(1) (4%, 1% each) If the input and output of a causal LTI system satisfy the difference equation

$$y[n] = ay[n-1] + x[n]$$

for a real number a, then the impulse response of the system must be $h[n] = a^n u[n]$.

(a) For what values of a is this system stable?

(b) Consider a causal LTI system for which the input and output are related by the difference equation

$$y[n] = ay[n-1] + x[n] - a^N x[n-N]$$

where N is a positive integer and finite-valued. Determine the impulse response of this system.

(c) Is the system in part (b) an FIR or an IIR system? Explain.

(d) For what values of a is the system in part (b) stable? Explain.

Sol: (a) LTI systems are stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (the summation should converge).

Then

$$\begin{aligned} S &= \sum_{n=-\infty}^{\infty} |a|^n u[n] \\ &= \sum_{n=0}^{\infty} |a|^n \end{aligned}$$

S is finite-valued only when $|a| < 1$ and in the case $S = \frac{1}{1-|a|} < \infty$.
Therefore the system is stable for $|a| < 1$.

(b) By $y[n] = ay[n-1] + x[n] - a^N x[n-N]$, the impulse response $h[n]$ satisfies $h[n] = ah[n-1] + \delta[n] - a^N \delta[n-N]$. Since the system is causal, $h[-1] = 0$. Thus, $h[0] = 0 + 1 - 0 = 1$. Also, $h[1] = a$ if $N > 1$ and $h[2] = a^2$ if $N > 2$. By recursion, $h[n] = a^n$ for $1 \leq n < N$. Also, $h[N] = a^N - a^N = 0$ and $h[n] = 0$ for $n > N$. In summary, $h[n] = a^n$ for $0 \leq n < N$ and $h[n] = 0$ for $n \geq N$.

(c) We see that even though it is a recursive system (with feedback), its impulse response $h[n]$ has$N$ terms and is finite in length. Hence, this system is FIR.

(d) FIR systems are <u>always</u> stable as the sum $\sum_{n=-\infty}^{\infty} |h[n]|$ has at most a finite number of finite-valued terms. Thus, the system in part (b) is stable for $|a| < \infty$.

(2) (8%) Consider the linear constant-coefficient difference equation

$$y[n] - \frac{1}{4} y[n-2] = x[n].$$

(a) (1%) Determine the general form of the homogeneous solution to this equation.

(b) (2%) Both a causal and a noncausal LTI system are characterized by the given difference equation. Find the impulse responses of the two systems.

(c) (2%) Determine whether the causal LTI system and the noncausal LTI system are respectively stable or unstable. Prove it if the system is stable and explain it if the system is unstable.

(d) (2%) Find a particular solution to the causal LTI system in part (b) when $x[n] = (1/2)^n u[n]$.

(e) (1%) Consider the causal LTI system in part (b). Find the input $x[n]$ that yields the output $y[n] = \delta[n]$.

Sol: (a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum_k A_k c_k^n$, where the c_k's solve the quadratic equation

$$c_k^2 - \frac{1}{4} = 0.$$

With the roots $c_1 = -1/2$ and $c_2 = 1/2$, the general form for the homogeneous solution is

$$y_h[n] = A_1 \left(\frac{-1}{2}\right)^n + A_2 \left(\frac{1}{2}\right)^n.$$

(b) By use of z-transform in association with different ROC's, the causal impulse response $h_c[n]$ and noncausal impulse response $h_{nc}[n]$ can be obtained as

$$\begin{aligned} H(z) &= \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{1/2}{1+\frac{1}{2}z^{-1}} + \frac{1/2}{1-\frac{1}{2}z^{-1}} \\ h_c[n] &= \frac{1}{2}\left[\left(\frac{-1}{2}\right)^n + \left(\frac{1}{2}\right)^n\right] u[n] \qquad \text{for } |z| > \frac{1}{2} \\ h_{nc}[n] &= -\frac{1}{2}\left[\left(\frac{-1}{2}\right)^n + \left(\frac{1}{2}\right)^n\right] u[-n-1] \qquad \text{for } |z| < \frac{1}{2}. \end{aligned}$$

(c) For the causal system,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h_c[n]| &= \frac{1}{2}\sum_{n=0}^{\infty} \left| \left(\frac{-1}{2}\right)^n + \left(\frac{1}{2}\right)^n \right| \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \\ &= \frac{4}{3} \end{aligned}$$

which is absolutely summable. Thus, the causal system is stable in the bounded-input bounded-output sense.

For the noncausal system,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |h_{nc}[n]| &= \frac{1}{2}\sum_{n=-\infty}^{-1} \left| \left[\left(\frac{-1}{2}\right)^n + \left(\frac{1}{2}\right)^n \right] \right| \\
&= \frac{1}{2}\sum_{n=1}^{\infty} |[(-2)^n + 2^n]| \\
&= \sum_{n=1}^{\infty} 4^n
\end{aligned}$$

which does not exist. Thus, the noncausal system is unstable in the bounded-input bounded-output sense.

(d) Now, the output sequence has the z-transform

$$\begin{aligned}
Y(z) &= X(z)H(z) \qquad \text{for } |z| > \frac{1}{2} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} \\
&= \frac{1/4}{1-\frac{1}{2}z^{-1}} + \frac{1/2}{(1-\frac{1}{2}z^{-1})^2} + \frac{1/4}{1+\frac{1}{2}z^{-1}}.
\end{aligned}$$

We thus have

$$\begin{aligned}
y[n] &= \mathcal{Z}^{-1}\{Y(z)\} \qquad \text{for } |z| > \frac{1}{2} \\
&= \frac{1}{4}\left(\frac{1}{2}\right)^n u[n] + (n+1)\left(\frac{1}{2}\right)^{n+1} u[n] + \frac{1}{4}\left(\frac{-1}{2}\right)^n u[n]
\end{aligned}$$

because (1) $\mathcal{Z}\{a^n u[n]\} = \frac{1}{1-az^{-1}}$ for $|z| > |a|$ and (2)

$$\begin{aligned}
\frac{1}{(1-az^{-1})^2} &= \frac{1}{-az^{-2}}\frac{d}{dz}\frac{1}{1-az^{-1}} \\
&= -a^{-1}z^2\frac{d}{dz}\frac{1}{1-az^{-1}} \qquad \text{for } |z| > |a|
\end{aligned}$$

yields

$$\begin{aligned}
\mathcal{Z}^{-1}\left\{\frac{1}{(1-az^{-1})^2}\right\} &= a^{-1}\cdot\mathcal{Z}^{-1}\left\{z\cdot(-z)\frac{d}{dz}\frac{1}{1-az^{-1}}\right\} \qquad \text{for } |z| > |a| \\
&= a^{-1}\cdot\mathcal{Z}^{-1}\{z\cdot\mathcal{Z}\{na^n u[n]\}\} \text{ (from property 4 of Table 3.2)} \\
&= a^{-1}(n+1)a^{n+1}u[n+1] \text{ (from property 2 of Table 3.2)} \\
&= (n+1)a^n u[n].
\end{aligned}$$

(e) For the causal system with $H(z) = \frac{1}{1-\frac{1}{4}z^{-2}}$, the output sequence $y[n] = \delta[n]$ has the z-transform $Y(z) = 1$. Thus, the z-transform of the corresponding input sequence $x[n]$ is

$$X(z) = \frac{1}{H(z)} = 1 - \frac{1}{4}z^{-2}.$$

This requests the input sequence

$$x[n] = \delta[n] - \frac{1}{4}\delta[n-2].$$

(3) (4%, 2% each) For each of the following sequences, determine the z-transform, ROC, and all poles and zeros:

(a) $x[n] = a^n u[n] + b^n u[n] + c^n u[-n-1]$, $|a| < |b| < |c|$

(b) $x[n] = e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right] u[n] - e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right] u[n-1]$

Sol: (a)

$$\begin{aligned} x[n] &= a^n u[n] + b^n u[n] + c^n u[-n-1] \qquad |a| < |b| < |c| \\ X(z) &= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} - \frac{1}{1-cz^{-1}} \qquad |b| < |z| < |c| \\ X(z) &= \frac{1 - 2cz^{-1} + (bc+ac-ab)z^{-2}}{(1-az^{-1})(1-bz^{-1})(1-cz^{-1})} \qquad |b| < |z| < |c| \end{aligned}$$

Poles: a, b, c,
Zeros: z_1, z_2, 0 where z_1 and z_2 are roots of numerator quadratic $z^2 - 2cz + (bc + ac - ab) = 0$.

(b) Now,

$$\begin{aligned} x[n] &= e^{n^4}\left(\cos\frac{\pi}{12}n\right) u[n] - e^{n^4}\left(\cos\frac{\pi}{12}n\right) u[n-1] \\ &= e^{n^4}\left(\cos\frac{\pi}{12}n\right)(u[n] - u[n-1]) = \delta[n] \end{aligned}$$

Therefore, $X(z) = 1$ for all $|z|$. There is no pole nor zero.

(4) (4%, 2% each) Determine the inverse z-transform of each of the following.

(a) Find a stable sequence for which $X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}}$.

(b) Determine the inverse z-transform of $Y(z) = \log(1-2z)$, $|z| < \frac{1}{2}$.

Sol: (a) Now,

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}} = \frac{3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{4}{1-\frac{1}{2}z^{-1}} - \frac{4}{1+\frac{1}{4}z^{-1}}$$

We have poles at $\frac{1}{2}$ and $-\frac{1}{4}$. Also, because $x[n]$ is stable, we have the ROC $|z| > \frac{1}{2}$, which shows that $x[n]$ is causal as well.
Therefore,

$$x[n] = 4(\frac{1}{2})^n u[n] - 4(\frac{-1}{4})^n u[n].$$

(b) Now,

$$\begin{aligned} Y(z) &= \log(1-2z) = -\Sigma_{i=1}^{\infty}\frac{(2z)^i}{i} \\ &= -\Sigma_{l=-\infty}^{-1}\frac{1}{-l}(2z)^{-l} = \Sigma_{l=-\infty}^{-1}\frac{1}{l}(\frac{1}{2})^l z^{-l} \end{aligned}$$

for $|z| < 1/2$. Therefore,

$$y[n] = \frac{1}{n}(\frac{1}{2})^n u[-n-1].$$

An alternative solution is as follows: From the z-transform properties, we have

$$\begin{aligned} ny[n] &\rightarrow -z\frac{d}{dz}\log(1-2z) = -z\left(\frac{1}{1-2z}\right)(-2) \\ &= \frac{-1}{1-\frac{1}{2}z^{-1}}, \ |z| < \frac{1}{2} \\ ny[n] &= (\frac{1}{2})^n u[-n-1] \\ \text{Thus, } y[n] &= \frac{1}{n}(\frac{1}{2})^n u[-n-1]. \end{aligned}$$

(5) (3%, 1% each) Let $h_c(t)$ denote the impulse response of an LTI continuous-time filter and $h_d[n]$ the impulse response of an LTI discrete-time filter.

(a) If

$$h_c(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where a is a positive real constant, determine the continuous-time filter frequency response and its magnitude response.

(b) If $h_d[n] = Th_c(nT)$ with $h_c(t)$ as in part (a), determine the discrete-time filter frequency response and its magnitude response.

(c) For a given value of a, determine, as a function of T, the minimum magnitude of the discrete-time filter frequency response.

Sol: (a) The Fourier transform of the filter impulse response

$$\begin{aligned} H_c(j\varpi) &= \int_{-\infty}^{\infty} h_c(t)e^{-j\varpi t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\varpi t}dt \\ &= \frac{1}{a+j\varpi}. \end{aligned}$$

So, we take the magnitude

$$|H_c(j\varpi)| = (\frac{1}{a^2+\varpi^2})^{\frac{1}{2}}.$$

(b) Sampling the filter impulse response in (a), the discrete-time filter is described by

$$h_d[n] = Te^{-anT}u[n]$$

$$\begin{aligned} H_d(e^{j\omega}) &= \sum_{n=0}^{\infty} Te^{-anT}e^{-j\omega n} \\ &= \frac{T}{1-e^{-aT}e^{-j\omega}}. \end{aligned}$$

Taking the magnitude of this response

$$|H_d(e^{j\omega})| = \frac{T}{(1-2e^{-aT}\cos(\omega)+e^{-2aT})^{\frac{1}{2}}}.$$

Note that the frequency response of the discrete-time filter is periodic, with period 2π.

(c) The minimum occurs at $\omega = \pi$. The corresponding value of the frequency response magnitude is

$$\begin{aligned} |H_d(e^{j\pi})| &= \frac{T}{(1+2e^{-aT}+e^{-2aT})^{\frac{1}{2}}} \\ &= \frac{T}{1+e^{-aT}}. \end{aligned}$$

(6) (3%, 1% each) Consider a complex baseband analog signal $x_c(t)$ with the Fourier transform $X_c(j\varpi) = 1$ for $a < \varpi < b$ and $X_c(j\varpi) = 0$ elsewhere, where $a = \frac{3}{4}b$ and $a > 0$. Suppose that we sample $x_c(t)$ with a sampling time T seconds and produce the sequence $x[n] = x_c(nT)$. In the following sampling times, what are the sampling times that can be used without incurring any aliasing distortion (i.e., so that $x_c(t)$ can be recovered from $x[n]$ by some means): (a) $T = \frac{2\pi}{3b}$; (b) $T = \frac{2\pi}{3b/4}$; (c) $T = \frac{2\pi}{b/2}$? Explain or illustrate your answer.

Sol:

(a) For $T = \frac{2\pi}{3b}$, the sampling frequency $\varpi_s = \frac{2\pi}{T} = 3b$ is greater than the Nyquist rate $2\varpi_N = 2b$. There is no aliasing distortion with $T = \frac{2\pi}{3b}$.

(b) For $T = \frac{2\pi}{3b/4}$, the sampling frequency $\varpi_s = \frac{2\pi}{T} = 3b/4$ produces the Fourier transform $X_s(j\varpi)$ of the sampling signal $x_s(t)$ as

$$\begin{aligned} X_s(j\varpi) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\varpi - jk\varpi_s) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} 1_{\frac{3}{4}b<\varpi-k\frac{3}{4}b<b} \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} 1_{(k+1)\frac{3}{4}b<\varpi<k\frac{3}{4}b+b} \end{aligned}$$

where $1_{condition}$ is an indicator function with $1_{condition} = 1$ if the *condition* is met and $1_{condition} = 0$ otherwise. Note that $X_s(j\varpi)$ is equal to $\frac{1}{T}$ when ϖ is in the frequency ranges ..., $(0, \frac{1}{4}b)$, $(\frac{3}{4}b, \frac{4}{4}b)$, $(\frac{6}{4}b, \frac{7}{4}b)$,... Thus, the original $x_c(t)$ which is bandlimited white over $(\frac{3}{4}b, \frac{4}{4}b)$ can be recovered by ideally bandpass filtering $x_s(t)$. There is no aliasing distortion with $T = \frac{2\pi}{3b/4}$.

(c) For $T = \frac{2\pi}{b/2}$, the sampling frequency $\varpi_s = \frac{2\pi}{T} = b/2$ produces the Fourier transform $X_s(j\varpi)$ of the sampling signal $x_s(t)$ as

$$\begin{aligned} X_s(j\varpi) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\varpi - jk\varpi_s) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} 1_{\frac{3}{4}b<\varpi-k\frac{1}{2}b<b} \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} 1_{(2k+3)\frac{b}{4}<\varpi<(2k+4)\frac{b}{4}}. \end{aligned}$$

Note that $X_s(j\varpi)$ is equal to $\frac{1}{T}$ when ϖ is in the frequency ranges ..., $(\frac{1}{4}b, \frac{2}{4}b)$, $(\frac{3}{4}b, \frac{4}{4}b)$, $(\frac{5}{4}b, \frac{6}{4}b)$,... Thus, the original $x_c(t)$ which is bandlimited white over $(\frac{3}{4}b, \frac{4}{4}b)$ can be recovered by ideally bandpass filtering $x_s(t)$. There is no aliasing distortion with $T = \frac{2\pi}{b/2}$.

(7) (4%, 2% each) For each of the following systems, determine whether the system is (1) stable, (2) causal, (3) linear, and (4) time-invariant. Prove or explain it if the system has a property and give a counter-example if the system does not have a property.

(a) $\mathcal{T}\{x[n]\} = x[-n]$

(b) $\mathcal{T}\{x[n]\} = x[n]\,u[-n]$.

Sol: (a) This system reverses the input sequence. It is stable, noncausal, linear, and time-variant.

- The system is stable in that if $x[n]$ is bounded for n, $x[-n]$ is also bounded for all n.
- It is not causal. For example, $y[-4] = \mathcal{T}\{x[-4]\} = x[4]$, the output at $n = -4$ depends on the future input.
- It is linear. Let $y_1[n] = \mathcal{T}\{x_1[n]\} = x_1[-n]$, and $y_2[n] = \mathcal{T}\{x_2[n]\} = x_2[-n]$. Now

$$\mathcal{T}\{ax_1[n] + bx_2[n]\} = ax_1[-n] + bx_2[-n] = ay_1[n] + by_2[n]$$

for arbitrary a and b.

- It is not time-invariant. If $y[n] = \mathcal{T}\{x[n]\} = x[-n]$, then the new sequence $w[n] = x[n-1]$ yields

$$\mathcal{T}\{x[n-1]\} = \mathcal{T}\{w[n]\} = w[-n] = x[-n-1] \neq y[n-1]$$

in general.

(b) This system simply "retains" $x[n]$ for nonpositive n. This system is stable, causal, linear, but not time-invariant.

- The system is stable in that if $x[n]$ is bounded for all n, $x[n]\,u[-n]$ is also bounded for all n.
- It is causal since $y[n] = \mathcal{T}\{x[n]\} = x[n]\,u[-n]$ depends only on the current input.
- It is linear. Let $y_1[n] = \mathcal{T}\{x_1[n]\} = x_1[n]\,u[-n]$, and $y_2[n] = \mathcal{T}\{x_2[n]\} = x_2[n]\,u[-n]$. Now

$$\mathcal{T}\{ax_1[n] + bx_2[n]\} = \{ax_1[n] + bx_2[n]\}u[-n] = ay_1[n] + by_2[n]$$

  for arbitrary a and b.
- It is not time-invariant. If $y[n] = \mathcal{T}\{x[n]\} = x[n]\,u[-n]$, then the new sequence $w[n] = x[n-1]$ yields

$$\mathcal{T}\{x[n-1]\} = \mathcal{T}\{w[n]\} = w[n]\,u[-n] = x[n-1]\,u[-n] \neq y[n-1]$$

  in general.