Digital Signal Processing

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- Text Book:

  A. V. Oppenheim and R. W. Schafer, "Discrete-Time Signal Processing," 3rd ed. Pearson Prentice-Hall, 2010.
- Prerequisite: Calculus, Engineering Mathematics (Complex Variables, Transform Theory)
- Grading Policy: Two Quizs (9%), Two Midterms (60%, 30% each), One Final (40%)
- Course Outline: Chaps. 1-9 (excluding all summary Sections, Sections 2.10, 4.8-4.10, 6.7-6.10, 7.7-7.10, 9.4-9.7).
- Reading Assignment: Chap. 1
  - Applications of DSP: digital communication systems, speech and video signal processings, biomedical engineering, acoustic signal processing, radar and sonar signal processing, numerical analysis etc.

2 Discrete-Time Signals and Systems

2.1 Discrete-Time Signals

- Defn: A *discrete-time signal* x is a sequence of numbers (real, complex, integer), which is formally written as

  $$x = \{x[n]\}, \ -\infty < n < \infty, \ n \text{ integer}$$

  with $x[n]$ representing the n-th number.
- A discrete-time signal x may be generated by periodically sampling an analog (i.e., continuous-time) signal $x_a(t)$ by

  $$x[n] = x_a(nT), \ -\infty < n < \infty$$

where T is called the *sampling period*, and $1/T$ is called the *sampling frequency.*

Figs. 2.1-2.2

- Basic Sequence Operations: Let x and y be two discrete-time signals. The following operations are conveniently adopted.

  (1) $\alpha x = \{\alpha x[n]\}$ with α a number.

  (2) $x + y = \{x[n] + y[n]\}$.

  (3) $xy = \{x[n]y[n]\}$.

  (4) y is said to be a delayed or shifted version of x if

  $$y[n] = x[n - n_0]$$

  with n_0 an integer.

- Some Basic Sequences:

  1. $\delta[n]$ denotes the *unit sample sequence*, defined by

  $$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

  which is referred to as a *discrete-time impulse* or simply an *impulse*, for convenience.

  Fig. 2.3(a)

  Note: Any sequence can be expressed as

  $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].$$

  For example,

  $$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7].$$

  Fig. 2.4

  2. $u[n]$ denotes the *unit step sequence*, defined by

  $$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}.$$

  Notes:

(a) $u[n]$ is related to $\delta[n]$ by

$$\begin{aligned} u[n] &= \delta[n] + \delta[n-1] + \delta[n-2] + \ldots \\ &= \sum_{k=0}^{\infty} \delta[n-k] \\ &= \sum_{k'=-\infty}^{n} \delta[k'] \;\; (\text{with } k' = n-k) \end{aligned}$$

i.e., the value of $u[n]$ at time n is equal to the accumulated sum of the value at time n and all previous values of the impulse sequence $\delta[k']$.

Fig. 2.3(b)

(b) $\delta[n]$ is related to $u[n]$ by

$$\delta[n] = u[n] - u[n-1]$$

i.e., the first backward difference of the unit step sequence.

3. A *sinusoidal sequence* is defined as

$$x[n] = A\cos(\omega_0 n + \phi)$$

with A, ω_0, ϕ real (ω_0 is called frequency, ϕ is called phase).

Fig. 2.3(d)

4. An *exponential sequence* is defined as

$$x[n] = A\alpha^n$$

with A, α real. If $A > 0$ and $0 < \alpha < 1$, an example sequence can be depicted as

Fig. 2.3(c)

5. A *complex exponential sequence* is defined by

$$x[n] = |A|e^{j(\omega_0 n + \phi)}$$

with A complex, ω_0 and ϕ real. Here, $|A|$ denotes the magnitude of a complex value A.

Note that

$$\begin{aligned} x[n] &= |A|e^{j[(\omega_0+2\pi k)n+\phi]} \\ &= |A|e^{j(\omega_0 n+\phi)} \text{ for } k \text{ integer.} \end{aligned}$$

That is, the complex exponential sequences with frequency $\omega_0 + 2\pi k$ are indistinguishable from one another. Therefore, when considering sinusoidal and complex exponential sequences, we only need to consider frequencies in an interval of length 2π, such as $-\pi \leq \omega_0 < \pi$ or $0 \leq \omega_0 < 2\pi$.

6. A *periodic sequence* is said to have a period N if

$$x[n] = x[n+N] \qquad \forall n$$

Ex 1: If $N = 2\pi k/\omega_0$ is an integer, both $\cos(\omega_0 n + \phi)$ and $|A|e^{j(\omega_0 n+\phi)}$ are periodic with period N.

Ex 2: $x_1[n] = \cos(\pi n/4)$ has a period of $N = 8$. $x_2[n] = \cos(3\pi n/8)$ has a higher frequency than $x_1[n]$, and a period of $N = 16$ which is also higher than $x_1[n]$.

Notes:

(a) A continuous-time sinusoidal signal $\cos(\varpi_0 t + \phi)$ has a smaller period (i.e., $2\pi/\varpi_0$) when its frequency ϖ_0 increases. The property does not, however, hold for a discrete-time sinusoidal signal $\cos(\omega_0 n + \phi)$. This occurs because discrete-time sinusoidal signals are only defined for integer indices n.

(b) A continuous-time sinusoidal signal $\cos(\varpi_0 t + \phi)$ oscillates progressively more rapidly as ϖ_0 increases. However, a discrete-time sinusoidal signal $\cos(\omega_0 n + \phi)$ oscillates progressively more rapidly as ω_0 increases from $\omega_0 = 0$ to $\omega_0 = \pi$, but progressively slower as ω_0 increases from $\omega_0 = \pi$ to $\omega_0 = 2\pi$.

Fig. 2.5

Because of the periodicity in ω_0 of $A\cos(\omega_0 n + \phi)$ and $|A|e^{j(\omega_0 n+\phi)}$, frequencies around $\omega_0 = 2\pi k$ are indistinguishable from frequencies around $\omega_0 = 2\pi l$ for integers $k \neq l$.

For sequences $A\cos(\omega_0 n + \phi)$ and $|A|e^{j(\omega_0 n+\phi)}$, frequencies around $\omega_0 = 2\pi k$ are referred to as low frequencies (relatively slow oscillations), and frequencies around $\omega_0 = 2\pi k + \pi$ are referred to as high frequencies (relatively rapid oscillations), for any integer k.

2.2 Discrete-Time Systems

- Defn: A discrete-time system is a transformation or an operator $T\{\bullet\}$ that maps an input sequence $\{x[n]\}$ into an output sequence $\{y[n]\}$.

  Throughout, we consider a discrete-time system $T\{\bullet\}$ with input sequence $x[n]$ and output sequence $y[n]$, with input/output relation denoted mathematically by

  $$y[n] = T\{x[n]\}$$

  and graphically by

  Fig. 2.6

- Ex: Ideal Delay System: For a fixed positive integer (delay) n_d,

  $$y[n] = x[n - n_d].$$

- Ex: Ideal Advance System: For a fixed negative integer n_d (with advance $|n_d|$),

  $$y[n] = x[n - n_d].$$

- Ex: Moving-Average System:

  $$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

  is the average of $(M_1 + M_2 + 1)$ samples of the input sequence $\{x[n]\}$ around the n-th sample.

  Fig. 2.7

- Discrete-time systems can be categorized into various classes by placing constraints on the properties of $T\{\bullet\}$.

2.2.1 Memoryless Systems

- A system is called *memoryless* iff (if and only if) the n-th sample $y[n]$ of the output sequence depends only on the n-th sample $x[n]$ of the input sequence.

- Ex: $y[n] = T\{x[n]\} = (x[n])^2$ is a memoryless system.

- Ex: $y[n] = x[n - n_d]$ for $n_d \neq 0$ is not memoryless.

2.2.2 Linear Systems

- Let $y_1[n] = T\{x_1[n]\}$ and $y_2[n] = T\{x_2[n]\}$. The system T is called linear iff the *additivity property*, i.e.,

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$$

  and the *homogeneity* (or *scaling*) property, i.e.,

$$T\{ax[n]\} = aT\{x[n]\}$$

  for an arbitrary constant a, are both satisfied. Both *additivity* and *homogeneity* properties comprise the *principle of superposition*, i.e.,

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= aT\{x_1[n]\} + bT\{x_2[n]\} \\ &= ay_1[n] + by_2[n] \end{aligned}$$

  for arbitrary constants a and b.

- Defn: A system T is *linear* iff "if $y_k[n] = T\{x_k[n]\}$ for any permissible k and $x[n] = \sum\limits_k a_k x_k[n]$ for any a_k, then $y[n] = T\{x[n]\} = \sum\limits_k a_k y_k[n]$".

- Ex: The moving-average system

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

  is linear since

$$\begin{aligned} y[n] &= \frac{1}{M_1 + M_2 + 1} \{x[n - M_2] + x\,[n - M_2 + 1] + ... + x\,[n] + ... + x\,[n + M_1]\} \\ &= \sum_{k=-M_1}^{M_2} \frac{1}{M_1 + M_2 + 1} y_k[n] \end{aligned}$$

  where $y_k[n] = x[n-k] = T\,\{x_k[n]\}$ with T an identity operator and $x_k[n] = x[n-k]$.

- Ex: The accumulator system defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is linear. This can be proved as follows: Let

$$y_i[n] = \sum_{k=-\infty}^{n} x_i[k] \quad \text{for } i = 1, 2, 3.$$

When $x_3[n] = ax_1[n] + bx_2[n]$ for arbitrary constants a and b,

$$\begin{aligned} y_3[n] &= \sum_{k=-\infty}^{n} x_3[k] \\ &= \sum_{k=-\infty}^{n} (ax_1[k] + bx_2[k]) \\ &= a \sum_{k=-\infty}^{n} x_1[k] + b \sum_{k=-\infty}^{n} x_2[k] \\ &= ay_1[n] + by_2[n]. \quad \text{Q.E.D.} \end{aligned}$$

- Ex: $y[n] = x^2[n]$ and $y[n] = \log_\alpha(x[n])$ are nonlinear.

2.2.3 Time-Invariant Systems

- Let $y[n] = T\{x[n]\}$. A system T is *time-invariant* iff

  $$y[n - n_0] = T\{x[n - n_0]\}$$

  for any n_0, i.e., a time shift of the input sequence yields a corresponding shift in the output sequence.

- Ex: The moving-average system is time-invariant since

  $$y[n - n_0] = \sum_{k=-M_1}^{M_2} \frac{1}{M_1 + M_2 + 1} x[n - k - n_0]$$

  for any n_0.

- Ex: The accumulator system $y[n] = \sum\limits_{k=-\infty}^{n} x[k]$ is time-invariant since

$$\begin{aligned} y[n - n_0] &= \sum_{k=-\infty}^{n-n_0} x[k] \\ &= \sum_{k'=-\infty}^{n} x[k' - n_0] \quad \text{with } k' = k + n_0. \end{aligned}$$

- Ex: The nonlinear system $y[n] = x^2[n]$ is time-invariant since

$$y[n - n_0] = x^2[n - n_0]$$

for any n_0.

- Ex: The compressor system

$$y[n] = x[Mn], \quad -\infty < n < \infty$$

with M a positive integer greater than one creates the output sequence by selecting every M-th sample of the input sequence. It is not time-invariant since

$$y[n - n_0] = x[Mn - Mn_0]$$

is not the output sequence $x[Mn - n_0]$ of the compressor system corresponding to the input sequence $x[n - n_0]$.

2.2.4 Causal Systems

- A system is *causal* iff, for any n_0, the output sample $y[n]$ at $n = n_0$ depends only on the input samples $x[m]$ for $m \leq n_0$. It is called noncausal otherwise.

  Note: For a causal system T, if $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$ where $y_i[n] = T\{x_i[n]\}$, $i = 1, 2$. The system is *nonanticipative.*

- Ex: The system $y[n] = x[n - n_d]$ is causal for $n_d \geq 0$ and noncausal otherwise.

- Ex: In the moving-average example, if $-M_1 \geq 0$ and $M_2 \geq 0$, i.e.,

$$y[n] = \sum_{k=-M_1}^{M_2} x[n-k] \frac{1}{M_2 + M_1 + 1} \text{ for } M_2 + M_1 \geq 0$$

then the system is causal. In this case, $y[n]$ depends on $x[n-M_2]$,...,$x[n+M_1]$, not on any future input sample.

- Ex: The forward difference system $y[n] = x[n+1] - x[n]$ is noncausal. This can be shown as follows: Let $x_1[n] = \delta[n-1]$ and $x_2[n] = 0$. Also, the corresponding output sequences are $y_1[n] = \delta[n] - \delta[n-1]$ and $y_2[n] = 0$. Thus, $x_1[n] = x_2[n]$ for $n \leq 0$. However, $y_1[n] = y_2[n]$ holds only for $n < 0$ but not for $n = 0$. This violates the causality property.

2.2.5 Stable Systems

- A system is called *stable* in the bounded-input bounded-output (BIBO) sense iff "every bounded input sequence produces a bounded output sequence". In other words, if there exists a fixed positive finite value B_x such that
$$|x[n]| \leq B_x < \infty \; \forall n$$
then the stable system output has
$$|y[n]| \leq B_y < \infty \; \forall n$$
for a fixed positive finite value B_y.

- Ex: The moving-average system is stable since
$$\begin{aligned} \text{"If } |x[n]| &\leq B_x, \text{ then} \\ |y[n]| &= \frac{1}{M_2+M_1+1}\left|\sum_{k=-M_1}^{M_2} x[n-k]\right| \\ &\leq \frac{1}{M_2+M_1+1}\sum_{k=-M_1}^{M_2} |x[n-k]| \\ &\leq B_x.\text{"} \end{aligned}$$

- Ex: The accumulator system is not stable. This can be shown by letting $x[n] = u[n]$. Now, $|x[n]| \leq 1$ and thus $B_x = 1$. However,
$$\begin{aligned} |y[n]| &= \left|\sum_{k=-\infty}^{n} u[k]\right| \\ &= \begin{cases} 0, & n < 0 \\ n+1, & \text{otherwise} \end{cases}. \end{aligned}$$
There is no finite B_y value to bound $|y[n]|$ as n approaches to the infinity.

- Note: The above five properties (memoryless, linear, time-invariant, causal, stable) are defined over *systems*, not over *inputs*. For example, for a stable system; *all* bounded inputs produce bounded outputs. As long as we can find an example (an input-output pair) for which a certain system property does not hold, then we can show that the system does not have that property.

2.3 Linear and Time-Invariant (LTI) Systems

- An LTI system is a system that has linearity and time-invariance properties. Note that any sequence $\{x[n]\}$ can be represented by

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].$$

This shows an equivalent relation for $y[n] = T\{x[n]\}$ as

$$y[n] = T\left\{ \sum_{k=-\infty}^{\infty} \underbrace{x[k]\,\delta[n-k]}_{\text{regarded as a new sequence}} \right\}.$$

From the linearity property (principle of superposition),

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} T\left\{ \underbrace{x[k]}_{\text{a constant}} \underbrace{\delta[n-k]}_{\text{a sequence}} \right\} \quad \text{(by additivity)} \\ &= \sum_{k=-\infty}^{\infty} x[k]\,T\{\delta[n-k]\}. \quad \text{(by homogeneity)} \end{aligned}$$

Defining $h_k[n] = T\{\delta[n-k]\}$, we come out with an equivalent relation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n].$$

Since T is time-invariant, then $h_k[n]$ can be rewritten as follows. If we let $h[n]$ be the output to $\delta[n]$, i.e., $h[n] = T\{\delta[n]\}$, then from time-invariant property,

$$h[n-k] = T\{\delta[n-k]\}$$

which implies $h_k[n] = h[n-k]$. Thus, an LTI system can be described as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \qquad (\bigstar)$$

which is commonly referred to as the *convolution sum*.

- Here, $h[n] = T\{\delta[n]\}$, the output of an LTI system corresponding to an impulse input, is called the *impulse response* of the LTI system T.

For notational convenience, we define

$$\begin{aligned} y[n] &= x[n] * h[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\ &\underset{k'=n-k}{\overset{\uparrow}{=}} \sum_{k'=-\infty}^{\infty} x\left[n-k'\right] h\left[k'\right] \\ &= h[n] * x[n] \end{aligned}$$

with $*$ being called a convolution operator. Here, we notice that $*$ is commutative.

- Notes:

  1. The above convolution sum form for $y[n]$ in terms of $x[n]$ and $h[n]$ is a direct result of linearity and time-invariance.
  2. The shorthand notation $y[n] = x[n] * h[n]$ should be used with caution. For example, $y[n-n_0] = x[n] * h[n-n_0] = x[n-n_0] * h[n]$ denotes the convolution sum relation

     $$y[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k-n_0]$$

     but not

     $$\begin{aligned} y[n-n_0] &= x[n-n_0] * h[n-n_0] \\ &= \sum_{k=-\infty}^{\infty} x[k-n_0]\, h[n-k-n_0]. \end{aligned}$$

- See Figure 2.8 for the illustration of convolution sum, and the approach of computing $y[n]$.

Fig. 2.8

- Note that $h[n-k] = h[-(k-n)]$ can be regarded as an n-shifted reverse sequence in index k.

Fig. 2.9

This shows that $\sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[k]\, h[-(k-n)]$ can be computed as a sum of the termwise products of two sequences in index

k, namely the input sequence $x[k]$ and the n-shifted reverse impulse response sequence $h[-(k-n)]$.

Ex: Consider an LTI system with $h[n] = u[n] - u[n-N]$ with a positive integer N and the input sequence $x[n] = a^n u[n]$ ($|a| < 1$).

Fig. 2.10

The output sequence is given by

$$y[n] = \begin{cases} 0, & n < 0 \\ \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, & 0 \le n \le N-1 \\ \sum_{k=n-N+1}^{n} a^k = a^{n-N+1}(\frac{1-a^N}{1-a}) & n > N-1 \end{cases}.$$

2.4 Properties of LTI Systems

- The relation (★) holds for any input/output pair for an LTI system. Thus, the impulse response $h[n]$ completely characterizes an LTI system.

  Thus, to study $h[n]$ suffices to define the properties of an LTI system.

- First, $*$ has the following properties:

  1. $*$ is commutative (shown earlier).
  2. $*$ is distributive over addition, i.e.,

  $$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

  Pf:

  $$\begin{aligned} x[n] * (h_1[n] + h_2[n]) &= \sum_k x[k](h_1[n-k] + h_2[n-k]) \\ &= \sum_k x[k] h_1[n-k] + \sum_k x[k] h_2[n-k] \\ &= x[n] * h_1[n] + x[n] * h_2[n]. \quad \text{QED} \end{aligned}$$

  This property enables the parallel combination of LTI systems, as

  Fig. 2.11

  Thus, $h_1[n] + h_2[n]$ is the impulse response of the single equivalent system.

3. $*$ is associative, i.e.,

$$\begin{aligned}(x[n] * h_1[n]) * h_2[n] &= x[n] * (h_1[n] * h_2[n]) \\ &= (x[n] * h_2[n]) * h_1[n].\end{aligned}$$

Pf:

$$\begin{aligned}(x[n] * h_1[n]) * h_2[n] &= \sum_{k_2}\sum_{k_1} x[k_1]\, h_1[k_2 - k_1]\, h_2[n - k_2] \\ &= \sum_{k_1} x[k_1] \sum_{k_2} h_1[k_2 - k_1]\, h_2[n - k_2] \\ &= \sum_{k_1} x[k_1] \sum_{\substack{k_3 \\ (k_3 = k_2 - k_1)}} h_1[k_3]\, h_2[n - (k_3 + k_1)] \\ &= \sum_{k_1} x[k_1] \sum_{k_3} h_1[k_3]\, h_2[(n - k_1) - k_3] \\ &= \sum_{k_1} x[k_1]\, h_3[n - k_1] \\ &= x[n] * (h_1[n] * h_2[n]).\end{aligned}$$

where $h_3[n] \triangleq h_1[n] * h_2[n]$. Similarly,

$$(x[n] * h_2[n]) * h_1[n] = x[n] * (h_2[n] * h_1[n]).$$

Since $h_2[n] * h_1[n] = h_1[n] * h_2[n]$ (i.e., $*$ is commutative),

$$(x[n] * h_2[n]) * h_1[n] = x[n] * (h_1[n] * h_2[n]). \quad \text{QED}$$

This property enables the cascade combination of LTI systems, as

Fig. 2.12

Thus, $h_1[n] * h_2[n]$ is the impulse response of the single equivalent system.

- The impulse response has the following properties:

1. An LTI system is stable iff the impulse response is absolutely summable, i.e., iff

$$B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

Pf: "$\Leftarrow$"

$$\begin{aligned}
|y[n]| &= \left|\sum_k x[k]\,h[n-k]\right| = \left|\sum_k h[k]\,x[n-k]\right| \\
&\leq \sum_k |h[k]|\,|x[n-k]| \\
&\leq B_x \sum_k |h[k]|
\end{aligned}$$

since the system input is bounded. Thus, if

$$\sum_k |h[k]| = B_h < \infty$$

with B_h a constant, then

$$|y[n]| \leq B_x B_h < \infty.$$

"$\Rightarrow$" By contradiction, if $B_h = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$, then we want to show that a bounded input can be found and will yield an unbounded output. Now let

$$x[n] = \begin{cases} \frac{h^*[-n]}{|h[-n]|}, & h[n] \neq 0 \\ 0, & h[n] = 0 \end{cases}$$

with superscript $*$ the complex conjugate operator. First, $\{x[n]\}$ is bounded since

$$|x[n]| \leq 1 \;\forall n.$$

However, at $n = 0$

$$\begin{aligned}
y[0] &= \sum_{k=-\infty}^{\infty} x[-k]\,h[k] = \sum_{\text{admissible } k} \frac{|h[k]|^2}{|h[k]|} \\
&\qquad = \sum_{k=-\infty}^{\infty} |h[k]| \\
&\qquad = \infty.
\end{aligned}$$

Thus, if $B_h = \infty$, it is impossible to have a stable system in the BIBO sense. QED

2. An LTI system is said to have finite-duration impulse response (FIR) iff $h[n]$ is nonzero only for a finite number of n's.

(a) The moving-average system is an FIR LTI system since

$$\begin{aligned}
y[n] &= \sum_{k=-M_1}^{M_2} x[n-k] \frac{1}{M_1+M_2+1} \\
&= \sum_{k=-M_1}^{M_2} \sum_{k'=-\infty}^{\infty} x\left[k'\right] \delta\left[(n-k)-k'\right] \frac{1}{M_1+M_2+1} \\
&= \sum_{k'=-\infty}^{\infty} x\left[k'\right] \underbrace{\sum_{k=-M_1}^{M_2} \delta\left[(n-k)-k'\right] \frac{1}{M_1+M_2+1}}_{h\left[n-k'\right]} \\
\Rightarrow h[n] &= \sum_{k=-M_1}^{M_2} \frac{1}{M_1+M_2+1} \delta[n-k].
\end{aligned}$$

Note that $h[n]$ can be simply obtained from $h[n] = T\{\delta[n]\}$ if the moving average system is known to be LTI *a priori.*

(b) The ideal delay system is an FIR LTI system since $h[n] = \delta[n-n_d]$ with n_d a fixed positive integer.

(c) The forward difference system is an FIR LTI system since $h[n] = \delta[n+1] - \delta[n]$.

(d) The backward difference system is an FIR LTI system since $h[n] = \delta[n] - \delta[n-1]$.

Note: An FIR LTI system is always stable as long as each of impulse response values is finite in magnitude.

3. An LTI system is said to have an infinite-duration impulse response (IIR) iff $h[n]$ is nonzero for an infinite number of n's.

(a) The accumulator system is an IIR LTI system with

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n].$$

This is because

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} x[k] \\ &= \sum_{k=0}^{\infty} x[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[n-k]\, u[k] \\ &= x[n] * u[n] \\ \Rightarrow \quad h[n] &= u[n]. \end{aligned}$$

(b) The system with $h[n] = a^n u[n]$ with $|a| < 1$ is an IIR LTI system.

Note: An IIR LTI system may be stable or unstable.

4. An LTI system is causal iff $h[n] = 0$, $n < 0$.

Pf: The LTI system satisfies

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

First, if $h[n] = 0$, $n < 0$, then

$$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

depends only on $x[n]$, $x[n-1]$,..., i.e., current and all past input samples.

Second, if the system is causal, then $y[n]$ depends only on current and all past input samples, and thus

$$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k] = \sum_{k=0}^{\infty} x[n-k]\, h[k]$$

which means that $h[n] = 0$ for $n < 0$. QED

Notes:

(a) For a causal LTI system,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} x[k]\, h[n-k] \\ &= \sum_{k=0}^{\infty} x[n-k]\, h[k]. \end{aligned}$$

(b) We call a sequence $h[n]$ causal iff $h[n] = 0$, $n < 0$, with an implication that it is suitable to be an impulse response of a causal LTI system. It is called noncausal otherwise.

(c) The ideal delay system is causal since $h[n] = \delta[n - n_d]$ with n_d a fixed positive integer.

(d) The backward difference system is causal since $h[n] = \delta[n] - \delta[n-1]$.

(e) The accumulator system is causal since $h[n] = u[n]$.

(f) The forward difference system is noncausal since $h[n] = \delta[n+1] - \delta[n]$.

5. The output of the ideal delay system with impulse response $h[n] = \delta[n - n_d]$ and input $x[n]$ is given by

$$y[n] = x[n] * \delta[n - n_d] = \delta[n - n_d] * x[n] = x[n - n_d].$$

6. The cascade of forward difference system with $h_1[n] = \delta[n+1] - \delta[n]$ and one-sample delay system with $h_2[n] = \delta[n-1]$ can be described by an equivalent system with

$$\begin{aligned} h[n] &= (\delta[n+1] - \delta[n]) * \delta[n-1] \\ &= \delta[n-1] * (\delta[n+1] - \delta[n]) \\ &= \delta[n] - \delta[n-1] \end{aligned}$$

which is the backward difference system.

Fig. 2.13

Notes:

(a) From Fig. 2.13(a), we have

$$\begin{aligned} y[n] &= (x[n] * h_1[n]) * h_2[n] \\ &= x[n] * (h_1[n] * h_2[n]) \\ &= x[n] * (\delta[n+1] * \delta[n-1] - \delta[n] * \delta[n-1]) \\ &= x[n] * (\delta[n] - \delta[n-1]). \end{aligned}$$

Thus, handling $h_1[n] * h_2[n]$ suffices to describe the equivalent system in Fig. 2.13(c).

(b) Any noncausal FIR LTI system can be made causal by cascading it with a sufficiently long delay. For example, if $h[n] = 0$ for $n < -N$ for a positive integer N, we can obtain a causal system with $h_{new}[n]$ by delaying $h[n]$ with $\delta[n - N]$, i.e., $h_{new}[n] = h[n] * \delta[n - N] = h[n - N]$ which is zero for $n < 0$.

7. The cascade of two ideal delay systems, one with delay a and the other with delay b, constructs an ideal delay system with delay $a+b$.

   Pf: Let $h_1[n] = \delta[n-a]$, $h_2[n] = \delta[n-b]$. Then

$$\begin{aligned}
\delta[n-a] * \delta[n-b] &= h_1[n] * h_2[n] \\
&= \sum_{k=-\infty}^{\infty} h_1[k]\, h_2[n-k] \\
&= \sum_{k=-\infty}^{\infty} \delta[k-a]\, \delta[n-k-b] \\
&= \delta[n-(a+b)]. \\
\Rightarrow y[n] &= x[n] * h[n] = x[n-(a+b)]
\end{aligned}$$

   with $h[n] = \delta[n-(a+b)]$.

8. If $y[n] = x[n] * h[n] = T\{x[n]\}$, then $h_i[n]$ is called the impulse response of the *inverse* system for T iff

$$x[n] = y[n] * h_i[n].$$

   Notes:

   (a) $h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]$ since $x[n] = y[n] * h_i[n] = x[n] * (h[n] * h_i[n])$.

   (b) The backward difference and accumulator systems form an inverse system pair since

$$\begin{aligned}
& u[n] * (\delta[n] - \delta[n-1]) \\
=\ & (\delta[n] - \delta[n-1]) * u[n] \\
=\ & u[n] - u[n-1] \\
=\ & \delta[n].
\end{aligned}$$

   Fig. 2.14

   This can be shown alternatively as follows: Now, given $h[n] = \delta[n] - \delta[n-1]$, we want to find $h_i[n]$ which satisfies

$$\begin{aligned}
& h_i[n] * h[n] = \delta[n] \\
\Rightarrow\ & h_i[n] * (\delta[n] - \delta[n-1]) = \delta[n] \\
\Rightarrow\ & h_i[n] - h_i[n-1] = \delta[n] \ \ \forall n
\end{aligned}$$

   Since $\delta[n] = u[n] - u[n-1] \Rightarrow h_i[n] = u[n]$. Here comes the question: Is it unique? We defer the discussion till Chapter 3 on z-transform.

2.5 Linear Constant-Coefficient Difference Equations

- We are interested in LTI systems whose input $x[n]$ and output $y[n]$ satisfy N-th order linear constant-coefficient difference (LCCD) equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

where the left-hand side of the equality consists of a linear combination of $y[n-N], y[n-N+1], ..., y[n]$ and the right-hand side of the equality consists of a linear combination of $x[n-M], x[n-M+1], ..., x[n]$.

Note: $a_0 \neq 0$ in general.

- Ex: Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k].$$

Thus,

$$\begin{aligned} y[n] - y[n-1] &= \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] \\ &= x[n] \\ \Rightarrow \quad y[n] &= x[n] + y[n-1] \qquad (\$) \\ \Rightarrow \quad N = 1,\ M = 0,\ \text{and} & \begin{cases} a_0 = 1 \\ a_1 = -1 \\ b_0 = 1 \end{cases}. \end{aligned}$$

Note: (\$) suggests a simple implementation of the accumulator system, as

Fig. 2.15

- Ex: Moving-Average System

$$y[n] = \frac{1}{M_2+1} \sum_{m=0}^{M_2} x[n-m] \Rightarrow \begin{cases} a_0 = 1, & N = 0 \\ b_m = \frac{1}{M_2+1} \text{ for } 0 \leq m \leq M_2, & M = M_2 \end{cases}.$$

The impulse response of the system is

$$
\begin{aligned}
h[n] &= \sum_{m=0}^{M_2} \frac{1}{M_2+1} \delta[n-m] \\
&= \frac{1}{M_2+1} \left(u[n] - u[n-M_2-1] \right) \\
&= \frac{1}{M_2+1} \left(\delta[n] - \delta[n-M_2-1] \right) * \underbrace{u[n]}_{\text{accumulator}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \left[x[n] * \left(\frac{1}{M_2+1} \right) \left(\delta[n] - \delta[n-M_2-1] \right) \right] * u[n].
\end{aligned}
$$

The block diagram of the moving-average system is

Fig. 2.16

Note that the input $x[n]$ and the output $y[n]$ also satisfy

$$
\begin{aligned}
y[n] - y[n-1] &= \frac{1}{M_2+1} \left\{ \sum_{m=0}^{M_2} x[n-m] - x[n-1-m] \right\} \\
&= \frac{1}{M_2+1} \left\{ x[n] - x[n-1-M_2] \right\}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & a_0 = 1, a_1 = -1, N = 1 \\
& b_0 = -b_{M_2+1} = \frac{1}{M_2+1}, M = M_2 + 1.
\end{aligned}
$$

This implies that the linear constant-coefficient difference equation for discrete LTI systems does not necessarily provide a unique specification of the output for a given input.

- Now, let $x_p[n]$ be a given input, and $y_p[n]$ be the output satisfying

$$
\sum_{k=0}^{N} a_k y_p[n-k] = \sum_{k=0}^{M} b_k x_p[n-k] \qquad (*)
$$

for some $\{a_k\}_{k=0}^{N}$, $\{b_k\}_{k=0}^{M}$, N, and M.

Also, let $y_h[n]$ be any solution of

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0 \qquad (+)$$

then $y[n] = y_p[n] + y_h[n]$ is also a solution to $(*)$. We call the equation $(+)$ the *homogeneous difference equation* and $y_h[n]$ the *homogeneous solution* for an N-th order linear constant-coefficient equation with coefficients $\{a_k\}_{k=0}^{N}$.

Note: For $N \geq 1$, the homogeneous solution is not unique.

- The linear constant-coefficient difference equation with $N \geq 1$ can be rewritten as

  $$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k].$$

  Provided with $y[-1]$, $y[-2]$,...,$y[-N]$ and the input sequence $\{x[n]\}$, $y[n]$ for $n \geq 0$ can be computed *recursively* if $\{a_k\}_{k=0}^{N}$, $\{b_k\}_{k=0}^{M}$ are specified for an LTI system.

  In other words, if the initial values for the output $y[-1]$, $y[-2]$,...,$y[-N]$ are given, the output $y[n]$ can be *uniquely* determined and *recursively* computed by the linear constant-coefficient difference equation for a given input sequence $\{x[n]\}$.

- If we delimit the LTI system to be causal and have causal input and output sequences, i.e., $y[n] = 0$ and $x[n] = 0$ for $n < 0$, then for a given input, the linear constant-coefficient difference equation can *uniquely* determine the output of an LTI system.

  Ex: Consider the causal LTI system with equation

  $$y[n] = ay[n-1] + x[n]$$

  and with causal $x[n]$ and $y[n]$. Now, for $n \geq 1$

  $$\begin{aligned} y[n] &= a(ay[n-2] + x[n-1]) + x[n] \\ &= a^2 y[n-2] + ax[n-1] + x[n] \\ &= a^3 y[n-3] + a^2 x[n-2] + ax[n-1] + x[n] \\ &= \ldots \\ &= \sum_{k=0}^{n-1} a^k x[n-k] + a^n y[0] \end{aligned}$$

Since $y[0] = ay[-1] + x[0] = x[0]$,

$$y[n] = \sum_{k=0}^{n} a^k x[n-k] \text{ for } n \geq 0.$$

The output is uniquely specified by the input!!

- The linear constant-coefficient difference equation with $N = 0$ can be rewritten as

$$\begin{aligned} y[n] &= \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \\ &= x[n] * h[n] \end{aligned}$$

with $h[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} \delta[n-k]$. This shows that the output $y[n]$ can be *uniquely* determined and *nonrecursively* computed by the linear constant-coefficient difference equation with $N = 0$ for a given input sequence $\{x[n]\}$.

2.6 Frequency-Domain Representation of Discrete-Time Signals and Systems

2.6.1 Eigenfunctions for LTI Systems

- *Complex exponential sequences are eigenfunctions of LTI systems*:

Consider an LTI system with impulse response $h[n]$ and input

$$x[n] = e^{j\omega n} \; \forall n.$$

Then, the output is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} \\ &= e^{j\omega n} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right). \end{aligned}$$

Defn: $H(e^{j\omega}) \triangleq \sum\limits_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$ is called the frequency response of the LTI system with impulse response $h[n]$.

For an LTI system with real-valued impulse response $h[n]$, $H(e^{j\omega})$ is *conjugate-symmetric* since $H(e^{-j\omega}) = H^*(e^{j\omega})$.

By this definition,

$$\begin{aligned} & y[n] = H\left(e^{j\omega}\right) e^{j\omega n} = H\left(e^{j\omega}\right) x[n] \\ \Leftrightarrow \ & T\{e^{j\omega n}\} = H\left(e^{j\omega}\right) e^{j\omega n} \end{aligned}$$

which means that $e^{j\omega n}$ is an *eigenfunction* of the LTI system, with the associated *eigenvalue* $H\left(e^{j\omega}\right)$.

- Because $|H\left(e^{j\omega}\right)| = |\sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}| \leq \sum_{k=-\infty}^{\infty} |h[k]|$, $H\left(e^{j\omega}\right)$ exists if the LTI system is stable in the BIBO sense, i.e., $B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$. Thus, system stability is a *sufficient* condition for the existence of the frequency response.

- Quadrature (Rectangular) representation of frequency response:

$$H\left(e^{j\omega}\right) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

where H_R and H_I are both real, and are called real and imaginary parts of $H\left(e^{j\omega}\right)$, respectively.

- Polar representation of frequency response:

$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\angle H\left(e^{j\omega}\right)}$$

where $|H\left(e^{j\omega}\right)|$ and $\angle H\left(e^{j\omega}\right)$ are the amplitude and phase of $H\left(e^{j\omega}\right)$, respectively.

- Ex: Ideal Delay System

For the ideal delay system with $h[n] = \delta[n - n_d]$, the frequency response is given by

$$H\left(e^{j\omega}\right) = e^{-j\omega n_d}$$

with $H_R(e^{j\omega}) = \cos(\omega n_d)$, $H_I(e^{j\omega}) = -\sin(\omega n_d)$, $|H\left(e^{j\omega}\right)| = 1$, and $\angle H\left(e^{j\omega}\right) = -\omega n_d$.

- An input signal of the form

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}$$

(i.e., a linear combination of complex exponential sequences with different frequencies ω_k's) entering into the LTI system will yield the output

$$\begin{aligned} y[n] &= T\{x[n]\} = T\left\{\sum_k \alpha_k e^{j\omega_k n}\right\} \\ &= \sum_k \alpha_k T\left\{e^{j\omega_k n}\right\} \text{ (from the principle of superposition)} \\ &= \sum_k \alpha_k H\left(e^{j\omega_k}\right) e^{j\omega_k n}. \text{ (by the definition of frequency response)} \end{aligned}$$

Therefore, if the frequency response of the LTI system is known, then the output for any input of the form of a linear combination of complex exponentials can be found.

- Ex: Consider the LTI system with real-valued impulse response $h[n]$ and frequency response $H(e^{j\omega})$. Also, it has the input

$$\begin{aligned} x[n] &= A\cos(\omega_0 n + \phi) \\ &= \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n} \\ \Rightarrow y[n] &= \frac{A}{2}e^{j\phi}H\left(e^{j\omega_0}\right)e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}H\left(e^{-j\omega_0}\right)e^{-j\omega_0 n} \\ &= A|H\left(e^{j\omega_0}\right)|\cos\left(\omega_0 n + \phi + \angle H\left(e^{j\omega_0}\right)\right) \end{aligned}$$

since $H(e^{-j\omega_0}) = H^*(e^{j\omega_0})$.

- The frequency response of a discrete LTI system is periodic with period 2π.

$$\begin{aligned} H\left(e^{j(\omega+2k\pi)}\right) &= \sum_{n=-\infty}^{\infty} h[n]\, e^{-j(\omega+2k\pi)n} \\ &= \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} \\ &= H\left(e^{j\omega}\right) \qquad \forall k \text{ integer.} \end{aligned}$$

Thus, it suffices to specify $H(e^{j\omega})$ over an interval of length 2π, e.g., $0 \le \omega < 2\pi$ or $-\pi \le \omega < \pi$.

- Ex: Ideal Lowpass Filter

Fig. 2.17

- Ex: Ideal Highpass, Bandstop, and Bandpass Filters

Fig. 2.18

- Ex: The frequency response of the moving-average system with $h[n] = \sum_{k=-M_1}^{M_2} \frac{1}{M_1+M_2+1}\delta[n-k]$ is

$$H\left(e^{j\omega}\right) = \sum_{k=-M_1}^{M_2} \frac{1}{M_1+M_2+1} e^{-j\omega k}.$$

Fig. 2.19

2.6.2 Suddenly Applied Complex Exponential Inputs

- Consider the complex exponential input applied at $n = 0$

$$x[n] = e^{j\omega n} u[n].$$

The corresponding output of a causal LTI system with causal impulse response $h[n]$ is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \left(\sum_{k=0}^{n} h[k]x[n-k]\right) u[n] \\ &= \left(\sum_{k=0}^{n} h[k]e^{-j\omega k}\right) e^{j\omega n} u[n] \\ &= \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} u[n] - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} u[n] \\ &= y_{SS}[n]u[n] + y_t[n]u[n]. \end{aligned}$$

Here, $y_{SS}[n] = H\left(e^{j\omega}\right) e^{j\omega n}$ in the first term is the *steady-state response* which is identical to the output from $x[n] = e^{j\omega n}$ for all n. In the second term,

$$y_t[n] = -\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}$$

is called the *transient response* and has a magnitude bounded by

$$|y_t[n]| \leq \sum_{k=n+1}^{\infty} |h[k]|.$$

For an FIR LTI system with $h[n] = 0$ except for $0 \leq n \leq M$, $y_t[n] = 0$ for $n > M - 1$ and thus

$$y[n] = y_{SS}[n]u[n - M]$$

for $n \geq M$. For a stable and causal IIR LTI system with bounded impulse response, i.e., $B_h = \sum_{k=0}^{\infty} |h[k]| < \infty$, the transient response becomes increasingly smaller as n increases to the infinity. Thus, system stability is a *sufficient* condition for the transient response to decay asymptotically.

2.7 Representation of Sequences by Fourier Transforms

- A broad class of sequences can be represented by the form

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega \qquad (*)$$

where $X(e^{j\omega})$ is defined by

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}. \qquad (+)$$

$(*)$ and $(+)$ together form the Fourier representation for the sequence $x[n]$. $X(e^{j\omega})$ is called the Fourier transform of $x[n]$, while $x[n]$ is called the inverse Fourier transform of $X(e^{j\omega})$. Thus, it follows that $H(e^{j\omega})$, defined previously, is the Fourier transform of $h[n]$.

Notes:

1. The $X(e^{j\omega})$ is sometimes called the Fourier spectrum or simply the spectrum.
2. $x[n]$ can be regarded as a superposition of infinitesimally small complex exponentials of the form $\frac{1}{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$ with ω ranging over $[-\pi, \pi)$. Thus, $x[n]$ can be synthesized from $X(e^{j\omega})$ by $(*)$, with $X(e^{j\omega})$ computed from $(+)$.

- Similar to $H(e^{j\omega})$,

$$\begin{aligned} X(e^{j\omega}) &= X_R(e^{j\omega}) + jX_I(e^{j\omega}) \\ &= \left|X(e^{j\omega})\right| e^{j\angle X(e^{j\omega})} \end{aligned}$$

are the rectangular and polar representations of $X(e^{j\omega})$, respectively. $|X(e^{j\omega})|$ is sometimes called the magnitude (amplitude) spectrum, while $\angle X(e^{j\omega})$ the phase spectrum.

Notes:

1. $X(e^{j\omega})$ is periodic with period 2π, as are $X_R(e^{j\omega})$, $X_I(e^{j\omega})$, $|X(e^{j\omega})|$, and $\angle X(e^{j\omega})$.
2. By analogy, we have the transform

$$h\,[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

for certain $h\,[n]$.

- We show that "$(*)$ and $(+)$ are inverse of each other if both $X(e^{j\omega})$ and $x\,[n]$ exist."

Pf: Defining

$$\widehat{x}\,[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n}d\omega$$

and

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x\,[n]\,e^{-j\omega n}$$

Now,

$$\begin{aligned} \widehat{x}\,[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} x\,[k]\,e^{-j\omega k}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} x\,[k]\,e^{j\omega(n-k)}d\omega \\ &= \sum_{k=-\infty}^{\infty} x\,[k]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega\right]. \end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega &= Sa\left(\pi\left(n-k\right)\right) \\
&\triangleq \begin{cases} \frac{\sin(\pi(n-k))}{\pi(n-k)}, & \text{if } n-k \neq 0 \\ 1, & \text{if } n-k = 0 \end{cases} \\
&= \begin{cases} 0, & \text{if } n-k \neq 0 \\ 1, & \text{if } n-k = 0 \end{cases} \\
&= \delta\left[n-k\right]
\end{aligned}$$

we have

$$\widehat{x}\left[n\right] = \sum_{k=-\infty}^{\infty} x\left[k\right]\delta\left[n-k\right] = x\left[n\right].$$

QED

- Given a sequence $x[n]$, the existence condition for its Fourier transform is

  $$\left|X\left(e^{j\omega}\right)\right| < \infty \qquad \forall\omega.$$

  A *sufficient* condition for this existence is

  $$\begin{aligned}
  \left|X\left(e^{j\omega}\right)\right| &= \left|\sum_{n=-\infty}^{\infty} x\left[n\right]e^{-j\omega n}\right| \\
  &\leq \sum_{n=-\infty}^{\infty} \left|x\left[n\right]\right| < \infty
  \end{aligned}$$

  i.e., $x[n]$ is absolutely summable.

  Thus, for an *absolutely summable* sequence, its Fourier transform exists.

  Ex: Consider the sequence $x[n] = a^n u[n]$ with $|a| < 1$. Because $\sum\limits_{n=-\infty}^{\infty} |x[n]| = \frac{1}{1-|a|} < \infty$, its Fourier transform exists and is given by

  $$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1-ae^{-j\omega}}.$$

  Notes:

  1. Absolute summability is only a *sufficient* condition for the existence of a Fourier transform.

2. If $x[n]$ is absolutely summable, $X_M(e^{j\omega})$ with $X_M(e^{j\omega}) \triangleq \sum_{n=-M}^{M} x[n]\, e^{-j\omega n}$ can be shown to converge uniformly (with respect to M) to a continuous function of ω.
3. Some sequences are not absolutely summable, but are square summable, i.e., $\sum\limits_{n=-\infty}^{\infty} |x[n]|^2 < \infty$. Such sequences can be represented by a Fourier transform in the following sense: If $x[n]$ is square summable, $X_M(e^{j\omega})$ with $X_M(e^{j\omega}) \triangleq \sum_{n=-M}^{M} x[n]\, e^{-j\omega n}$ can be shown to converge to $X(e^{j\omega})$ in the mean-square sense, i.e.,

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 d\omega = 0.$$

Note that the convergence does not guarantee $\lim_{M\to\infty} X_M(e^{j\omega}) = X(e^{j\omega})$ for each value of ω.

Ex: Consider the ideal lowpass filter with

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}.$$

The inverse Fourier transform of $H_{lp}(e^{j\omega})$ gives

$$h_{lp}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{lp}(e^{j\omega}) e^{j\omega n} d\omega = \frac{\sin(\omega_c n)}{\pi n} \qquad \forall n.$$

Note that $h_{lp}[n]$ is not causal nor absolutely summable. However, $h_{lp}[n]$ is square summable.
4. There are other Fourier transformable pairs, with the sequences not absolutely summable nor square summable, e.g.,
   (a) $x[n] = 1$, $\forall n$

   $$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} 2\pi\delta\left(\omega + 2n\pi\right)$$

   where $\delta(\omega)$ is the Dirac delta function which is a generalized function defined by $\int_{-\infty}^{\infty} g(x)\delta(x-\omega)\, dx = g(\omega)$ for any well-defined function $g(\omega)$. Note that the infinite sum does not converge in any regular sense.

   (b) $x[n] = e^{j\omega_0 n}$, $\forall n$

   $$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} 2\pi\delta\left(\omega - \omega_0 + 2n\pi\right).$$

(c) $x[n] = u[n]$

$$X\left(e^{j\omega}\right) = \frac{1}{1-e^{-j\omega}} + \sum_{n=-\infty}^{\infty} \pi\delta\left(\omega + 2n\pi\right).$$

5. Representing sequences and system responses in frequency domain by Fourier transform can facilitate the analysis of LTI systems.

2.8 Symmetry Properties of the Fourier Transform

- Motivation: Symmetry properties are very useful for simplifying the solution of problems.

- Defns:

  (1) A sequence $x_e[n]$ is called conjugate-symmetric iff

  $$x_e[n] = x_e^*[-n] \qquad \forall n.$$

  (2) A sequence $x_o[n]$ is called conjugate-antisymmetric iff

  $$x_o[n] = -x_o^*[-n] \qquad \forall n.$$

  (3) If a conjugate-symmetric sequence $x_e[n]$ is real, it is called an even sequence.

  (4) If a conjugate-antisymmetric sequence $x_o[n]$ is real, it is called an odd sequence.

  Similar definitions can be applied to the conjugate-symmetric and anti-symmetric functions $X_e\left(e^{j\omega}\right)$ and $X_o\left(e^{j\omega}\right)$, as well as the even and odd functions.

- Properties:

  (1) Any sequence can be represented by

  $$x[n] = x_e[n] + x_o[n]$$

  with

  $$\begin{aligned} x_e[n] &= \frac{1}{2}\left(x[n] + x^*[-n]\right) = x_e^*[-n] \\ x_o[n] &= \frac{1}{2}\left(x[n] - x^*[-n]\right) = -x_o^*[-n]. \end{aligned}$$

(2) Any function $X\left(e^{j\omega}\right)$ can be decomposed into

$$X\left(e^{j\omega}\right) = X_e\left(e^{j\omega}\right) + X_o\left(e^{j\omega}\right)$$

with

$$\begin{aligned} X_e\left(e^{j\omega}\right) &= \frac{1}{2}\left[X\left(e^{j\omega}\right) + X^*\left(e^{-j\omega}\right)\right] = X_e^*\left(e^{-j\omega}\right) \\ X_o\left(e^{j\omega}\right) &= \frac{1}{2}\left[X\left(e^{j\omega}\right) - X^*\left(e^{-j\omega}\right)\right] = -X_o^*\left(e^{-j\omega}\right). \end{aligned}$$

(3) Table 2.1 lists important symmetry properties of the Fourier transform, which are useful for manipulating Fourier transforms. (You prove them as self exercise)

Table 2.1

- Ex: Recall that the Fourier transform of real-valued $x[n] = a^n u[n]$ with $|a| < 1$ is

$$X\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}.$$

It follows from the properties of complex numbers that

$$\begin{aligned} X\left(e^{j\omega}\right) &= \frac{1}{1 - ae^{-j\omega}} = X^*\left(e^{-j\omega}\right) \qquad \text{(Property 7)} \\ X_R\left(e^{j\omega}\right) &= \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R\left(e^{-j\omega}\right) \qquad \text{(Property 8)} \\ X_I\left(e^{j\omega}\right) &= \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I\left(e^{-j\omega}\right) \qquad \text{(Property 9)} \\ |X\left(e^{j\omega}\right)| &= \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} = |X\left(e^{-j\omega}\right)| \qquad \text{(Property 10)} \\ \angle X\left(e^{j\omega}\right) &= \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = -\angle X\left(e^{-j\omega}\right) \qquad \text{(Property 11).} \end{aligned}$$

2.9 Fourier Transform Theorems

- Let us adopt the following notation

$$\begin{aligned} &X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\}. \\ &x[n] = \mathcal{F}^{-1}\{X\left(e^{j\omega}\right)\}. \\ &x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right). \end{aligned}$$

and address the following theorems: (You prove them as self exercise)

Theorems:

1. Linearity: If

$$x_i[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_i\left(e^{j\omega}\right), \ i = 1, 2, ..., M$$

then

$$\sum_{i=1}^{M} a_i x_i[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \sum_{i=1}^{M} a_i X_i\left(e^{j\omega}\right) \text{ for } \forall a_i.$$

2. Time Shifting and Frequency Shifting: If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$x[n-n_d] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} X\left(e^{j\omega}\right)$$
$$e^{j\omega_0 n} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right).$$

3. Time Reversal: If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{-j\omega}\right).$$

If $x[n]$ is real,

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*\left(e^{j\omega}\right).$$

4. Differentiation in Frequency Domain: If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{dX\left(e^{j\omega}\right)}{d\omega}.$$

5. Parseval's Theorem: If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then it has an *energy density spectrum*

$$E \triangleq \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega.$$

Moreover, if

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$
$$y[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y\left(e^{j\omega}\right)$$

then

$$\sum_{n=-\infty}^{\infty} x[n]\, y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) Y^*\left(e^{j\omega}\right) d\omega.$$

6. Convolution Theorem: If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$
$$h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} H\left(e^{j\omega}\right)$$
$$y[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y\left(e^{j\omega}\right)$$

and if

$$y[n] = x[n] * h[n]$$

then

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right).$$

Pf:

$$\begin{aligned}
Y\left(e^{j\omega}\right) &= \mathcal{F}\{x[n] * h[n]\} \\
&= \sum_{n=-\infty}^{\infty} (x[n] * h[n]) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]\, e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[n-k]\, e^{-j\omega(n-k)} h[k]\, e^{-j\omega k} \\
&= \sum_{k=-\infty}^{\infty} \sum_{\substack{l=-\infty \\ (l=n-k)}}^{\infty} x[l]\, e^{-j\omega l} h[k]\, e^{-j\omega k} \\
&= X\left(e^{j\omega}\right) H\left(e^{j\omega}\right) \qquad \text{QED.}
\end{aligned}$$

Notes:

(a) The time-shifting property is a special case of the convolution property. First,

$$\delta[n-n_d] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d}$$

and if $h[n] = \delta[n-n_d]$, then $y[n] = x[n] * \delta[n-n_d] = x[n-n_d]$. Second, since $H(e^{j\omega}) = e^{-j\omega n_d}$, we have $Y(e^{j\omega}) = e^{-j\omega n_d} X(e^{j\omega})$. This gives the proof for the time-shifting property.

(b) The convolution property can be interpreted as a direct consequence of the eigenfunction property. Recall from the eigenfunction property that, if $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega}) e^{j\omega n}$. Now, from the inverse Fourier transform,

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega \\ &= \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_k X\left(e^{jk\Delta\omega}\right) e^{jk\Delta\omega n} \Delta\omega. \end{aligned}$$

Quoting the eigenfunction property, we have

$$\begin{aligned} y[n] &= T\{x[n]\} \\ &= \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_k X\left(e^{jk\Delta\omega}\right) T\{e^{jk\Delta\omega n}\}\Delta\omega \\ &= \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_k X\left(e^{jk\Delta\omega}\right) H\left(e^{jk\Delta\omega}\right) e^{jk\Delta\omega n}\Delta\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) H\left(e^{j\omega}\right) e^{j\omega n} d\omega \\ \Leftrightarrow Y\left(e^{j\omega}\right) &= X\left(e^{j\omega}\right) H\left(e^{j\omega}\right). \end{aligned}$$

This provides an alternative proof for the convolution property.

7. Modulation or Windowing Theorem: If

$$\begin{gathered} x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j\omega}\right) \\ w[n] \xleftrightarrow{\mathcal{F}} W\left(e^{j\omega}\right) \\ y[n] = x[n]\, w[n] \end{gathered}$$

then

$$Y\left(e^{j\omega}\right) = \mathcal{F}\{y[n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right) W\left(e^{j(\omega-\theta)}\right) d\theta.$$

Note that modulation and convolution theorems are a duality.

- The above theorems are summarized in

Table 2.2

- See Table *2.3* for useful Fourier transform pairs.

Table 2.3

- Ex:

$$\begin{aligned} x[n] = a^n u[n-n_0] &= a^{n_0}\left\{a^{n-n_0}u[n-n_0]\right\} \quad |a|<1 \\ \Rightarrow \quad X\left(e^{j\omega}\right) &\underset{\text{linearity}}{=} a^{n_0}\mathcal{F}\left\{a^{n-n_0}u[n-n_0]\right\} \\ &\underset{\text{time-shifting}}{=} a^{n_0}e^{-j\omega n_0}\mathcal{F}\left\{a^n u[n]\right\} \\ &\underset{\text{pair 4 of Table 2.3}}{=} a^{n_0}e^{-j\omega n_0}\frac{1}{1-ae^{-j\omega}}. \end{aligned}$$

- Ex: Determine the impulse response of an LTI system with the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1].$$

First, applying Fourier transform to both sides yields

$$Y\left(e^{j\omega}\right) - \frac{1}{2}e^{-j\omega}Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) - \frac{1}{4}e^{-j\omega}X\left(e^{j\omega}\right)$$

(time-shifting)

$$\Rightarrow \quad H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}.$$

(convolution)

Now, from pair 4 of Table 2.3,

$$\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

and

$$\begin{aligned} -\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1}u[n-1] \overset{\mathcal{F}}{\longleftrightarrow} & -\left(\frac{1}{4}\right)e^{-j\omega}\mathcal{F}\left\{\left(\frac{1}{2}\right)^n u[n]\right\} \\ &= -\left(\frac{1}{4}\right)e^{-j\omega}\frac{1}{1-\frac{1}{2}e^{-j\omega}} \end{aligned}$$

we have

$$h[n] = \mathcal{F}^{-1}\left\{H\left(e^{j\omega}\right)\right\} = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1}u[n-1].$$

- Self-exercise: Prove Tables 2.1, 2.2, and 2.3.