1. (0.5%) Page 167, Problem 3.2

   Determine the z-transform of the sequence

$$x[n] = \begin{cases} n, & 0 \le n \le N-1, \\ N, & N \le n. \end{cases}$$

   Sol:

$$\begin{aligned} x[n] &= \begin{cases} n, & 0 \le n \le N-1, \\ N, & N \le n. \end{cases} = nu[n] - (n-N)u[n-N] \\ nx[n] &\Leftrightarrow -z\frac{d}{dz}X(z) \Rightarrow nu[n] \Leftrightarrow -z\frac{d}{dz}\frac{1}{1-z^{-1}} \quad |z|>1 \\ nu[n] &\Leftrightarrow \frac{z^{-1}}{(1-z^{-1})^2} \quad |z|>1 \\ x[n-n_0] &\Leftrightarrow X(z)\cdot z^{-n_0} \Rightarrow (n-N)u[n-N] \Leftrightarrow \frac{z^{-N-1}}{(1-z^{-1})^2} \quad |z|>1 \end{aligned}$$

   Therefore,

$$X(z) = \frac{z^{-1} - z^{-N-1}}{(1-z^{-1})^2} = \frac{z^{-1}\left(1-z^{-N}\right)}{(1-z^{-1})^2}$$

2. (0.5%) Page 168, Problem 3.7

   The input to a causal LTI system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n].$$

   The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}.$$

   (a) Determine $H(z)$, the z-transform of the system impulse response. Be sure to specify the ROC.

   (b) What is the ROC for $Y(z)$?

   (c) Determine $y[n]$.

   Sol:

   (a)

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$$

$$\Rightarrow X(z) = \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \quad \frac{1}{2} < |z| < 1$$

Now to find $H(z)$ we simply use $H(z) = Y(z)/X(z)$; i.e.,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})} \cdot \frac{(1-z^{-1})\left(1-\frac{1}{2}z^{-1}\right)}{-\frac{1}{2}z^{-1}} = \frac{(1-z^{-1})}{(1+z^{-1})}$$

$H(z)$ causal $\Rightarrow$ ROC $|z| < 1$.

(b) Since one of the poles of $X(z)$, which limited the ROC of $X(z)$ to be less than 1, is cancelled by the zero of $H(z)$, the ROC of $Y(z)$ is the region in the z-plane that satisfies the remaining two constraints $|z| > \frac{1}{2}$ and $|z| > 1$. Hence $Y(z)$ converges on $|z| > 1$.

(c)

$$Y(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1+z^{-1}} \qquad |z| > 1$$

Therefore,

$$y[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}(-1)^n u[n]$$

3. (0.5%) Page 170, Problem 3.16

When the input to an LTI system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + (2)^n u[-n-1],$$

the corresponding output is

$$y[n] = 5\left(\frac{1}{3}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the pole(s) and zero(s) of $H(z)$ and indicate the ROC.

(b) Find the impulse response $h[n]$ of the system.

(c) Write a difference equation that is satisfied by the given input and output.

(d) Is the system stable? Is it causal?

Sol:

(a) To determine $H(z)$, we first find $X(z)$ and $Y(z)$:

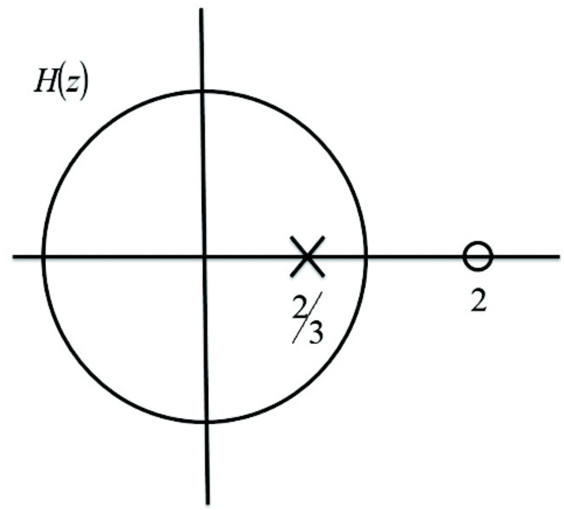
$$\begin{aligned} X(z) &= \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-2z^{-1}} \\ &= \frac{-\frac{5}{3}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)(1-2z^{-1})}, \quad \frac{1}{3} < |z| < 2 \end{aligned}$$

$$\begin{aligned} Y(z) &= \frac{5}{1-\frac{1}{3}z^{-1}} - \frac{5}{1-\frac{2}{3}z^{-1}} \\ &= \frac{-\frac{5}{3}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)}, \quad |z| > \frac{2}{3} \end{aligned}$$

Now

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}} \quad |z| > \frac{2}{3} \end{aligned}$$

The pole-zero plot of $H(z)$ is plotted below.



(b) Taking the inverse z-transform of $H(z)$, we get

$$\begin{aligned} h[n] &= \left(\frac{2}{3}\right)^n u[n] - 2\left(\frac{2}{3}\right)^{n-1} u[n-1] \\ &= \left(\frac{2}{3}\right)^n (u[n] - 3u[n-1]) \end{aligned}$$

(c) Since

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}},$$

we can write

$$Y(z)\left(1-\frac{2}{3}z^{-1}\right) = X(z)\left(1-2z^{-1}\right),$$

whose inverse z-transform leads to

$$y[n] - \frac{2}{3}y[n-1] = x[n] - 2x[n-1]$$

(d) The system is stable becase the ROC includes the unit circle. It is also causal since the impulse response $h[n] = 0$ for $n < 0$.

4. (0.5%) Page 172, Problem 3.23

An LTI system is characterized by the system function

$$H(z) = \frac{\left(1-\frac{1}{2}z^{-2}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}, \quad |z| > \frac{1}{2}.$$

(a) Determine the impulse response of the system.

(b) Determine the difference equation relating the system input $x[n]$ and the system output $y[n]$.

Sol:

(a)

$$\begin{aligned} H(z) &= \frac{1-\frac{1}{2}z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \\ &= -4+\frac{5+\frac{7}{2}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \\ &= -4-\frac{2}{1-\frac{1}{2}z^{-1}}+\frac{7}{1-\frac{1}{4}z^{-1}} \end{aligned}$$

$$h[n] = -4\delta[n]-2\left(\frac{1}{2}\right)^n u[n]+7\left(\frac{1}{4}\right)^n u[n]$$

(b)

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=x[n]-\frac{1}{2}x[n-2]$$

5. (0.5%) Page 172, Problem 3.25

Consider a right-sided sequence $x[n]$ with z-transform

$$X(z)=\frac{1}{(1-az^{-1})(1-bz^{-1})}=\frac{z^2}{(z-a)(z-b)}.$$

In Section 3.3, we considered the determination of $x[n]$ by carrying out a partial fraction expansion, with $X(z)$ considered as a ratio of polynomials in z^{-1}. Carrry out a partial fraction expansion of $X(z)$, considered as a ratio of polynomials in z, and determine $x[n]$ from this expansion.

Sol:

$$X(z)=\frac{z^2}{(z-a)(z-b)}=\frac{z^2}{z^2-(a+b)z+ab}$$

Obtain a proper fraction:

$$\begin{array}{r|l} & \quad 1 \\ \hline z^2-(a+b)z+ab & z^2 \\ & \underline{z^2-(a+b)z+ab} \\ & (a+b)z-ab \end{array}$$

$$\begin{aligned} X(z) &= 1+\frac{(a+b)z-ab}{(z-a)(z-b)}=1+\frac{\frac{(a+b)a-ab}{a-b}}{z-a}+\frac{\frac{(a+b)a-ab}{b-a}}{z-b} \\ &= 1+\frac{\frac{a^2}{a-b}}{z-a}-\frac{\frac{b^2}{a-b}}{z-b}=1+\frac{1}{a-b}\left(\frac{a^2z^{-1}}{1-az^{-1}}-\frac{b^2z^{-1}}{1-bz^{-1}}\right) \end{aligned}$$

$$\begin{aligned} x[n] &= \delta[n] + \frac{a^2}{a-b} a^{n-1} u[n-1] - \frac{b^2}{a-b} b^{n-1} u[n-1] \\ &= \delta[n] + \left(\frac{1}{a-b}\right)\left(a^{n+1} - b^{n+1}\right) u[n-1] \end{aligned}$$

6. (0.5%) Page 173, Problem 3.29

A causal LTI system has system function

$$H(z) = \frac{1-z^{-1}}{1-0.25z^{-2}} = \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})}.$$

(a) Determine the output of the system when the input is $x[n] = u[n]$.

(b) Determine the input $x[n]$ so that the corresponding output of the above system is $y[n] = \delta[n] - \delta[n-1]$.

(c) Determine the output $y[n]$ when the input is $x[n] = \cos(0.5\pi n)$ for $-\infty < n < \infty$. You may leave your answer in any convenient form.

Sol:

$$H(z) = \frac{1-z^{-1}}{1-0.25z^{-2}} = \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})}$$

(a) Given $x[n] = u[n]$, we have $X(z) = \frac{1}{1-z^{-1}}$, $1 < |z|$. Then

$$\begin{aligned} Y(z) &= H(z) X(z) \\ &= \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})} \cdot \frac{1}{1-z^{-1}} \\ &= \frac{1}{(1-0.5z^{-1})(1+0.5z^{-1})} \\ &= \frac{\frac{1}{2}}{(1-0.5z^{-1})} + \frac{\frac{1}{2}}{(1+0.5z^{-1})}, \quad 0.5 < |z|. \end{aligned}$$

(The ROC for $Y(z)$ includes the intersection of the ROC of $H(z)$ with the ROC of $X(z)$.)

Inverse z-transform gives

$$y[n] = \frac{1}{2}(0.5)^n u[n] + \frac{1}{2}(-0.5)^n u[n].$$

(b) If $y[n] = \delta[n] - \delta[n-1]$, then $Y(z) = 1 - z^{-1}$, $0 < |z|$. We have

$$\begin{aligned} X(z) &= \frac{Y(z)}{H(z)} \\ &= \frac{1-z^{-1}}{\left(\frac{1-z^{-1}}{1-0.25z^{-2}}\right)} \\ &= 1 - 0.25z^{-2}, \quad 0 < |z|. \end{aligned}$$

Inverse z-transform gives

$$x[n] = \delta[n] - 0.25\delta[n-2].$$

(c) Now $x[n] = \cos(0.5\pi n)$, $-\infty < n < \infty$. At $\omega = 0.5\pi$ we have

$$\begin{aligned} H\left(e^{j0.5\pi}\right) &= \frac{1-e^{-j0.5\pi}}{1-0.25e^{-j\pi}} \\ &= 1.13e^{j\frac{\pi}{4}}. \end{aligned}$$

Then

$$y[n] = 1.13\cos\left(0.5\pi + \frac{\pi}{4}\right).$$

7. (0.5%) Page 174, Problem 3.31

Using any method, determine the inverse z-transform for each of the following:

(a) $X(z) = \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)^2\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}$,
($x[n]$ is a stable sequence)

(b) $X(z) = e^{z^{-1}}$

(c) $X(z) = \frac{z^3-2z}{z-2}$, ($x[n]$ is a left-sided sequence)

Sol:

(a)

$$\begin{aligned} X(z) &= \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)^2\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)} \quad \frac{1}{2} < |z| < 2 \\ &= \frac{\frac{1}{35}}{\left(1+\frac{1}{2}z^{-1}\right)^2} + \frac{\frac{58}{1225}}{\left(1+\frac{1}{2}z^{-1}\right)} - \frac{\frac{1568}{1225}}{\left(1-2z^{-1}\right)} + \frac{\frac{2700}{1225}}{\left(1-3z^{-1}\right)} \end{aligned}$$

Therefore,

$$\begin{aligned} x[n] &= \frac{1}{35}(n+1)\left(\frac{-1}{2}\right)^n u[n+1] + \frac{58}{(35)^2}\left(\frac{-1}{2}\right)^n u[n] \\ &\quad + \frac{1568}{(35)^2}(2)^n u[-n-1] - \frac{2700}{(35)^2}(3)^n u[-n-1] \end{aligned}$$

(b)

$$X(z) = e^{z^{-1}} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \ldots$$

Therefore, $x[n] = \frac{1}{n!}u[n]$.

(c)

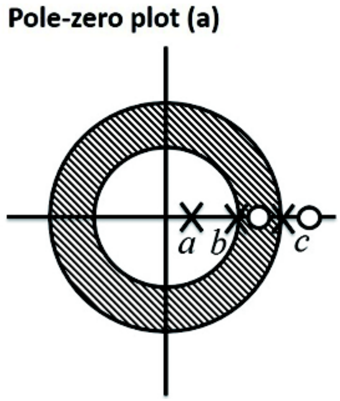
$$X(z) = \frac{z^3-2z}{z-2} = z^2 + 2z + \frac{2}{1-2z^{-1}} \qquad |z| < 2$$

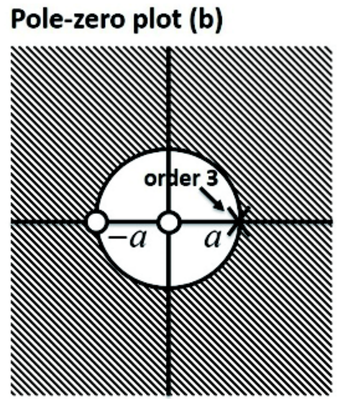
Therefore,

$$x[n] = \delta[n+2] + 2\delta[n+1] - 2(2)^n u[-n-1]$$

8. (0.5%) Page 174, Problem 3.35

For each of the following sequences, determine the z-transform and ROC, and sketch the pole-zero diagram:





(a) $x[n] = a^n u[n] + b^n u[n] + c^n u[-n-1]\,,\ |a| < |b| < |c|$

(b) $x[n] = n^2 a^n u[n]$

(c) $x[n] = e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right]u[n] - e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right]u[n-1]$

Sol:

(a)

$$\begin{aligned}
x[n] &= a^n u[n] + b^n u[n] + c^n u[-n-1] \qquad |a| < |b| < |c| \\
X(z) &= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} - \frac{1}{1-cz^{-1}} \qquad |b| < |z| < |c| \\
X(z) &= \frac{1 - 2cz^{-1} + (bc + ac - ab)\,z^{-2}}{(1-az^{-1})(1-bz^{-1})(1-cz^{-1})} \qquad |b| < |z| < |c|
\end{aligned}$$

Poles: a, b, c,

Zeros: z_1, z_2, ∞ where z_1 and z_2 are roots of numerator quadratic.

(b)

$$\begin{aligned}
x[n] &= n^2 a^n u[n] \\
x_1[n] &= a^n u[n] \Leftrightarrow X_1(z) = \frac{1}{1-az^{-1}} \qquad |z| > a \\
x_2[n] &= n x_1[n] = n a^n u[n] \Leftrightarrow X_2(z) = -z\frac{d}{dz}X_1(z) = \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > a \\
x[n] &= n x_2[n] = n^2 a^n u[n] \Leftrightarrow -z\frac{d}{dz}X_2(z) = -z\frac{d}{dz}\left(\frac{az^{-1}}{(1-az^{-1})^2}\right) \qquad |z| > a \\
X(z) &= \frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3} \qquad |z| > a
\end{aligned}$$

(c)

$$\begin{aligned}
x[n] &= e^{n^4}\left(\cos\frac{\pi}{12}n\right)u[n] - e^{n^4}\left(\cos\frac{\pi}{12}n\right)u[n-1] \\
&= e^{n^4}\left(\cos\frac{\pi}{12}n\right)(u[n] - u[n-1]) = \delta[n]
\end{aligned}$$

Therefore, $X(z) = 1$ for all $|z|$.

9. (0.5%) Page 267, Problem 4.5

Consider the system of Figure 4.10, with the discrete-time system an ideal lowpass filter with cutoff frequency $\pi/8$ radians/s.

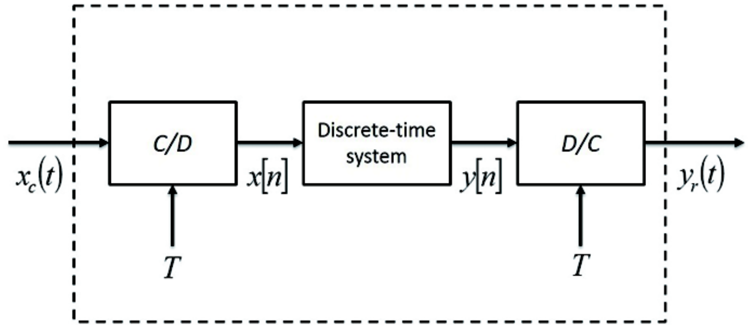


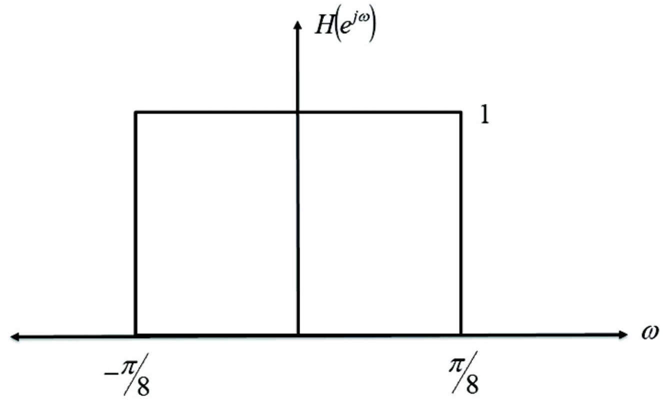
Figure 4.10 Discrete-time processing of continuous-time signals

(a) If $x_c(t)$ is bandlimited to 5kHz, what is the maximum value of T that will avoid aliasing in the C/D converter?

(b) If $1/T = 10\text{kHz}$, what will the cutoff frequency of the effective continuous-time filter be?

(c) Repeat part (b) for $1/T = 20\text{kHz}$.

Sol: A plot of $H(e^{j\omega})$ appears below.



(a)

$$x_c(t) = 0, \quad |\Omega| \geq 2\pi \cdot 5000$$

The Nyquist rate is 2 times the highest frequency. $\Rightarrow T = \frac{1}{10{,}000}$sec. This avoids all aliasing in the C/D converter.

(b)

$$\begin{aligned} \frac{1}{T} &= 10\text{kHz} \\ \omega &= T\Omega \\ \frac{\pi}{8} &= \frac{1}{10,000}\Omega_c \\ \Omega_c &= 2\pi \cdot 625\text{rad/sec} \\ f_c &= 625\text{Hz} \end{aligned}$$

(c)

$$\begin{aligned}
\frac{1}{T} &= 20\text{kHz} \\
\omega &= T\Omega \\
\frac{\pi}{8} &= \frac{1}{20,000}\Omega_c \\
\Omega_c &= 2\pi \cdot 1250\text{rad/sec} \\
f_c &= 1250\text{Hz}
\end{aligned}$$

10. (0.5%) Page 271, Problem 4.21

Consider a continuous-time signal $x_c(t)$ with Fourier transform $X_c(j\Omega)$ shown in Figure P4.21-1.

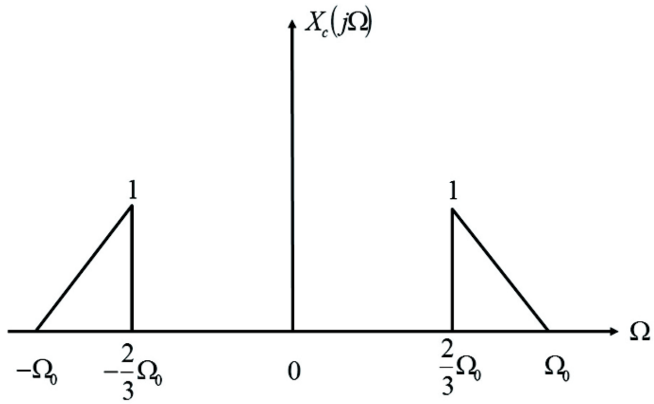


Figure P4.21-1 Fourier transform $X_c(j\Omega)$

(a) A continuous-time signal $x_r(t)$ is obtained through the process shown in Figure P4.21-2. First, $x_c(t)$ is multiplied by an impulse train of period T_1 to produce the waveform $x_s(t)$, i.e.,

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x[n]\,\delta(t - nT_1)$$

Next, $x_s(t)$ is passed through a low pass filter with frequency response $H_r(j\Omega)$. $H_r(j\Omega)$ is shown in Figure P4.21-3.

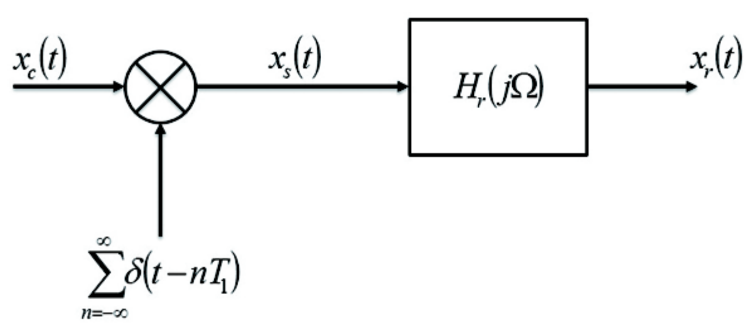
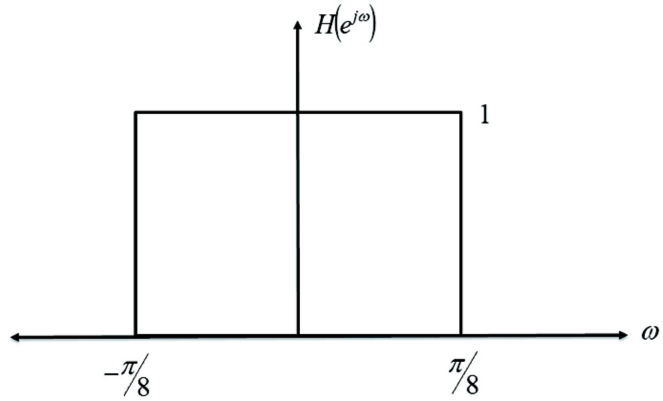


Figure P4.21-2 Conversion system for part (a)



Determine the range of values for T_1 for which $x_r(t) = x_c(t)$.

(b) Consider the system in Figure P4.21-4. The system in this case is the same as the one in part (a), except that the sampling period is now T_2. The system $H_s(j\Omega)$ is some continuous-time ideal LTI filter. We want $x_o(t)$ to be equal to $x_c(t)$ for all t, i.e., $x_o(t) = x_c(t)$ for some choice of $H_s(j\Omega)$. Find all values of T_2 for which $x_o(t) = x_c(t)$ is possible. For the largest T_2 you determined that would still allow recovery of $x_c(t)$, choose $H_s(j\Omega)$ so that $x_o(t) = x_c(t)$. Sketch $H_s(j\Omega)$.

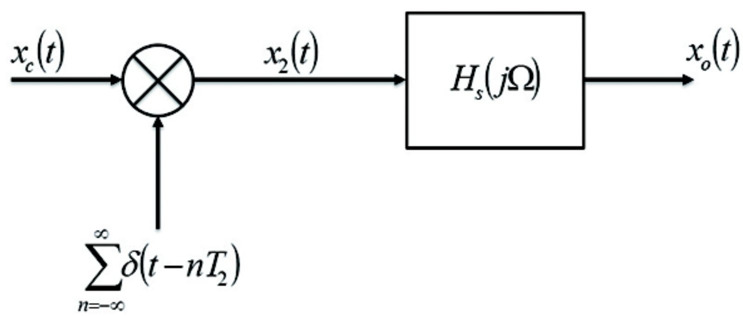


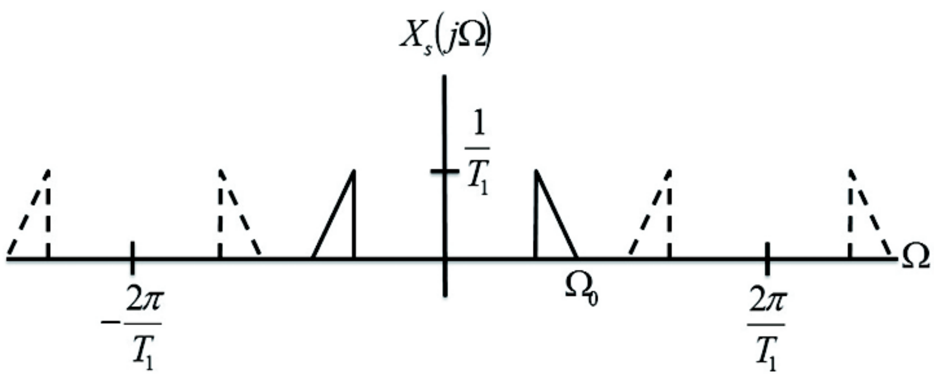
Figure P4.21-4 Conversion system for part (b)

Sol:

(a) The impulse-train signal $x_s(t)$ has spectrum $X_s(j\Omega)$ given by

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left[j\left(\Omega - k\frac{2\pi}{T_1}\right)\right].$$

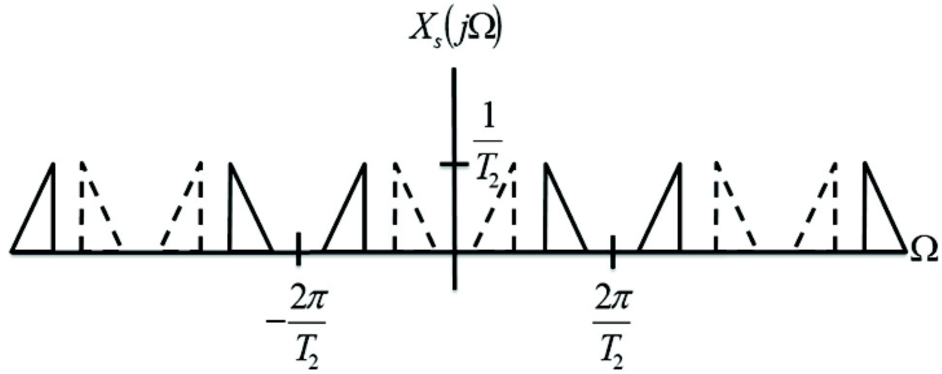
An example is shown below.



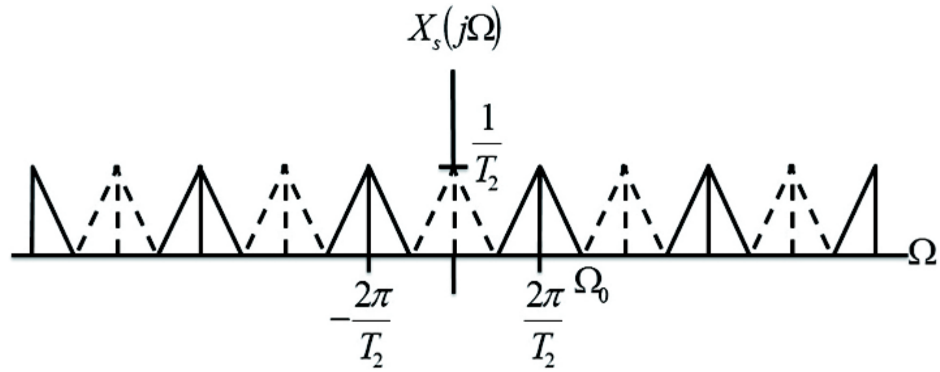
We will have $x_r(t) = x_c(t)$ provided $T_1 \leq \frac{\pi}{\Omega_0}$.

(b) We will have $x_0(t) = x_c(t)$ under any of the following circumstances:

1. As illustrated above, $T_2 \leq \frac{\pi}{\Omega_0}$.
2. As illustrated below, $\frac{1.5\pi}{\Omega_0} \leq T_2 \leq \frac{2\pi}{\Omega_0}$.



3. As illustrated below, $T_2 = \frac{3\pi}{\Omega_0}$.



The frequency response of the filter that is needed to recover $x_c(t)$ is shown below.