1. (0.5%) Page 99, Problem 2.2

(a) The impulse response $h[n]$ of an LTI system is known to be zero, except in the interval $N_0 \le n \le N_1$. The input $x[n]$ is known to be zero, except in the interval $N_2 \le n \le N_3$. As a result, the output is constrained to be zero, except in some interval $N_4 \le n \le N_5$. Determine N_4 and N_5 in terms of N_0, N_1, N_2, and N_3.

(b) If $x[n]$ is zero, except for N consecutive points, and $h[n]$ is zero, except for M consecutive points, what is the maximum number of consecutive points for which $y[n]$ can be nonzero?

Sol: For an LTI system, the output is obtained from the convolution of the input with the impulse response of the system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

(a) Since $h[k] \neq 0$, for $(N_0 \le n \le N_1)$,

$$y[n] = \sum_{k=N_0}^{N_1} h[k]\,x[n-k]$$

The input, $x[n] \neq 0$, for $(N_2 \le n \le N_3)$, so

$$x[n-k] \neq 0, \text{ for } N_2 \le n-k \le N_3$$

Note that the minimum value of $(n-k)$ is N_2. Thus, the lower bound on n, which occurs for $k = N_0$ is

$$N_4 = N_0 + N_2.$$

Using a similar argument,

$$N_5 = N_1 + N_3.$$

Therefore, the output is nonzero for

$$(N_0 + N_2) \le n \le (N_1 + N_3).$$

(b) If $x[n] \neq 0$, for some $n_0 \le n \le (n_0 + N - 1)$, $h[k] \neq 0$, for some $n_1 \le n \le (n_1 + M - 1)$, the results of part (a) imply that the output is nonzero for:

$$(n_0 + n_1) \le n \le (n_0 + n_1 + M + N - 2)$$

So the output sequence is $M + N - 1$ samples long. This is an important quality of the convolution for finite length sequences as we shall see in Chapter 8.

2. (0.5%) Page 99, Problem 2.7

Determine whether each of the following signals is periodic. If the signal is periodic, state its period.

(a) $x[n] = e^{j(\pi n/6)}$

(b) $x[n] = e^{j(3\pi n/4)}$

(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$

(d) $x[n] = e^{j\pi n/\sqrt{2}}$

Sol: $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) $x[n]$ is periodic with period 12:

$$e^{j\left(\frac{\pi}{6}n\right)} = e^{j\left(\frac{\pi}{6}n\right)(n+N)} = e^{j\left(\frac{\pi}{6}n+2\pi k\right)}$$
$$\Longrightarrow 2\pi k = \tfrac{\pi}{6}N, \text{ for intergers } k, N$$

Making $k = 1$ and $N = 12$ shows that $x[n]$ has period 12.

(b) $x[n]$ is periodic with period 8:

$$e^{j\left(\frac{3\pi}{4}n\right)} = e^{j\left(\frac{3\pi}{4}n\right)(n+N)} = e^{j\left(\frac{3\pi}{4}n+2\pi k\right)}$$
$$\Longrightarrow 2\pi k = \tfrac{3\pi}{4}N, \text{ for intergers } k, N$$
$$\Longrightarrow N = \tfrac{8}{3}k, \text{ for intergers } k, N$$

The smallest k for which both k and N are integers is 3, resulting in the period N being 8.

(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$ is not periodic because the denominator term is linear in n.

(d) We will show that $x[n]$ is not periodic. Suppose that $x[n]$ is periodic for some period N:

$$e^{j\left(\frac{\pi}{\sqrt{2}}n\right)} = e^{j\left(\frac{\pi}{\sqrt{2}}n\right)(n+N)} = e^{j\left(\frac{\pi}{\sqrt{2}}n+2\pi k\right)}$$
$$\Longrightarrow 2\pi k = \tfrac{\pi}{\sqrt{2}}N, \text{ for intergers } k, N$$
$$\Longrightarrow N = 2\sqrt{2}k, \text{ for intergers } k, N$$

There is no integer k for which N is an integer. Hence $x[n]$ is not periodic.

3. (0.5%) Page 101, Problem 2.15

Consider the system illustrated in Figure P2.15. The output of an LTI system with an impulse response $h[n] = (\frac{1}{4})^n u[n+10]$ is multiplied by a unit step function $u[n]$ to yield the output of the overall system. Answer each of the following questions, and briefly justify your answers:

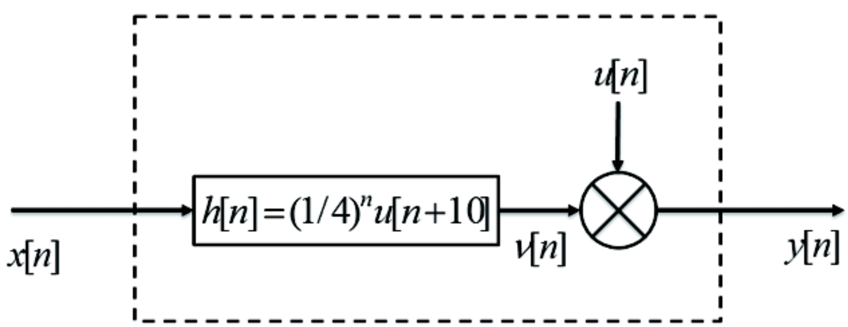


Figure P2.15

(a) Is the overall system LTI?

(b) Is the overall system causal?

(c) Is the overall system stable in the BIBO sense?

Sol: $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) No. Consider the following input/output:

$$x_1[n] = \delta[n] \Longrightarrow y_1[n] = \left(\tfrac{1}{4}\right)^n u[n]$$
$$x_2[n] = \delta[n-1] \Longrightarrow y_2[n] = \left(\tfrac{1}{4}\right)^{n-1} u[n]$$

Even though $x_2[n] = x_1[n-1]$, $y_2[n] \neq y_1[n-1] = \left(\tfrac{1}{4}\right)^{n-1} u[n-1]$.

(b) No. Consider the input/output pair $x_2[n]$ and $y_2[n]$ above. $x_2[n] = 0$ for $n < 1$, but $y_2[0] \neq 0$.

(c) Yes. Since $h[n]$ is stable and multiplication with $u[n]$ will not cause any sequences to become unbounded, the entire system is stable.

4. (0.5%) Page 103, Problem 2.23

For each of the following systems, determine whether the system is (1) stable, (2) causal, (3) linear, and (4) time invariant.

(a) $T(x[n]) = (\cos \pi n)\, x[n]$

(b) $T(x[n]) = x[n^2]$

(c) $T(x[n]) = x[n] \sum_{k=0}^{\infty} \delta[n-k]$

(d) $T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$.

Sol:

(a) Since $\cos(\pi n)$ only takes on values of $+1$ or -1, this transformation outputs the current value of $x[n]$ multiplied by either ± 1. $T(x[n]) = (-1)^n x[n]$.

- Hence, it is stable, because it doesn't change the magnitude of $x[n]$ and hence satisfies bounded-in/bounded-out stability.
- It is causal, because each output depends only on the current value of $x[n]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = \cos(\pi n)\, x_1[n]$, and $y_2[n] = T(x_2[n]) = \cos(\pi n)\, x_2[n]$. Now

$$T(ax_1[n] + bx_2[n]) = \cos(\pi n)(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = (-1)^n x[n]$, then $T(x[n-1]) = (-1)^n x[n-1] \neq y[n-1]$.

(b) This transformation simply "samples" $x[n]$ at location which can be expressed as k^2.

- The system is stable, since if $x[n]$ is bounded, $x[n^2]$ is also bounded.

- It is not causal. For example, $T(x[4]) = x[16]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = x_1[n^2]$, and $y_2[n] = T(x_2[n]) = x_2[n^2]$. Now

$$T(ax_1[n] + bx_2[n]) = ax_1\left[n^2\right] + bx_2\left[n^2\right] = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = x[n^2]$, then $T(x[n-1]) = x[n^2-1] \neq y[n-1]$.

(c) First notice that

$$\sum_{k=0}^{\infty} \delta[n-k] = u[n]$$

So $T(x[n]) = x[n]u[n]$. This transformation is therefore stable, causal, linear, but not time-invariant.

To see that it is not time invariant, notice that $T(\delta[n]) = \delta[n]$, but $T(\delta[n+1]) = 0$.

(d) $T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$.

- This is not stable. For example, for $T(u[n]) = \infty$ all $n \geq 1$.
- It is not causal, since it sums *forward* in time.
- It is linear, since

$$\sum_{k=n-1}^{\infty} ax_1[k] + bx_2[k] = a\sum_{k=n-1}^{\infty} x_1[k] + b\sum_{k=n-1}^{\infty} x_2[k]$$

- It is time-invariant. Let

$$y[n] = T(x[n]) = \sum_{k=n-1}^{\infty} x[k],$$

then

$$T(x[n-n_0]) = \sum_{k=n-n_0-1}^{\infty} x[k] = y[n-n_0].$$

5. (0.5%) Page 104, Problem 2.28

Consider the difference equation

$$y[n] + \frac{1}{15}y[n-1] - \frac{2}{15}y[n-2] = x[n].$$

(a) Determine the general form of the homogeneous solution to this equation.

(b) Both a causal and an anticausal LTI system are characterized by the given difference equation. Find the impulse responses of the two systems.

(c) Show that the causal LTI system is stable and the anticausal LTI system is unstable.

(d) Find a particular solution to the difference equation when $x[n] = (3/5)^n u[n]$.

Sol:

(a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 + \frac{1}{15}c - \frac{2}{5} = 0$$

So for $c = 1/3$ and $c = -2/5$, the general form for the homogeneous solution is :

$$y_h[n] = A_1\left(\frac{1}{3}\right)^n + A_2\left(-\frac{2}{5}\right)^n$$

(b) We use the z-transform, and use different ROCs to generate the causal and anti-causal impulses responses:

$$\begin{aligned} H(z) &= \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{2}{5}z^{-1}\right)} = \frac{5/11}{1-\frac{1}{3}z^{-1}} + \frac{6/11}{1+\frac{2}{5}z^{-1}} \\ h_c[n] &= \frac{5}{11}\left(\frac{1}{3}\right)^n u[n] + \frac{6}{11}\left(-\frac{2}{5}\right)^n u[n] \\ h_{ac}[n] &= -\frac{5}{11}\left(\frac{1}{3}\right)^n u[-n-1] - \frac{6}{11}\left(-\frac{2}{5}\right)^n u[-n-1] \end{aligned}$$

(c) Since $h_c[n]$ is causal, and the two exponential bases in $h_c[n]$ are both less than 1, it is absolutely summable. $h_{ac}[n]$ grows without bounds as n approaches $-\infty$.

(d)

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= \frac{1}{1-\frac{3}{5}z^{-1}} \cdot \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{2}{5}z^{-1}\right)} \\ &= \frac{-25/44}{1-1/3z^{-1}} + \frac{12/55}{1+2/5z^{-1}} + \frac{27/20}{1-3/5z^{-1}} \\ y[n] &= \frac{-25}{44}\left(\frac{1}{3}\right)^n u[n] + \frac{55}{12}\left(-\frac{2}{5}\right)^n u[n] + \frac{27}{20}\left(\frac{3}{5}\right)^n u[n] \end{aligned}$$

6. (0.5%) Page 105, Problem 2.29

Three systems A, B, and C have the inputs and outputs indicated in Figure P2.29. Determine whether each system could be LTI. If your answer is yes, specify whether there could be more than one LTI system with the given input-output pair. Explain your answer.

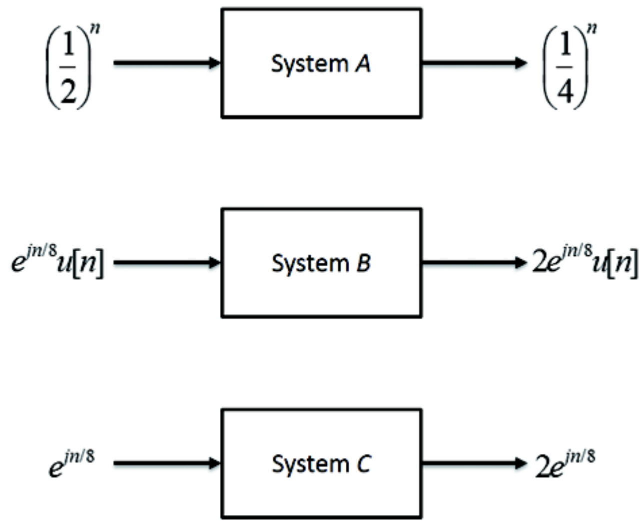


Figure P2.29

Sol: $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

- System A:

$$x[n] = \left(\frac{1}{2}\right)^n$$

  This input is an eigenfunction of an LTI system. That is, if the system is linear, the output will be a replica of the input, scaled by a complex constant.
  Since $y[n] = \left(\frac{1}{4}\right)^n$, System A is NOT LTI.

- System B:

$$x[n] = e^{jn/8} u[n]$$

  The Fourier transform of $x[n]$ is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} e^{jn/8} u[n] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} e^{-j\left(\omega - \frac{1}{8}\right)n} \\ &= \frac{1}{1 - e^{-j\left(\omega - \frac{1}{8}\right)}}. \end{aligned}$$

  The output is $y[n] = 2x[n]$, thus

$$Y\left(e^{j\omega}\right) = \frac{2}{1 - e^{-j\left(\omega - \frac{1}{8}\right)}}.$$

  Therefore, the frequency response of the system is

$$\begin{aligned} H\left(e^{j\omega}\right) &= \frac{Y\left(e^{j\omega}\right)}{X(e^{j\omega})} \\ &= 2. \end{aligned}$$

  Hence, the system is a linear amplifier. We conclude that System B is LTI, and unique.

- System C: Since $x[n] = e^{jn/8}$ is an eigenfunction of an LTI system, we would expect the output to be given by

$$y[n] = \gamma e^{jn/8},$$

where γ is some complex constant, if System C were indeed LTI. The given output, $y[n] = 2e^{jn/8}$, indicates that this is so.
Hence, System C is LTI. However, it is not unique, since the only constraint is that

$$H\left(e^{j\omega}\right)|_{\omega=1/8} = 2.$$

7. (0.5%) Page 105, Problem 2.30

If the input and output of a causal LTI system satisfy the difference equation

$$y[n] = ay[n-1] + x[n],$$

then the impulse response of the system must be $h[n] = a^n u[n]$.

(a) For what values of a is this system stable?

(b) Consider a causal LTI system for which the input and output are related by the difference equation

$$y[n] = ay[n-1] + x[n] - a^N x[n-N],$$

where N is a positive integer. Determine and sketch the impulse response of this system.
Hint: Use linearity and time-invariance to simplify the solution.

(c) Is the system in part (b) an FIR or an IIR system? Explain.

(d) For what values of a is the system in part (b) stable? Explain.

Sol:

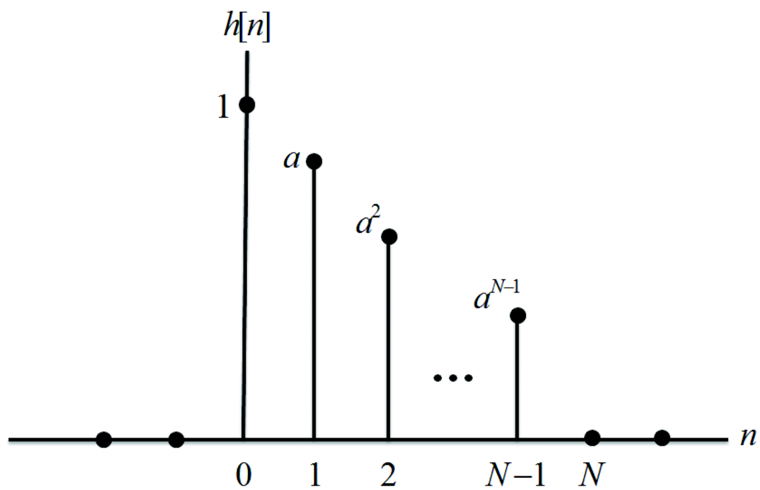
(a) LTI systems are stable iff $\sum\limits_{n=-\infty}^{\infty} |h[n]| < \infty$ (the summation should converge).

Then

$$\begin{aligned} S &= \sum_{n=-\infty}^{\infty} |a|^n u[n] \\ &= \sum_{n=0}^{\infty} |a|^n \end{aligned}$$

S will converge only when $|a| < 1$ and $S = \frac{1}{1-|a|} < \infty$.
Therefore the system is stable for $|a| < 1$.

(b) $y[n] = ay[n-1] + x[n] - a^N x[n-N]$. Therefore,
$h[n] = ah[n-1] + \delta[n] - a^N \delta[n-N]$.
Since the system is causal, $h[-1] = 0$. Then
$h[0] = 0 + 1 - 0 = 1$
$h[1] = a$, $h[2] = a^2$, $h[N] = a^N - a^N = 0$
$h[N+1] = a \times 0 + 0 - 0 = 0$.



(c) We see that even though it is a recursive system (with feedback), its impulse response finite in length. The length of $h[n]$ is N terms. Hence this system is FIR.

(d) FIR systems are always stable as the sum $\sum_{n=-\infty}^{\infty} |h[n]|$ has at most a finite number of nonzero terms.

8. (0.5%) Page 107, Problem 2.36

An LTI discrete-time system has frequency response given by

$$H(e^{j\omega}) = \frac{(1 - je^{-j\omega})(1 + je^{-j\omega})}{1 - 0.8e^{-j\omega}} = \frac{1 + e^{-j2\omega}}{1 - 0.8e^{-j\omega}} = \frac{1}{1 - 0.8e^{-j\omega}} + \frac{e^{-j2\omega}}{1 - 0.8e^{-j\omega}}.$$

(a) Use one of the above forms of the frequency response to obtain an equation for the impulse response $h[n]$ of the system.

(b) From the frequency response, determine the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$ of the system.

(c) If the input to this system is

$$x[n] = 4 + 2\cos(\omega_0 n) \text{ for } -\infty < n < \infty$$

for what value of ω_0 will the output be of the form

$$y[n] = A = \text{constant}$$

for $-\infty < n < \infty$? What is the constant A?

Sol:

(a)

$$
\begin{aligned}
H(e^{j\omega}) &= \frac{1}{1-0.8e^{j\omega}} + \frac{e^{-j2\omega}}{1-0.8e^{j\omega}} \\
h\left[n\right] &= (0.8)^{n}\, u\left[n\right] + (0.8)^{n-2}\, u\left[n-2\right]
\end{aligned}
$$

(b)

$$
\begin{aligned}
H(e^{j\omega}) &= \frac{Y\left(e^{j\omega}\right)}{X(e^{j\omega})} = \frac{1+e^{-j2\omega}}{1-0.8e^{-j\omega}} \\
Y\left(e^{j\omega}\right) - 0.8e^{j\omega} Y\left(e^{j\omega}\right) &= X\left(e^{j\omega}\right) + e^{-j2\omega} X\left(e^{j\omega}\right) \\
y\left[n\right] - 0.8y\left[n-1\right] &= x\left[n\right] + x\left[n-2\right] \\
y\left[n\right] &= 0.8y\left[n-1\right] + x\left[n\right] + x\left[n-2\right]
\end{aligned}
$$

(c) Using the frequency response we can write the output as

$$
y\left[n\right] = H(e^{j0})4 + 2\left|H\left(e^{j\omega_0}\right)\right| \cos\left(\omega_0 n + \angle H\left(e^{j\omega_0}\right)\right).
$$

To get $y\left[n\right]$ =constant we need $\left|H\left(e^{j\omega_0}\right)\right| = 0$, which means $1 + e^{-j2\omega_0} = 0$, or $\omega_0 = \pi/2$.

Then $y\left[n\right] = 4\frac{1+1}{1-0.8} = 40$.

9. (0.5%) Page 110, Problem 2.44

Consider the cascade of LTI discrete-time systems shown in Figure P2.44.

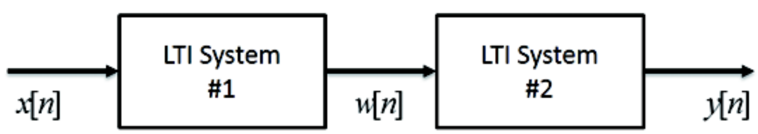


Figure P2.44

The first system is described by the equation

$$
H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < 0.5\pi, \\ 0, & 0.5\pi \le |\omega| < \pi, \end{cases}
$$

and the second system is described by the equation

$$
y[n] = w[n] - w[n-1].
$$

The input to this system is

$$
x[n] = \cos(0.6\pi n) + 3\delta[n-5] + 2.
$$

Determine the output $y[n]$. With careful thought, you will be able to use the properties of LTI systems to write down the answer by inspection.

Sol: First $x[n]$ goes through a lowpass filter with cutoff frequency 0.5π. Since the cosine has a frequency of 0.6π, it will be filtered out. The delayed impulse will be filtered to a delayed sinc and the constant will remain unchanged. We thus get:

$$w[n] = 3\frac{\sin(0.5\pi(n-5))}{\pi(n-5)} + 2.$$

$y[n]$ is then given by:

$$y[n] = 3\frac{\sin(0.5\pi(n-5))}{\pi(n-5)} - 3\frac{\sin(0.5\pi(n-6))}{\pi(n-6)}.$$

10. (0.5%) Page 114, Problem 2.54

Consider the system in Figure P2.54.

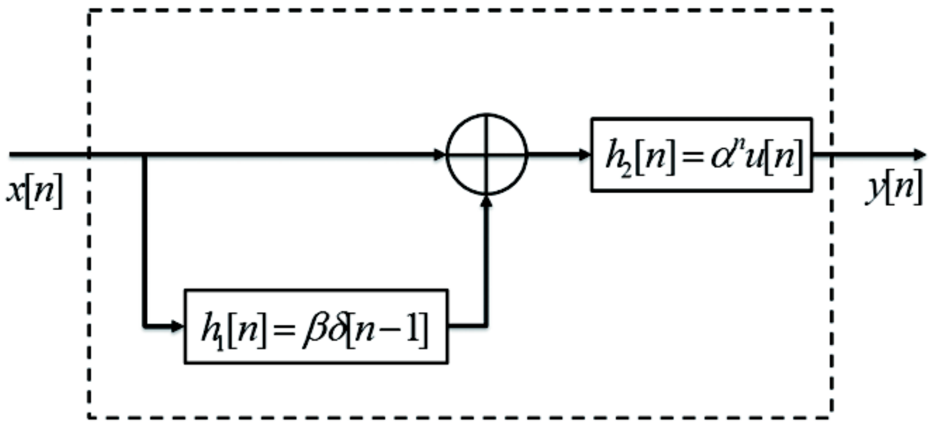


Figure P2.54

(a) Determine the impulse response $h[n]$ of the overall system.

(b) Determine the frequency response of the overall system.

(c) Specify a difference equation that relates the output $y[n]$ to the input $x[n]$.

(d) Is this system causal? Under what condition would the system be stable?

Sol:

(a) From the figure,

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= (x[n] * (\delta[n] + h_1[n])) * h_2[n]. \end{aligned}$$

Let $h[n]$ be the impulse response of the overall system,

$$y[n] = x[n] * h[n].$$

Comparing with the above expression,

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= h_2[n] + h_1[n] * h_2[n] \\ &= \alpha^n u[n] + \beta\alpha^{(n-1)} u[n-1]. \end{aligned}$$

(b) Taking the Fourier transform of $h[n]$ from part (a),

$$\begin{aligned}
H\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} h\left[n\right] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \alpha^{n} u\left[n\right] e^{-j\omega n} + \beta \sum_{n=-\infty}^{\infty} \alpha^{(n-1)} u\left[n-1\right] e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \alpha^{n} e^{-j\omega n} + \beta \sum_{\ell=0}^{\infty} \alpha^{(\ell-1)} e^{-j\omega \ell},
\end{aligned}$$

where we have used $\ell = (n-1)$ in the second sum.

$$\begin{aligned}
H\left(e^{j\omega}\right) &= \frac{1}{1-\alpha e^{-j\omega}} + \frac{\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} \\
&= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \text{ for } |\alpha| < 1
\end{aligned}$$

Note that the Fourier transform of $\alpha^{n} u\left[n\right]$ is well known, and the second term of $h[n]$ (see part (a)) is just a scaled and shifted version of $\alpha^{n} u\left[n\right]$. So, we could have used the properties of the Fourier transform to reduce the algebra.

(c) We have

$$\begin{aligned}
H\left(e^{j\omega}\right) &= \frac{Y\left(e^{j\omega}\right)}{X(e^{j\omega})} \\
&= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}},
\end{aligned}$$

cross multiplying,

$$Y\left(e^{j\omega}\right)\left[1-\alpha e^{-j\omega}\right] = X(e^{j\omega})\left[1+\beta e^{-j\omega}\right]$$

taking the inverse Fourier transform, we have

$$y\left[n\right] - \alpha y\left[n-1\right] = x\left[n\right] + \beta x\left[n-1\right].$$

(d) From part (a):

$$h\left[n\right] = 0, \text{ for } n < 0$$

This implies that the system is CAUSAL.

If the system is stable, its Fourier transform exists. Therefore, the condition for stability is the same as the condition imposed on the frequency response of part (b). That is, STABLE, if $|\alpha| < 1$.