8 The Discrete Fourier Transform (DFT)

- Discrete-Time Fourier Transform and Z-transform are defined over infinite-duration sequence. Both transforms are functions of continuous variables (ω and z).

  For finite-duration sequences, one can define an alternative Fourier representation, called *Discrete Fourier Transform*, which is a function of discrete variables. In general, we represent DFT as a sequence.

  Consider complex sequences below.

8.1 Representation of Periodic Sequences: The Discrete Fourier Series

- Let a complex sequence $\widetilde{x}[n]$ be periodic with period N (i.e., $\widetilde{x}[n] = \widetilde{x}[n + rN]$ for any integer values of n and r). Any complex periodic sequence of period N can be represented by a linear sum of complex exponentials with frequencies being integer multiples of the *fundamental frequency* $\frac{2\pi}{N}$, as

$$\widetilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{X}[k] \, e^{\frac{j2\pi k}{N} n}. \qquad \circledast$$

  This expansion is called the *Fourier series representation* of $\widetilde{x}[n]$ with coefficients $\widetilde{X}[k]$'s.

  Note: Since there are N different complex exponential sequences $e^{\frac{j2\pi kn}{N}}$, the summation extends over frequency variable k from 0 to $N-1$.

- How can we obtain the coefficient $\widetilde{X}[k]$ from $\widetilde{x}[n]$?

Now, for an integer r,

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \widetilde{x}[n]\, e^{-\frac{j2\pi r}{N}n} \\
=\;& \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{X}[k]\, e^{\frac{j2\pi}{N}(k-r)n} \\
=\;& \sum_{k=0}^{N-1} \widetilde{X}[k] \cdot \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} e^{\frac{j2\pi}{N}(k-r)n}}_{=\begin{cases} 1, & \text{if } k=r \\ 0, & \text{otherwise} \end{cases}} \\
&\left(\sum_{n=0}^{N-1} e^{\frac{j2\pi}{N}\alpha n} = \frac{1-e^{\frac{j2\pi}{N}\alpha N}}{1-e^{\frac{j2\pi}{N}\alpha}} = 0 \text{ for integer } a \neq 0 \right) \\
=\;& \widetilde{X}[r]
\end{aligned}
$$

$$\Rightarrow \widetilde{X}[k] = \sum_{n=0}^{N-1} \widetilde{x}[n]\, e^{-\frac{j2\pi n}{N}k}.$$

Note: $\widetilde{X}[k]$ is also periodic with period N in general and thus can be represented by a Fourier series expansion.

- We shall define hereafter

$$W_N \triangleq e^{-\frac{j2\pi}{N}}$$

with $\frac{2\pi}{N}$ being the fundamental frequency and refer to the following

$$\widetilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]\, W_N^{-kn} \quad \text{(Synthesis Equation)}$$

and

$$\widetilde{X}[k] = \sum_{n=0}^{N-1} \widetilde{x}[n]\, W_N^{kn} \quad \text{(Analysis Equation)}$$

as the *Discrete Fourier Series* (DFS) pair, denoted by

$$\widetilde{x}[n] \xleftrightarrow{\mathcal{DFS}} \widetilde{X}[k].$$

- Ex: DFS of a Periodic Impulse Train

$$\begin{aligned}
\widetilde{x}[n] &= \sum_{r=-\infty}^{\infty} \delta[n - rN] \\
\Rightarrow \quad \widetilde{X}[k] &= \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = 1 \text{ for all } k \\
\widetilde{x}[n] &= \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{j2\pi}{N} nk}
\end{aligned}$$

Fig. 81-F1

Note that $\sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{j2\pi}{N} nk}$.

- Ex: Duality in the DFS

  Consider the DFS coefficients as a periodic impulse train

$$\begin{aligned}
\widetilde{Y}[k] &= \sum_{r=-\infty}^{\infty} N\delta[k - rN] \\
\Rightarrow \quad \widetilde{y}[n] &= \frac{1}{N} \sum_{k=0}^{N-1} N\delta[k] W_N^{-kn} = 1 \text{ for all } n \\
\widetilde{Y}[k] &= \sum_{n=0}^{N-1} W_N^{kn} = \sum_{n=0}^{N-1} e^{-\frac{j2\pi}{N} nk}.
\end{aligned}$$

- Ex*: The DFS of a Periodic Rectangular Pulse Train

  Let N be even and consider

$$\widetilde{x}[n] = \begin{cases} 1, & Nl \le n < Nl + \frac{N}{2} \\ 0, & Nl + \frac{N}{2} \le n < N(l+1) \end{cases} ;\forall l \text{ integer}$$

$$\Rightarrow \quad \widetilde{X}[k] = \sum_{n=0}^{N/2-1} W_N^{kn} = \frac{1 - W_N^{kN/2}}{1 - W_N^k}$$

Fig. 8.1

Now, we further have

$$
\begin{aligned}
\widetilde{X}[k] &= \frac{W_{2N}^{kN/2}\left(W_{2N}^{-kN/2} - W_{2N}^{kN/2}\right)}{W_{2N}^{k}\left(W_{2N}^{-k} - W_{2N}^{k}\right)} \\
&\quad \left(W_N = e^{-\frac{j2\pi}{N}} = e^{-\frac{j2\pi}{2N}} \cdot e^{-\frac{j2\pi}{2N}} = W_{2N}^2\right) \\
&= \frac{W_{2N}^{k(N/2-1)} \sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{N}\right)} \\
&\quad \left(W_{2N}^{-k} - W_{2N}^{k} = e^{\frac{j2\pi k}{2N}} - e^{-\frac{j2\pi k}{2N}} = 2j \sin\left(\frac{\pi k}{N}\right)\right) \\
&= e^{-j\pi k\left(\frac{1}{2}-\frac{1}{N}\right)} \frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{N}\right)}.
\end{aligned}
$$

Fig. 8.2

- Now, let

$$
x[n] = \begin{cases} \widetilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}
$$

and find its DT Fourier transform

$$
\begin{aligned}
X\left(e^{j\omega}\right) &= \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n} = \sum_{n=0}^{N-1} \widetilde{x}[n]\, e^{-j\omega n} \\
\Rightarrow \quad \widetilde{X}[k] &= X\left(e^{j\omega}\right)\Big|_{\omega = \frac{2\pi k}{N}}.
\end{aligned}
$$

Thus, the DFS $\widetilde{X}[k]$ is a sequence of equally-spaced samples of $X\left(e^{j\omega}\right)$, with a frequency spacing of $\frac{2\pi}{N}$.

Note: In Ex*, if we let

$$
x[n] = \begin{cases} 1, & 0 \le n \le \frac{N}{2} - 1 \\ 0, & \text{elsewhere} \end{cases}
$$

$$
\Rightarrow \quad X\left(e^{j\omega}\right) = \sum_{n=0}^{N/2-1} e^{-j\omega n} = e^{-j\left(\frac{N}{2}-1\right)\omega} \frac{\sin\left(\frac{N}{4}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}
$$

$$
\Rightarrow \quad \widetilde{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}}.
$$

8.2 Properties of DFS

- Assume that all sequences below have period N. The DFS has the following properties:

8.2.1 Linearity:

If

$$\widetilde{x}_1[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \widetilde{X}_1[k]$$
$$\widetilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \widetilde{X}_2[k]$$

then

$$a\widetilde{x}_1[n] + b\widetilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} a\widetilde{X}_1[k] + b\widetilde{X}_2[k].$$

(In general, superposition principle holds.)

8.2.2 Shifting of a Sequence:

If

$$\widetilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \widetilde{X}[k]$$

then

$$\widetilde{x}[n-m] = \frac{1}{N}\sum_{k=0}^{N-1}\widetilde{X}[k]\,W_N^{-k(n-m)} \overset{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\widetilde{X}[k],\ m \text{ integer.}$$

Note: $\widetilde{x}[n-m_1] = \widetilde{x}[n-m_1-lN]$, l integer.

Furthermore,

$$W_N^{-nl}\widetilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \widetilde{X}[k-l],\ l \text{ integer.}$$

8.2.3 Duality:

Now,

$$\begin{aligned}
\widetilde{X}[k] &= \sum_{n=0}^{N-1}\widetilde{x}[n]\,W_N^{kn} \\
&= N\cdot\frac{1}{N}\sum_{n=0}^{N-1}\widetilde{x}[n]\,W_N^{kn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\left(N\widetilde{x}[n]\right)W_N^{-k(-n)} \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\left(N\widetilde{x}[N-m]\right)W_N^{-km} \quad \begin{pmatrix} W_N^{-n} = W_N^{N-n} \\ m = N-n \end{pmatrix} \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\left(N\widetilde{x}[-m]\right)W_N^{-km}
\end{aligned}$$

and

$$\widetilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]\, W_N^{-kn}.$$

Thus, if

$$\widetilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \widetilde{X}[k]$$

then

$$\widetilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\widetilde{x}[-k].$$

Note: This property is unique for DFS!

8.2.4 Symmetry Properties: See items 9-17 in Table 8.1.

Table 8.1

8.2.5 Periodic Convolution:

$$\sum_{m=0}^{N-1} \widetilde{x}_1[m]\, \widetilde{x}_2[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} \widetilde{X}_1[k]\, \widetilde{X}_2[k]$$

Pf: The Fourier series coefficients of $\sum_{m=0}^{N-1} \widetilde{x}_1[m]\, \widetilde{x}_2[n-m]$ are

$$\begin{aligned} & \sum_{n=0}^{N-1} W_N^{kn} \left(\sum_{m=0}^{N-1} \widetilde{x}_1[m]\, \widetilde{x}_2[n-m]\right) \\ = & \sum_{m=0}^{N-1} \widetilde{x}_1[m] \underbrace{\left(\sum_{n=0}^{N-1} \widetilde{x}_2[n-m]\, W_N^{k(n-m)}\right)}_{=\widetilde{X}_2[k] \text{ (since } \widetilde{x}_2[n] \text{ is periodic with period } N)} W_N^{km} \\ = & \widetilde{X}_2[k]\, \widetilde{X}_1[k]. \end{aligned}$$

Notes:

1. We shall denote the periodic convolution

$$\widetilde{x}_1[n] \circledast_P \widetilde{x}_2[n] = \sum_{m=0}^{N-1} \widetilde{x}_1[m]\, \widetilde{x}_2[n-m].$$

2. $\widetilde{x}_1[n] \circledast_P \widetilde{x}_2[n] = \widetilde{x}_2[n] \circledast_P \widetilde{x}_1[n]$, i.e., $\circledast_P$ is commutative.
3. The periodic convolution of two periodic sequences of period N is also a periodic sequence of period N.

4. Procedure comparison between periodic convolution and aperiodic convolution:

Fig. 8.4

Aperiodic: (Convolution Sum)

Fig. 84-B1

Periodic:

Fig. 85-F1

8.3 The DT Fourier Transform of Periodic Signals

- Recall that "absolute summability" is a sufficient condition for the existence of DT Fourier transform.

  No periodic sequence is absolutely summable!

- Defn: The DT Fourier transform $\widetilde{X}\left(e^{j\omega}\right)$ of a periodic sequence $\widetilde{x}[n]$ is defined as an impulse train in the frequency domain with impulse values proportional to the DFS coefficients of $\widetilde{x}[n]$.

  Mathematically,

$$\widetilde{X}\left(e^{j\omega}\right) \triangleq \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\widetilde{X}[k]\,\delta\left(\omega - \frac{2\pi k}{N}\right)$$

  where $\delta(\omega)$ is an impulse.

  Now, denoting x^- as a value smaller than but extremely close to x,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{0^-}^{2\pi^-} \widetilde{X}\left(e^{j\omega}\right) e^{j\omega n} d\omega \\
=\ & \frac{1}{2\pi}\int_{0^-}^{2\pi^-} \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\widetilde{X}[k]\,\delta\left(\omega - \frac{2\pi k}{N}\right) e^{j\omega n} d\omega \\
=\ & \frac{1}{N}\sum_{k=-\infty}^{\infty} \widetilde{X}[k] \int_{0^-}^{2\pi^-} \delta\left(\omega - \frac{2\pi k}{N}\right) e^{j\omega n} d\omega \\
& \left(\int_{0^-}^{2\pi^-} \delta\left(\omega - \frac{2\pi k}{N}\right) e^{j\omega n} d\omega = \begin{cases} e^{j\frac{2\pi}{N}nk}, & \text{if } 0 \le \frac{2\pi k}{N} < 2\pi \\ 0, & \text{otherwise} \end{cases} \right) \\
=\ & \frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]\, e^{j\frac{2\pi}{N}nk} \\
=\ & \widetilde{x}[n].
\end{aligned}$$

Thus,

$$\widetilde{x}[n] \stackrel{\mathcal{DTFT}}{\longleftrightarrow} \widetilde{X}\left(e^{j\omega}\right)$$

in a sense that

$$\widetilde{x}[n] = \frac{1}{2\pi}\int_{0^-}^{2\pi^-} \widetilde{X}\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

and

$$\begin{aligned} \widetilde{X}\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\widetilde{X}[k]\,\delta\left(\omega - \frac{2\pi k}{N}\right) \qquad \circledast \\ \widetilde{X}[k] &= \sum_{n=0}^{N-1} \widetilde{x}[n]\, e^{-j\frac{2\pi}{N}nk}. \end{aligned}$$

- Note: This $\widetilde{X}\left(e^{j\omega}\right)$ does not converge in any normal sense. However, this form $\circledast$ is useful and convenient for analysis purpose. In other words, the introduction of impulses permits us to include periodic sequences formally within the framework of discrete-time Fourier transform analysis.

- Ex: The DTFT of a Periodic Discrete-Time Impulse Train

  Consider the periodic discrete-time impulse train

  $$\widetilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN].$$

  Now, its DFS coefficient is

  $$\widetilde{P}[k] = 1 \text{ for all } k$$

  and the corresponding DTFT is

  $$\widetilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right).$$

- Consider a finite-length signal $x[n]$ such that $x[n] = 0$ except in the interval $0 \leq n \leq N-1$. Now, we produce a periodic signal $\widetilde{x}[n]$ by (aperiodically) convolving $x[n]$ with the periodic impulse train $\widetilde{p}[n]$ as

  $$\widetilde{x}[n] = x[n] * \widetilde{p}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]. \quad \text{(X)}$$

  Fig. 8.4

Note that we can express $x[n]$ in terms of $\widetilde{x}[n]$ as

$$x[n] = \begin{cases} \widetilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}.$$

From (X), the DT Fourier transforms of $x[n]$ and $\widetilde{x}[n]$ are related by

$$\begin{aligned} \widetilde{X}\left(e^{j\omega}\right) &= X\left(e^{j\omega}\right)\widetilde{P}\left(e^{j\omega}\right) \\ &= X\left(e^{j\omega}\right)\sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\delta\left(\omega-\frac{2\pi k}{N}\right) \\ &= \sum_{k=-\infty}^{\infty}\frac{2\pi}{N}X\left(e^{j\frac{2\pi k}{N}}\right)\delta\left(\omega-\frac{2\pi k}{N}\right). \end{aligned}$$

Thus, $\widetilde{X}[k]$ is related to $X\left(e^{j\frac{2\pi k}{N}}\right)$ by

$$\widetilde{X}[k] = X\left(e^{j\frac{2\pi k}{N}}\right) = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}}.$$

That is, $\widetilde{X}[k]$ is obtained by sampling the DTFT $X\left(e^{j\omega}\right)$ at N equally spaced frequencies between $\omega = 0$ and $\omega = 2\pi$ with a frequency spacing of $\frac{2\pi}{N}$. This is exactly the same relation as we have shown in Section 8.1.

- Ex: Relationship Between the DFS Coefficients and the DTFT of One Period

  Consider the finite-length signal with $N = 10$

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & 5 \leq n \leq 9 \end{cases}$$

$$\Rightarrow \quad X\left(e^{j\omega}\right) = \sum_{n=0}^{4} e^{-j\omega n} = e^{-j2\omega}\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\Rightarrow \quad \widetilde{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{10}} = e^{-j2\pi k/5}\frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{10}\right)}$$

  where $\widetilde{X}[k]$ is the DFS coefficient of the periodic signal $\widetilde{x}[n]$, given by

$$\widetilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-10r].$$

Figs. 8.5-8.6

8.4 Sampling The DT Fourier Transform

- Consider an aperiodic sequence $x[n]$ with DT Fourier transform $X(e^{j\omega}) = \mathcal{F}\{x[n]\}$, i.e.,

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}.$$

  Note that $x[n]$ may be of finite duration or of infinite duration.

  Let us pick one number N and sample $X(e^{j\omega})$ at N equally spaced frequencies between $\omega = 0$ and $\omega = 2\pi$ with a frequency spacing of $\frac{2\pi}{N}$, as

$$\begin{aligned}\widetilde{X}[k] &= X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi}{N}k} = X\left(e^{j\frac{2\pi}{N}k}\right) \\ &= X(z)\big|_{z=e^{j\frac{2\pi}{N}k}}\end{aligned}$$

  which is periodic with period N.

Fig. 8.7

- Now, let $\widetilde{x}[n]$ be the periodic sequence expanded by $\widetilde{X}[k]$, i.e.,

$$\widetilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]\, W_N^{-kn}$$

$$\begin{aligned}\Rightarrow\quad \widetilde{x}[n] &= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]\, \underbrace{e^{-j\frac{2\pi}{N}km}}_{W_N^{km}}\right] W_N^{-kn} \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[m]\, W_N^{k(m-n)} \\ &= \sum_{m=-\infty}^{\infty} x[m]\, \frac{1}{N}\sum_{k=0}^{N-1} W_N^{k(m-n)}.\end{aligned}$$

  Since

$$\begin{aligned}\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k\alpha} &= \begin{cases} 1, & \text{if } \alpha = N\cdot\text{integer} \\ \frac{1}{N}\frac{1-W_N^{-N\alpha}}{1-W_N^{-\alpha}} = 0, & \text{otherwise}\end{cases} \\ &= \sum_{k=-\infty}^{\infty} \delta[\alpha - kN]\end{aligned}$$

$$\Rightarrow \quad \widetilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m] \sum_{k=-\infty}^{\infty} \delta[(n-m)-kN]$$

$$= x[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

$$\Rightarrow \quad \boxed{\widetilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n-kN]}$$

i.e., $\widetilde{x}[n]$ represents the periodic sequence resulting from repeating $x[n]$ every N samples.

Figs. 8.8-8.9

- Notes:

1. In other words, the samples of DT Fourier transform of $x[n]$ can be thought of as DFS coefficients of $\widetilde{x}[n]$.
2. If $x[n]$ is of finite duration, it can be recovered from $\widetilde{x}[n]$ provided that N is larger than its duration. Specifically,

$$x[n] = \begin{cases} \widetilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases}.$$

3. Given a finite-length $x[n]$, pick N larger than or equal to the length and conduct the transforms

Fig. 87-B1

Under this circumstance, $\widetilde{X}[k]$ for $k = 0, 1, ..., N-1$ is called the discrete Fourier transform (DFT) of the finite-length sequence $x[n]$.

(a) Key Point: $x[n]$ is one period of $\widetilde{x}[n]$ (may not start from $n = 0$!) since

$$\widetilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n-kN].$$

(b) Note: N has to be chosen not smaller than the length of $x[n]$!

8.5 Fourier Representation of Finite-Duration Sequences: The DFT

- Consider a finite-duration sequence $x[n]$ with $x[n] = 0$ outside the range $0 \leq n \leq N-1$. Define

$$\widetilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

which is periodic with period N. Then, it is clear that $x[n]$ can be retrieved from $\widetilde{x}[n]$

$$x[n] = \begin{cases} \widetilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}.$$

- An alternative form to define $\widetilde{x}[n]$ is

$$\widetilde{x}[n] = x[(n \text{ modulo } N)] \triangleq x[((n))_N].$$

A cylindrical visualization of $\widetilde{x}[n] = x[((n))_N]$ is

Fig. 88-B1

- Let $\widetilde{X}[k]$ be the DFS of $\widetilde{x}[n]$ and define

$$X[k] = \begin{cases} \widetilde{X}[k], & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}.$$

$$\Rightarrow \quad \widetilde{X}[k] = X[((k))_N].$$

Since

$$\begin{aligned} \widetilde{X}[k] &= \sum_{n=0}^{N-1} \widetilde{x}[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] W_N^{kn} \\ \widetilde{x}[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{X}[k] W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \end{aligned}$$

$$\Rightarrow \quad X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases} \qquad (*)$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}. \qquad (+)$$

(*) and (+) form a Discrete Fourier Transform pair; (*) is called an analysis equation, while (+) is called a synthesis equation. We shall use the notation

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$$

and $X[k] = \mathcal{DFT}\{x[n]\}$ throughout.

- Usually, we use for notational brevity

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

and

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

as the definition of the DFT pair, keeping in mind that $X[k] = 0$ for $k \notin [0,\ N-1]$ and $x[n] = 0$ for $n \notin [0,\ N-1]$.

- Ex: The DFT of a Rectangular Pulse

Consider the finite-duration sequence with length N $(N \geq 5)$

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & 5 \leq n \leq N-1 \end{cases}$$

Fig. 8.10 for $N = 5$

Fig. 8.11 for $N = 10$

8.6 Properties of DFT

- Consider the properties of the DFT for finite-duration sequences. We let $x_i[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_i[k]$ defined over length N_i throughout.

8.6.1 Linearity:

If

$$x_3[n] = ax_1[n] + bx_2[n]$$

then

$$X_3[k] = aX_1[k] + bX_2[k]$$

where $x_1[n]$ and $x_2[n]$ are of length N_1 and N_2 respectively; and $x_3[n]$ is of length $N_3 = \max\{N_1, N_2\}$. The DFT's $X_1[k]$, $X_2[k]$, and $X_3[k]$ are defined over length N_3.

Fig. 89-B1

In other words, the shorter sequence should be zero-padded, and the corresponding DFT defined over the largest length.

8.6.2 Circular Shift of A Sequence:

Let

$$\widetilde{x}[n] = x[((n))_N]$$

and

$$\widetilde{x}_1[n] = \widetilde{x}[n-m].$$

Then,

$$\begin{aligned}
x_1[n] &= \begin{cases} \widetilde{x}_1[n] = \widetilde{x}[n-m], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \widetilde{x}[n-m], & m \le n \le N-1 \\ \widetilde{x}[n-m+N], & 0 \le n \le m-1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} x[n-m], & m \le n \le N-1 \\ x[n-m+N], & 0 \le n \le m-1 \quad (\widetilde{x}[n] = x[((n))_N]) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Fig. 8.12

Thus, we say

$$x_1[n] = \begin{cases} x[((n-m))_N], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

a circular shift of $x[n]$ by m samples. Since

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$$
\begin{aligned}
\Rightarrow X_1[k] &= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\, W_N^{kn} \text{ for } 0 \le k \le N-1 \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x\left[((n-m))_N\right] W_N^{kn} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \widetilde{x}[n-m]\, W_N^{kn} \\
&= W_N^{km}\cdot\frac{1}{N}\underbrace{\sum_{n'=-m}^{N-1-m}}_{n'=n-m} \underbrace{\widetilde{x}\left[n'\right]}_{\text{periodic}} \underbrace{W_N^{kn'}}_{\text{periodic}} \\
&= W_N^{km}\cdot\frac{1}{N}\sum_{n'=0}^{N-1} \widetilde{x}\left[n'\right] W_N^{kn'} \\
&= W_N^{km}\cdot\frac{1}{N}\sum_{n=0}^{N-1} x\left[((n))_N\right] W_N^{kn} \\
&= W_N^{km}\cdot\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, W_N^{kn} \\
&= W_N^{km} X[k].
\end{aligned}
$$

In summary,

$$x_1[n] = x\left[((n-m))_N\right],\ 0 \le n \le N-1 \overset{\mathcal{DFT}}{\longleftrightarrow} W_N^{km} X[k].$$

8.6.3 Duality:

Since

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

we have

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \left(N\widetilde{x}[n]\right) W_N^{kn} \\
&= \frac{1}{N} \sum_{n^{'}=1}^{N} \left(N\underbrace{\widetilde{x}\left[N-n^{'}\right]}_{\text{periodic}} \right) \underbrace{W_N^{k\left(N-n^{'}\right)}}_{\text{periodic}} \quad \left(n^{'} = -n+N\right) \\
&= \frac{1}{N} \sum_{n^{'}=0}^{N-1} \left(N\widetilde{x}\left[N-n^{'}\right]\right) W_N^{k\left(N-n^{'}\right)} \\
&= \frac{1}{N} \sum_{n^{'}=0}^{N-1} \left(Nx\left[\left(\left(N-n^{'}\right)\right)_N\right]\right) W_N^{-kn^{'}} \\
&= \frac{1}{N} \sum_{n^{'}=0}^{N-1} \left(Nx\left[\left(\left(-n^{'}\right)\right)_N\right]\right) W_N^{-kn^{'}}.
\end{aligned}
$$

Thus, if

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$$

then

$$X[n] \overset{\mathcal{DFT}}{\longleftrightarrow} Nx\left[((-k))_N\right] \text{ for } 0 \le k \le N-1.$$

Fig. 8.13

8.6.4 Symmetry Property:

Since

$$
\begin{aligned}
\widetilde{x}[n] &= x\left[((n))_N\right] \\
\widetilde{X}[k] &= X\left[((k))_N\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{x}^*[n] &\overset{\mathcal{DFS}}{\longleftrightarrow} \widetilde{X}^*[-k] \\
\widetilde{x}^*[-n] &\overset{\mathcal{DFS}}{\longleftrightarrow} \widetilde{X}^*[k]
\end{aligned}
$$

we have

$$x^*\left[((n))_N\right] \text{ for } 0 \le n \le N-1 \overset{\mathcal{DFT}}{\longleftrightarrow} X^*\left[((-k))_N\right] \text{ for } 0 \le k \le N-1$$

$$\Rightarrow \boxed{x^*[n] \text{ for } 0 \le n \le N-1 \overset{\mathcal{DFT}}{\longleftrightarrow} X^*\left[((-k))_N\right] \text{ for } 0 \le k \le N-1}$$

and

$$x^*\left[((-n))_N\right] \text{ for } 0 \le n \le N-1 \overset{\mathcal{DFT}}{\longleftrightarrow} X^*\left[((k))_N\right] \text{ for } 0 \le k \le N-1$$

$$\Rightarrow \boxed{x^*\left[((-n))_N\right] \text{ for } 0 \le n \le N-1 \overset{\mathcal{DFT}}{\longleftrightarrow} X^*[k] \text{for } 0 \le k \le N-1}$$

See Table 8.2 for more symmetry properties.

Table 8.2

8.6.5 Circular Convolution:

Let $\widetilde{x}_3[n] = \widetilde{x}_1[n] \circledast_P \widetilde{x}_2[n]$. Then

$$\begin{aligned} x_3[n] &\triangleq \widetilde{x}_3[n] \text{ for } 0 \le n \le N-1 \\ &= \widetilde{x}_1[n] \circledast_P \widetilde{x}_2[n] \\ &= \sum_{m=0}^{N-1} \widetilde{x}_1[m]\, \widetilde{x}_2[n-m] \\ &= \sum_{m=0}^{N-1} x_1\left[((m))_N\right] x_2\left[((n-m))_N\right] \\ &= \sum_{m=0}^{N-1} x_1[m]\, x_2\left[((n-m))_N\right] \text{ for } 0 \le n \le N-1. \end{aligned}$$

This last expression is called the circular convolution between $x_1[n]$ and $x_2[n]$ (both finite-duration sequences of length N). We shall denote

$$x_3[n] = x_1[n] \circledast^N x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]\, x_2\left[((n-m))_N\right].$$

Furthermore,

$$\begin{aligned} x_3[n] &= \sum_{m=0}^{N-1} \widetilde{x}_1[n-m]\, \widetilde{x}_2[m] \text{ for } 0 \le n \le N-1 \quad \left(\begin{array}{r} \text{since } \widetilde{x}_1[n] \circledast_P \widetilde{x}_2[n] \\ = \widetilde{x}_2[n] \circledast_P \widetilde{x}_1[n] \end{array}\right) \\ &= \sum_{m=0}^{N-1} x_1\left[((n-m))_N\right] x_2[m] \end{aligned}$$

which tells us

$$\begin{aligned} x_3[n] &= x_1[n] \circledast^N x_2[n] = x_2[n] \circledast^N x_1[n]. \\ &\quad \left(\text{i.e., } \circledast^N \text{ is commutative}\right) \end{aligned}$$

- Ex: Circular Convolution With a Delayed Impulse Sequence

  Let $x_2[n]$ be a finite-duration sequence of length N and

$$\begin{aligned} & x_1[n] = \delta[n-n_0] \text{ for } 0 < n_0 < N \\ \Rightarrow\ & X_1[k] = W_N^{kn_0}. \end{aligned}$$

  Defining $X_3[k] = W_N^{kn_0} X_2[k]$, we find from the circular shifting property, $x_3[n] = x_2[((n-n_0))_N]$ for $0 \le n \le N-1$. Also,

$$\begin{aligned} x_1[n] \circledast^N x_2[n] &= \sum_{m=0}^{N-1} x_1[m]\, x_2[((n-m))_N] \\ &= \sum_{m=0}^{N-1} \delta[m-n_0]\, x_2[((n-m))_N] \\ &= x_2[((n-n_0))_N] \text{ for } 0 \le n \le N-1. \end{aligned}$$

  Thus, this example implies

$$x_1[n] \circledast^N x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k]\, X_2[k].$$

- In general, letting $N = \max\{N_1, N_2\}$, we have for $x_i[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_i[k]$ defined over length N that

$$\begin{aligned} X_1[k]\, X_2[k] &= \sum_{n_1=0}^{N-1} x_1[n_1]\, W_N^{kn_1} \sum_{n_2=0}^{N-1} x_2[n_2]\, W_N^{kn_2} \\ &= \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \widetilde{x}_1[n_1]\, \widetilde{x}_2[n_2]\, W_N^{k(n_1+n_2)} \\ &= \sum_{n_1=0}^{N-1} \sum_{n=n_1}^{N-1+n_1} \underset{x_1[n_1]}{\underset{\shortparallel}{\widetilde{x}_1[n_1]}}\, \underset{x_2[((n-n_1))_N]}{\underset{\shortparallel}{\widetilde{x}_2[n-n_1]}}\, W_N^{kn} \quad \left(\begin{array}{r} n = n_1 + n_2 \\ \Rightarrow n_2 = n - n_1 \end{array} \right) \\ &= \sum_{n_1=0}^{N-1} \sum_{n=0}^{N-1} x_1[n_1]\, \underbrace{x_2[((n-n_1))_N]}_{\text{periodic}}\, \underbrace{W_N^{kn}}_{\text{periodic}} \\ &= \sum_{n=0}^{N-1} \left(\sum_{n_1=0}^{N-1} x_1[n_1]\, x_2[((n-n_1))_N] \right) W_N^{kn} \\ &= \sum_{n=0}^{N-1} \left(x_1[n] \circledast^N x_2[n] \right) W_N^{kn}. \end{aligned}$$

Similarly, one can show that

$$x_1[n] \circledast^N x_2[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]\, X_2[k]\, W_N^{-kn}.$$

Thus,

$$x_1[n] \circledast^N x_2[n] \text{ for } 0 \le n \le N-1 \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k]\, X_2[k] \text{ for } 0 \le k \le N-1.$$

- A duality to the previous property is

$$x_1[n]\, x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} \frac{1}{N} X_1[k] \circledast^N X_2[k]$$

Pf:

$$\begin{aligned}
x_1[n]\, x_2[n] &= \frac{1}{N}\sum_{k_1=0}^{N-1} X_1[k_1]\, W_N^{-k_1 n} \cdot \frac{1}{N}\sum_{k_2=0}^{N-1} X_2[k_2]\, W_N^{-k_2 n} \\
&= \left(\frac{1}{N}\right)^2 \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} \widetilde{X}_1[k_1]\, \widetilde{X}_2[k_2]\, W_N^{-(k_1+k_2)n} \\
&= \left(\frac{1}{N}\right)^2 \sum_{k_1=0}^{N-1}\sum_{k=k_1}^{N-1+k_1} \underset{X_1[k_1]}{\underset{\shortparallel}{\overset{\text{periodic}}{\widetilde{X}_1[k_1]}}} \underset{X_2[((k-k_1))_N]}{\underset{\shortparallel}{\overset{\text{periodic}}{\widetilde{X}_2[k-k_1]}}} W_N^{-kn} \\
&\quad \left(\begin{array}{c} k = k_1 + k_2 \\ \Rightarrow k_2 = k - k_1 \end{array}\right) \\
&= \left(\frac{1}{N}\right)^2 \sum_{k_1=0}^{N-1}\sum_{k=0}^{N-1} X_1[k_1]\, X_2[((k-k_1))_N]\, W_N^{-kn} \\
&= \left(\frac{1}{N}\right)^2 \sum_{k=0}^{N-1}\left(\sum_{k_1=0}^{N-1} X_1[k_1]\, X_2[((k-k_1))_N]\right) W_N^{-kn} \\
&= \frac{1}{N}\cdot\frac{1}{N}\sum_{k=0}^{N-1} X_1[k] \circledast^N X_2[k]\, W_N^{-kn} \\
&= \frac{1}{N}\mathcal{IDFT}\left\{X_1[k] \circledast^N X_2[k]\right\}.
\end{aligned}$$

Similarly, the other direction

$$\frac{1}{N} X_1[k] \circledast^N X_2[k] = \mathcal{DFT}\left\{x_1[n]\, x_2[n]\right\}$$

can be proved. QED

- See Table 8.2 for a complete list of DFT properties (prove them for your own fun).

8.7 Computing Linear Convolution Using The DFT

- Since efficient algorithms exist for computing DFT (e.g., Fast Fourier Transform (FFT)), it is desirable to compute linear convolution of two sequences (e.g., implementing an LTI system) by DFT approach. That is, we want to perform

  Fig. 94-B1

  Question: How is $x_1[n] * x_2[n]$ (linear convolution) related to $x_1[n] \circledast^N x_2[n]$ (circular convolution)?

- Now, consider causal sequences $x_1[n]$ of length L and $x_2[n]$ of length P. We want to find the output of the linear convolution

$$x_3[n] = x_1[n] * x_2[n].$$

Taking discrete-time Fourier transform both-sided,

$$X_3\left(e^{j\omega}\right) = X_1\left(e^{j\omega}\right) X_2\left(e^{j\omega}\right).$$

Since for $N \geq \max\{L, P\}$,

$$\begin{aligned} X[k] &= \begin{cases} \widetilde{X}[k], & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} X\left(e^{j\frac{2\pi}{N}k}\right), & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$\Rightarrow \quad X_3[k] = X_3\left(e^{j\frac{2\pi}{N}k}\right) = X_1\left(e^{j\frac{2\pi}{N}k}\right) X_2\left(e^{j\frac{2\pi}{N}k}\right) \text{ for } 0 \leq k \leq N-1$$
$$\Rightarrow \quad X_3[k] = X_1[k] X_2[k] \text{ for } 0 \leq k \leq N-1.$$

Denoting

$$x_{3p}[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_3[k]$$

we have

$$x_{3p}[n] = x_1[n] \circledast^N x_2[n] \text{ for } 0 \leq n \leq N-1$$

where $N \geq \max\{L, P\}$ and

$$x_1[n] \underset{N \text{ points}}{\overset{\mathcal{DFT}}{\longleftrightarrow}} X_1[k], \quad x_2[n] \underset{N \text{ points}}{\overset{\mathcal{DFT}}{\longleftrightarrow}} X_2[k].$$

- Question: How large N should be, such that $x_3[n] = x_{3p}[n]$ for all n?

  Answer:

1.

$$\begin{aligned}
x_3[n] &= x_1[n] * x_2[n] = \sum_{m=-\infty}^{\infty} x_1[m]\, x_2[n-m] \\
&= \sum_{m=\max\{0,n-P+1\}}^{\min\{L-1,n\}} x_1[m]\, x_2[n-m] \quad \left(\begin{array}{l} 0 \le m \le L-1 \\ 0 \le n-m \le P-1 \\ n-P+1 \le m \le n \end{array} \right)
\end{aligned}$$

$\Rightarrow x_3[n]$ is nontrivial when

$$\min\{L-1, n\} \ge \max\{0, n-P+1\}$$

or equivalently when $\boxed{0 \le n \le L+P-2.}$

$$\left(\begin{array}{l} \text{(a) Assume } L \ge P \text{ without loss of generality:} \\ \text{If } n \ge L-1\text{, then } L-1 \ge n-P+1 \Rightarrow, \boxed{n \le L+P-2}. \\ \text{If } L-1 > n \ge P-1\text{, then } n \ge n-P+1 \Rightarrow P \ge 1 \text{ and } \boxed{n \ge 0}. \\ \text{(b) The same result can be obtained for } P \ge L \text{ similarly.} \end{array} \right)$$

Thus, $x_3[n]$ has length $L+P-1$ and is nontrivial when $n = 0, 1, ..., L+P-2$.

$\Rightarrow N$ has to be no smaller than $L+P-1$!!

Fig. 8.17

2. Next, for $0 \le n \le N-1$,

$$x_{3p}[n] = \sum_{m=0}^{N-1} x_1[n]\, x_2[((n-m))_N].$$

If we choose $N = L+P-1$, then

$$\begin{aligned}
x_{3p}[n] &= \sum_{m=0}^{L+P-2} x_1[m]\, x_2\left[((n-m))_{L+P-1}\right] \\
&= \sum_{m=0}^{L-1} x_1[m]\, x_2\left[((n-m))_{L+P-1}\right].
\end{aligned}$$

(a)

$$\begin{aligned}
&((n-m))_{L+P-1} \\
= \; & n-m \bmod L+P-1 \\
= \; & \begin{cases} n-m, & 0 \le m \le n \\ n-m+(L+P-1), & n < m \le n+(L+P-1) \end{cases}
\end{aligned}$$

noting that $0 \le n \le L+P-2$ and $0 \le m \le L-1$.

(b) $x_2[n-m]$ is nontrivial for $n-m \in [0, P-1] \Rightarrow m \in [n-P+1, n]$.

(c) $x_2[n-m+(L+P-1)]$ is nontrivial for $0 \leq n-m+(L+P-1) \leq P-1$.
$\Rightarrow n+L \leq m \leq n+L+P-1$. This case is not possible since $0 \leq n \leq L+P-2$ and $0 \leq m \leq L-1$.

(d) Therefore,

$$\begin{aligned} x_{3p}[n] &= \sum_{m=\max\{0,n-P+1\}}^{\min\{L-1,n\}} x_1[m]\, x_2[n-m] \\ &= x_3[n] \text{ for } 0 \leq n \leq N-1. \end{aligned}$$

3. Trivially, $x_{3p}[n] = x_3[n] = 0$ for $n \notin [0, N-1]$. In summary, if we let $N = L+P-1$, then the linear convolution of two finite length sequences $x_1[n]$ and $x_2[n]$ of length L and P, respectively, is equal to their circular convolution with N. In this case, one can use DFT and inverse DFT to obtain the linear convolution $x_1[n] * x_2[n]$.

Notes:

1. Let $N = L+P-1$. Then we can perform

Fig. 97-F1

2. An Illustration: Since $x_{3p}[n] = \sum_{r=-\infty}^{\infty} x_3[n-rN]$ for a predetermined period N

Fig. 8.19

Fig. 8.21

- Implementing LTI Systems Using The DFT

Consider a causal FIR system with $h[n]$ of length P and an semi-infinite-length causal sequence $x[n]$. Let $x[n]$ be segmented into nonoverlapping blocks of length L, i.e.,

$$x[n] = \sum_{r=0}^{\infty} x_r[n-rL]\, 1_{rL \leq n < (r+1)L}$$

where $1_{\text{condition}} = 1$ if the condition is met and $1_{\text{condition}} = 0$ otherwise.

Fig. 97-F2

Now, we want to implement

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{r=0}^{\infty} x_r[n-rL]\, 1_{rL\le n<(r+1)L} * h[n] \\
&= \sum_{r=0}^{\infty} \left[\sum_{m=-\infty}^{\infty} x_r[m-rL]\, 1_{rL\le m<(r+1)L} h[n-m] \right] \\
&= \sum_{r=0}^{\infty} \left[\sum_{m'=0}^{L-1} x_r\left[m'\right] h\left[n-m'-rL\right] \right] \quad \left(m'=m-rL\right) \\
&= \sum_{r=0}^{\infty} \left[\sum_{m'=0}^{L-1} x_r\left[m'\right] h\left[(n-rL)-m'\right] \right] \\
&= \sum_{r=0}^{\infty} x_r[n] * h[n-rL].
\end{aligned}$$

If we choose $N = L + P - 1$, we have

$$y[n] = \sum_{r=0}^{\infty} x_r[n] \circledast^N h[n-rL].$$

Note here that $x_r[n] \circledast^N h[n-rL]$ is nontrivial for $0 \le n - rL \le N - 1$. Note that

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} y_r[n-rL]$$

with $y_r[n-rL] = x_r[n] \circledast^N h[n-rL]$. This implies an implementation approach called the *overlap-add* approach. The following is an illustration:

Fig. 8.22

Fig. 8.23

An alternative approach is *overlap-save* approach. Assuming $P < L$, the approach is to divide $x[n]$ into overlapping blocks of length L with each block overlapping the preceding block by $P - 1$ symbols, i.e., the rth block containing

$$x_r[n] = x[n + r(L-P+1) - P + 1] \quad \text{for } 0 \le n \le L-1.$$

The L-point circular convolution of $x_r[n]$ with $h[n]$ is denoted by $y_{rp}[n]$. Now, deleting the first $P-1$ symbols in $y_{rp}[n]$ yields

$$y_r[n] = \begin{cases} y_{rp}[n], & P-1 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, the output sequence $y[n] = x[n] * h[n]$ is given by

$$y[n] = \sum_{r=0}^{\infty} y_r\left[n - r(L-P+1) + P - 1\right].$$

(Prove it yourself) The following is an illustration:

Fig. 8.24

- Note: Efficient algorithms, like Fast Fourier Transform (FFT), can be applied to implement the circular convolution, and, thus, the LTI system operation!