1. (0.5%) Page 99, Problem 2.3

   By direct evaluation of the convolution sum, determine the unit step response ($x[n] = u[n]$) of an LTI system whose impulse response is

$$h[n] = a^{-n}u[-n],\ 0 < a < 1.$$

Sol: We desire the step response to a system whose impulse reponse is

$$h[n] = a^{-n}u[-n], \text{ for } 0 < a < 1.$$

The convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k].$$

The step response results when the inpuut is the unit step:

$$x[n] = u[n] = \begin{cases} 1, \text{ for } n \geq 0 \\ 0, \text{ for } n < 0 \end{cases}.$$

Substitution into the convolution sum yields

$$y[n] = \sum_{k=-\infty}^{\infty} a^{-k}u[-k]\,u[n-k].$$

For $n \leq 0$:

$$y[n] = \sum_{k=-\infty}^{n} a^{-k} = \sum_{k=-n}^{\infty} a^{k} = \frac{a^{-n}}{1-a}.$$

For $n > 0$:

$$y[n] = \sum_{k=-\infty}^{0} a^{-k} = \sum_{k=0}^{\infty} a^{k} = \frac{1}{1-a}.$$

2. (0.5%) Page 99, Problem 2.6

   (a) Determine the frequency response $H(e^{j\omega})$ of the LTI system whose input and output satisfy the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2].$$

   (b) Write a difference equation that characterizes a system whose frequency response is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}.$$

Sol:

(a) The difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2].$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \end{aligned}$$

(b) A system with frequency response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} \\ &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3].$$

3. (0.5%) Page 100, Problem 2.14

A single input-output relationship is given for each of the following three systems:

(a) System A: $x[n] = (1/3)^n,\ \ y[n] = 2(1/3)^n$.

(b) System B: $x[n] = (1/2)^n,\ \ y[n] = (1/4)^n$.

(c) System C: $x[n] = (2/3)^n u[n],\ \ y[n] = 4(2/3)^n u[n] - 3(1/2)^n u[n]$.

Based on this information, pick the strongest possible conclusion that you can make about each system from the following list of statements:

(i) The system cannot possibly be LTI.

(ii) The system must be LTI.

(iii) The system can be LTI, and there is only one LTI system that satisfies this input-output constraint.

(iv) The system can be LTI, but cannot be uniquely determined from the information in this input-output constraint.

If you chose option (iii) from this list, specify either the impulse response $h[n]$ or the frequency response $H(e^{j\omega})$ for the LTI system.

Sol:

(a) The information given shows that the system satisfies the eigenfunction property of exponential sequences for LTI systems for one particular eigenfunction input. However, we do not know the system response for any other eigenfunction. Hence, we can say that the system may be LTI, but we cannot uniquely determine it.
$\rightarrow$ (iv)

(b) If the system were LTI, the output should be in the form of $A(1/2)^n$, since $(1/2)^n$ would have been an eigenfunction of the system. Since this is not true, the system cannot be LTI.
$\rightarrow$ (i)

(c) Given the information, the system may be LTI, but does not have to be. For example, for any input other than the given one, the system may output zero, making this system non-LTI.
$\rightarrow$ (iii)
If it were LTI, its system function can be found by using DTFT:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

$$h[n] = (\frac{1}{2})^n u[n].$$

4. (0.5%) Page 102, Problem 2.20

Consider the difference equation representing a causal LTI system

$$y[n] + (1/a)y[n-1] = x[n-1].$$

(a) Find the inpulse response of the system, $h[n]$, as a function of the constant a.

(b) For what range of values a will the system be stable?

Sol:

(a) Taking the difference equation $y[n] = (1/a)y[n-1] + x[n-1]$ and assuming $h[0] = 0$ for $n < 0$:

$$\begin{aligned} h[0] &= 0 \\ h[1] &= 1 \\ h[2] &= 1/a \\ h[3] &= (1/a)^2 \\ &\vdots \\ h[n] &= (1/a)^{n-1}u[n-1] \end{aligned}$$

(b) $h[n]$ is absolutely summable if $|1/a| < 1$ or if $|a| > 1$.

5. (0.5%) Page 104, Problem 2.27

A linear time-invariant system has impulse response $h[n] = u[n]$. Determine the response of this system to the input $x[n]$ shown in Figure P2.27 and described as

$$x[n] = u[n] = \begin{cases} 0, & n < 0, \\ a^n, & 0 \le n \le N_1, \\ 0, & N_1 < n < N_2, \\ a^{n-N_2}, & N_2 \le n \le N_2 + N_1, \\ 0, & N_2 + N_1 < n, \end{cases}.$$

where $0 < a < 1$.

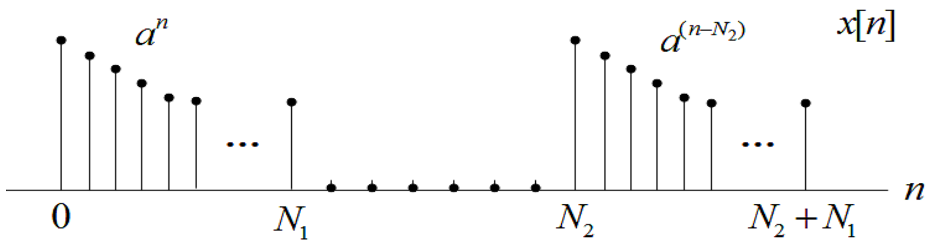

Figure P2.27

Sol: The output is obtained from the convolution sum:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k]\, u[n-k] \end{aligned}$$

The convolution may be broken into five regions over the range of n:
For $n < 0$,

$$y[n] = 0$$

For $0 \le n \le N_1$,

$$y[n] = \sum_{k=0}^{n} a^k = \frac{1-a^{(n+1)}}{1-a}$$

For $N_1 < n < N_2$,

$$y[n] = \sum_{k=0}^{N_1} a^k = \frac{1-a^{(N_1+1)}}{1-a}$$

For $N_1 \leq n \leq (N_1 + N_2)$,

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{n} a^{(k-N_2)} \\ &= \frac{1-a^{(N_1+1)}}{1-a} + \frac{1-a^{(n+1)}}{1-a} \\ &= \frac{2-a^{(N_1+1)}-a^{(n+1)}}{1-a} \end{aligned}$$

For $n > (N_1 + N_2)$,

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{N_1+N_2} a^{(k-N_2)} \\ &= \sum_{k=0}^{N_1} a^k + \sum_{m=0}^{N_1} a^m = 2\sum_{k=0}^{N_1} a^k \\ &= 2\left(\frac{1-a^{(N_1+1)}}{1-a}\right). \end{aligned}$$

6. (0.5%) Page 105, Problem 2.31

For $X(e^{j\omega}) = 1/(1 - ae^{-j\omega})$, with $-1 < a < 0$, determine and sketch the following as a function of ω:

(a) $\mathrm{Re}\{X(e^{j\omega})\}$

(b) $\mathrm{Im}\{X(e^{j\omega})\}$

(c) $|X(e^{j\omega})|$

(d) $\angle X(e^{j\omega})$.

Sol: For $(-1 < a < 0)$, we have

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

(a) Real part of $X(e^{j\omega})$:

$$\begin{aligned} X_R(e^{j\omega}) &= \frac{1}{2}\cdot[X(e^{j\omega}) + X^*(e^{j\omega})] \\ &= \frac{1-a\cos(\omega)}{1-2a\cos(\omega)+a^2}. \end{aligned}$$

(b) Imaginary part of $X(e^{j\omega})$:

$$\begin{aligned} X_I(e^{j\omega}) &= \frac{1}{2j}\cdot[X(e^{j\omega}) - X^*(e^{j\omega})] \\ &= \frac{-a\sin(\omega)}{1-2a\cos(\omega)+a^2}. \end{aligned}$$

(c) Magnitude of $X(e^{j\omega})$:

$$
\begin{aligned}
|X(e^{j\omega})| &= [X(e^{j\omega})X^*(e^{j\omega})]^{\frac{1}{2}} \\
&= \left(\frac{1}{1-2a\cos(\omega)+a^2}\right)^{\frac{1}{2}}.
\end{aligned}
$$

(d) Phase of $X(e^{j\omega})$:

$$
\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin(\omega)}{1-a\cos(\omega)}\right).
$$

7. (0.5%) Page 106, Problem 2.33

Which of the following discrete-time signals could be eigenfunctions of any stable LTI system?

(a) $5^n u[n]$

(b) $e^{j2\omega n}$

(c) $e^{j\omega n} + e^{j2\omega n}$

(d) 5^n

(e) $5^n \cdot e^{j2\omega n}$

Sol: Recall that an eigenfunction of a system is an input signal which appears at the output of the system scaled by a complex constant.

(a) $x[n] = 5^n u[n]$:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, 5^{(n-k)} u[n-k] \\
&= 5^n \sum_{k=-\infty}^{n} h[k]\, 5^{-k}.
\end{aligned}
$$

Because the summation depends on n, $x[n]$ is NOT an eigenfunction.

(b) $x[n] = e^{j2\omega n}$:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, e^{j2\omega(n-k)} \\
&= e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j2\omega k} \\
&= e^{j2\omega n} \cdot H(e^{j2\omega}).
\end{aligned}
$$

Yes, $x[n]$ is an eigenfunction!

(c) $x[n] = e^{j\omega n} + e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega(n-k)} + y[n] = \sum_{k=-\infty}^{\infty} h[k]\, e^{j2\omega(n-k)} \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k} + e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j2\omega k} \\ &= e^{j\omega n} H(e^{j\omega}) + e^{j2\omega n} \cdot H(e^{j2\omega}). \end{aligned}$$

Since the input cannot be extracted from the above expression, the sum of complex exponentials is NOT an eigenfunction. (Although, separately the input are eigenfunctions. In general, complex exponential signals are always eigenfunctions of LTI systems.)

(d) $x[n] = 5^n$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, 5^{(n-k)} \\ &= 5^n \sum_{k=-\infty}^{\infty} h[k]\, 5^{-k} \end{aligned}$$

Yes, $x[n]$ is an eigenfunction!

(e) $x[n] = 5^n \cdot e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, 5^{(n-k)} e^{j2\omega(n-k)} \\ &= 5^n e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]\, 5^{-k} e^{-j2\omega k} \end{aligned}$$

Yes, $x[n]$ is an eigenfunction!

8. (0.5%) Page 106, Problem 2.35

Consider an LTI system with frequency response

$$H(e^{j\omega}) = e^{-j(\omega - \frac{\pi}{4})} \left(\frac{1 + e^{-j2\omega} + 4e^{-j4\omega}}{1 + \frac{1}{2}e^{-j2\omega}} \right), \quad -\pi < \omega \le \pi.$$

Determine the output $y[n]$ for all n if the input for all n is

$$x[n] = \cos(\frac{\pi n}{2}).$$

Sol: We first rewrite the system function $H(e^{j\omega})$:

$$\begin{aligned} H(e^{j\omega}) &= e^{j\pi/4} \cdot e^{-j\omega} \left(\frac{1 + e^{-j2\omega} + 4e^{-j4\omega}}{1 + \frac{1}{2}e^{-j2\omega}} \right) \\ &= e^{j\pi/4} G(e^{j\omega}) \end{aligned}$$

Let $y_1[n] = x[n] * g[n]$, then

$$\begin{aligned} x[n] &= \cos(\frac{\pi n}{2}) = \frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2} \\ y_1[n] &= \frac{G(e^{j\pi/2})e^{j\pi n/2} + G(e^{-j\pi/2})e^{-j\pi n/2}}{2} \end{aligned}$$

Evaluating the frequency response at $\omega = \pm\pi/2$:

$$\begin{aligned} G(e^{j\pi/2}) &= e^{-j\pi/2}\left(\frac{1 + e^{-j\pi} + 4e^{-j2\pi}}{1 + \frac{1}{2}e^{-j\pi}}\right) = 8e^{-j\pi/2} \\ G(e^{-j\pi/2}) &= 8e^{j\pi/2} \end{aligned}$$

Therefore,

$$\begin{aligned} y_1[n] &= (8e^{j(\pi n/2 - \pi/2)} + 8e^{j(-\pi n/2 + \pi/2)})/2 \\ &= 8\cos(\frac{\pi n}{2} - \frac{\pi}{2}) \end{aligned}$$

and

$$\begin{aligned} y[n] &= e^{j\pi/4} y_1[n] \\ &= 8e^{j\pi/4}\cos(\frac{\pi n}{2} - \frac{\pi}{2}). \end{aligned}$$

9. (0.5%) Page 108, Problem 2.40

   Determine which of the following signals is periodic. If a signal is periodic, determine its period.

   (a) $x[n] = e^{j(2\pi n/5)}$
   (b) $x[n] = \sin(\pi n/19)$
   (c) $x[n] = ne^{j\pi n}$
   (d) $x[n] = e^{jn}$.

   Sol: $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

   (a) $x[n]$ is periodic with period 5:

   $$e^{j(\frac{2\pi}{5}n)} = e^{j(\frac{2\pi}{5})(n+N)} = e^{j(\frac{2\pi}{5}n + 2\pi k)}$$

   $$\Rightarrow 2\pi k = \frac{2\pi}{5}N, \text{ for some integers } k,\ N$$

   Making $k = 1$ and $N = 5$ shows that $x[n]$ has period 5.

   (b) $x[n]$ is periodic with period 38. Since the sin function has period of 2π:

   $$x[n+38] = \sin(\pi(n+38)/19) = \sin(\pi n/19 + 2\pi) = x[n].$$

   (c) This is NOT periodic because the linear term n is not periodic.

(d) This is NOT periodic. $e^{j\omega}$ is periodic over 2π, so we have to find k,N such that

$$x[n+N] = e^{j(n+N)} = e^{j(n+2\pi k)}.$$

Since we cannot make k,N integers at the same time, $x[n]$ is not periodic.

10. (0.5%) Page 113, Problem 2.50

Consider a system with input $x[n]$ and output $y[n]$. The input-output relation for the system is defined by the following two properties:

(1) $y[n] - ay[n-1] = x[n]$,
(2) $y[0] = 1$.

(a) Determine whether the system is time-invariant.
(b) Determine whether the system is linear.
(c) Assume that the difference equation (property 1) remains the same, but the value $y[0]$ is specified to be zero. Does this change your answer to either part (a) or part (b)?

Sol:

(a) For $x_1[n] = \delta[n]$,

$$\begin{aligned} y_1[0] &= 1, \\ y_1[1] &= ay_1[0] = a. \end{aligned}$$

or $x_2[n] = \delta[n-1]$,

$$\begin{aligned} y_2[0] &= 1, \\ y_2[1] &= ay_2[0] + x_2[1] = a + 1 \neq y_1[0]. \end{aligned}$$

Even though $x_2[n] = x_1[n-1]$, $y_2[n] \neq y_1[n-1]$. Hence, the system is NOT time-invariant.

(b) A linear system has the property that

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}.$$

Hence, if the input is doubled, the output must also double at each value of n. Because $y[0] = 1$, always, the system is NOT linear.

(c) Let $x_3[n] = \alpha x_1[n] + \beta x_2[n]$.
For $n \geq 0$,

$$\begin{aligned} y_3[n] &= x_3[n] + ay_3[n-1] \\ &= \alpha x_1[n] + \beta x_2[n] + a(x_3[n-1] + y_3[n-2]) \\ &= \alpha \sum_{k=0}^{n-1} a^k x_1[n-k] + \beta \sum_{k=0}^{n-1} a^k x_2[n-k] \\ &= \alpha(h[n] * x_1[n]) + \beta(h[n] * x_2[n]) \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

For $n < 0$,

$$\begin{aligned} y_3[n] &= a^{-1}(y_3[n+1] + x_3[n]) \\ &= -\alpha \sum_{k=-1}^{n} a^k x_1[n-k] - \beta \sum_{k=-1}^{n} a^k x_2[n-k] \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

For $n = 0$,

$$y_3[n] = y_2[n] = y_1[n] = 0.$$

Conclude,

$$y_3[n] = \alpha y_1[n] + \beta y_2[n], \text{ for all } n.$$

Therefore, the system is linear. The system is still NOT time-invariant.