5 Transform Analysis of LTI Systems

- For an LTI system with input $x[n]$, output $y[n]$, and impulse response $h[n]$:

Fig. 48-F1

1. $y[n] = h[n] * x[n]$.
2. $Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$.
3. From the "Convolution of Sequences" property of the z-transform,

$$Y(z) = H(z) X(z).$$

- Notes:

1. $h[n]$ and $H(z)$ form a unique z-transform pair.
2. $H(z)$ is called the *system function* of the LTI system, which completely characterizes the system in conjunction with its ROC.
3. $H(z)$ is related to the *frequency response* $H(e^{j\omega})$ by $z \to e^{j\omega}$.

5.1 The Frequency Response of LTI Systems

- The frequency response $H(e^{j\omega})$ is in general complex and can be described alternatively by its *magnitude response* $|H(e^{j\omega})|$ and *phase response* $\angle H(e^{j\omega})$.

$$\begin{aligned} & Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right) \\ \Rightarrow \quad & \left|Y\left(e^{j\omega}\right)\right| = \underbrace{\left|H\left(e^{j\omega}\right)\right|}_{\text{magnitude response, or gain, or magnitude distortion}} \left|X\left(e^{j\omega}\right)\right| \\ & \angle Y\left(e^{j\omega}\right) = \underbrace{\angle H\left(e^{j\omega}\right)}_{\text{phase response, or phase shift, or phase distortion}} + \angle X\left(e^{j\omega}\right). \end{aligned}$$

- The principal value of $\angle H(e^{j\omega})$ is denoted by ARG$[H(e^{j\omega})]$ and defined to take value in the range

$$-\pi < \mathrm{ARG}[H\left(e^{j\omega}\right)] \leq \pi.$$

Thus, $\angle H(e^{j\omega})$ can be represented as

$$\angle H\left(e^{j\omega}\right) = \mathrm{ARG}[H\left(e^{j\omega}\right)] + 2\pi r(\omega)$$

where $r(\omega)$ is an integer function of ω. Since $\mathrm{ARG}[H(e^{j\omega})]$ takes the modulo-2π value of $\angle H(e^{j\omega})$ in $(-\pi, \pi]$, it wraps the phase response on a circle and is thus called the *wrapped phase* of $\angle H(e^{j\omega})$. We also use $\arg[H(e^{j\omega})]$ to denote a continuous (unwrapped) phase function of $\angle H(e^{j\omega})$.

Fig. 5.1

Ex: For an ideal filter, $\angle H(e^{j\omega}) = 0$, $\forall\omega$.

- The group delay of an LTI system with $H(e^{j\omega})$ is defined as

$$\begin{aligned}
\tau(\omega) &\equiv \mathrm{grd}[H\left(e^{j\omega}\right)] \\
&\triangleq -\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\} \text{ (continuous and differentiable phase response)} \\
&= -\frac{d}{d\omega}\left\{\angle H\left(e^{j\omega}\right)\right\} \text{ (except jump points)} \\
&= -\frac{d}{d\omega}\left\{\mathrm{ARG}[H\left(e^{j\omega}\right)]\right\}. \text{ (except jump points)}
\end{aligned}$$

- Ex: Ideal Delay System

$$\begin{aligned}
& h_{id}[n] = \delta[n - n_d] \\
\Rightarrow\ & H_{id}\left(e^{j\omega}\right) = e^{-j\omega n_d}
\end{aligned}$$

Now, we have

$$\begin{aligned}
|H_{id}\left(e^{j\omega}\right)| &= 1 \\
\angle H_{id}\left(e^{j\omega}\right) &= -\omega n_d \text{ for } |\omega| < \pi \\
\tau_{id}(\omega) &= n_d \text{ for } |\omega| < \pi.
\end{aligned}$$

- Ex: Ideal Lowpass Filter (LPF)

$$\begin{aligned}
& h_{ilp}[n] = \frac{\sin\omega_c n}{\pi n} \\
\Rightarrow\ & H_{ilp}\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
|H_{ilp}\left(e^{j\omega}\right)| &= 1 \text{ for } |\omega| < \omega_c \\
\angle H_{ilp}\left(e^{j\omega}\right) &= 0 \text{ for } |\omega| < \omega_c \\
\tau_{ilp}(\omega) &= 0 \text{ for } |\omega| < \omega_c.
\end{aligned}$$

It has zero phase response. For most applications, a linear phase response can be tolerated since the phase response can be compensated for by introducing delay in other parts of a larger system. In the case, we can also accept an ideal LPF with linear phase response as

$$h_{lp}[n] = \frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}$$
$$\Rightarrow \quad H_{lp}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

which has

$$\begin{aligned} \left|H_{lp}\left(e^{j\omega}\right)\right| &= 1 \text{ for } |\omega| < \omega_c \\ \angle H_{lp}\left(e^{j\omega}\right) &= -\omega n_d \text{ for } |\omega| < \omega_c \\ \tau_{lp}(\omega) &= n_d \text{ for } |\omega| < \omega_c. \end{aligned}$$

- Ex: Ideal highpass filter (HPF)

$$H_{ihp}\left(e^{j\omega}\right) = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & \omega_c < |\omega| \le \pi \end{cases}$$
$$= 1 - H_{ilp}\left(e^{j\omega}\right)$$
$$\Rightarrow \quad h_{ihp}[n] = \delta[n] - h_{ilp}[n].$$

Now, we have

$$\begin{aligned} \left|H_{ihp}\left(e^{j\omega}\right)\right| &= 1 \text{ for } \omega_c < |\omega| \le \pi \\ \angle H_{ihp}\left(e^{j\omega}\right) &= 0 \text{ for } \omega_c < |\omega| \le \pi \\ \tau_{ihp}(\omega) &= 0 \text{ for } \omega_c < |\omega| \le \pi. \end{aligned}$$

- Notes:

1. The impulse responses for ideal filters (with ideal cutoff) is not computationally realizable since they extend from $-\infty$ to ∞.
$$\begin{pmatrix} \text{limited in time domain} \longleftrightarrow \text{unlimited in frequency domain} \\ \text{unlimited in time domain} \longleftrightarrow \text{limited in frequency domain} \end{pmatrix}$$
2. Recall that
$$\delta[n-n_d] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d}$$
Thus,

Fig.49-F1

i.e., all the input samples suffer the same amount of delay n_d, called "group delay".

3. If $\angle H\left(e^{j\omega}\right) = -\alpha - \omega n_d$, then $\tau(\omega) = n_d$ represents the system's group delay to the input sequence. Therefore, the group delay represents a convenient measure of the linearity of the phase response.

4. Read Subsection 5.1.2 (pp. 306-311) for the illustration of effects of group delay and attenuation.

5.2 LTI Systems Characterized by LCCD Equations

- Consider an LTI system with input $x[n]$ and output $y[n]$ characterized by the Nth-order LCCD equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \ a_0 \neq 0.$$

Taking z-transform both-sided,

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}$$

where $\{c_k\}_{k=1}^{M}$ is the set of zeros and $\{d_k\}_{k=1}^{N}$ is the set of poles.

- Ex: Second-Order System

Consider the LTI system characterized by the second-order LCCD equation

$$y[n] + \frac{1}{4} y[n-1] - \frac{3}{8} y[n-2] = x[n] + 2x[n-1] + x[n-2]$$

$$\Rightarrow \quad H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4} z^{-1} - \frac{3}{8} z^{-2}} = \frac{\left(1 + z^{-1}\right)^2}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{3}{4} z^{-1}\right)}$$

which has two first-order poles at $\frac{1}{2}$ and $-\frac{3}{4}$, and one second-order zero at -1.

Notes: (1) The ROC is not specified yet! (2) The LCCD equation can not uniquely determine an LTI system!

5.2.1 Stability and Causality

- A sufficient and necessary (iff) condition for (BIBO) stable LTI systems is
$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \text{ (absolutely summable)}$$
Now, for $|z| = 1$,
$$|H(z)| \leq \sum_{n=-\infty}^{\infty} \left|h[n] z^{-n}\right| = \sum_{n=-\infty}^{\infty} |h[n]|.$$
Hence, whether $\sum_{n=-\infty}^{\infty} |h[n]|$ is finite or not determines whether the ROC of $H(z)$ includes unit circle or not. Thus, an LTI system is stable iff the ROC of $H(z)$ includes unit circle. Note that stable systems have a frequency response.

- A system is causal iff $h[n] = 0$ for $n < 0$. Thus, $H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$ exists for $|z|$ approaching infinity. This implies that a sufficient and necessary condition for $h[n]$ being causal is that the ROC of $H(z)$ should be outside the pole with the largest magnitude.

- Ex: Consider the LTI system with input and output related by
$$y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]$$
$$\Rightarrow \quad H(z) = \frac{1}{1 - \frac{5}{2} z^{-1} + z^{-2}} = \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - 2z^{-1}\right)} = \frac{z^2}{\left(z - \frac{1}{2}\right)(z-2)}.$$
The pole-zero plot for $H(z)$ is indicated in

  Fig. 5.7

  There are three possible choices for the ROC:

  1. If ROC $= \{z \,||z| < \frac{1}{2}\}$, the system is neither causal nor stable.
  2. If ROC $= \{z \,|\frac{1}{2} < |z| < 2\}$, the system is stable but not causal.
  3. If ROC $= \{z \,||z| > 2\}$, the system is causal but not stable.

- Notes:

  1. Stability and Causality are not necessarily compatible!
  2. If an LTI system is stable and causal, its ROC should take the form $\{z : |z| > |z_0|\}$ with $|z_0| < 1$.

5.2.2 Inverse Systems

- The inverse system of an LTI system with $H(z)$ has system function $H_i(z)$ such that

  Fig. 51-F1 (∗)

  This implies

$$H_i(z) = \frac{1}{H(z)} \text{ and } h[n] * h_i[n] = \delta[n].$$

  Note: Not every system has an inverse. For example, the ideal lowpass system does not have an inverse.

  Fig. 51-F2

- Consider a rational system function

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

  with $a_0 \neq 0$ and $b_0 \neq 0$. Its inverse is

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}.$$

  For (∗) to hold, the ROC's of $H(z)$ and $H_i(z)$ must overlap. For example, if $H(z)$ is causal, its ROC is

$$|z| > \max_k |d_k|.$$

  Then, any appropriate ROC for $H_i(z)$ that overlaps the above region is a valid ROC for $H_i(z)$. On the other hand, $H_i(z)$ is causal iff the ROC of $H_i(z)$ is given by

$$|z| > \max_k |c_k|.$$

  If $\max_k |c_k| < 1$ (i.e., all the zeros of $H(z)$ locate inside the unit circle), then $H_i(z)$ is stable as well.

  Therefore, both $H(z)$ and $H_i(z)$ are stable and causal iff both the poles and the zeros of $H(z)$ locate inside the unit circle. Such systems are referred to as *minimum-phase* systems.

Ex:

$$
\begin{aligned}
H(z) &= \frac{1-0.5z^{-1}}{1-0.9z^{-1}}, \quad |z|>0.9 \\
\Rightarrow \quad H_i(z) &= \frac{1-0.9z^{-1}}{1-0.5z^{-1}}, \quad |z|>0.5
\end{aligned}
$$

Both systems are causal and stable. Thus, $H(z)$ is a minimum-phase system.

Ex:

$$
\begin{aligned}
H(z) &= \frac{z^{-1}-0.5}{1-0.9z^{-1}} = (\frac{-1}{2})\frac{1-2z^{-1}}{1-0.9z^{-1}}, \quad |z|>0.9 \\
\Rightarrow \quad H_i(z) &= \frac{1-0.9z^{-1}}{z^{-1}-0.5} = (-2)\frac{1-0.9z^{-1}}{1-2z^{-1}}
\end{aligned}
$$

Two ROC's $|z|>2$ and $|z|<2$ for $H_i(z)$ are allowable.

1. $|z|>2$: $H_i(z)$ is causal but unstable.
2. $|z|<2$: $H_i(z)$ is stable but noncausal.

This indicates that the inverse system for $H(z)$ may not be unique! Also, $H(z)$ is not a minimum-phase system.

5.2.3 Impulse Response for Rational System Functions

- Consider the rational $H(z)$ specified

$$
H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}
$$

with $M \geq N$ and different d_k's which are nonzero. If we want $H(z)$ to be causal, its ROC should be $|z| > \max_k |d_k|$; in this case

$$
h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n].
$$

There are two classes of LTI systems:

First, at least one nonzero pole is not canceled by a zero. $h[n]$ has at least one term of the form $A_k d_k^n u[n]$ and thus of infinite length. Such systems are IIR systems.

Ex: For $a \neq 0$,

$$H(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a| \Rightarrow h[n] = a^n u[n].$$

Second, if the rational $H(z)$ has $A_k = 0$ for all k, i.e., $H(z)$ has the form ($M - N \to M$ and $B_k \to b_k$)

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

then its impulse response has the form

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k].$$

Such systems are FIR systems.

- An LCCD equation of the form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

can definitely define an FIR system. However, an LCCD equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

does not necessarily specify an IIR (or FIR) system.

Ex:

$$y[n] = x[n] + x[n-1]$$

is an FIR system with $h[n] = \delta[n] + \delta[n-1]$. Note that this system can also be expressed as

$$\Rightarrow y[n] - y[n-1] = x[n] - x[n-2].$$

5.3 Frequency Response for Rational System Functions

- For a stable LTI system with

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}},$$

it has a frequency response (since unit circle $\in$ ROC)

$$\begin{aligned} H\left(e^{j\omega}\right) &= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \\ \Rightarrow \quad H\left(e^{j\omega}\right) &= \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} \left(1 - c_k e^{-j\omega}\right)}{\prod_{k=1}^{N} \left(1 - d_k e^{-j\omega}\right)}. \end{aligned}$$

- Defns:

1. $|H(e^{j\omega})|$ is the magnitude response for $H(e^{j\omega})$, given by

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N} |1 - d_k e^{-j\omega}|}.$$

2. $|H(e^{j\omega})|^2$ is the magnitude-squared response for $H(e^{j\omega})$, given by

$$\left|H\left(e^{j\omega}\right)\right|^2 = \left|\frac{b_0}{a_0}\right|^2 \frac{\prod_{k=1}^{M} \left(1 - c_k e^{-j\omega}\right)\left(1 - c_k^* e^{j\omega}\right)}{\prod_{k=1}^{N} \left(1 - d_k e^{-j\omega}\right)\left(1 - d_k^* e^{j\omega}\right)}.$$

3. $20\log_{10}|H(e^{j\omega})|$ is the log magnitude response, or gain in dB, for $H(e^{j\omega})$.

$$\begin{aligned} 20\log_{10}\left|H\left(e^{j\omega}\right)\right| &= 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}|1 - c_k e^{-j\omega}| \\ &\quad - \sum_{k=1}^{N} 20\log_{10}|1 - d_k e^{-j\omega}|. \end{aligned}$$

4. $-20\log_{10}|H(e^{j\omega})|$ is called attenuation in dB.
5. $\angle H(e^{j\omega})$ is the phase response for $H(e^{j\omega})$, given by

$$\begin{aligned} \angle H\left(e^{j\omega}\right) &= \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M} \angle\left(1 - c_k e^{-j\omega}\right) \\ &\quad - \sum_{k=1}^{N} \angle\left(1 - d_k e^{-j\omega}\right). \end{aligned}$$

6. $\text{ARG}\left[H\left(e^{j\omega}\right)\right]$ is the principal phase response for $H\left(e^{j\omega}\right)$, with $-\pi < \text{ARG}\left[H\left(e^{j\omega}\right)\right] \leq \pi$ and given by

$$\begin{aligned}\text{ARG}\left[H\left(e^{j\omega}\right)\right] &= \text{ARG}\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \text{ARG}\left[1 - c_k e^{-j\omega}\right] \\ &\quad - \sum_{k=1}^{N} \text{ARG}\left[1 - d_k e^{-j\omega}\right] + 2\pi r\left(\omega\right).\end{aligned}$$

Note: $\angle H\left(e^{j\omega}\right) = \text{ARG}\left[H\left(e^{j\omega}\right)\right] + 2\pi r\left(\omega\right)$ with $r\left(\omega\right)$ an integer function of ω.

7. $\arg\left[H\left(e^{j\omega}\right)\right]$ is the continuous phase response for $H\left(e^{j\omega}\right)$, given by

$$\begin{aligned}\arg\left[H\left(e^{j\omega}\right)\right] &= \arg\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \arg\left[1 - c_k e^{-j\omega}\right] \\ &\quad - \sum_{k=1}^{N} \arg\left[1 - d_k e^{-j\omega}\right].\end{aligned}$$

Notes:

(a)

Fig.5.1

(b)

$$ARG\left[H\left(e^{j\omega}\right)\right] = \tan^{-1}\left(\frac{\text{Im}\left\{H\left(e^{j\omega}\right)\right\}}{\text{Re}\left\{H\left(e^{j\omega}\right)\right\}}\right).$$

8. $\text{grd}\left[H\left(e^{j\omega}\right)\right]$ is the group delay for $H\left(e^{j\omega}\right)$, given by

$$\begin{aligned}\text{grd}\left[H\left(e^{j\omega}\right)\right] &= \sum_{k=1}^{N} \frac{d}{d\omega}\left(\arg\left[1 - d_k e^{-j\omega}\right]\right) \\ &\quad - \sum_{k=1}^{M} \frac{d}{d\omega}\left(\arg\left[1 - c_k e^{-j\omega}\right]\right) \\ &= -\frac{d}{d\omega} \arg\left[H\left(e^{j\omega}\right)\right] \\ &= \sum_{k=1}^{N} \frac{|d_k|^2 - \text{Re}\{d_k e^{-j\omega}\}}{1 + |d_k|^2 - 2\,\text{Re}\{d_k e^{-j\omega}\}} \\ &\quad - \sum_{k=1}^{M} \frac{|c_k|^2 - \text{Re}\{c_k e^{-j\omega}\}}{1 + |c_k|^2 - 2\,\text{Re}\{c_k e^{-j\omega}\}}.\end{aligned}$$

Notes:

(a)

$$\begin{aligned}\mathrm{ARG}\left[1-ae^{-j\omega}\right] &= \mathrm{ARG}\left[\left(1-\mathrm{Re}\left\{ae^{-j\omega}\right\}\right)-j\,\mathrm{Im}\left\{ae^{-j\omega}\right\}\right] \\ &= \tan^{-1}\left(\frac{-\mathrm{Im}\left\{ae^{-j\omega}\right\}}{1-\mathrm{Re}\left\{ae^{-j\omega}\right\}}\right).\end{aligned}$$

$$\Rightarrow \frac{d}{d\omega}\mathrm{ARG}\left[1-ae^{-j\omega}\right] = \frac{|a|^2-\mathrm{Re}\left\{ae^{-j\omega}\right\}}{1+|a|^2-2\,\mathrm{Re}\left\{ae^{-j\omega}\right\}}$$

since $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$. (Show this yourself.)

(b) Except at jumps from π to $-\pi$ or $-\pi$ to π,

$$\begin{aligned}\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\} &= \frac{d}{d\omega}\left\{\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]\right\} \\ &= \frac{d}{d\omega}\tan^{-1}\left(\frac{\mathrm{Im}\left\{H\left(e^{j\omega}\right)\right\}}{\mathrm{Re}\left\{H\left(e^{j\omega}\right)\right\}}\right).\end{aligned}$$

5.3.1 Frequency Response of First-Order Systems

- Consider a factor $1-re^{j\theta}e^{-j\omega}$ with $0<r<1$:

1.

$$\begin{aligned}\text{log magnitude} &= 20\log_{10}\left|1-re^{j\theta}e^{-j\omega}\right| \\ &= 10\log_{10}\left[1+r^2-2r\cos\left(\omega-\theta\right)\right]\end{aligned}$$

2.

$$\begin{aligned}\text{principal phase} &= \mathrm{ARG}\left[1-re^{j\theta}e^{-j\omega}\right] \\ &= \tan^{-1}\left(\frac{r\sin\left(\omega-\theta\right)}{1-r\cos\left(\omega-\theta\right)}\right)\end{aligned}$$

3. Except at discontinuities,

$$\begin{aligned}\text{group delay} &= -\frac{d}{d\omega}\tan^{-1}\left(\frac{r\sin\left(\omega-\theta\right)}{1-r\cos\left(\omega-\theta\right)}\right) \\ &= \frac{r^2-r\cos\left(\omega-\theta\right)}{1+r^2-2r\cos\left(\omega-\theta\right)}\end{aligned}$$

4. Notes:

(a) Log magnitude, phase, and group delay are all periodic with 2π in ω. Thus, it suffices to consider the frequency range $0 \le \omega < 2\pi$ for these functions.

(b) maximum log magnitude (occurs at $\omega - \theta = \pi$) $= 20\log_{10}(1+r)$
minimum log magnitude (occurs at $\omega - \theta = 0$) $= 20\log_{10}|1-r|$

Fig. 54-F1

For $r \approx 1$, the dip is sharp!

Fig. 5.11(a)

(c) When $\omega = \theta$ and $\omega = \theta + \pi$, phase response $= 0$.

Fig. 54-F2

For $r \approx 1$, there is a jump at $\omega = \theta$. (infinity slope)

Fig. 5.11(b)

(d) When $\omega = \theta$, there is a dip for group delay.

Fig. 54-B1

For $r \approx 1$, the dip is sharp!

Fig. 5.11(c)

(e) At $\omega = \theta + \pi$, we have maximum magnitude and zero phase response.

- Geometric Construction: In inferring the frequency response characteristics from pole-zero plots of LTI systems, the associated vector diagram in the complex plane are helpful.

  Now, consider the LTI system with first-order system function

$$H(z) = \frac{z - re^{j\theta}}{z} \text{ (It has } H\left(e^{j\omega}\right) = 1 - re^{j\theta}e^{-j\omega}\text{)}$$

  with $r < 1$ (a stable system). The relative position of a point on unit circle (i.e., $e^{j\omega}$) from the poles/zeros of a system function carries important information of $H(e^{j\omega})$, i.e., the frequency response of the LTI system.

Fig. 5.10

  where the three vectors are defined by $v_1 = e^{j\omega}$, $v_2 = re^{j\theta}$, and $v_3 = v_1 - v_2$.

  The three vectors have the following interpretation:

1.
$$\text{pole vector} = v_p = \underbrace{e^{j\omega}}_{\text{a point on unit circle}} - \underbrace{0}_{\text{pole}} = e^{j\omega} = v_1$$

2.
$$\text{zero vector} = v_o = \underbrace{e^{j\omega}}_{\text{a point on unit circle}} - \underbrace{re^{j\theta}}_{\text{zero}} = v_3.$$

3.
$$v_2 = re^{j\theta} \equiv \text{vector from origin to zero at } re^{j\theta}.$$

Now, the responses of $H\left(e^{j\omega}\right)$ can be interpreted in terms of vector relationship as follows:

1. The magnitude response at ω is

$$\left|H\left(e^{j\omega}\right)\right| = \left|1 - re^{j\theta}e^{-j\omega}\right| = \left|\frac{v_o}{v_p}\right|.$$

   Since $|v_p| = 1$, $|H(e^{j\omega})| = |v_o|$ is minimum as $\omega = \theta$.
   On the other hand, $|H(e^{j\omega})|$ is maximum as $\omega = \theta + \pi$.
   As ω increases from θ to $\theta + \pi$, $|v_o| / |v_p|$ increases from minimum to maximum.
   As ω increases from $\theta + \pi$ to 0, and to θ, $|v_o| / |v_p|$ decreases from maximum to minimum.
   As r approaches 1, the magnitude dips sharply as ω is around θ.

   Fig. 55-B1

   Fig. 5.10

2. The phase response is $\angle H\left(e^{j\omega}\right) = \angle v_o - \angle v_p = \phi_3 - \omega$ which is zero as $\omega = \theta$ and $\omega = \theta + \pi$ (as we have learned previously).

   Fig. 56-F1

   Fig. 5.10

   As $r = 1$, $\angle H\left(e^{j\omega}\right) = \angle v_o - \angle v_p$ has a jump.

3. The group delay is

$$-\frac{d}{d\omega}\left\{\text{ARG}\left[H\left(e^{j\omega}\right)\right]\right\} = grd\left[H\left(e^{j\omega}\right)\right]$$

   Fig. 56-F3

Note: If

$$\begin{aligned}
H\left(e^{j\omega}\right) &= \frac{1}{1-re^{j\theta}e^{-j\omega}} \quad \left(H\left(z\right)=\frac{z}{z-re^{j\theta}}\right) \\
v_2 &= re^{j\theta},\ v_o=e^{j\omega},\ v_p=e^{j\omega}-re^{j\theta}
\end{aligned}$$

Fig. 56-B1

then

$$\begin{aligned}
\left|H\left(e^{j\omega}\right)\right| &= \frac{|v_o|}{|v_p|} \\
\angle H\left(e^{j\omega}\right) &= \angle v_o - \angle v_p.
\end{aligned}$$

Then, all the previous discussion (for $r<1$) on log magnitude, phase, and group delay will apply with signs reversed.

- Examples of Geometric Construction with Multiple Poles and Zeros

Ex 5.8: Consider the stable LTI system with

$$H\left(z\right)=\frac{0.05634\left(1+z^{-1}\right)\left(1-1.0166z^{-1}+z^{-2}\right)}{\left(1-0.683z^{-1}\right)\left(1-1.466z^{-1}+0.7957z^{-2}\right)}$$

This system has

* poles at $z=0.683,\ 0.892e^{\pm j0.6257}$
* zeros at $z=-1,\ e^{\pm j1.0376}$.

Fig. 5.14
Fig. 5.15

The system has

$$\begin{pmatrix} -\infty \text{ log magnitude} \\ \text{phase jumps} \end{pmatrix} \text{ at } \omega=\pi,\ 1.0376,\ 2\pi-1.0376.$$

At $\omega \approx 0.22\pi$ and $2\pi-0.22\pi$, there are phase jumps due to the use of the principal value in plotting.

See Exs 5.6 & 5.7 for self-study.

5.4 Relationship Between Magnitude and Phase

- Consider the magnitude square of the frequency response

$$
\begin{aligned}
\left|H\left(e^{j\omega}\right)\right|^2 &= H\left(e^{j\omega}\right) H^*\left(e^{j\omega}\right) \\
&= H(z) H^*(z)\big|_{z=e^{j\omega}} \\
&= H(z) H^*\left(\frac{1}{z^*}\right)\bigg|_{z=e^{j\omega}} \\
&= C(z)\big|_{z=e^{j\omega}}
\end{aligned}
$$

where we have defined

$$C(z) \triangleq H(z) H^*\left(\frac{1}{z^*}\right).$$

For an LTI system specified by an LCCD equation, its $H(z)$ has the form

$$H(z) = \frac{b_0 \prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{a_0 \prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}$$

which yields

$$C(z) = \left|\frac{b_0}{a_0}\right|^2 \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)\left(1 - c_k^* z\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)\left(1 - d_k^* z\right)}.$$

Thus, a pole d_k of $H(z)$ gives two poles d_k and $(d_k^*)^{-1}$ of $C(z)$; similarly, a zero c_k of $H(z)$ gives two zeros c_k and $(c_k^*)^{-1}$ of $C(z)$.

Fig. 57-B1

For a stable and causal $H(z)$, all poles are located inside the unit circle, and thus the poles of $C(z)$ can uniquely specify the poles of $H(z)$; but this does not apply to zeros.

Note: $C(z)|_{z=e^{j\omega}}$ carries only the information of the magnitude response of $H(e^{j\omega})$.

- Ex: Different System With the Same $C(z)$

Consider two stable systems with

$$
\begin{aligned}
H_1(z) &= \frac{2(1 - z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} \\
H_2(z) &= \frac{(1 - z^{-1})(1 + 2z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}
\end{aligned}
$$

which can be described by the pole-zero plots

Fig. 5.16(a)-(b)

Both systems have the same $C(z)$

$$C(z) = \frac{4(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-z)(1+\frac{1}{2}z)}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z)(1-0.8e^{j\pi/4}z)}$$

described by

Fig. 5.16(c)

- Ex: Determination of $H(z)$ From $C(z)$

  Given $C(z)$ with the pole-zero plot

  Fig. 5.17

  The conjugate reciprocal pairs of poles and zeros are

  $$\begin{array}{c}(p_1, p_4),\ (p_2, p_5),\ (p_3, p_6)\\ (z_1, z_4),\ (z_2, z_5),\ (z_3, z_6).\end{array}$$

  For $H(z)$ to be causal and stable, we can only choose p_1, p_2, p_3 as poles of $H(z)$. However, there are 8 possible choices for zeros, i.e., (z_1, z_2, z_3), (z_1, z_2, z_6)...

  If, furthermore, we want the coefficients in LCCD equation to be real (real input sequence yields real output sequence in this case); then zeros are either real or in complex conjugate pairs. In this case, we have the following choices for zeros: (z_3, z_1, z_2), (z_3, z_4, z_5), (z_6, z_1, z_2), (z_6, z_4, z_5).

- Consider the all-pass system

  $$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}.$$

  Fig. 58-B1

  Note that

  $$H_{ap}(z)\, H_{ap}^*\left(\frac{1}{z^*}\right) = \frac{z^{-1} - a^*}{1 - az^{-1}} \cdot \frac{z - a}{1 - a^* z} = 1.$$

Then,

$$\begin{aligned} C(z) &= H(z)\, H^*\left(\frac{1}{z^*}\right) \\ &= H(z)\, H_{ap}(z)\, H^*\left(\frac{1}{z^*}\right) H_{ap}^*\left(\frac{1}{z^*}\right). \end{aligned}$$

The same $C(z)$ results! This implies that the number of poles and zeros of $C(z)$ does not uniquely specify the number of poles and zeros of $H(z)$. Since the pole and zero of $H_{ap}(z)$ are cancelled in $C(z)$, the number of poles and zeros of $H(z)$ is not restricted by the number of poles and zeros of $C(z)$.

5.5 Allpass Systems

- Consider a stable all-pass system

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

with $|a| < 1$ and $|z| > |a|$. It has a magnitude response that is independent of ω since

$$\begin{aligned} \left|H_{ap}(z)\right|\Big|_{z=e^{j\omega}} &= \left|\frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}\right| = \left\{\left|\frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}\right|^2\right\}^{\frac{1}{2}} \\ &= \left\{\frac{\left(e^{-j\omega} - a^*\right)\left(e^{j\omega} - a\right)}{\left(1 - ae^{-j\omega}\right)\left(1 - a^* e^{j\omega}\right)}\right\}^{\frac{1}{2}} \\ &= 1. \end{aligned}$$

A system for which the frequency-response magnitude is a constant is referred to as an all-pass system. Thus, an all-pass system passes all of the frequency components of its input with constant gain or attenuation.

Because

$$H_{ap}(z)\, H_{ap}^*\left(\frac{1}{z^*}\right) = \frac{z^{-1} - a^*}{1 - az^{-1}} \cdot \frac{z - a}{1 - a^* z} = 1$$

holds for arbitrary nonzero a values, an all-pass system can be generally defined as the system with $H_{ap}(z)$ satisfying $H_{ap}(z)\, H_{ap}^*\left(\frac{1}{z^*}\right) = A$ with $A > 0$.

Note that an all-pass system can be described by a product of multiple rational z-functions in the above form of $H_{ap}(z)$, perhaps with different a values, i.e.,

$$H_{ap}(z) = C \prod_i \frac{z^{-1} - a_i^*}{1 - a_i z^{-1}}$$

with C a constant. Here, a_i's can be restricted to $|a_i| < 1$ in order to define a stable and causal all-pass system.

- Consider a real system with a rational $H(z)$. Now, if $h[n]$ is real, then

$$\begin{aligned} & H\left(e^{j\omega}\right) = H^*\left(e^{-j\omega}\right) \text{ (pair 7 of Table 2.1)} \\ \Rightarrow \; & H(z) = H^*(z^*) \text{ for } z = e^{j\omega}. \end{aligned}$$

Further, if $H(z)$ is rational, then

$$\begin{aligned} H(z) &= \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})} = \frac{b_0^* \prod_{k=1}^{M} \left(1 - c_k (z^*)^{-1}\right)^*}{a_0^* \prod_{k=1}^{N} \left(1 - d_k (z^*)^{-1}\right)^*} \\ &= \frac{b_0^* \prod_{k=1}^{M} (1 - c_k^* z^{-1})}{a_0^* \prod_{k=1}^{N} (1 - d_k^* z^{-1})}. \end{aligned}$$

This implies that (1) $(\frac{b_0}{a_0}) = (\frac{b_0}{a_0})^*$; (2) c_k's (and d_k's) are either real or in a complex conjugate pair, in general.

Thus, allpass systems with real impulse response have a general form of

$$H_{ap}(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - \rho_k^*)(z^{-1} - \rho_k)}{(1 - \rho_k z^{-1})(1 - \rho_k^* z^{-1})} \qquad \circledast$$

where d_k's are real and ρ_k's are complex.

If the allpass system is stable and causal, then $|d_k| < 1$ and $|\rho_k| < 1$ for all k.

Note: In an all-pass system with real impulse response, *each pole is paired with a complex conjugate reciprocal zero.*

Fig. 5.18

- The phase response for $H_{ap}(z) = \frac{z^{-1}-a^*}{1-az^{-1}}$ with $a = re^{j\theta}$ is

$$\begin{aligned}
\angle\left[\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right] &= \angle\left[e^{-j\omega}-re^{-j\theta}\right] - \angle\left[1-re^{j\theta}e^{-j\omega}\right] \\
&= -\omega + \angle\left[1-re^{j\omega}e^{-j\theta}\right] - \angle\left[1-re^{j\theta}e^{-j\omega}\right] \\
&= -\omega + \tan^{-1}\left(\frac{-r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right) \\
&\quad - \tan^{-1}\left(\frac{-r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}\right) \\
&= -\omega - 2\tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right).
\end{aligned}$$

  Likewise, the phase response for a system with $\circledast$ is

$$\begin{aligned}
\angle\left[H_{ap}\left(e^{j\omega}\right)\text{ in }\circledast\right] &= -\left(M_r+2M_c\right)\omega - 2\sum_{k=1}^{M_r}\tan^{-1}\left(\frac{d_k\sin\omega}{1-d_k\cos\omega}\right) \\
&\quad -2\sum_{k=1}^{M_c}\left[\tan^{-1}\left(\frac{r_k\sin(\omega-\theta_k)}{1-r_k\cos(\omega-\theta_k)}\right)\right. \\
&\quad \left.+\tan^{-1}\left(\frac{r_k\sin(\omega+\theta_k)}{1-r_k\cos(\omega+\theta_k)}\right)\right]
\end{aligned}$$

  where we let $\rho_k = r_k e^{j\theta_k}$.

- Ex: First- and Second-Order All-Pass Systems

Fig. 5.19-5.20

- Properties of Stable and Causal Allpass systems:

  Since an allpass system consists of a product of $\frac{z^{-1}-a^*}{1-az^{-1}}$, it suffices to study the phase response and group delay for $\frac{z^{-1}-a^*}{1-az^{-1}}$ in order to characterize those for the allpass systems.

  1. $grd\left[H_{ap}\left(e^{j\omega}\right)\right] \geq 0$ for a stable and causal allpass system.

Pf:

$$
\begin{aligned}
grd\left[\frac{e^{-j\omega}-a^*}{1-ae^{-j\omega}}\right] \underset{a=re^{j\theta}}{=} grd\left[\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j(\theta-\omega)}}\right] &= -\frac{d}{d\omega}\left[-\omega-2\tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)\right] \\
&\text{(except those on jumps of } 2\pi) \\
&= \frac{1-r^2}{\left|1-re^{j\theta}e^{-j\omega}\right|^2} \text{ (you show it!)} \\
&\geq 0 \text{ (since } |a|<1) \text{ QED}
\end{aligned}
$$

2. $\arg\left[H_{ap}\left(e^{j\omega}\right)\right] \leq 0$, $0 \leq \omega < \pi$, for a real-valued (which has an odd phase response, i.e., $\arg\left[H_{ap}\left(e^{j\omega}\right)\right] = -\arg\left[H_{ap}\left(e^{-j\omega}\right)\right]$), stable and causal allpass system.

   Pf: Since

$$
grd\left[H_{ap}\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\arg\left[H_{ap}\left(e^{j\omega}\right)\right]
$$

$$
\Rightarrow \arg\left[H_{ap}\left(e^{j\omega}\right)\right] = -\int_0^{\omega} grd\left[H_{ap}\left(e^{j\omega'}\right)\right] d\omega' + \arg\left[H_{ap}\left(e^{j0}\right)\right]
$$

   for $0 \leq \omega < \pi$. Now,

   (a)

$$
\arg\left[H_{ap}\left(e^{j0}\right)\right] \underset{\circledast}{=} \arg\left\{\left[\prod_{k=1}^{M_r}\frac{1-d_k}{1-d_k}\right]\left[\prod_{k=1}^{M_c}\frac{|1-\rho_k|^2}{|1-\rho_k|^2}\right]\right\} = 0.
$$

   (b) $grd\left[H_{ap}\left(e^{j\omega}\right)\right] \geq 0$.

   Thus, $\arg\left[H_{ap}\left(e^{j\omega}\right)\right] \leq 0$. QED

- Thus, the phase and group delay responses of a stable and causal allpass system can be designed to enable different applications. The design of phase response and group delay response may be different for various applications, for example, compensator for phase distortion or compensator for group delay, transforming the phase response or group delay response of a filter into another form without changing the magnitude response, and in the theory of minimum-phase systems.

5.6 Minimum-Phase Systems

- A minimum-phase system $H_{\min}(z)$ is the one that both its $H(z)$ as well as $1/H(z)$ (the system function of its inverse system *exists*!) should have poles and zeros inside unit circle. That is, both the system and its inverse are causal and stable.

- Note: For a minimum-phase system with $H_{\min}(z)$, $C(z) = H_{\min}(z) H^*_{\min}(1/z^*)$ uniquely determines $H_{\min}(z)$, which will consist of all the poles and zeros of $C(z)$ that lie inside the unit circle.

5.6.1 Minimum-Phase and All-Pass Decomposition

- Any rational system function can be expressed as

$$\boxed{H(z) = H_{\min}(z) H_{ap}(z).}$$

Pf: First, for a minimum-phase system, it is trivial (since $H_{ap}(z) = 1$). Next, consider a nonminimum-phase system with

$$H(z) = H_1(z) \frac{\prod_{k=1}^{M_d} (1 - c_k z^{-1})}{\prod_{k=1}^{M_n} (1 - d_k z^{-1})}$$

where $H_1(z)$ is minimum-phase and c_k and d_k are outside unit circle. Then

$$H(z) = \underbrace{H_1(z) \frac{\prod_{k=1}^{M_d} \left(1 - (c_k^*)^{-1} z^{-1}\right)}{\prod_{k=1}^{M_n} \left(1 - (d_k^*)^{-1} z^{-1}\right)}}_{H_{\min}(z)} \underbrace{\frac{\prod_{k=1}^{M_d} (1 - c_k z^{-1}) \left(1 - (d_k^*)^{-1} z^{-1}\right)}{\prod_{k=1}^{M_n} (1 - d_k z^{-1}) \left(1 - (c_k^*)^{-1} z^{-1}\right)}}_{H_{ap}(z)}. \qquad (S)$$

This proves the property. QED

Notes:

1. A nonminimum-phase system can be formed from a minimum-phase system from (S), and vice versa.
2. The corresponding minimum-phase system of a nonminimum-phase system is obtained by reflecting the zeros (poles) lying outside unit circle to their conjugate reciprocal locations inside.

   Fig. 61-F1

   By such transformation, both minimum-phase and nonminimum-phase systems have the *same* magnitude response but different phase responses.

- Ex: Minimum-Phase/All-Pass Decomposition

Consider two stable and causal systems with

$$\begin{aligned} H_1(z) &= \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}} \\ H_2(z) &= \frac{(1+\frac{3}{2}e^{j\pi/4}z^{-1})(1+\frac{3}{2}e^{-j\pi/4}z^{-1})}{1-\frac{1}{3}z^{-1}}. \end{aligned}$$

Both system functions are nonminimum-phase. Now, we want to convert them into the corresponding minimum-phase system function. For $H_1(z)$, we reflect the zero $z=-3$ to $z=-1/3$ by the all-pass system

$$H_{ap}(z) = \frac{z^{-1}+\frac{1}{3}}{1+\frac{1}{3}z^{-1}}$$

(which has a zero at $z=-3$) to give the minimum-phase counterpart as

$$H_{1,\min}(z) = 3\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}$$

through the relation

$$H_1(z) = H_{1,\min}(z)H_{ap}(z) = 3\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}} \cdot \frac{z^{-1}+\frac{1}{3}}{1+\frac{1}{3}z^{-1}}.$$

For $H_2(z)$, we reflect the zeros $z=\frac{-3}{2}e^{\pm j\pi/4}$ to $z=\frac{-2}{3}e^{\pm j\pi/4}$ by the all-pass system

$$H_{ap}(z) = \frac{(z^{-1}+\frac{2}{3}e^{-j\pi/4})(z^{-1}+\frac{2}{3}e^{j\pi/4})}{(1+\frac{2}{3}e^{+j\pi/4}z^{-1})(1+\frac{2}{3}e^{-j\pi/4}z^{-1})}$$

(which has complex conjugate zeros at $z=\frac{-3}{2}e^{\pm j\pi/4}$) to give the minimum-phase counterpart as

$$H_{2,\min}(z) = \frac{9}{4}\frac{(1+\frac{2}{3}e^{-j\pi/4}z^{-1})(1+\frac{2}{3}e^{j\pi/4}z^{-1})}{1-\frac{1}{3}z^{-1}}$$

through the relation

$$\begin{aligned} H_2(z) &= H_{2,\min}(z)H_{ap}(z) \\ &= \frac{9}{4}\frac{(1+\frac{2}{3}e^{-j\pi/4}z^{-1})(1+\frac{2}{3}e^{j\pi/4}z^{-1})}{1-\frac{1}{3}z^{-1}} \\ &\quad \cdot \frac{(z^{-1}+\frac{2}{3}e^{-j\pi/4})(z^{-1}+\frac{2}{3}e^{j\pi/4})}{(1+\frac{2}{3}e^{+j\pi/4}z^{-1})(1+\frac{2}{3}e^{-j\pi/4}z^{-1})}. \end{aligned}$$

5.6.2 Frequency-Response Compensation of Non-Minimum-Phase Systems

- We want to compensate for the distortion introduced by a distorting system with $H_d(z)$ and recover back the original signal $s[n]$ *to some extent* through a *stable and causal* compensating system with $H_c(z)$. Thus, consider

  Fig. 5.22

  If perfect compensation is achieved, then $s_c[n] = s[n]$. However, if the distorting system is stable and causal and we require the compensating system to be stable and causal, then perfect compensation is possible only if $H_d(z)$ is a minimum-phase system (so that it has a stable and causal inverse). In this case, $G(z) = H_d(z) H_c(z) = 1$.

- When $H_d(z)$ is a nonminimum-phase system, a stable and causal $H_c(z)$ is not available to achieve $G(z) = 1$. In the case, our goal is modified to design $H_c(z)$ so that $|G(z)| = 1$, $\forall z$, i.e., the frequency-magnitude response of $H_d(z)$ is compensated. Now, consider the distorting system with a nonminimum-phase system function

  $$H_d(z) = H_{d,\min}(z) H_{ap}(z).$$

  If we set

  $$H_c(z) = \frac{1}{H_{d,\min}(z)}$$

  then the whole system becomes an all-pass system with system function

  $$G(z) = H_d(z) H_c(z) = H_{ap}(z).$$

  Thus, $|G(z)| = 1$ and $\angle G(z) = \angle H_{ap}(z)$.

- Ex: Compensation of an FIR System

  Consider the real-valued FIR distorting system with

  $$\begin{aligned} H_d(z) &= (1 - 0.9e^{j0.6\pi} z^{-1})(1 - 0.9e^{-j0.6\pi} z^{-1}) \\ &\quad \cdot(1 - 1.25e^{j0.8\pi} z^{-1})(1 - 1.25e^{-j0.8\pi} z^{-1}). \end{aligned}$$

  Fig. 5.23

  Fig. 5.24

which is a nonminimum phase system. Now, reflect the zeros $z = 1.25e^{\pm j0.8\pi}$ to $z = 0.8e^{\pm j0.8\pi}$, we have

$$\begin{aligned} H_{d,\min}(z) &= (1.25)^2(1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1}) \\ &\quad \cdot(1 - 0.8e^{j0.8\pi}z^{-1})(1 - 0.8e^{-j0.8\pi}z^{-1}) \end{aligned}$$

Fig. 5.25

and can represent $H_d(z) = H_{d,\min}(z) H_{ap}(z)$ with

$$H_{ap}(z) = \frac{(z^{-1} - 0.8e^{-j0.8\pi})(z^{-1} - 0.8e^{j0.8\pi})}{(1 - 0.8e^{j0.8\pi}z^{-1})(1 - 0.8e^{-j0.8\pi}z^{-1})}$$

(which has complex conjugate zeros at $z = 1.25e^{\pm j0.8\pi}$).

Fig. 5.26

This indicates that the stable and causal compensating system achieving $|G(z)| = 1$ is

$$H_c(z) = \frac{1}{H_{d,\min}(z)}.$$

5.6.3 Properties of Minimum-Phase Systems

- We consider real-valued stable and causal systems (which have odd phase responses, i.e., $\angle H(e^{j\omega}) = -\angle H(e^{-j\omega})$) and shall develop interesting and important properties of stable and causal minimum-phase systems with $H_{\min}(z)$ relative to all other stable and causal systems (with all poles located inside the unit circle) having the same frequency-response magnitude and

$$H(z) = H_{\min}(z) H_{ap}(z) = H_{\min}(z) \frac{\prod_{k=1}^{M_n}(1 - c_k z^{-1})}{\prod_{k=1}^{M_n}\left(1 - (c_k^*)^{-1} z^{-1}\right)}$$

where the zeros in $H_{ap}(z)$ are located outside the unit circle. Note that $H_{ap}(z)$ here has poles locating inside the unit circle and is assumed to be stable and causal.

(1) Minimum *Phase-Lag (Minus Phase Delay)* Property

Any nonminimum-phase stable and causal system with $H(e^{j\omega})$ has an unwrapped phase described by

$$\begin{aligned}\arg\left[H\left(e^{j\omega}\right)\right] &= \arg\left[H_{\min}\left(e^{j\omega}\right)\right] + \underbrace{\arg\left[H_{ap}\left(e^{j\omega}\right)\right]}_{\substack{\text{stable \& causal allpass systems } \Rightarrow \\ \arg\left[H_{ap}(e^{j0})\right]=0 \text{ and } \arg\left[H_{ap}(e^{j\omega})\right]\le 0 \\ \text{for } 0\le\omega<\pi}} \\ &\le \arg\left[H_{\min}\left(e^{j\omega}\right)\right] \\ &\quad \text{(not correct without the assumption } \arg[H_{ap}(e^{j0})]=0\text{)}\end{aligned}$$

for $0 \le \omega < \pi$. Thus, the reflection of zeros of $H_{\min}(z)$ from inside the unit circle to conjugate reciprocal locations outside the unit circle always *decreases* the (unwrapped) phase or *increases* the negative of the phase (noting that *the negative of the phase* is usually called the *phase-lag* function). Thus, the causal and stable system having $|H_{\min}(e^{j\omega})|$ as its magnitude response and also having all its zeros and poles inside the unit circle has the minimum phase-lag function (for $0 \le \omega < \pi$) among all the stable and causal systems having that same magnitude response. Thus, the stable and causal system with $H_{\min}(e^{j\omega})$ is called the *minimum-phase* or *minimum-phase-lag* system.

Usual Assumption: It is commonly assumed that, for all system functions, $H(e^{j0}) = \sum_n h[n] > 0$, i.e., $\arg[H(e^{j0})] = 0$. This assumption is necessary to ensure $\arg[H_{ap}(e^{j\omega})] \le 0$ for $0 \le \omega < \pi$ (see the proof in Section 5.5).

(2) Minimum *Group-Delay (Positive Delay)* Property

Since $grd[H_{ap}(e^{j\omega})] \ge 0$ for a real-valued stable and causal allpass system,

$$\begin{aligned}\Rightarrow \quad grd\left[H\left(e^{j\omega}\right)\right] &= grd\left[H_{\min}\left(e^{j\omega}\right)\right] + grd\left[H_{ap}\left(e^{j\omega}\right)\right] \\ &\ge grd\left[H_{\min}\left(e^{j\omega}\right)\right]\end{aligned}$$

for a stable and causal real-valued $H(e^{j\omega})$.

(3) Minimum Energy-Delay Property

A minimum-phase system has the minimum energy-delay than any other stable and causal real systems that have the same magnitude function.

(a) First, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}|h[n]|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H\left(e^{j\omega}\right)\right|^2 d\omega \text{ (Parseval's theorem)}\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_{\min}\left(e^{j\omega}\right)\right|^2 d\omega\\ &= \sum_{n=0}^{\infty}|h_{\min}[n]|^2\end{aligned}$$

(b) Second, defining

$$E[n] = \sum_{m=0}^{n}|h[m]|^2, \ E_{\min}[n] = \sum_{m=0}^{n}|h_{\min}[m]|^2$$

we have

$$E[n] \le E_{\min}[n].$$

(See Figs. 5.27 and 5.28)
Pf:

Fig. 62-B1

Let

$$H_{\min}(z) = Q(z)\left(1 - z_k z^{-1}\right), \quad |z_k| < 1.$$

Now, obtain

$$H(z) = H_{\min}(z)\frac{z^{-1} - z_k^*}{1 - z_k z^{-1}} = Q(z)\left(z^{-1} - z_k^*\right).$$

(Note: $\left|H\left(e^{j\omega}\right)\right| = \left|Q\left(e^{j\omega}\right)\right|\left|e^{-j\omega} - z_k^*\right| = \left|Q\left(e^{j\omega}\right)\right|\left|1 - z_k^* e^{j\omega}\right| = \left|H_{\min}\left(e^{j\omega}\right)\right|$)
Taking inverse z-transform, we have

$$\begin{aligned}h[n] &= q[n-1] - z_k^* q[n]\\ h_{\min}[n] &= q[n] - z_k q[n-1]\end{aligned}$$

where $q[n] = 0$ for $n < 0$ (since $h_{\min}[n]$ is causal).
Define

$$\epsilon = \sum_{m=0}^{n}|h_{\min}[m]|^2 - \sum_{m=0}^{n}|h[m]|^2.$$

We have for $n > 0$ that

$$\begin{aligned}
\epsilon &= \sum_{m=0}^{n} \left\{ |q[m] - z_k q[m-1]|^2 - |q[m-1] - z_k^* q[m]|^2 \right\} \\
&= \sum_{m=0}^{n} \{ |q[m]|^2 + |z_k|^2 |q[m-1]|^2 - q^*[m] z_k q[m-1] \\
&\quad -q[m] z_k^* q^*[m-1] - |q[m-1]|^2 - |z_k^*|^2 |q[m]|^2 \\
&\quad +q^*[m-1] z_k^* q[m] + q[m-1] z_k q^*[m] \} \\
&= \sum_{m=0}^{n} \left\{ \left(1 - |z_k|^2\right) |q[m]|^2 - \left(1 - |z_k|^2\right) |q[m-1]|^2 \right\} \\
&= \left(1 - |z_k|^2\right) |q[n]|^2 \;\; (\because \text{causal}) \\
&\geq 0 \; (\because |z_k| < 1) \qquad\qquad \text{QED}
\end{aligned}$$

(c) $|h[0]| < |h_{\min}[0]|$.
Pf: For causal sequences,

$$\begin{aligned}
h[0] &= \lim_{z\to\infty} H(z) \\
&= \lim_{z\to\infty} H_{\min}(z) H_{ap}(z).
\end{aligned}$$

Now,

$$|h[0]| = \left| \lim_{z\to\infty} H_{\min}(z) H_{ap}(z) \right| \leq \left| \lim_{z\to\infty} H_{\min}(z) \right| \left| \lim_{z\to\infty} H_{ap}(z) \right|.$$

Since

$$H_{ap}(z) = \prod_{k=1}^{K} \frac{z^{-1} - a_k^*}{1 - a_k z^{-1}}, \;\; |a_k| < 1 \text{ (for a stable and causal system)}$$

$$\Rightarrow \; \left| \lim_{z\to\infty} H_{ap}(z) \right| = \left| \prod_{k=1}^{K} (-a_k^*) \right| = \prod_{k=1}^{K} |-a_k^*| < 1.$$

Thus,

$$|h[0]| < \left| \lim_{z\to\infty} H_{\min}(z) \right| = |h_{\min}[0]|. \qquad\qquad \text{QED}$$

(d) From (a), (b) and (c), we show that a minimum-phase system has minimum energy-delay property.

Figs. 5.27-2.29

5.7 Linear Systems With Generalized Linear Phase

- A "desired" digital filter is the one with constant in-band gain and zero phase response as well. In the cases where zero-phase is not attainable, linear phase response is desirable since linear phase results in a constant group delay.

5.7.1 Systems With Linear Phase

- Consider an LTI system with $H_{id}\left(e^{j\omega}\right) = e^{-j\omega\alpha}$, $|\omega| < \pi$ with α a real constant.

  $\Rightarrow$ $\left|H_{id}\left(e^{j\omega}\right)\right| = 1,\ \angle H_{id}\left(e^{j\omega}\right) = -\omega\alpha,\ grd\left[H_{id}\left(e^{j\omega}\right)\right] = \alpha$ for $|\omega| < \pi$ and $h_{id}[n] = Sa\left(\pi\left(n-\alpha\right)\right)\ \forall n$

  Recall:

  Fig. 64-F1

  $$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, Sa\left(\pi\left(n-k-\alpha\right)\right)$$

  If $\alpha =$ integer $= n_d$, $y[n] = x[n-n_d]$.

  If $\alpha \neq$ integer, $y[n] = x_c\left(\left(n-\alpha\right)T\right)$. (previous discussion)

  Fig. 5.30

  Thus, αT is the amount of delay the system $h_{id}[n]$ introduces to the input $x_c(t)$!

  In general, an LTI system with $H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{-j\omega\alpha}$, $|\omega| < \pi$ reshapes the magnitude of $x[n]$ by $\left|H\left(e^{j\omega}\right)\right|$ and delays the output by α.

  Fig. 5.31

  One example is the linear-phase ideal lowpass filter with

  $$\begin{aligned} H_{lp}\left(e^{j\omega}\right) &= \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \\ h_{lp}[n] &= \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)}. \end{aligned}$$

  Ex: Ideal Lowpass Filter With Linear Phase

1. Let $\alpha = n_d$, an integer. Then, $h_{lp}[2n_d - n] = h_{lp}[n]$. In the case, we can define a *zero-phase* system with

$$\begin{aligned}\widehat{H}_{lp}\left(e^{j\omega}\right) &= H_{lp}\left(e^{j\omega}\right)e^{j\omega n_d} = |H_{lp}\left(e^{j\omega}\right)| \\ \widehat{h}_{lp}[n] &= \frac{\sin(\omega_c n)}{\pi n} = \widehat{h}_{lp}[-n]\end{aligned}$$

   The even sequence $\widehat{h}_{lp}[n]$ that has zero phase can be simply obtained by shifting $h_{lp}[n]$ to the left by n_d samples.

   Fig. 5.32(a)

2. Let $2\alpha = n_d$ be an odd integer. Then, $h_{lp}[2\alpha - n] = h_{lp}[n]$.

   Fig. 5.32(b)

   Since α is not an integer, it is not possible to shift $h_{lp}[n]$ to obtain an even sequence that has zero phase.

3. There are cases without symmetry in sequence.

   Fig. 5.32(c)

- In general, a real linear-phase system (which has constant group delay) has frequency response

$$H\left(e^{j\omega}\right) = |H\left(e^{j\omega}\right)|e^{-j\omega\alpha}.$$

  If 2α is an integer, the corresponding real $h[n]$ has even symmetry about α, i.e., $h[2\alpha - n] = h[n]$ (You can show this by inverse Fourier transform); otherwise, $h[n]$ will not have symmetry.

5.7.2 Generalized Linear Phase

- An LTI system with $H\left(e^{j\omega}\right)$ is said to have a generalized linear phase response (with constant group delay) if

$$H\left(e^{j\omega}\right) = A\left(e^{j\omega}\right)e^{-j\omega\alpha + j\beta}$$

  where α, β are constants and $A\left(e^{j\omega}\right)$ is a real function of ω.

  Note:

$$\begin{aligned}\arg\left[H\left(e^{j\omega}\right)\right] &= \beta - \omega\alpha, \quad |\omega| < \pi \\ \operatorname{grd}\left[H\left(e^{j\omega}\right)\right] &= \alpha\end{aligned}$$

- Now, $H\left(e^{j\omega}\right)$ has a generalized linear phase response

$$
\begin{aligned}
\Rightarrow \quad H\left(e^{j\omega}\right) &= A\left(e^{j\omega}\right) e^{j(\beta-\omega\alpha)} \\
&= A\left(e^{j\omega}\right)\cos\left(\beta-\omega\alpha\right) + jA\left(e^{j\omega}\right)\sin\left(\beta-\omega\alpha\right) \\
&= \sum_{n=-\infty}^{\infty} h\left[n\right] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} h\left[n\right]\cos\left(\omega n\right) - j\sum_{n=-\infty}^{\infty} h\left[n\right]\sin\left(\omega n\right)
\end{aligned}
$$

$$
\Rightarrow \quad -\tan^{-1}\left(\frac{\sum_{n=-\infty}^{\infty} h\left[n\right]\sin\left(\omega n\right)}{\sum_{n=-\infty}^{\infty} h\left[n\right]\cos\left(\omega n\right)}\right) = \tan^{-1}\left(\frac{A\left(e^{j\omega}\right)\sin\left(\beta-\omega\alpha\right)}{A\left(e^{j\omega}\right)\cos\left(\beta-\omega\alpha\right)}\right)
$$

$$
\Rightarrow \quad \sum_{n=-\infty}^{\infty} h\left[n\right] A\left(e^{j\omega}\right)\left[\sin\left(\beta-\omega\alpha\right)\cos\left(\omega n\right) + \cos\left(\beta-\omega\alpha\right)\sin\left(\omega n\right)\right] = 0
$$

$$
\Rightarrow \quad \sum_{n=-\infty}^{\infty} h\left[n\right]\sin\left(\beta-\omega\alpha+\omega n\right) A\left(e^{j\omega}\right) = 0
$$

$$
\Rightarrow \quad \boxed{\sum_{n=-\infty}^{\infty} h\left[n\right]\sin\left(\omega\left(n-\alpha\right)+\beta\right) = 0 \ \forall\omega} \qquad (I)
$$

which is a necessary condition for generalized linear phase real-valued system.

- A class of generalized linear-phase systems are those with

$$
\begin{aligned}
\beta &= 0 \text{ or } \pi \\
2\alpha &= M \quad \text{an integer} \\
h[2\alpha-n] &= h[n] \quad \text{(i.e., } h[n] \text{ has even symmetry about } \alpha\text{)}
\end{aligned}
$$

For this class of systems, (I) becomes

$$
\sum_{n=-\infty}^{\infty} h\left[n\right]\sin\left(\omega\left(n-\alpha\right)\right) = 0 \quad \forall\omega.
$$

- Another class of generalized linear-phase systems are those with

$$
\begin{aligned}
\beta &= \pi/2 \text{ or } 3\pi/2 \\
2\alpha &= M \quad \text{an integer} \\
h[2\alpha-n] &= -h[n] \quad \text{(i.e., } h[n] \text{ has odd symmetry about } \alpha\text{)}
\end{aligned}
$$

For this class of systems, (I) becomes

$$\sum_{n=-\infty}^{\infty} h[n] \cos(\omega(n-\alpha)) = 0 \quad \forall\omega.$$

5.7.3 Causal Generalized Linear Phase FIR Systems

- If a generalized linear phase system is causal, then

$$\boxed{\sum_{n=0}^{\infty} h[n] \sin(\omega(n-\alpha)+\beta) = 0 \ \ \forall\omega.}$$

- Let us consider real-valued FIR systems, with $h[n] = 0$ for $n < 0$ or $n > M$, below.

1. Type I FIR Linear Phase Systems:

$$\boxed{h[n] = h[M-n]} \text{ for } 0 \leq n \leq M$$

with M an even integer. See $Fig.$ 5.33(a). Its frequency response is of the form

$$\begin{aligned}
H\left(e^{j\omega}\right) &= \sum_{n=0}^{M} h[n] e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n] e^{-j\omega n} + \sum_{n=M/2}^{M/2+M/2} h[M-n] e^{-j\omega n} \\
& \left(\sum_{n=M/2}^{M/2+M/2} h[M-n] e^{-j\omega n} \underset{n'=M-n}{=} \sum_{n'=0}^{M/2} h\left[n'\right] e^{-j\omega M} e^{j\omega n'} \right) \\
&= \sum_{n=0}^{M/2-1} h[n] \left(e^{-j\omega n} + e^{-j\omega(M-n)}\right) + h\left[\frac{M}{2}\right] e^{-j\omega\frac{M}{2}} \\
&= e^{-j\omega\frac{M}{2}} \sum_{n=0}^{M/2-1} h[n] \underbrace{\left(e^{-j\omega\left(n-\frac{M}{2}\right)} + e^{-j\omega\left(\frac{M}{2}-n\right)}\right)}_{2\cos\left(\omega\left(n-\frac{M}{2}\right)\right)} \\
& \quad + h\left[\frac{M}{2}\right] e^{-j\omega\frac{M}{2}} \\
&= \underbrace{e^{-j\omega\frac{M}{2}}}_{\text{linear phase}} \sum_{k=0}^{M/2} a[k] \cos(\omega k)
\end{aligned}$$

where

$$\begin{aligned} a[0] &= h\left[\frac{M}{2}\right] \\ a[k] &= 2h\left[\frac{M}{2}-k\right],\ k=1,...,\frac{M}{2}. \end{aligned}$$

Note: $\sum_{k=0}^{M/2} a[k]\cos(\omega k)$ is real, which implies that the system $H(e^{j\omega})$ is a generalized linear phase system.

2. Type II FIR Linear-Phase Systems:

$$\boxed{h[n]=h[M-n]} \text{ for } 0\leq n\leq M$$

with M an odd integer. See $Fig.$ 5.33(b). Similarly, its frequency response is of the form

$$H\left(e^{j\omega}\right)=e^{-j\omega\frac{M}{2}}\sum_{k=1}^{(M+1)/2} b[k]\cos\left(\omega\left(k-\frac{1}{2}\right)\right)$$

where

$$b[k]=2h\left[\frac{M+1}{2}-k\right],\ k=1,2,...,\frac{M+1}{2}.$$

3. Type III FIR Linear-Phase Systems:

$$\boxed{h[n]=-h[M-n]} \text{ for } 0\leq n\leq M$$

with M an even integer and $h\left[\frac{M}{2}\right]=0$. See $Fig.$ 5.33(c). Its frequency response is of the form

$$\begin{aligned} H\left(e^{j\omega}\right) &= \sum_{n=0}^{M/2-1} h[n]\left(e^{-j\omega n}-e^{-j\omega(M-n)}\right)+h\left[\frac{M}{2}\right]e^{-j\omega\frac{M}{2}} \\ &= e^{-j\omega\frac{M}{2}}\sum_{n=0}^{M/2-1} h[n]\left(e^{-j\omega\left(n-\frac{M}{2}\right)}-e^{-j\omega\left(\frac{M}{2}-n\right)}\right) \\ &\quad +h\left[\frac{M}{2}\right]e^{-j\omega\frac{M}{2}} \\ &= je^{-j\omega\frac{M}{2}}\sum_{k=1}^{M/2} c[k]\sin(\omega k) \end{aligned}$$

with

$$c[k]=2h\left[\frac{M}{2}-k\right],\ k=1,2,...,\frac{M}{2}.$$

4. Type IV FIR Linear Phase Systems:

$$\boxed{h[n] = -h[M-n]} \text{ for } 0 \leq n \leq M$$

with M an odd integer. See *Fig.* 5.33(d). Its frequency response is of the form

$$H\left(e^{j\omega}\right) = e^{-j\omega\frac{M}{2}} \sum_{k=1}^{(M+1)/2} d[k] \sin\left(\omega\left(k - \frac{1}{2}\right)\right)$$

with

$$d[k] = 2h\left[\frac{M+1}{2} - k\right], \ k = 1, 2, ..., \frac{M+1}{2}.$$

- Ex: Type I FIR Linear Phase System

$$\begin{aligned} h[n] &= \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases} \\ H\left(e^{j\omega}\right) &= e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)} \end{aligned}$$

Fig. 5.34

- Ex: Type II FIR Linear Phase System

$$\begin{aligned} h[n] &= \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{otherwise} \end{cases} \\ H\left(e^{j\omega}\right) &= e^{-j5\omega/2}\frac{\sin(3\omega)}{\sin(\omega/2)} \end{aligned}$$

Fig. 5.35

- Ex: Type III FIR Linear Phase System

$$\begin{aligned} h[n] &= \delta[n] - \delta[n-2] \\ H\left(e^{j\omega}\right) &= (j2\sin\omega)e^{-j\omega} \end{aligned}$$

Fig. 5.36

- Ex: Type IV FIR Linear Phase System

$$\begin{aligned} h[n] &= \delta[n] - \delta[n-1] \\ H\left(e^{j\omega}\right) &= (j2\sin(\omega/2))e^{-j\omega/2} \end{aligned}$$

Fig. 5.37

- Zero Locations for Causal, Real, Linear Phase FIR Systems:

  For a real and causal FIR system with $h[n] = 0$ for $n < 0$ or $n > M$, there are one pole of order M at $z = 0$ and M zeros. Moreover, zeros are either real or in a complex conjugate pair.

  1. For type I and II systems,

  $$\begin{aligned} H(z) &= \sum_{n=0}^{M} h[n] z^{-n} = \sum_{n=0}^{M} h[M-n] z^{-n} = \sum_{n=0}^{M} h[n] z^{n} z^{-M} \\ &= z^{-M} H\left(z^{-1}\right). \end{aligned}$$

  Thus, $H(z_0) = 0 \Rightarrow z_0^{-M} H\left(z_0^{-1}\right) = 0$. This means that

  Fig. 67-F1

  Fig. 5.38(a)-(b)

  (a) If z_0 is complex and not on unit circle, z_0^{-1}, z_0^*, and $(z_0^*)^{-1}$ are also zeros.

  (b) If z_0 is complex and on unit circle, z_0^* is also zero. $(z_0^{-1} = z_0^*)$

  (c) If z_0 is real and not on unit circle, z_0^{-1} is also zero.

  (d) If $z_0 = 1$ (or $z_0 = -1$), then there is no other new zero defined by it.

  (e) If $z = -1$ and M is odd,

  $$\begin{aligned} & H(-1) = (-1)^M H(-1) = -H(-1) \\ \Rightarrow\ & H(-1) = 0. \end{aligned}$$

  In other words, -1 has to be a zero for a type II system.

  2. For type III and IV systems,

  $$\begin{aligned} H(z) &= \sum_{n=0}^{M} h[n] z^{-n} = \sum_{n=0}^{M} -h[M-n] z^{-n} = -\sum_{n=0}^{M} h[n] z^{n} z^{-M} \\ &= -z^{-M} H\left(z^{-1}\right) \end{aligned}$$

  Thus, $H(z_0) = 0 \Rightarrow z_0^{-M} H\left(z_0^{-1}\right) = 0$. Therefore, (a), (b), (c), (d) in 1. apply.

  e. If $z = 1$, $H(1) = -H(1) \Rightarrow H(1) = 0$, i.e., 1 has to be a zero for both type III and IV systems

f. If $z = -1$ and M is even, $H(-1) = -(-1)^{-M} H(-1) = -H(-1) \Rightarrow H(-1) = 0$, i.e., -1 has to be a zero for a type III systems.

Notes:

1. The above constraints on the zero locations of real and causal linear phase FIR systems impose limitations on the magnitude response that can be designed.
2. FIR systems can be easily designed to have "exact" linear phase (or generalized linear phase). However, IIR systems are more efficient in the system design of obtaining desired magnitude response (see Chap. 7).

5.7.4 Relation of FIR Linear-Phase Systems to Minimum-Phase Systems

- Because all real causal FIR linear-phase systems have M zeros either on the unit circle or at conjugate reciprocal locations, their system functions can be factored into

$$H(z) = H_{\min}(z)H_{uc}(z)H_{\max}(z)$$

where $H_{\min}(z)$ is a minimum-phase term with M_i zeros locating *inside* the unit circle, $H_{\max}(z)$ a maximum-phase term defined by

$$H_{\max}(z) = H_{\min}(z^{-1})z^{-M_i}$$

with M_i zeros locating *outside* the unit circle, and $H_{uc}(z)$ a term containing only M_o zeros *on* the unit circle. Thus, $M = 2M_i + M_o$.

- Ex: Decomposition of a Linear-Phase Systems

Consider the minimum-phase term with $M_i = 4$ zeros

$$\begin{aligned} H_{\min}(z) &= (1.25)^2(1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1}) \\ &\quad \cdot(1 - 0.8e^{-j0.8\pi}z^{-1})(1 - 0.8e^{j0.8\pi}z^{-1}). \end{aligned}$$

The corresponding maximum-phase system is

$$\begin{aligned} H_{\max}(z) &= (0.9)^2(1 - 1.1111e^{j0.6\pi}z^{-1})(1 - 1.1111e^{-j0.6\pi}z^{-1}) \\ &\quad \cdot(1 - 1.25e^{-j0.8\pi}z^{-1})(1 - 1.25e^{j0.8\pi}z^{-1}). \end{aligned}$$

Thus, the composite system $H(z) = H_{\min}(z)\, H_{\max}(z)$ has linear phase and the following responses

$$20 \log_{10} |H(e^{j\omega})| = 40 \log_{10} |H_{\min}(e^{j\omega})|$$

since $|H_{\min}(e^{j\omega})| = |H_{\max}(e^{j\omega})|$,

$$\angle H(e^{j\omega}) = -\omega M_i$$

since $\angle H_{\max}(e^{j\omega}) = -\omega M_i - \angle H_{\min}(e^{j\omega})$, and

$$\mathrm{grd}[H(e^{j\omega})] = M_i.$$

Fig. 5.25

Fig. 5.39

Fig. 5.40