Definition

- **Arbitrary** is a number which could be any number it is defined to be but for which no specific value is chosen.
- An **even number** is an integer of the form
$$x = 2k \quad \text{where } k \text{ is an integer.}$$
- An **odd number** is an integer of the form
$$x = 2k + 1 \quad \text{where } k \text{ is an integer.}$$
- A function F is **odd** if $F(-x) = -F(x)$.
- A function F is **even** if $F(-x) = F(x)$.
- $y(x)$ means y is a function with variable x.
$$y = x + 1$$
- Factorial
$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$
- Combinations
$${}^{n}C_{r} = C(n,r) = \frac{n!}{(n-r)!r!}$$
- Permutation
$${}^{n}P_{r} = P(n,r) = \frac{n!}{(n-r)!}$$
- Natural number
$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n} = e$$

Derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}f(ax+b) = af'(ax+b)$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}cf(x) = cf'(x) \text{ (c constant)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x))g'(x) \text{ (Chain rule)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] = f'(x)+g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)-g(x)] = f'(x)-g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x)+f(x)g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}c = 0 \text{ (c constant)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^a = ax^{a-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a \quad (a>0, a\neq 1)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(|x|) = \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{\log_a e}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccot}(x) = -\frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh(x) = \cosh(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \sinh(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh(x) = \operatorname{sech}^2(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \sec(x)\tan(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = -\csc^2(x)$$

Integral

$$\int f'(ax+b)dx = \frac{f(ax+b)}{a} + c$$

$$\int cf(x)\ dx = c\int f(x)\ dx\ (c \text{ constant})$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int [f(x)+g(x)]\ dx = \int f(x)\ dx + \int g(x)\ dx$$

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx \text{ (by parts)}$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\int_a^b f(\varphi(x))\varphi'(x)dx = \int_{\varphi(a)}^{\varphi(b)} f(u)du \text{ (Substitution)}$$

$$\int 0\ dx = c\ (c \text{ constant})$$

$$\int x^n\ dx = \frac{1}{n+1}x^{n+1} + c\ (n \neq -1)$$

$$\int \frac{1}{x}\ dx = \ln|x|$$

$$\int e^x\ dx = \frac{1}{a}e^x + c$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + c$$

$$\int \sin(x)\ dx = -\cos(x) + c$$

$$\int \cos(x)\ dx = \sin(x) + c$$

$$\int \tan(x)\ dx = -\ln|\cos(x)| + c$$

$$\int \sin^2(x)\ dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

$$\int \cos^2(x)\ dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

$$\int \cot(x)\csc(x)\ dx = -\csc(x) + c$$

$$\int \tan(x)\sec(x)\ dx = \sec(x) + c$$

$$\int \csc^2(x)\mathrm{d}x = -\cot(x) + c$$

$$\int \sec^2(x)\mathrm{d}x = \tan(x) + c$$

$$\int \cot^2(x)\ dx = -\cot x - x + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \arcsin\frac{x}{a} + c$$

$$\int -\frac{1}{\sqrt{a^2-x^2}}\ dx = \arccos\frac{x}{a} + c$$

$$\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \tan^2(x)\ dx = \tan x - x + c$$

$$\int \ln(x)\ dx = x\ln x - x + c$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\ dx = \operatorname{arcsinh}\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\ dx = \operatorname{arccosh}\frac{x}{a} + c$$

$$\int \sec(x)\ dx = \ln|\sec(x)+\tan(x)| + c$$

$$\int \csc(x)\ dx = \ln|\csc(x)-\cot(x)| + c$$

$$\int \cot(x)\ dx = \ln|\sin(x)| + c$$

Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$$\iint r\ dr\ d\theta = \iint dx\ dy$$

Differential Equations

Second Order Homogeneous

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \text{ where } a,\ b \text{ and} c \text{ are real constants}$$

The **general solution**, y, of the differential equation will depend on the nature of the roots, α and β, of the **auxiliary equation**.

$$am^2 + bm + c = 0$$

If $\alpha,\ \beta$ are real and distinct then $y = Ae^{\alpha x} + Be^{\beta x}$

If $\alpha,\ \beta$ are real and repeated then $y = e^{\alpha x}(Ax + B)$

If $\alpha,\ \beta$ are imaginary $p + iq$ then $y = e^{px}(A\cos qx + B\sin qx)$

Second Order Non-Homogeneous Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \text{ where } f(x) \neq 0$$

1. Find the **homogeneous solution**, y_h, of
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
2. Find the **particular solution**, y_p, of
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$
3. The **general solution**, y, of the differential equation is $y = y_h + y_p$.

If $f(x) = de^{kx}$ try $y_p = De^{kx}$

If $f(x) = d\cos qx + e\sin qx$ try $y_p = D\cos qx + E\sin qx$

If $f(x) = dx^2 + ex + f$ try $y_p = Dx^2 + Ex + F$

$f(x)$ lower case constants given, and y_p (upper case constants to be found).

Series

Taylor Series at a point x_0

$$f(x-x_0)=f(x-x_0)+f'(x-x_0)x+\frac{f''(x-x_0)}{2!}x^2+\frac{f'''(x-x_0)}{3!}x^3+\cdots+\frac{f^{(n)}(x-x_0)}{n!}x^n+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(x-x_0)}{k!}x^k$$

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\ldots$$

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^nx^{n+1}}{n+1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots$$

Arithmetic Series

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ a+(n-1)d$$

$a_n=a+(n-1)d$

$S_n=\frac{n}{2}(a+a_n)=\frac{n}{2}[2a+(n-1)d]$

Geometric Series

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1}$$

$a_n=ar^{n-1}$

$S_n=\frac{a\left(1-r^n\right)}{(1-r)}$, when $r\neq 1$

$S_\infty=\frac{1}{1-r}$, when $|r|<1$. Note that $\frac{1}{1-r}=\sum_{k=0}^{\infty}r^k$.

Binomial Series

$$(a+b)^n=\binom{n}{0}a^nb^0+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{n}a^0b^n$$

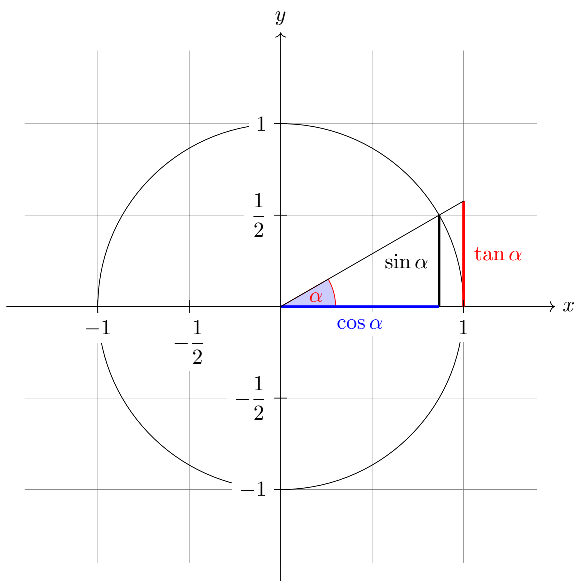
Trigonometry

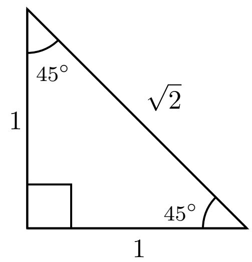
$$2\pi = 360^\circ$$

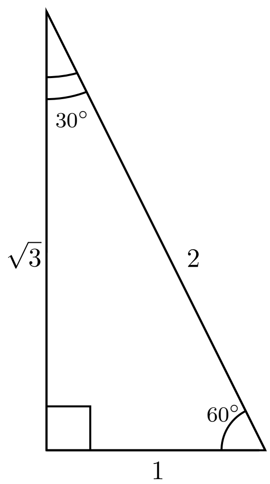
$\frac{\pi}{6} = 30^\circ$	$\frac{\pi}{2} = 90^\circ$	$\frac{5\pi}{6} = 150^\circ$
$\frac{\pi}{4} = 45^\circ$	$\frac{2\pi}{3} = 120^\circ$	$\pi = 180^\circ$
$\frac{\pi}{3} = 60^\circ$	$\frac{3\pi}{4} = 135^\circ$	$\frac{3\pi}{2} = 270^\circ$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$	$\tan\left(\frac{\pi}{4}\right) = 1$	$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$	$\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$	$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$
$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$	$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$	$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
$\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$	$\tan\left(\frac{3\pi}{4}\right) = -1$	$\tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$







Trigonometry Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$\csc\theta = \dfrac{1}{\sin\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$
$\sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta)$	$\cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta)$	$\cot\left(\frac{\pi}{2}-\theta\right) = \tan(\theta)$
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$	$\cot(-\theta) = -\cot$
$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$	$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$	$\tan(A\pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\sin(\pi+\theta) = -\sin\theta$	$\cos(\pi+\theta) = -\cos\theta$	$\tan(\pi+\theta) = \tan\theta$
$\sin(\pi-\theta) = \sin\theta$	$\cos(\pi-\theta) = -\cos\theta$	$\tan(\pi-\theta) = -\tan\theta$
$\sec(\pi+\theta) = -\sec\theta$	$\csc(\pi+\theta) = -\csc\theta$	$\cot(\pi+\theta) = \cot\theta$
$\sec(\pi-\theta) = -\sec\theta$	$\csc(\pi-\theta) = \csc\theta$	$\cot(\pi-\theta) = -\cot\theta$
$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$

Hyperbolic

$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

$\cosh^2 x - \sinh^2 x = 1$ $\tanh^2 x + \operatorname{sech}^2 x = 1$ $\coth^2 x = 1 + \operatorname{csch}^2 x$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$ $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$\cosh 2x = \cosh^2 x + \sinh^2 x$ $\sinh 2x = 2 \sinh x \cosh x$

Its the same as the trigonometric but if appears $\sin^2(x) \to -\sinh^2(x)$

Algebra

Exponential

$a^x \times a^y = a^{x+y}$

$\frac{a^x}{a^y} = a^{x-y}$

$(ab)^x = a^x b^x$

$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

$a^{\frac{x}{y}} = \sqrt[y]{a^x}$

$(a^x)^y = a^{xy}$

$a^0 = 1$

$a^{-1} = \frac{1}{a}$

Logarithms

$$y = b^x \implies x = \log_b y \quad \text{with} \quad \log_b(b^k) = k$$

$\log(xy) = \log(x) + \log(y)$

$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

$\log_a(b) = \frac{\log_c b}{\log_c a}$

$\log_a b = \frac{1}{\log_b a}$

$\log(x^n) = n\log(x)$

$b^{\log_b(k)} = k$

$\log_b(1) = 0$

$\log_b(b) = 1$

Absolute Value

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases} \quad \text{and} \quad |x| < a \implies -a < x < a$$

$|a \times b| = |a| \times |b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$|a|^2 = a^2$

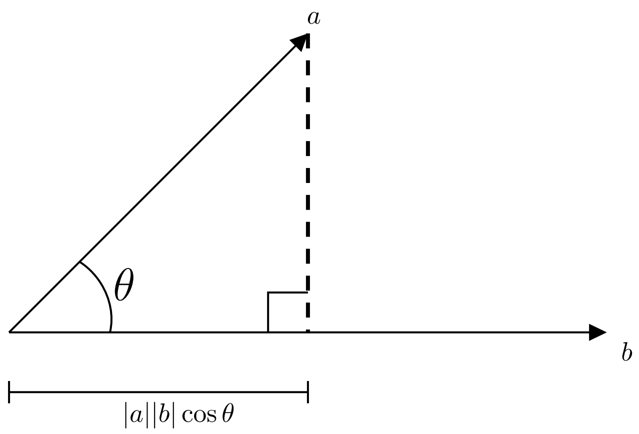
$|b - a| = |a - b|$

$|a + b| \leq |a| + |b|$

Dot Product

The dot product of two vectors $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]$ is defined as:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$$



$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

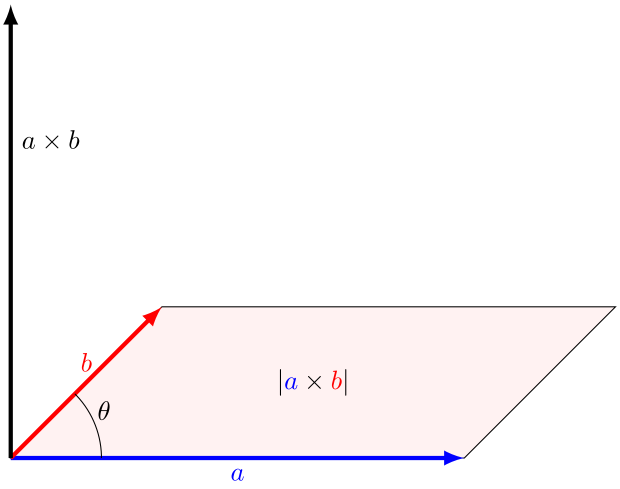
Cross Product

The cross product of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is defined by the formula

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\mathbf{a}|\,|\mathbf{b}|\sin(\theta)\mathbf{n}$$

where $\mathbf{n}$ is a unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$, in the direction given by the right-hand rule.

- If the angle made from $\mathbf{a}$ to $\mathbf{b}$ is **clockwise** then the direction of the normal is **negative (downward)**.
- If the angle made from $\mathbf{a}$ to $\mathbf{b}$ is **anticlockwise** then the direction of the normal is **positive (upward)**.



$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$,

Geometry Formulas

Perimeter

Square	$P = 4s$
Rectangle	$P = 2(l + w)$

Circumference

Circle	$C = 2\pi r$

Area

Square	$A = s^2$
Rectangle	$A = lw$
Triangle	$A = bh/2$
Trapezoid	$A = (b_1 + b_2)h/2$
Circle	$A = \pi r^2$

Surface Area

Cube	$SA = 6s^2$
Cylinder	$SA = 2\pi rh + 2\pi r^2$
Cone	$SA = \pi rl$
Sphere	$SA = 4\pi r^2$

Volume

Cube	$V = s^3$
Cylinder	$V = \pi r^2 h$
Cone	$V = \pi r^2 h/3$
Sphere	$V = 4\pi r^3/3$
Pyramid	$V = lwh/3$

The volume of a cone is

$$\frac{1}{3}bh$$

where b is the area of the base and h the height from the base to the apex