4 Sampling of Continuous-Time Signals

- In most applications, discrete-time signals represent sampled continuous-time signals. It is thus essential to learn the relationship of both signals.

4.1 Periodic Sampling

- Consider the *ideal continuous-to-discrete-time (C/D) converter*

  Fig. 4.1

  The output sequence is obtained by periodically sampling the input continuous-time signal $x_c(t)$, i.e.,

$$x[n] = x_c(nT)$$

  where T is the *sampling period*, or the *sampling time*, with $f_s = 1/T$ called the *sampling frequency* in samples per second and $\varpi_s = 2\pi/T$ the *sampling frequency* in radians per second.

- Mathematically, it is convenient to represent $x[n]$ as an output of impulse-train-to-sequence converter with two inputs $x_c(t)$ and $s(t)$ which is the periodic impulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

  ($\delta(t)$ being the Dirac delta function), and the intermediate output $x_s(t)$ defined by

$$x_s(t) = x_c(t)\, s(t) \triangleq \sum_{n=-\infty}^{\infty} x[n]\, \delta(t - nT).$$

  Now,

$$\begin{aligned} x_s(t) &= x_c(t)\, s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT). \end{aligned}$$

  Thus, the sequence output $x[n]$ represents $x_c(nT)$.

  Fig. 4.2

- On Dirac Delta Function $\delta(x)$:

  (Materials come from D.S. Jones, "The Theory of Generalized Functions," Cambridge University Press, 1980)

  $\delta(x)$ is a generalized function which has the property

  $$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$

  for *any* suitably continuous function $f(x)$. Note that no function in the ordinary sense has the above property. However, one can imagine a sequence of functions which have progressively taller and thinner peaks at $x = 0$, with the area under the curve remaining equal to one, while the value of the function tends to 0 at every point except $x = 0$ where it tends to infinity. For example, the limit of a sequence $g_1(x), g_2(x), ...$ with $g_n(x) = n$ for $|x| < \frac{1}{2n}$ and $g_n(x) = 0$ otherwise, can be used to imagine $\delta(x)$. The following is a list of features on $\delta(x)$:

  1. $\delta(x) = 0$ for all $x \neq 0$, but $\delta(x)$ does not really exist at $x = 0$.
  2. $\int_{-\varepsilon}^{\varepsilon} \delta(x) dx = 1$ if $\varepsilon > 0$.
  3. $\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$ for any suitably continuous function $f(x)$ which is continuous at $x = x_0$.
  4. $\int_{-\infty}^{\varepsilon} \delta(x) dx = u(\varepsilon)$ if $\varepsilon \neq 0$ where $u(\varepsilon)$ is a unit step function defined by $u(\varepsilon) = 1$ if $\varepsilon > 0$ and $u(\varepsilon) = 0$ if $\varepsilon < 0$, but $u(\varepsilon)$ does not really exist if $\varepsilon = 0$. Note that $u(\varepsilon)$ is not an ordinary function and $u(0) = 1$ is used to be defined in an ordinary sense.
  5. $\delta(x) = \frac{du(x)}{dx}$ is a convenient notation commonly adopted. Note that the derivative of $u(x)$ does not really exist for $x = 0$.
  6. $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$ can be conveniently represented for any function $f(x)$ which is continuous at $x = x_0$.
  7. $f(x) * \delta(x - x_0) = f(x - x_0)$ for any suitably continuous function $f(x)$, where $*$ denotes the continuous-time convolution.

4.2 Frequency-Domain Representation of Sampling

- Consider $x_s(t) = x_c(t)\, s(t)$. Taking Fourier transform for continuous-time signals,

  $$X_s(j\varpi) = \frac{1}{2\pi} X_c(j\varpi) * S(j\varpi) \text{ (windowing theorem)}$$

where the continuous-time Fourier transform $Y(j\varpi)$ of a continuous-time signal $y(t)$ is defined by

$$Y(j\varpi) = \int_{-\infty}^{\infty} y(t)e^{-j\varpi t}dt.$$

Since the continuous-time Fourier transform of the periodic impulse train $s(t)$ is the periodic impulse train

$$S(j\varpi) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\varpi - k\varpi_s)$$

(by Poisson's sum formula, i.e., $\mathcal{F}\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\varpi - k\frac{2\pi}{T}\right))$ with $\varpi_s = \frac{2\pi}{T}$, we have

$$X_s(j\varpi) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\varpi - jk\varpi_s).$$

- Consider the bandlimited spectrum for $X_c(j\varpi)$,

$$|X_c(j\varpi)| = 0 \text{ for } |\varpi| > \varpi_N.$$

Figs. 4.3 (a) and (b)

Case 1: If $\varpi_s \geq 2\varpi_N$,

Fig. 4.3(c)

Case 2: If $\varpi_s < 2\varpi_N$,

Fig. 4.3(d)

If we employ an ideal lowpass filter with $H_r(j\varpi)$ to filter $x_s(t)$

Fig. 4.4(a)

where $H_r(j\varpi)$ is given by

Fig. 4.4(d)

For Case 1, if $\varpi_N < \varpi_c < \varpi_s - \varpi_N$, then

$$X_r(j\varpi) = X_c(j\varpi).$$

Figs. 4.4 (c) and (e)

Thus, $X_c(j\varpi)$ can be recovered by lowpass filtering $x_s(t)$.

For Case 2,

$$X_r(j\varpi) \neq X_c(j\varpi)$$

(i.e., $X_c(j\varpi)$ is no longer recoverable by lowpass filtering). In this case, $x_r(t)$ is a distorted version of $x_c(t)$. Such distortion is called *aliasing distortion,* or simply *aliasing.* For example, consider $x_c(t) = \cos(\varpi_0 t)$ with $X_c(j\varpi) = \pi\delta(\varpi - \varpi_0) + \pi\delta(\varpi + \varpi_0)$ with $\varpi_0 > 0$.

Fig. 4.5

Without aliasing, $x_r(t) = \cos(\varpi_0 t)$ (*Fig. 4.5(d)*). With aliasing, $x_r(t) = \cos((\varpi_s - \varpi_0)t)$ (*Fig. 4.5(e)*).

- Thm: (Nyquist-Shannon Sampling Theorem)

  Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\varpi) = 0 \text{ for } |\varpi| > \varpi_N.$$

  Then $x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT), \ n = 0, \pm 1, \pm 2, ...$$

  if the sampling frequency in radians per second, ϖ_s, is

$$\varpi_s = \frac{2\pi}{T} > 2\varpi_N$$

  where we call ϖ_N the *Nyquist frequency* and $2\varpi_N$ the *Nyquist rate.*

- Note: If the sampling frequency is larger than the Nyquist rate, samples of a continuous-time bandlimited signal are sufficient to represent the signal exactly in the sense that the signal can be recovered from the samples.

- The discrete-time (DT) Fourier transform of $x[n]$ is

$$\begin{aligned} X\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\frac{\omega}{T}nT} \\ &= X_s\left(j\frac{\omega}{T}\right) \end{aligned}$$

where the last equality comes from

$$\begin{aligned}
X_s(j\varpi) &= \mathcal{F}\{x_c(t)\,s(t)\} \\
&= \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)\right\} \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_c(nT)\,\delta(t-nT)\,e^{-j\varpi t}dt \\
&= \sum_{n=-\infty}^{\infty} x_c(nT)\,e^{-jn\varpi T}.
\end{aligned}$$

Moreover, because

$$X_s(j\varpi) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\varpi - jk\varpi_s)$$

we have

$$\boxed{\underset{\substack{\uparrow \\ \text{DTFT}}}{X\left(e^{j\omega}\right)} = \frac{1}{T}\sum_{k=-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{CTFT}}}{X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)}.}$$

This equality relates $X\left(e^{j\omega}\right)$ to the continuous-time Fourier transform of the original signal $x_c(t)$.

- Note: $X\left(e^{j\omega}\right)$ is a frequency-scaled version of $X_s(j\varpi)$ with the frequency scaling specified by $\omega = \varpi T$. That is,

  Fig. 21-F1

  where

$$\begin{aligned}
X_s(j\varpi) &= \int_{-\infty}^{\infty} x_s(t)\,e^{-j\varpi t}dt \\
X\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}.
\end{aligned}$$

- Ex: Consider $x_c(t) = \cos(4000\pi t)$ with $T = 1/6000$. Now, $X_s(j\varpi)$ and $X\left(e^{j\omega}\right)$ are given by

  Fig. 4.6

4.3 Reconstruction of a Band-Limited Signal From its Samples

- Question: How can we reconstruct $x_c(t)$ from $x[n]$, provided that the sampling frequency is chosen larger than Nyquist rate?

  Answer: We adopt the reconstruction system

  Fig. 4.7

  The output is

$$\begin{aligned} x_r(t) &= x_s(t) * h_r(t) \\ &= \left[\sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT)\right] * h_r(t) \\ &= \sum_{n=-\infty}^{\infty} x[n]\,h_r(t-nT). \end{aligned}$$

  Since

$$h_r(t) = \mathcal{F}^{-1}\{H_r(j\varpi)\} = \frac{\sin\left(\pi\frac{t}{T}\right)}{\pi\frac{t}{T}} \triangleq Sa\left(\pi\frac{t}{T}\right)$$

$$\Rightarrow x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,Sa\left(\frac{\pi}{T}(t-nT)\right).$$

  Fig. 4.8

  Note that

$$\begin{aligned} x_r(mT) &= \sum_{n=-\infty}^{\infty} x[n]\,Sa((m-n)\pi) = x[m] \\ &= x_c(mT) \end{aligned}$$

  regardless of the sampling period T.

- Let us check if $x_r(t) = x_c(t)$ for all t:

$$\begin{aligned} X_r(j\varpi) &= \mathcal{F}\{x_r(t)\} = \sum_{n=-\infty}^{\infty} x[n]\,\mathcal{F}\left\{Sa\left(\frac{\pi}{T}(t-nT)\right)\right\} \\ &= \sum_{n=-\infty}^{\infty} x[n]\,e^{-jnT\varpi}\mathcal{F}\left\{Sa\left(\frac{\pi}{T}t\right)\right\} \text{ (time-shifting)} \\ &= \sum_{n=-\infty}^{\infty} x[n]\,e^{-jnT\varpi}H_r(j\varpi) \\ &= H_r(j\varpi)\,X\left(e^{j\varpi T}\right). \end{aligned}$$

If $\frac{2\pi}{T} > 2\varpi_N$,

Fig. 22-F1

then $X_r(j\varpi) = X_c(j\varpi)$, i.e., the continuous-time signal can be reconstructed if the Nyquist condition is followed.

The ideal reconstruction process can be formalized as an ideal system for converting a discrete-time sequence back to a *bandlimited* continuous-time signal, as

Fig.4.9

which is called the *ideal discrete-to-continuous-time (D/C) converter.*

4.4 Discrete-Time Processing of Continuous-Time Signals

- The discrete-time processing of continuous-time signals is useful for realizing continuous-time systems.
- Consider the discrete-time processing of continuous-time signals

Fig. 4.10

The C/D input/output relationship is

$$\begin{aligned} x[n] &= x_c(nT) \\ X\left(e^{j\omega}\right) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right). \end{aligned}$$

The D/C input/output relationship is

$$\begin{aligned} y_r(t) &= \sum_{n=-\infty}^{\infty} y[n]\, Sa\left(\frac{\pi}{T}(t-nT)\right) \\ Y_r(j\varpi) &= H_r(j\varpi)\, Y\left(e^{j\varpi T}\right) = \begin{cases} TY\left(e^{j\varpi T}\right), & |\varpi| < \frac{\pi}{T} \\ 0, & \text{elsewhere} \end{cases}. \end{aligned}$$

Fig. 22-B2

The discrete-time LTI system I/O relationship is

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right).$$

(In general, it is easier to deal with frequency-domain functions.)

Therefore, the I/O relationship for the whole system is

$$\begin{aligned} Y_r(j\varpi) &= H_r(j\varpi)\, H\left(e^{j\varpi T}\right) X\left(e^{j\varpi T}\right) \\ &= H_r(j\varpi)\, H\left(e^{j\varpi T}\right) \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\varpi - j\frac{2\pi k}{T}\right). \end{aligned}$$

If $X_c(j\varpi)$ is bandlimited and the sampling rate is above the Nyquist rate

Fig. 23-F1

$$Y_r(j\varpi) = H_{eff}(j\varpi)\, X_c(j\varpi)$$

where

$$H_{eff}(j\varpi) = \begin{cases} H\left(e^{j\varpi T}\right), & |\varpi| < \frac{\pi}{T} \\ 0, & \text{elsewhere} \end{cases}.$$

Equivalent Continuous-Time LTI System: Thus, the whole system has an *effective frequency response* $H_{eff}(j\varpi)$, which is bandlimited within $|\varpi| < \frac{\pi}{T}$.

Fig. 23-F2

Note: The above conclusion is based on two assumptions:

1. The discrete-time system is LTI.
2. The input signal must be bandlimited, and the sampling rate must be higher than the Nyquist rate (such that no aliasing effect exists).

- Ex: The ideal continuous-time lowpass filter with the discrete-time cutoff frequency ω_c:

$$H\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$\underset{\omega=\varpi T}{\Rightarrow} H_{eff}(j\varpi) = \begin{cases} 1, & |\varpi| < \frac{\omega_c}{T} \\ 0, & \text{elsewhere} \end{cases}.$$

Fig. 4.11

The intermediate signals processed within the system is illustrated as

Fig. 4.12

Notes:

1. To avoid aliasing, we need

$$2\pi - \varpi_N T > \omega_c.$$

If $\omega_c = \pi$, the system is equivalent to an all-pass filter.

2. The equivalent continuous-time cutoff frequency corresponding to the discrete-time cutoff frequency ω_c is

$$\varpi_c = \frac{\omega_c}{T}.$$

3. Thus, for a fixed discrete-time LTI filter, we can vary the cutoff frequency of the equivalent continuous-time LTI filter by varying the sampling period T, provided that T satisfies the constraint

$$\boxed{2\pi - \varpi_N T > \omega_c.}$$

In this case, ϖ_c is bounded by

$$\frac{2\pi}{T} - \varpi_N > \varpi_c \geq 0.$$

- Question: How can we build a discrete-time LTI system $h[n]$ such that the equivalent continuous-time LTI system $h_c(t)$ has a given frequency response $H_c(j\varpi)$?

Fig. 4.14

Answer: We want to equate

$$H_{eff}(j\varpi) = H_c(j\varpi) = \begin{cases} H\left(e^{j\varpi T}\right), & |\varpi| < \frac{\pi}{T} \\ 0, & |\varpi| \geq \frac{\pi}{T} \end{cases}.$$

Thus, T has to be chosen so that $H_c(j\varpi) = 0$ for $|\varpi| \geq \frac{\pi}{T}$. For the input bandlimited signal which is limited in the frequency range $|\varpi| \leq \varpi_N$, T has to satisfy $\frac{\pi}{T} > \varpi_N$ also (i.e., the sampling frequency should be larger than the Nyquist rate to avoid aliasing in the discrete-time processing). Under this constraint of T, we proceed as follow:

For $|\varpi| < \frac{\pi}{T}$,

$$H_c(j\varpi) = H\left(e^{j\varpi T}\right)$$

$$\underset{\omega=\varpi T}{\Rightarrow} \boxed{H_c\left(j\frac{\omega}{T}\right) = H\left(e^{j\omega}\right) \text{ for } |\omega| < \pi.}$$

Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_c\left(j\frac{\omega}{T}\right)e^{j\omega n}d\omega$$

$$\begin{aligned}
\Rightarrow\quad h[n] &\underset{\varpi=\frac{\omega}{T}}{=} \frac{1}{2\pi}T\int_{-\pi/T}^{\pi/T} H_c(j\varpi)\, e^{j\varpi nT}d\varpi \\
&= T\frac{1}{2\pi}\int_{-\infty}^{\infty} H_c(j\varpi)\, e^{j\varpi(nT)}d\varpi \;\; (\because H_c(j\varpi)=0 \text{ for } |\varpi| \geq \frac{\pi}{T}) \\
&= Th_c(nT)
\end{aligned}$$

$$\Rightarrow \boxed{h[n] = Th_c(nT)\,.}$$

Fig. 25-B1

Thus, for T's that make $H_c(j\varpi) = 0$ for $|\varpi| \geq \frac{\pi}{T}$, the discrete-time LTI system with $h[n] = Th_c(nT)$ is an *impulse-invariant* version of the continuous-time LTI system with $h_c(t)$, because in this case

$$\boxed{H\left(e^{j\omega}\right) = H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi.}$$

- Ex: Suppose that we want to have an ideal lowpass discrete-time filter with cutoff frequency ω_c. This can be done by sampling an ideal lowpass continuous-time filter with cutoff frequency $\varpi_c = \omega_c/T < \pi/T$

$$H_c(j\varpi) = \begin{cases} 1, & |\varpi| < \varpi_c \\ 0, & |\varpi| \geq \varpi_c \end{cases}.$$

Because

$$h_c(t) = \frac{\sin(\varpi_c t)}{\pi t}$$

we have

$$\begin{aligned}
h[n] &= Th_c(nT) = \frac{\sin(n\omega_c)}{n\pi} \\
H\left(e^{j\omega}\right) &= \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases} = H_c\left(j\frac{\omega}{T}\right).
\end{aligned}$$

4.5 Continuous-Time Processing of Discrete-Time Signals

- The continuous-time processing of discrete-time signals can provide a useful interpretation of certain discrete-time systems that have no simple interpretation in the discrete-time domain.

- Consider the continuous-time processing of discrete-time signals:

Fig. 4.15

The D/C I/O relationship is

$$\begin{aligned} x_c(t) &= \sum_{n=-\infty}^{\infty} x[n]\, Sa\left(\frac{\pi}{T}(t-nT)\right) \\ X_c(j\varpi) &= \begin{cases} TX\left(e^{j\varpi T}\right), & |\varpi| < \frac{\pi}{T} \\ 0, & |\varpi| \geq \frac{\pi}{T} \end{cases}. \end{aligned}$$

The C/D I/O relationship is

$$\begin{aligned} y[n] &= y_c(nT) \\ Y\left(e^{j\omega}\right) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right). \end{aligned}$$

The continuous-time LTI system I/O relationship is

$$Y_c(j\varpi) = X_c(j\varpi)\, H_c(j\varpi).$$

Thus,

$$Y\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) H_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right).$$

Since $X_c(j\varpi) = 0$ for $|\varpi| \geq \frac{\pi}{T}$ and $Y\left(e^{j\omega}\right)$ is periodic with period 2π, then for $|\omega| \leq \pi$

$$\begin{aligned} Y\left(e^{j\omega}\right) &= \frac{1}{T} X_c\left(j\frac{\omega}{T}\right) H_c\left(j\frac{\omega}{T}\right) \\ &= X\left(e^{j\omega}\right) H_c\left(j\frac{\omega}{T}\right) \end{aligned}$$

$$\Rightarrow \boxed{H\left(e^{j\omega}\right) = H_c\left(j\frac{\omega}{T}\right) \text{ for } |\omega| \leq \pi.}$$

Note that $H\left(e^{j\omega}\right)$ is periodic with period 2π.

The overall system behaves as a discrete-time LTI system with frequency response $H_c\left(j\frac{\omega}{T}\right)$, $|\omega| \leq \pi$.

Note: Since $X_c(j\varpi) = 0$ for $|\varpi| \geq \frac{\pi}{T}$, $H_c(j\varpi)$ can be chosen arbitrarily for $|\varpi| \geq \frac{\pi}{T}$.

- Ex: Noninteger Delay System (Interpolation)

  Let
  $$H\left(e^{j\omega}\right) = e^{-j\omega\Delta}, \ \ |\omega| < \pi$$
  where Δ is not an integer. Now, let us look at the equivalent system as follows:

  Fig. 4.15
  $$\begin{aligned} H_c\left(j\varpi\right) &= H\left(e^{j\varpi T}\right) \text{ for } |\varpi| < \frac{\pi}{T} \\ &= e^{-j\varpi\Delta T} \text{ for } |\varpi| < \frac{\pi}{T} \end{aligned}$$
  Since $H_c\left(j\varpi\right)$ can be chosen arbitrarily for $|\varpi| \geq \frac{\pi}{T}$, we can make $H_c\left(j\varpi\right) = e^{-j\varpi\Delta T}$ for all ϖ, which is an all-pass continuous-time system with group delay ΔT. In the case, we have the continuous-time version of I/O relationship
  $$y_c\left(t\right) = x_c\left(t - \Delta T\right)$$
  which denotes that $y_c\left(t\right)$ is a ΔT-delayed version of $x_c\left(t\right)$. Now, the discrete-time version of I/O relationship is
  $$\begin{aligned} y\left[n\right] &= y_c\left(nT\right) = x_c\left(nT - \Delta T\right) \\ &= \sum_{k=-\infty}^{\infty} x\left[k\right] Sa\left(\pi\left(n - \Delta - k\right)\right). \end{aligned}$$
  Here, $y\left[n\right]$ is the bandlimited interpolated version of $x\left[n\right]$ if $0 < \Delta < 1$.

  Fig. 4.16

4.6 Changing the Sampling Rate Using Discrete-Time Processing

4.6.1 Sampling Rate Reduction by an Integer Factor

- The *downsampling* system

  Fig. 4.19

  reduces the sampling rate by a factor of M. It is also called a *sampling rate compressor*, or simply a *compressor*. Note that the input sequence $x[n]$ is the T-sampled sequence of continuous-time bandlimited signal $x_c(t)$ of Nyquist frequency ϖ_N under the constraint $\varpi_N T < \pi$ (in order to avoid aliasing).

- A mathematically-equivalent downsampling system is

Fig.27-F2

Now, C/D I/O is

$$
\begin{aligned}
& x_d[n] = x_c\left(nT^{'}\right) = x_c(nMT) = x[nM] \\
\Rightarrow \quad & X_d\left(e^{j\omega}\right) = \frac{1}{MT}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi k}{MT}\right) \\
& \underset{k=lM+i}{=} \frac{1}{MT}\sum_{l=-\infty}^{\infty}\sum_{i=0}^{M-1} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi(lM+i)}{MT}\right) \\
& = \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{l=-\infty}^{\infty} X_c\left(j\frac{\omega-2\pi i}{MT} - j\frac{2\pi l}{T}\right)\right] \\
& \left(X\left(e^{j\frac{\omega-2\pi i}{M}}\right) = \frac{1}{T}\sum_{l=-\infty}^{\infty} X_c\left(j\frac{\omega-2\pi i}{MT} - j\frac{2\pi l}{T}\right)\right) \\
\Rightarrow \quad & \boxed{X_d\left(e^{j\omega}\right) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi i}{M}}\right).}
\end{aligned}
$$

$X_d\left(e^{j\omega}\right)$ is composed of M copies of $X\left(e^{j\omega}\right)$ with frequencies scaled by $1/M$ and shifted by integer multiples of $2\pi/M$.

- Notes:

  1.

$$
\begin{aligned}
X_d\left(e^{j(\omega+2m\pi)}\right) &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi(i-m)}{M}}\right) \\
&= \frac{1}{M}\sum_{\substack{i^{'}=0 \\ \left(i^{'}=i-m\right)}}^{M-1} X\left(e^{j\frac{\omega-2\pi i^{'}}{M}}\right)
\end{aligned}
$$

because $X\left(e^{j\omega}\right)$ is periodic with period 2π. This yields

$$X_d\left(e^{j(\omega+2m\pi)}\right) = X_d\left(e^{j\omega}\right).$$

Thus, $X_d\left(e^{j\omega}\right)$ is periodic with period 2π.

2. Downsampling with $M = 2$ can be illustrated as

   Fig. 4.20

   where $\frac{2\pi}{T} = 4\varpi_N$, i.e., the original sampling rate is exactly twice the minimum rate to avoid aliasing. In general, if $\frac{2\pi}{M} > 2\omega_N \Leftrightarrow \frac{1}{M}\frac{2\pi}{T} > 2\varpi_N$, there is no aliasing. If $\frac{2\pi}{M} < 2\omega_N$, there is aliasing. For the previous example with $M = 3$ and $\frac{2\pi}{T} = 4\varpi_N$,

   Figs. 4.21(a)-(c)

   Thus, to avoid aliasing in downsampling by a factor of M requires that

   $$\boxed{\omega_N < \frac{\pi}{M}.}$$

   Simply put, the new sampling frequency $\frac{2\pi}{MT}$ should be larger than the Nyquist rate $2\varpi_N = 2\omega_N/T$ associated with input sequence $x[n]$ to avoid aliasing. Thus, since the new sampling frequency $\frac{2\pi}{MT}$ is smaller than the original sampling frequency, caution should be given to make sure that $2\varpi_N < \frac{2\pi}{MT}$, i.e.,

   $$\boxed{\omega_N < \frac{\pi}{M}.}$$

3. If $\omega_N > \frac{\pi}{M}$ and we still want to do downsampling without aliasing, a typical approach is to employ a prefilter to limit the output bandwidth within $\frac{\pi}{M}$.

   Fig. 4.22

   Figs. 4.21(d)-(f)

   In this case, $\widetilde{x}_d[n]$ no longer represents the original $x_c(t)$. Such system is generally referred to as a *decimator*.

4.6.2 Increasing the Sampling Rate by an Integer Factor

- The *upsampling* (or *interpolation*) system

  Fig. 4.23

  increases the sampling rate by a factor of L. The system on the left is called a *sampling rate expander*, or simply *expander*. The whole system is called an interpolator since it fills in missing samples. Now, considering the underlining continuous-time signal $x_c(t)$, we want to obtain samples

$$x_i[n] = x_c(nT_i)$$

with $T_i = T/L$, from the sequence of samples

$$x[n] = x_c(nT).$$

It follows that

$$x_i[n] = x_c(nT/L) = x[n/L], \quad n = 0, \pm L, \pm 2L, ...$$

We need to fill in missing samples as follow.

- Note first that the input sequence $x[n]$ is the T-sampled sequence of continuous-time bandlimited signal $x_c(t)$ of Nyquist frequency ϖ_N under the constraint $\varpi_N T < \pi$ (in order to avoid aliasing).
- Now, the output from the expander is

$$x_e[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = 0, \pm L, \pm 2L, ... \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n - kL].$$

The Fourier transform of $x_e[n]$ is

$$\begin{aligned} X_e\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]\right)e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} x[k]\,e^{-j\omega kL} \\ &= X\left(e^{j\omega L}\right) \end{aligned}$$

$$\Rightarrow \boxed{X_e\left(e^{j\omega}\right) = X\left(e^{j\omega L}\right).}$$

Thus, $X_e(e^{j\omega})$ is a frequency-scaled version of $X(e^{j\omega})$ by a factor of L, so that ω is now normalized by $\omega = \varpi T_i$.

Figs. 4.24(a)-(c)

Note that $X_e(e^{j\omega})$ is periodic with period 2π, because

$$\begin{aligned} X_e\left(e^{j(\omega+2m\pi)}\right) &= X\left(e^{j\omega L + j2mL\pi}\right) \\ &= X\left(e^{j\omega L}\right) \\ &= X_e\left(e^{j\omega}\right). \end{aligned}$$

- From

Figs. 4.24(d)-(e)

an ideal lowpass filtering with a gain of L and cutoff frequency π/L can provide the desired $X_i\left(e^{j\omega}\right)$. In other words, the ideal lowpass filtering fills in the missing samples in $x_e[n]$ for $n \neq 0, \pm L, \pm 2L, ...$ and generates $x_i[n]$.

The ideal discrete-time LPF has an impulse response

$$h_i[n] = Sa\left(\frac{\pi n}{L}\right).$$

Thus, the LPF output is

$$\begin{aligned} x_i[n] &= \sum_{k=-\infty}^{\infty} x_e[k]\, Sa\left(\frac{\pi(n-k)}{L}\right) \\ &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} x[l]\,\delta[k-lL]\, Sa\left(\frac{\pi(n-k)}{L}\right) \\ &= \sum_{l=-\infty}^{\infty} x[l]\, Sa\left(\frac{\pi(n-lL)}{L}\right). \end{aligned}$$

For $n = 0, \pm L, \pm 2L, ...$

$$x_i[n] = x\left[\frac{n}{L}\right] = x_c\left(\frac{n}{L}T\right) = x_c(nT_i).$$

Now,

$$\begin{aligned} X_e\left(e^{j\omega}\right) &= X\left(e^{j\omega L}\right) = \frac{1}{T}\sum_k X_c\left(j\frac{\omega L}{T} - j\frac{2\pi k}{T}\right) \\ &= \frac{1}{T}\sum_k X_c\left(j\frac{\omega}{T_i} - j\frac{2\pi k/L}{T_i}\right) \end{aligned}$$

$$\underset{\substack{\uparrow \\ Figs.\ 4.24(d)\text{-}(e)}}{\Rightarrow} \quad X_i\left(e^{j\omega}\right) = \frac{1}{T_i}\sum_k X_c\left(j\frac{\omega}{T_i} - j\frac{2\pi k}{T_i}\right)$$

(since $X_c(j\varpi) = 0$ for $|\varpi| \geq \frac{\pi}{T}$, and thus after filtering $X_c(j\frac{\omega}{T_i}) = 0$ for $|\omega| \geq \frac{\pi}{L}$). We have

$$\Rightarrow \boxed{x_i[n] = x_c(nT_i)}\ \forall n$$

Thus, $x_i[n]$ interpolates $x[n]$ in a sense that $x_i[n] = x_c(nT_i)$.

Fig. 30-B1

- Because the new sampling frequency $\frac{2\pi}{T/L}$ is larger than the Nyquist rate $2\varpi_N = 2\omega_N/T$ associated with input sequence $x[n]$, there is no threat of aliasing.

- The previous interpolator is ideal, but requires high complexity since an ideal LPF is required. A simpler interpolation procedure is desired in most cases. A typical one is a linear interpolator. See Subsection 4.6.3 (pp. 216∼219) for the relevant discussion.

4.6.4 Changing the Sampling Rate by a Noninteger Factor

- We want to change the sampling rate $\frac{1}{T}$ to $\frac{L}{M}\frac{1}{T}$ where $\frac{L}{M}$ is not an integer or the inverse of an integer by combining decimation and interpolation.

- Consider the system which shows an interpolator decreasing the sampling period from T to T/L, followed by a decimator increasing the sampling period from T/L to MT/L, as

  Fig. 4.29(a)

  which is equivalent to

  Fig. 4.29(b)

  Note first that the input sequence $x[n]$ is the T-sampled sequence of continuous-time bandlimited signal $x_c(t)$ of Nyquist frequency ϖ_N under the constraint $\varpi_N T < \pi$ (in order to avoid aliasing). Thus, the bandwidths for $x[n]$ and $x_i[n]$ are given respectively by $\omega_N = \varpi_N T$ and $\omega_N/L = \varpi_N T/L$.

  1. If $L \geq M$ (i.e., $\frac{\pi}{L} \leq \frac{\pi}{M}$),

  $$\widetilde{x}_d[n] = \widetilde{x}_i[nM] \underset{(\frac{\omega_N}{L} \leq \frac{\pi}{L} \leq \frac{\pi}{M})}{\underset{\uparrow}{=}} x_i[nM]$$

  and

  $$x_i[n] = x_c\left(n\frac{T}{L}\right)$$

  $$\Rightarrow \boxed{\widetilde{x}_d[n] = x_c\left(n\frac{M}{L}T\right)}.$$

  Thus, $\widetilde{x}_d[n]$ represents the sequence, taken by sampling the original analog signal with a rate of $\frac{L}{M}\frac{1}{T}$.

2. If $M > L$ (i.e., $\frac{\pi}{M} < \frac{\pi}{L}$),

$$\widetilde{x}_d[n] = \widetilde{x}_i[nM] = x_i[nM]$$

only when

$$\boxed{\frac{1}{L}\omega_N < \frac{\pi}{M}}$$

$$\frac{1}{L}\omega_N \quad : \quad \text{bandwidth of interpolated sequence } x_i[n]$$

(from previous discussion), i.e., when

$$\frac{L}{M}\frac{2\pi}{T} > 2\frac{\omega_N}{T} = 2\varpi_N$$

(namely, when the new sampling frequency $\frac{L}{M}\frac{2\pi}{T}$ is larger than the Nyquist rate); otherwise, $\widetilde{x}_d[n]$ is distorted from the original analog signal!

3. Simply put, the new sampling frequency $\frac{L}{M}\frac{2\pi}{T}$ should be larger than the Nyquist rate $2\varpi_N = 2\omega_N/T$ associated with input sequence $x[n]$ to avoid aliasing. Thus, if the new sampling frequency $\frac{L}{M}\frac{2\pi}{T}$ is smaller than the original sampling frequency, caution should be given to make sure that $2\varpi_N < \frac{L}{M}\frac{2\pi}{T}$, i.e.,

$$\boxed{\frac{1}{L}\omega_N < \frac{\pi}{M}.}$$

- Ex: Sampling Rate Conversion by a Noninteger Rational Factor

  Consider that a bandlimited signal with $X_c(j\varpi)$ is sampled at the Nyquist rate $2\varpi_N = \frac{2\pi}{T}$.

  Fig. 4.30

4.7 Multirate Signal Processing

- Multirate signal processing techniques are exceedingly useful in the sampling rate conversion (especially by a noninteger rational factor), in the A/D and D/A systems exploiting oversampling and noise shaping, and in constructing filter banks.

4.7.1 Interchange of Filtering With Compressor/Expander

- The Downsampling Identity: Consider

  Fig. 4.31

  The two systems are equivalent because in Fig. 4.31(b)

$$\begin{aligned}
X_b(e^{j\omega}) &= H(e^{j\omega M})X(e^{j\omega}) \\
Y(e^{j\omega}) &= \frac{1}{M}\sum_{i=0}^{M-1} X_b(e^{j(\omega/M - 2\pi i/M)}) \\
&= \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)})H(e^{j(\omega - 2\pi i)}) \\
&= H(e^{j\omega})\frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)}) \\
&= H(e^{j\omega})X_a(e^{j\omega})
\end{aligned}$$

  which is equivalent to the output of Fig. 4.31(a).

- The Upsampling Identity: Consider

  Fig. 4.32

  The two systems are equivalent because in Fig. 4.32(a)

$$\begin{aligned}
Y(e^{j\omega}) &= X_a(e^{j\omega L}) \\
&= H(e^{j\omega L})X(e^{j\omega L}) \\
&= H(e^{j\omega L})X_b(e^{j\omega})
\end{aligned}$$

  which is equivalent to the output of Fig. 4.32(b).

4.7.2 Multistage Decimation and Interpolation

- Consider the multistage decimation system

  Fig. 4.33

  which is equivalent to a single-stage decimation system with system function $H(z) = H_1(z)H_2(z^{M_1})$ and overall decimation ratio $M = M_1M_2$.

- Consider the multistage interpolation system

Fig. 4.34

which is equivalent to a single-stage interpolation system with system function $H(z) = H_1(z^{L_2})H_2(z)$ and overall interpolation ratio $L = L_1L_2$.

4.7.3 Polyphase Decomposition

- The polyphase decomposition of a sequence $h[n]$ is obtained by representing it as a superposition of M subsequences $h_k[n]$ with $k = 0, 1, ..., M-1$, each consisting of every Mth value of successively delayed versions of $h[n]$. Specifically, $h[n]$ is represented as

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

with $h_k[n]$ given by

$$h_k[n] = \begin{cases} h[n+k], & n = \text{integer multiple of } M \\ 0, & \text{otherwise} \end{cases}.$$

This decomposition leads to efficient implementation structures for the LTI systems with impulse response $h[n]$, as

Fig. 4.35

Fig. 4.36

where the intermediate sequences $e_k[n]$ are given by

$$e_k[n] = h[nM+k] = h_k[nM]$$

and are referred to as the *polyphase components* of $h[n]$.

Notes:

1. There are several other ways to derive the polyphase components of a sequence.
2. Figs. 4.35 and 4.36 show that the system with $h[n]$ can be decomposed into a bank of M parallel filters in a way that

$$H(z) = \sum_{k=0}^{M-1} z^{-k} E_k(z^M)$$

which decomposes the system function $H(z)$ into a sum of M delayed polyphase component filter responses, with $H(z) = \mathcal{Z}\{h[n]\}$ and $E_k(z) = \mathcal{Z}\{e_k[n]\}$. The filter bank structure looks like

Fig. 4.37

4.7.4 Polyphase Implementation of Decimation Filters

- Consider the decimation system

  Fig. 4.38

  A straightforward implementation is to take one output sample out of $y[n]$ but still compute other unused samples in $y[n]$ at the filter with $H(z)$.

- A polyphase decomposition of the filter with $H(z)$ can exploit a more efficient implementation:

  1. Decompose $h[n]$ as

  $$H(z) = \sum_{k=0}^{M-1} z^{-k} E_k(z^M)$$

     with polyphase components $e_k[n] = h[nM + k]$.
  2. Using the fact that downsampling commutes with addition, Fig. 4.38 in conjunction with Fig. 4.37 can be redrawn as

     Fig. 4.39

  3. Using the downsampling identity in Fig. 4.31 transforms Fig. 4.39 to

     Fig. 4.40

- The Complexity Issue: Suppose that the input $x[n]$ is clocked at a rate of one sample per unit time and that $H(z)$ is an N-point FIR filter with N divisible by M. In Fig. 4.38 (straightforward implementation), we require N multiplications and $N - 1$ additions for computing each $y[n]$ at each unit time. In Fig. 4.40, each $E_k(z)$ is an N/M-point FIR filter and its input is cloaked at a rate of one sample per M unit times. Thus, each polyphase component filter requires $\frac{1}{M}(\frac{N}{M})$ multiplications and $\frac{1}{M}(\frac{N}{M}-1)$ additions per unit time. The total complexity required by Fig. 4.40 is then $\frac{N}{M}$ multiplications and $\frac{N}{M} - 1$ additions per unit time. This shows that the polyphase implementation can save the complexity with a factor $\frac{1}{M}$.

4.7.5 Polyphase Implementation of Interpolation Filters

- Similarly, the interpolation system

  Fig. 4.41

  can be implemented more efficiently by first decomposing $H(z)$ by a polyphase structure as

  Fig. 4.42

  and then employing the interpolation identity in Fig. 4.32 to transform Fig. 4.42 into

  Fig. 4.43

  which can save the implementation complexity with respect to the straightforward implementation in Fig. 4.41 by a factor $\frac{1}{L}$.

- Reading Assignment: 4.7.6 Multirate Filter Banks.