1. (0.5%) Page 167, Problem 3.3

   Determine the z-transform of each of the following sequences. Include with your answer the ROC in the z-plane and a sketch of the pole-zero plot. Express all sums in closed form; α can be complex.

   (a) $x_a[n] = \alpha^{|n|}$, $0 < |\alpha| < 1$.

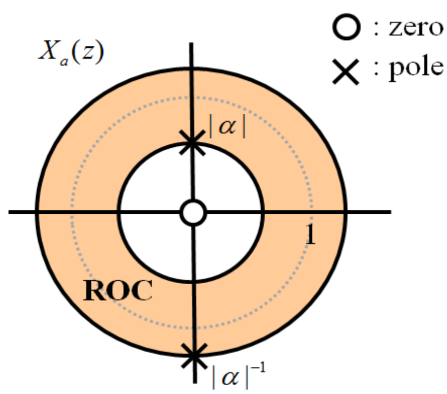
   (b) $x_b[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$

   (c) $x_c[n] = \begin{cases} n+1, & 0 \le n \le N-1, \\ 2N-1-n, & N \le n \le 2(N-1), \\ 0, & \text{otherwise.} \end{cases}$

   Sol:

   (a)

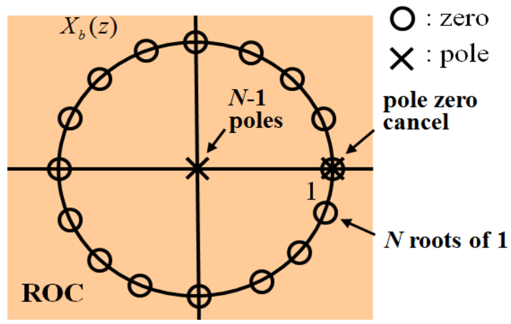
$$
\begin{aligned}
x_a[n] &= \alpha^{|n|}, 0 < |\alpha| < 1. \\
X_a(z) &= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \sum_{n=1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}} = \frac{(1-\alpha^2) z}{(1-\alpha z)(z-\alpha)}, \ |\alpha| < |z| < \frac{1}{|\alpha|}
\end{aligned}
$$



   (b)

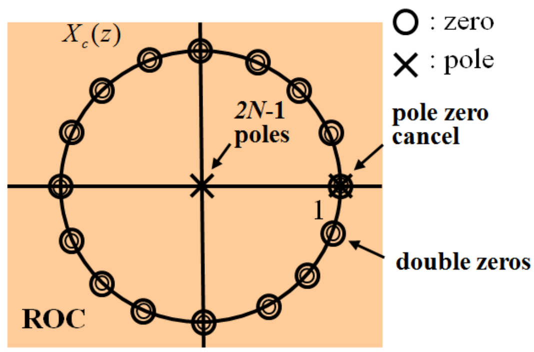
$$
x_b[n] = \begin{cases} 1, & 0 \le n \le N-1; \\ 0, & n \ge N \ ; \\ 0, & n < 0 \end{cases}
$$

$$
\Rightarrow X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)}, \quad z \neq 0
$$



(c)

$$
\begin{aligned}
x_c[n] &= x_b[n-1] * x_b[n] \Rightarrow X_c(z) = z^{-1}X_b(z) \cdot X_b(z) \\
X_c(z) &= z^{-1}(\frac{z^N - 1}{z^{N-1}(z-1)})^2 = \frac{1}{z^{2N-1}}(\frac{z^N-1}{z-1})^2,\ z \neq 0,\ 1
\end{aligned}
$$



2. (0.5%) Page 168, Problem 3.5

   Determine the sequence $x[n]$ with z-transform

$$
X(z) = (1 + 2z)(1 + 3z^{-1})(1 - z^{-1}).
$$

   Sol:

$$
\begin{aligned}
X(z) &= (1 + 2z)(1 + 3z^{-1})(1 - z^{-1}) \\
&= 2z + 5 - 4z^{-1} - 3z^{-2} \\
&= \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\end{aligned}
$$

   Therefore,

$$
x[n] = 2\delta[n+1] + 5\delta[n] - 4\delta[n-1] - 3\delta[n-2].
$$

3. (0.5%) Page 168, Problem 3.8

The system function of a causal LTI system is

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}.$$

The input to this system is

$$x[n] = (\frac{1}{3})^n u[n] + u[-n-1].$$

(a) Find the impulse response of the system, $h[n]$.

(b) Find the output $y[n]$.

(c) Is the system stable? That is, is $h[n]$ absolutely summable?

Sol: The causal system has system function

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}$$

and the input is $x[n] = (\frac{1}{3})^n u[n] + u[-n-1]$. Therefore the z-transform of the input is

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - z^{-1}} = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - z^{-1})}, \quad \frac{1}{3} < |z| < 1$$

(a) $h[n]$ is causal $\Rightarrow h[n]$

$$h[n] = (-\frac{3}{4})^n u[n] - (-\frac{3}{4})^{n-1} u[n-1].$$

(b)

$$\begin{aligned} Y(z) &= X(z)\,H(z) = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 + \frac{3}{4}z^{-1})}, \quad \frac{3}{4} < |z| \\ &= \frac{-\frac{8}{13}}{1 - \frac{1}{3}z^{-1}} + \frac{\frac{8}{13}}{1 + \frac{3}{4}z^{-1}}. \end{aligned}$$

Therefor the output is

$$y[n] = -\frac{8}{13}(\frac{1}{3})^n u[n] + \frac{8}{13}(-\frac{3}{4})^n u[n].$$

(c) For $h[n]$ to be causal ROC of $H(z)$ must be $\frac{3}{4} < |z|$ which includes the unit circle. Therefore, $h[n]$ absolutely summable.

4. (0.5%) Page 170, Problem 3.17

Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1].$$

Determine all possible values for the system's impulse response $h[n]$ at $n = 0$.

Sol: We solve this problem by finding the system function $H(z)$ of the system, and then looking at the different impulse responses which can result from our choice of the ROC.

Taking the z-transform of the difference equation, we get

$$Y(z)(1 - \frac{5}{2}z^{-1} + z^{-2}) = X(z)(1 - z^{-1}),$$

and thus

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \\ &= \frac{1 - z^{-1}}{(1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})} \\ &= \frac{2/3}{1 - 2z^{-1}} + \frac{1/3}{1 - \frac{1}{2}z^{-1}} \end{aligned}$$

If the ROC is

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}2^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^n u[-n-1]$$
$$\Rightarrow h[0] = 0.$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}2^n u[-n-1] + \frac{1}{3}(\frac{1}{2})^n u[n]$$
$$\Rightarrow h[0] = \frac{1}{3}.$$

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}2^n u[n] + \frac{1}{3}(\frac{1}{2})^n u[n]$$
$$\Rightarrow h[0] = 1.$$

5. (0.5%) Page 172, Problem 3.22

Consider an LTI system that is stable and for which $h[z]$, the $z-$transform of the impulse response, is given by

$$H(z) = \frac{3}{1 + \frac{1}{3}z^{-1}}.$$

Suppose $x[n]$, the input to the system, is a unit step sequence.

(a) Determine the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Determine the output $y[n]$ by computing the inverse z-transform of $Y(z)$.

Sol:

(a)

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} (3(-\frac{1}{3})^k u[k])u[n-k] \\
&= \sum_{k=0}^{n} 3(-\frac{1}{3})^k \\
&= \begin{cases} \frac{9}{4}(1-(-\frac{1}{3})^{n+1}), \; n \geq 0 \\ 0, \qquad\qquad\qquad \text{otherwise} \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
Y(z) &= X(z)H(z) \\
&= \frac{3}{1+\frac{1}{3}z^{-1}} \cdot \frac{1}{1-z^{-1}} \\
&= \frac{3/4}{1+\frac{1}{3}z^{-1}} + \frac{9/4}{1-z^{-1}}
\end{aligned}$$

$$\begin{aligned}
y[n] &= \frac{3}{4}(-\frac{1}{3})^n u[n] + \frac{9}{4}u[n] \\
&= \frac{9}{4}(1+\frac{1}{3}(-\frac{1}{3})^n)u[n] \\
&= \frac{9}{4}(1-(-\frac{1}{3})^{n+1})u[n].
\end{aligned}$$

6. (0.5%) Page 173, Problem 3.30

Determine the inverse z-transform of each of the following. In parts (a)-(c), use the methods specified. (In part (d), use any method you prefer.)

(a) Long division:

$$X(z) = \frac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}, \quad x[n] \text{ is a right-sided sequence}$$

(b) Partial fraction:

$$X(z) = \frac{3}{z-\frac{1}{4}-\frac{1}{8}z^{-1}}, \quad x[n] \text{ stable}$$

(c) Power series:

$$X(z) = \ln(1-4z), \quad |z| < \frac{1}{4}$$

(d) $X(z) = \frac{1}{1-\frac{1}{3}z^{-3}}, \quad |z| > (3)^{-\frac{1}{3}}$

Sol:

(a) $x[n]$ is right-sided sequence and

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

Long division:

$$\begin{array}{r|llll}
 & 1 - \frac{2}{3}z^{-1} & + \frac{2}{9}z^{-2} & + \ldots & \\
\hline
1 + \frac{1}{3}z^{-1} & 1 - \frac{1}{3}z^{-1} & & & \\
 & 1 - \frac{1}{3}z^{-1} & & & \\
\cline{2-2}
 & \quad - \frac{2}{3}z^{-1} & & & \\
 & \quad -\frac{2}{3}z^{-1} & - \frac{2}{9}z^{-2} & & \\
\cline{2-3}
 & & + \frac{2}{9}z^{-2} & &
\end{array}$$

Therefore, $x[n] = 2(-\frac{1}{3})^n - \delta[n]$.

(b)

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}} = \frac{3z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})} = \frac{4}{1 - \frac{1}{2}z^{-1}} - \frac{4}{1 + \frac{1}{4}z^{-1}}$$

Poles at $\frac{1}{2}$ and $-\frac{1}{4}$. $x[n]$ is stable. $\Rightarrow |x| > \frac{1}{2} \Rightarrow$ causal.
Therefore,

$$x\,[n] = 4(\frac{1}{2})^n u[n] - 4(-\frac{1}{4})^n u[n].$$

(c)

$$\begin{aligned}
x\,[n] &= \ln(1 - 4z), \ \ |z| < \frac{1}{4} \\
&= -\sum_{i=1}^{\infty} \frac{(4z)^i}{i} = \sum_{m=-\infty}^{-1} \frac{1}{m}(4z)^{-m}
\end{aligned}$$

Therefore,

$$x[n] = \frac{1}{n}(4)^{-n} u[-n-1].$$

(d)

$$\begin{aligned}
X(z) &= \frac{1}{1 - \frac{1}{3}z^{-1}}, \ |z| > (3)^{-\frac{1}{3}} \\
&\Rightarrow \ x[n] \text{ is causal.}
\end{aligned}$$

By long division:

$$
\begin{array}{r|l}
 & 1+\frac{1}{3}z^{-3} \quad +\frac{1}{9}z^{-6} \quad + ... \\
\hline
1-\frac{1}{3}z^{-3} & 1 \\
 & 1-\frac{1}{3}z^{-3} \\
\cline{2-2}
 & +\frac{1}{3}z^{-3} \\
 & +\frac{1}{3}z^{-3} \quad -\frac{1}{9}z^{-6} \\
\cline{2-2}
 & \qquad\qquad +\frac{1}{9}z^{-6}
\end{array}
$$

$$\Rightarrow x[n] = \begin{cases} (\frac{1}{3})^{\frac{n}{3}}, & n = 0,\ 3,\ 6,\ ... \\ 0, & \text{otherwise} \end{cases}$$

7. (0.5%) Page 174, Problem 3.33

   Determine a sequence $x[n]$ whose z-transform is $X(z) = e^z + e^{1/z}$, $z \neq 0$.

   Sol:

$$X(z) = e^z + e^{1/z} \quad z \neq 0$$

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n + \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{1}{z})^n \\ &= \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^{-n} + \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \\ \Rightarrow \quad & x[n] = \frac{1}{|n|!} + \delta[n]. \end{aligned}$$

8. (0.5%) Page 175, Problem 3.40

   If the input $x[n]$ to an LTI system is $x[n] = u[n]$, the output is

$$y[n] = (\frac{1}{2})^{n-1} u[n+1].$$

   (a) Find $H(z)$, the z-transform of the system impulse response, and plot its pole-zero diagram.
   (b) Find the impulse response $h[n]$.
   (c) Is the system stable?
   (d) Is the system causal?

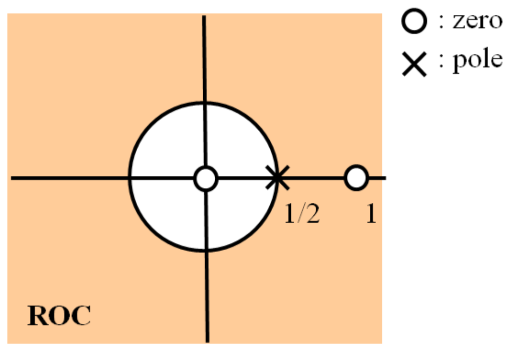
Sol:

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1-z^{-1}} \quad |z| > 1$$

$$y[n] = (\frac{1}{2})^{n-1}u[n+1] = 4(\frac{1}{2})^{n+1}u[n+1] \Leftrightarrow Y(z) = \frac{4z}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}.$$

(a)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4z(1-z^{-1})}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$



(b)

$$H(z) = \frac{4z}{1-\frac{1}{2}z^{-1}} - \frac{4}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

$$\begin{aligned} h[n] &= 4(\frac{1}{2})^{n+1}u[n+1] - 4(\frac{1}{2})^{n}u[n] \\ &= 4\delta[n+1] - 2(\frac{1}{2})^{n}u[n]. \end{aligned}$$

(c) The ROC of $H(z)$ includes $|z| = 1 \Rightarrow$ stable.

(d) From part (b) we see that $h[n]$ starts at $n = -1 \Rightarrow$ not causal.

9. (0.5%) Page 266, Problem 4.2

The sequence

$$x[n] = \cos(\frac{\pi}{4}n), \quad -\infty < n < \infty,$$

was obtained by sampling the continuous-time signal

$$x_c(t) = \cos(\Omega_0 t), \quad -\infty < t < \infty,$$

at a sampling rate of 1000 samples/s. What are two possible positive values of Ω_0 that could have resulated in the sequence $x[n]$?

Sol: The discrete-time sequence

$$x[n] = \cos(\frac{\pi}{4}n)$$

results by sampling the continuous-time signal

$$x_c(t) = \cos(\Omega_0 t),$$

Since $\omega = \Omega T$ and $T = 1/1000$ seconds, the signal frequency could be:

$$\Omega_0 = \frac{\pi}{4} \cdot 1000 = 250\pi.$$

Or possibly:

$$\Omega_0 = (2\pi + \frac{\pi}{4}) \cdot 1000 = 2250\pi.$$

10. (0.5%) Page 267, Problem 4.6

Let $h_c(t)$ denote the impulse response of an LTI continuous-time filter and $h_d[n]$ the impulse response of an LTI discrete-time filter.

(a) If

$$h_c(t) = \begin{cases} e^{-at}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where a is a positive real constant, determine the continuous-time filter frequency response and sketch its magnitude.

(b) If $h_d[n] = Th_c(nT)$ with $h_c(t)$ as in part (a), determine the discrete-time filter frequency response and sketch its magnitude..

(c) For a given value of a, determine, as a function of T, the minimum magnitude of the discrete-time filter frequency response.

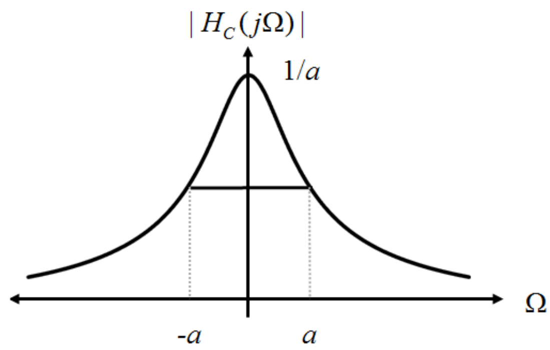
Sol:

(a) The Fourier transform of the filter impulse response

$$\begin{aligned} H_c(j\Omega) &= \int_{-\infty}^{\infty} h_c(t)e^{-j\Omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\Omega t}dt \\ &= \frac{1}{a+j\Omega}. \end{aligned}$$

So, we take the magnitude

$$|H_c(j\Omega)| = (\frac{1}{a^2+\Omega^2})^{\frac{1}{2}}.$$



(b) Sampling the filter impulse response in (a), the discrete-time filter is described by

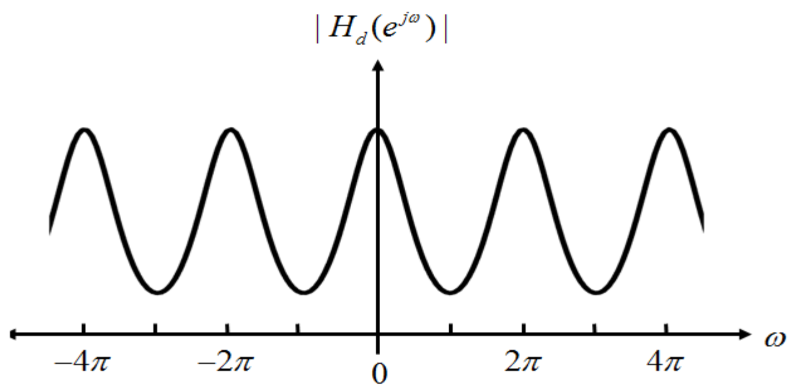
$$h_d[n] = Te^{-anT}u[n]$$

$$\begin{aligned} H_d(e^{j\omega}) &= \sum_{n=0}^{\infty} Te^{-anT}e^{-j\omega n} \\ &= \frac{T}{1 - e^{-aT}e^{-j\omega}}. \end{aligned}$$

Taking the magnitude of this response

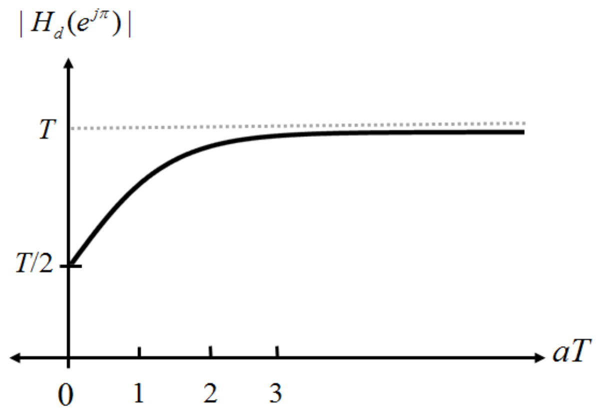
$$|H_d(e^{j\omega})| = \frac{T}{(1 - 2e^{-aT}\cos(\omega) + e^{-j2aT})^{\frac{1}{2}}}.$$

Note that the frequency response of the discrete-time filter is periodic, with period 2π.



(c) The minimum occurs at $\omega = \pi$. The corresponding value of the frequency response magnitude is

$$\begin{aligned} |H_d(e^{j\pi})| &= \frac{T}{(1 + 2e^{-aT} + e^{-j2aT})^{\frac{1}{2}}} \\ &= \frac{T}{1 + e^{-aT}}. \end{aligned}$$


$|H_d(e^{j\pi})|$
T
$T/2$
aT
0
1
2
3