Homework week 8

Problem 1 Recall the ISM loss

$$L_{ISM}(\theta) = \mathbb{E}_{p(x)}\left[\|S(x;\theta)\|^2 + 2\mathrm{div}S(x;\theta)\right] = \mathbb{E}_{p(x)}\left[\|S(x;\theta)\|^2 + 2\mathrm{trace}(\nabla S(x;\theta))\right]. \tag{0.1}$$

By using Hutchinson's trace estimate, we find

$$\begin{aligned} L_{ISM}(\theta) &= \mathbb{E}_{p(x)}\left[\|S(x;\theta)\|^2 + 2\mathrm{trace}(\nabla S(x;\theta))\right] \\ &= \mathbb{E}_{p(x)}\left[\|S(x;\theta)\|^2\right] + 2\mathbb{E}_{p(x)}\mathbb{E}_{p(v)}\left[v^T(\nabla S)v\right], \end{aligned} \tag{0.2}$$

where v is a random vector satisfies $\mathbb{E}\left[v^T v\right] = \mathrm{I}$.

Problem 2 Briefly explain *stochastic differential equation* (SDE)

To introduce SDEs, we first recall several definitions that clarify the concept.

Definition 0.1. *Let $(B_t, t \geq 0)$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. An Itô process $(X_t, t \geq 0)$ is a process of the form*

$$X_t = X_0 + \int_0^t V_s \,\mathrm{d}B_s + \int_0^t D_s \,\mathrm{d}s. \tag{0.3}$$

For convenience, we just write its as a differential form

$$\mathrm{d}X_t = V_s \mathrm{d}B_t + D_t \mathrm{d}t.$$

A natural question is how to interpret the stochastic integral $\int_0^t V_s, \mathrm{d}B_s$ with respect to Brownian motion. One may compare it with the Riemann–Stieltjes integral; however, Brownian paths have unbounded variation, so the classical Riemann–Stieltjes theory does not apply. We omit the technical details here.

We are now ready to formulate SDEs.

Definition 0.2. *Let $(B_t, t \geq 0)$ be a standard Brownian motion. An Itô process $(X_t, t \geq 0)$ of the form*

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + \mu(X_t)\mathrm{d}t, \quad X_0 = x, \tag{0.4}$$

where σ and μ are functions form $\mathbb{R}$ to $\mathbb{R}$, is called a time homogenous diffusion. An Itô process $(Y_t, t \geq 0)$ of the form

$$\mathrm{d}Y_t = \sigma(t, Y_t)\mathrm{d}B_t + \mu(t, Y_t)\mathrm{d}t, \quad Y_0 = y, \tag{0.5}$$

where σ and μ are functions form $[0, \infty) \times \mathbb{R}$ to $\mathbb{R}$, is called a time inhomogenous diffusion.

Equations (0.4) and (0.5) are the stochastic differential equations satisfied by the processes X_t and Y_t, respectively.

Remark 0.1. *SDEs generalize ordinary differential equations (ODEs). Indeed, in the absence of randomness—i.e., without the Brownian term—(0.4) reduces to*

$$\mathrm{d}X_t = \mu(X_t), \mathrm{d}t,$$

which, for $X_t = f(t)$*, can be written as* $\frac{\mathrm{d}f}{\mathrm{d}t} = \mu(f)$*.*

The SDEs provide a framework for modeling random phenomena that evolve over time—for example, asset returns in a portfolio or the dynamics of bond yields. The point is not that the physical quantity itself is Brownian motion, but rather that it is modeled as being driven by Brownian noise.

Problem 3

Question Given a SDEs (Ornstein-Uhlenbeck process)

$$\mathrm{d}Y_t = -kY_t\mathrm{d}t + \sigma\mathrm{d}B_t, \quad Y_0 = y,\, k \in \mathbb{R},\, \sigma > 0. \tag{0.6}$$

How to solve this SDE?