so this equilibrium is unstable.

The second eigenvalue are $[-2.1573, -0.1613 + 0.8567i, -0.1613 - 0.8567i]$, and all the $\text{Re}[\lambda] < 0$, so this equilibrium is stable.

1.3. Simulate the open loop system (with constant input $u^* = 0.8$) for different initial states $x(0)$. We can limit the investigation on to initial states of the form $x(0) = [x_1(0), 0, \sqrt{x_1(0)}]$ and vary $x_1(0)$ between 0 and 1. For any stable equilibrium, what is the region of attraction (in terms of $x_1(0)$)?

Solution:

when vary the value of x_0 between $(-1, 0)$, we can get the following figure:



When $u^*=0.8$, the equilibrium is shown as the above figure, the state is stabilized at the equilibrium point.

when the $x_1(0)$ bigger than the 0.59, the matlab could not solve the equation, because the equation do not converge, so the region of attraction is $[0 \ 0.59]$

State Feedback Control

The controller of the system, to control $x^*=0.6$

The eigenvalue of the transformed system

Linear Control

2.1. Linearize the system around the origin (in the transformed variables y and v). A complete answer shall provide the linearized form of the system written as variables, and numerical values of the linearized system.

Solution: The linearized system can be written as:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -b_2 & b_3 + 2y_3 \\ c_1 + c_{13}y_3 & 0 & c_3 + c_{13}y_1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\dot{y} = Ay + Bv$$

Then we can get the system written as variables:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -b & 2x_3^* + 2y_3 \\ cx_3^* + cy_3 & 0 & c(x_1^* - 1) + cy_1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\dot{y} = Ay + Bv$$

Given $x_1^* = 0.6$

$$x_3^* = \pm \sqrt{0.6}$$

Then plug in the numerical values:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2 & 2\sqrt{0.6} + 2y_3 \\ 2.6\sqrt{0.6} + 2.6y_3 & 0 & -1.04 + 2.6y_1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

or

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2 & -2\sqrt{0.6} + 2y_3 \\ -2.6\sqrt{0.6} + 2.6y_3 & 0 & -1.04 + 2.6y_1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\dot{y} = Ay + Bv$$

Around the origin they are:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2 & 2\sqrt{0.6} \\ 2.6\sqrt{0.6} & 0 & -1.04 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

or

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2 & -2\sqrt{0.6} \\ -2.6\sqrt{0.6} & 0 & -1.04 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Rank(A,B)=3, so it is controllable.

or

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -0.2 & -2\sqrt{0.6} \\ -2.6\sqrt{0.6} & 0 & -1.04 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\text{Rank}(A, B)=3$, so it is controllable.

For $x_3^* = \sqrt{0.6}$:

2.2. Design a controller using methods from linear control, for example pole placement (`place(A,B,P)`) or LQR control (`lqr(A,B,Q,R)`). Choose/tune the controller parameters so that $|u(t)| \leq 2$ when $x(0) = 0$. A complete solution shall include the key steps of derivation of the controller and the numerical values of the controller.

Solution: We list the functions f1, f2 and f3, and calculate the Jacobian matrix A. Then, using function `place(A,B,P)`, the controller gain K_v is calculated.

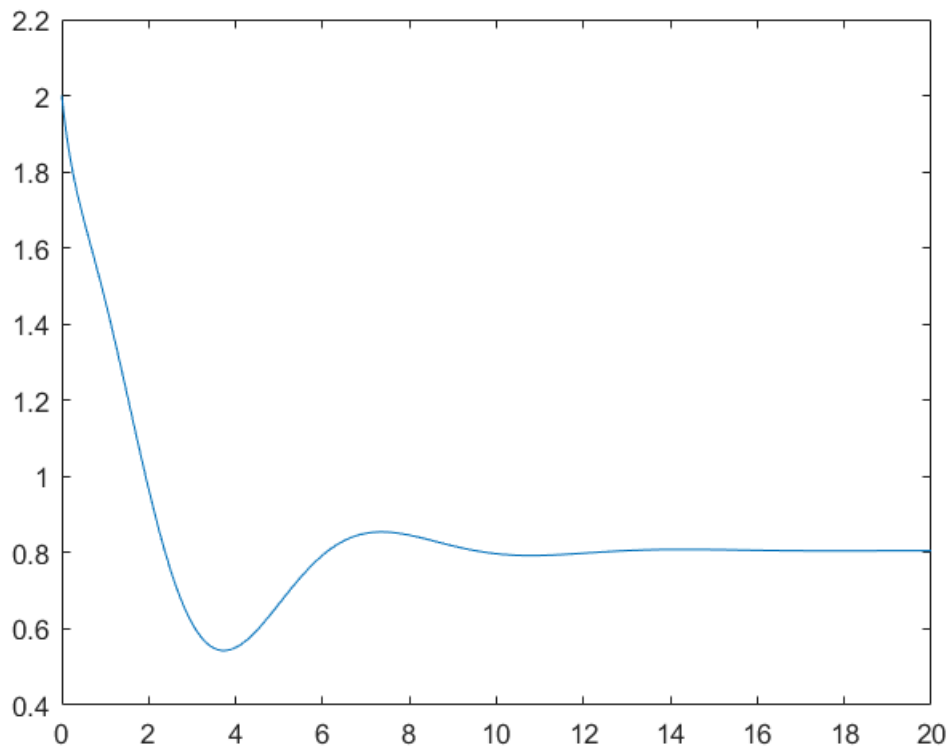
The poles we used are $[-1.3; -0.4-0.9i; -0.4+0.9i]$.

$$K_v = 1 \times 3$$

1.6015	0.4067	0.8600
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Then we can design the controller in the $u = -K_v x$ form.

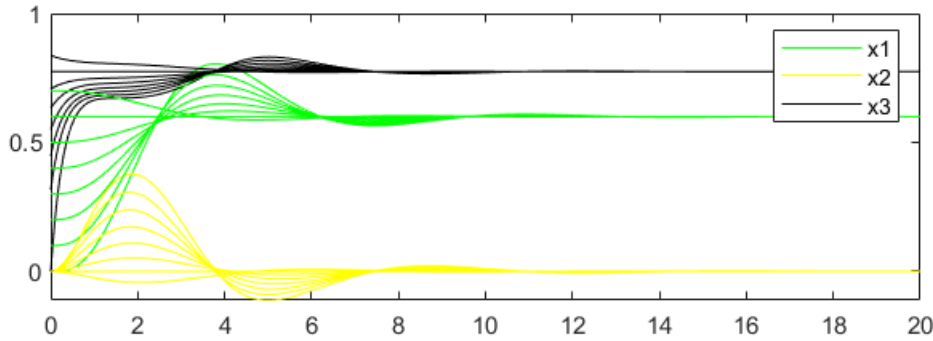
for $x(0) = (0.1, 0, \sqrt{0.1})$, plot the control input u as below:



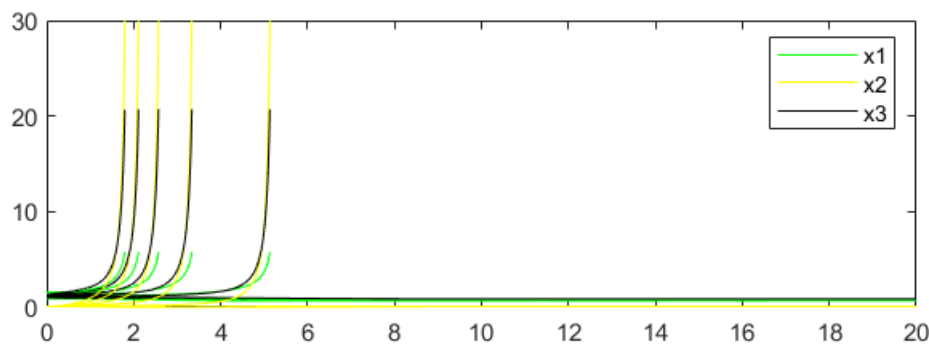
It can be shown that $|u(t)| \leq 2$ when $x(0) = [0.1, 0, \sqrt{0.1}]$, so the controller satisfies the requirement.

2.3. Simulate the system under feedback control. Vary the initial condition as in Exercise 1.3. Remember to plot the original state $x_1(t)$ and not $y_1(t)$. Comment on the performance and the region of attraction.

Solution: for $x(0) = (x_1, 0, \sqrt{x_1})$, vary x_1 from 0 to 0.7, with a step length 0.1, simulate the process and plot x_1 , x_2 and x_3 with respect to time as below:



vary x_1 from 0.8 to 1.5, with a step length 0.1, simulate the process and plot x_1 , x_2 and x_3 with respect to time as below:



For $x_3^* = -\sqrt{0.6}$:

2.2. Design a controller using methods from linear control, for example pole placement (`place(A,B,P)`) or LQR control (`lqr(A,B,Q,R)`). Choose/tune the controller parameters so that $|u(t)| \leq 2$ when $x(0) = 0$. A complete solution shall include the key steps of derivation of the controller and the numerical values of the controller.

Solution: We list the functions f_1 , f_2 and f_3 , and calculate the Jacobian matrix A . Then, using function `place(A,B,P)`, the controller gain K_v is calculated.

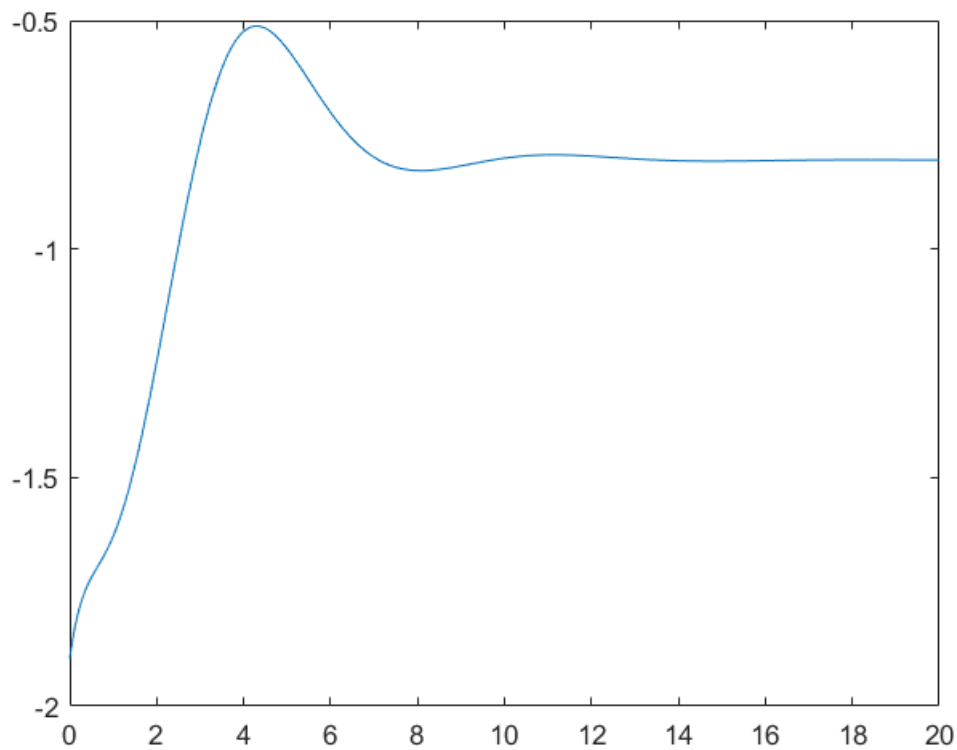
The poles we used are $[-0.7; -0.4-0.9i; -0.4+0.9i]$.

$$u_0 = -0.8056$$

$$K_v = \begin{matrix} 1 \times 3 \\ -1.6131 & -0.1743 & 0.2600 \end{matrix}$$

Then we can design the controller in the $u = -K_v x$ form.

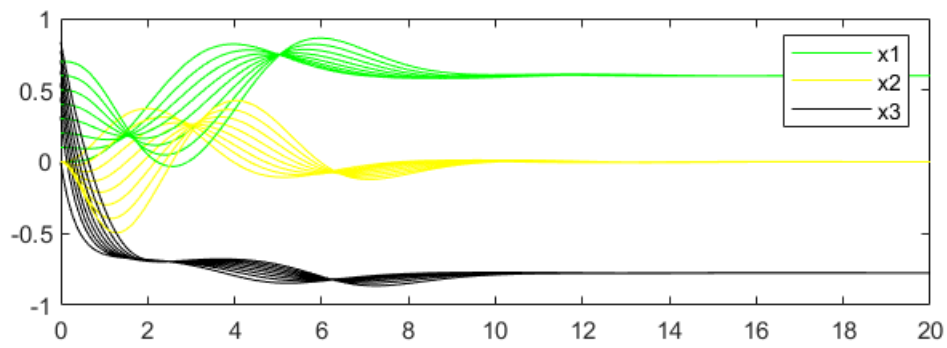
for $x(0) = (0.1, 0, \sqrt{0.1})$, plot the control input u as below:



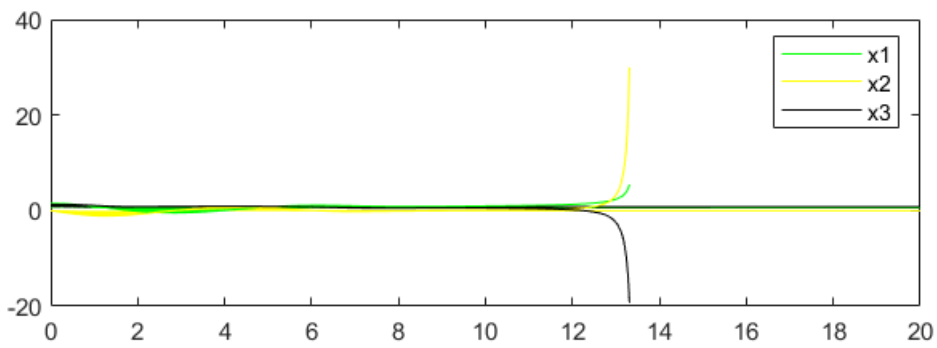
It can be shown that $\|()\|_{\leq 2}$ when $x(0) = [0.1, 0, \sqrt{0.1}]$, so the controller satisfies the requirement.

2.3. Simulate the system under feedback control. Vary the initial condition as in Exercise 1.3. Remember to plot the original state $x_1(t)$ and not $y_1(t)$. Comment on the performance and the region of attraction.

Solution: for $x(0) = (x_1, 0, \sqrt{x_1})$, vary x_1 from 0 to 0.7, with a step length 0.1, simulate the process and plot x_1 , x_2 and x_3 with respect to time as below:



vary x_1 from 0.8 to 1.5, with a step length 0.1, simulate the process and plot x_1 , x_2 and x_3 with respect to time as below:



During the tuning, we find that if the poles are further from the imaginary axis, the region of attraction will be larger, but the initial $\text{abs}(u)$ will become larger, causing more oscillation.

Exercise 3:

State Feedback Control

3.1. Verify that the system above is on controller form. What is A , B , $\psi(y)$ and $\gamma(y)$? Is (A, B) controllable? On what domain is T invertible?

Solution:

The linearized state equation can be transformed into:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b_2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [(b_3 + 2y_3) \cdot (c_1 \cdot y_1 + c_3 \cdot y_3 + c_{13}y_1y_3) + (b_3 + 2y_3)v]$$

In this equation, there exists:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b_2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\psi(y) = (b_3 + 2y_3) \cdot (c_1 \cdot y_1 + c_3 \cdot y_3 + c_{13}y_1y_3)$$

$$\gamma(y) = b_3 + 2y_3$$

By calculating the controllability matrix ($R = [B \quad AB \quad A^2B]$), we find that the controllability matrix is full rank. So (A, B) is controllable.

$$R = [B, A \cdot B, A^2 \cdot B]$$

$$\text{rank}(R) = 3$$

$$B = 3 \times 1$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

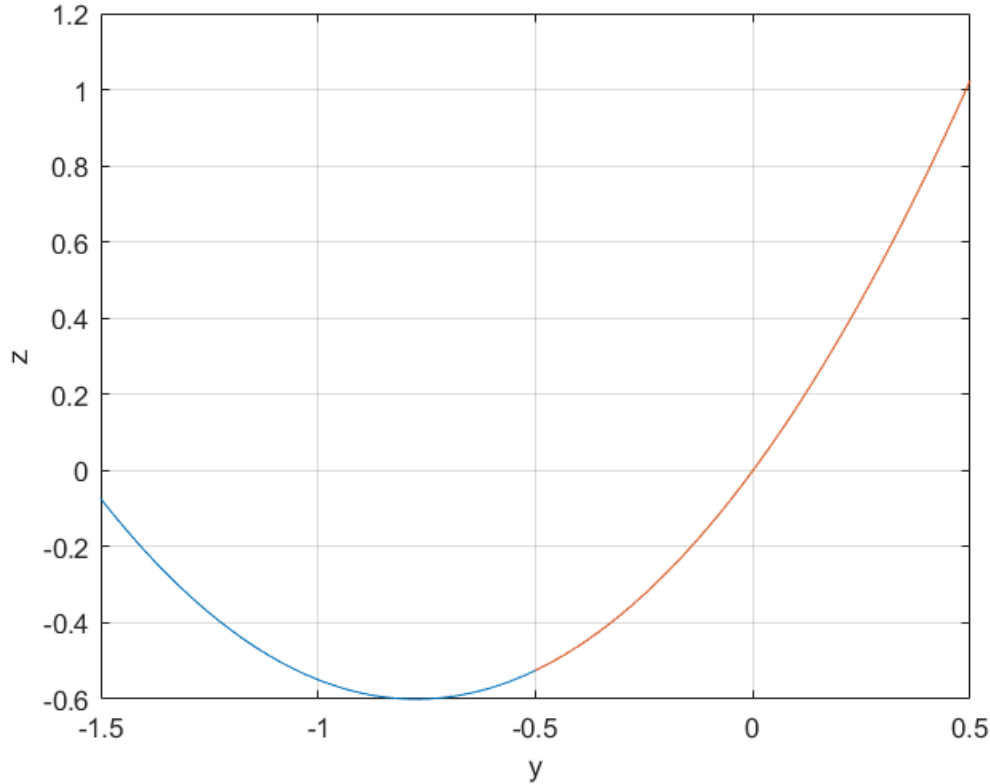
$$R = 3 \times 3$$

$$\begin{bmatrix} 0 & 0 & 1.0000 \\ 0 & 1.0000 & -0.2000 \\ 1.0000 & 0 & 0 \end{bmatrix}$$

$$\text{ans} = 3$$

For the third equation, plot the relation between y and z:

$$x_3 = 0.7746$$



So T is not globally invertible. But if $y_3 \geq -x_3^*$ or $y_3 \leq -x_3^*$ ($x_3 \geq 0$ or $x_3 \leq 0$) holds for the whole system running, we can regard the transformation T as invertible.

3.2. Derive the feedback linearizing controller. For the linear part, follow the same procedure as you did in Exercise 2.2. Explain the main steps and write out the resulting controller.

Solution: The process of deriving the linearizing controller:

In the first step, we can get the main form of the equation.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b_2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [(b_3 + 2y_3) \cdot (c_1 \cdot y_1 + c_3 \cdot y_3 + c_{13}y_1y_3) + (b_3 + 2y_3)u]$$

And then we chose

$$u = \frac{1}{(b_3 + 2y_3)} \cdot (-(b_3 + 2y_3) \cdot (c_1 \cdot y_1 + c_3 \cdot y_3 + c_{13}y_1y_3) + v)$$

Then the system is transformed into

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b_2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot v$$

Because (A, B) is controllable, so we can chose

$$v = -k \cdot z$$

So the final form can be expressed as

$$u = \frac{1}{(b_3 + 2y_3)} \cdot (-(b_3 + 2y_3) \cdot (c_1 \cdot y_1 + c_3 \cdot y_3 + c_{13}y_1y_3) - KT(y))$$

where

$$z = T(y) = \begin{bmatrix} y_1 \\ y_2 \\ b_3y_3 + y_3^2 \end{bmatrix}$$

3.3 Simulate the system from different initial conditions like in Exercise 2.3.

Solution:

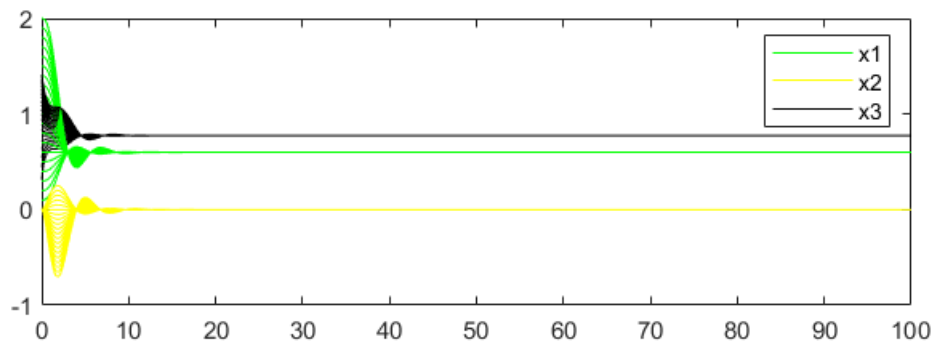
change the initial condition from 0-1 of x1, and the initial state is defined as

$$x(0) = [x_1(0) \quad 0 \quad \sqrt{x_1(0)}]$$

The simulation results is shown as:

$$1. \text{ when } x_3^* = \sqrt{0.6}$$

From this figure we can see that the state will converge when the initial value change between [0 2], so we can prove that it is a stable system.



2. when $x_3^* = -\sqrt{0.6}$

From this figure we can see that the state will converge when the initial value change between $[0 \ 2]$, so we can prove that it is a stable system.

