3 The z-Transform

- Two advantages with the z-transform:

  1. The z-transform is a generalization of the Fourier transform for discrete-time signals; which encompasses a broader class of sequences. The z-transform exists for some sequences for which the Fourier transform does not converge.
  2. The advantage and power of complex variable theory allow one to obtain z-transforms easily.

3.1 Z-Transform

- Defn: The (*two-sided* or *bilateral*) *z-transform* of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

  where z is a complex variable.

  We shall adopt the notation

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

  or

$$\begin{aligned} X(z) &= \mathcal{Z}\{x[n]\} \\ x[n] &= \mathcal{Z}^{-1}\{X(z)\} \end{aligned}$$

  throughout.

- Note that the Fourier transform $X(e^{j\omega})$ of $x[n]$ is only a special case of $X(z)$ since

$$X\left(e^{j\omega}\right) = X(z)\big|_{z=e^{j\omega}}.$$

  If we express z by a polar form, i.e., $z = re^{j\omega}$, then

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{r^{-n} x[n]\right\} e^{-jn\omega}$$

  whose *region of convergence* (ROC) may or may not contain the circle $r = 1$ (unit circle on z-plane).

Fig. 3.1

The *sufficient and necessary* condition for a z-transform to exist is

$$|X(z)| < \infty \qquad \forall z \in \text{ROC}.$$

A *sufficient* condition is that the sequence $\{x[n]z^{-n}\}$ is absolutely summable for $z \in$ ROC, i.e.,

$$|X(z)| \leq \sum_{n=-\infty}^{\infty} \left|x[n]z^{-n}\right| = \sum_{n=-\infty}^{\infty} |x[n]|\,|z|^{-n} = \sum_{n=-\infty}^{\infty} |x[n]|\,r^{-n} < \infty$$
$$\forall z \in \text{ROC}.$$

This sufficient condition implies that if $z = z_1 \in$ ROC, then all z on the circle defined by $|z| = |z_1|$ will also be in ROC. Throughout the course, the ROC is thus defined by the set of z values for which $\{x[n]z^{-n}\}$ is absolutely summable, i.e., a collection of concentric circles on z-plane. Therefore, the ROC consists of a *ring* in the z-plane.

Fig. 3.2

Note: if ROC does not contain the unit circle, then the Fourier transform of $x[n]$ does not exist.

- It can be shown that if the sequence $\{x[n]z^{-n}\}$ is absolutely summable for $z \in$ ROC, i.e.

  $$\sum_{n=-\infty}^{\infty} |x[n]|\,r^{-n} < \infty$$

  then $X_M(z)$ with

  $$X_M(z) = \sum_{n=-M}^{M} x[n]z^{-n}$$

  converges *uniformly* to a continuous function of z for $z \in$ ROC.

- It is shown by theory that the z-transform and all its derivatives are *continuous* functions of z within ROC.

  Thus if ROC contains unit circle, then the Fourier transform $X(e^{j\omega})$ and all its derivatives w.r.t. ω are continuous function of ω.

- Some useful z-transform pairs are summarized in Table 3.1.

Table 3.1

- Ex: Right-Sided Sequence

$$x[n] = a^n u[n], \ a \text{ real} \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n.$$

Convergence of $X(z)$ requires *sufficiently* that

$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

which implies the ROC $= \{z : |z| > |a|\}$. Thus, for $|z| > |a|$,

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}.$$

Notes:

1. if $|a| \geq 1$, $x[n] = a^n u[n]$ does not have a Fourier transform.
2. $X(z)$ simplifies to a rational function. Any rational z-transform can be described by its poles and zeros.
3. if $a > 0$, $z = a$ is the pole of $X(z)$, while $z = 0$ is the zero.

Fig. 3.3

- Ex: Left-Sided Sequence

$$\begin{aligned} & x[n] = -a^n u[-n-1], \ a \text{ real} \\ \Rightarrow \ & X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1}z\right)^n. \end{aligned}$$

For $|z| < |a|$, $X(z)$ converges to

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{z}{z-a}.$$

Notes:

1. The z-transform here equals to that in the previous example for $x[n] = a^n u[n]$ with the same pole-zero plot, while ROC is different.

Fig. 3.4

2. It is necessary to specify the algebraic expression for $X(z)$ and the ROC for the bilateral z-transform of a given sequence.

- Ex: Sum of Two Sequences

$$
\begin{aligned}
x[n] &= \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \\
\Rightarrow \quad X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \qquad \text{for } \left|\frac{1}{2}z^{-1}\right| < 1 \text{ and } \left|-\frac{1}{3}z^{-1}\right| < 1 \\
&= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} \qquad \text{for } |z| > \frac{1}{2}.
\end{aligned}
$$

Fig. 3.5

- Ex: Sum of Two Sequences (Revisited)

$$
\begin{aligned}
x[n] &= \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \\
\Rightarrow \quad & \left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2} \\
& \left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3} \\
& \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{2}
\end{aligned}
$$

This indicates that the z-transform is linear and that the resultant ROC is the intersection of two individual ROCs.

- Ex: Two-Sided Sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\Rightarrow \quad \left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3}$$

$$\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{-1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2}$$

$$\left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \frac{1}{3} < |z| < \frac{1}{2}$$

$$= \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})}.$$

Notes:

1. The rational function here is identical to that in the previous example, but the ROC is different.

Fig. 3.6

2. This sequence does not have a Fourier transform.

- Ex: Finite-Length Exponential Sequence

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}.$$

The ROC is determined by

$$\sum_{n=0}^{N-1} \left|az^{-1}\right|^n < \infty.$$

As long as $|az^{-1}|$ is finite, i.e., $|a| < \infty$ and $z \neq 0$, $X(z)$ exists. Assuming a finite $|a|$, $X(z)$ has a pole at $z = 0$ and $N-1$ zeros at $z_k = a\exp\{j2\pi k/N\}$ for $k = 1, 2, ..., N-1$.

Fig. 3.7

- See Table 3.1 for useful pairs.

3.2 Properties of the ROC for the z-Transform

- Assuming that $X(z)$ is rational and $x[n]$ has finite amplitude for $-\infty < n < \infty$.

    1. The ROC is generally of the form $0 \le r_R < |z| < r_L \le \infty$ (an annulus).
    2. For a given sequence $x[n]$, $X(e^{j\omega})$ exists iff the ROC of $X(z)$ includes the unit circle.
    3. The ROC can not contain any poles.
    4. If $x[n]$ is a *finite-duration sequence*, i.e., it is nonzero only for finite n's, then the ROC is the entire z-plane except possibly $z = 0$ or $|z| = \infty$.
    5. If $x[n]$ is a *right-sided sequence*, i.e., it is zero for $n < N_1 < \infty$, then the ROC is of the form either $r_{\max} < |z|$ or $r_{\max} < |z| < \infty$, with $r_{\max}$ the largest magnitude of a finite pole.
    6. If $x[n]$ is a *left-sided sequence*, i.e., it is zero for $n > N_2 > -\infty$, then the ROC is of the form either $|z| < r_{\min}$ or $0 < |z| < r_{\min}$, with $r_{\min}$ the smallest magnitude of a finite pole.
    7. If $x[n]$ is a *two-sided sequence*, i.e., it is zero at most at a finite number of finite values for n, then the ROC is of the form $0 \le r_R < |z| < r_L \le \infty$ with r_R and r_L the magnitudes of two distinct poles, and not containing any poles inside.
    8. The ROC must be a connected region (since $X(z)$ is continuous).

- See section 3.2 for proof and

Fig. 3.8

- Ex: Non-Overlapping Regions of Convergence

$$
\begin{aligned}
x[n] &= \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]. \\
\Rightarrow X(z) &= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \qquad \text{for } |z| > \frac{1}{2} \text{ and } |z| < \frac{1}{3}
\end{aligned}
$$

Thus, $x[n]$ has no z-transform.

3.3 The Inverse Z-Transform

- Defn: The inverse z-transform of $X(z)$ is defined by the contour integral

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz$$

where C is a counterclockwise closed contour in the ROC of $X(z)$ and encircling the origin $z = 0$.

- Derivation of this formal definition:

From Cauchy integral theorem,

$$\frac{1}{2\pi j}\oint_C z^{-k} dz = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases} = \delta[k-1]$$

for *such* C. Since

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \\ \Rightarrow \frac{1}{2\pi j}\oint_C X(z)\, z^{k-1} dz &= \frac{1}{2\pi j}\oint_C \sum_{n=-\infty}^{\infty} x[n]\, z^{-n+k-1} dz \\ &= \sum_{n=-\infty}^{\infty} x[n] \underbrace{\frac{1}{2\pi j}\oint_C z^{-n+k-1} dz}_{\delta[n-k+1-1]=\delta[n-k]} \\ &= x[k]. \end{aligned}$$

- If the ROC includes unit circle and we let C be unit circle,

$$\begin{aligned} x[n] &\underset{z=e^{j\omega}}{=} \frac{1}{2\pi j}\oint_{\text{unit circle}} X\left(e^{j\omega}\right) e^{j\omega(n-1)} de^{j\omega} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega. \end{aligned}$$

- We shall discuss below various useful procedures of evaluating inverse z-transform, i.e., finding $x[n]$ from $X(z)$.

3.3.1 Inspection Method

- The method is to inspect Table 3.1 to find the desired $x[n]$. For example, with pair 5 of Table 3.1

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

we can find the $x[n]$ of

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{2}$$

as

$$\begin{pmatrix} \left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \\ \left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3} \end{pmatrix}$$

$$\Longrightarrow x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n].$$

3.3.2 Partial Fraction Expansion

- The method is to applying partial fraction expansion to a rational $X(z)$ first, and use the inspection method to obtain $x[n]$. Let

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \text{ for } a_0 \neq 0 \text{ and } b_0 \neq 0.$$

Then,

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} \quad \left(\begin{array}{l} \text{implying } M \text{ zeros and } N \text{ poles} \\ \text{at nonzero locations at most} \end{array} \right).$$

- In general,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \text{ for } a_0 \neq 0 \text{ and } b_0 \neq 0$$

can be expressed as

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where c_k's are nonzero zeros of $X(z)$ and d_k's are nonzero poles of $X(z)$. Suppose that c_k's are different from d_k's and that $d_1, d_2, ..., d_{N-s+1}$ are different and $d_{N-s+1}, ..., d_N$ are the same. Then, we can expand $X(z)$ into

$$\begin{aligned} X(z) &= \sum_{r=0}^{M-N} B_r z^{-r} \text{ (exists if } M \geq N) \\ &\quad + \sum_{k=1}^{N-s} \frac{A_k}{1 - d_k z^{-1}} \text{ (for poles of order one)} \\ &\quad + \sum_{m=1}^{s} \frac{C_m}{(1 - d_{N-s+1} z^{-1})^m} \text{ (for pole of order } s). \end{aligned}$$

(For the cases with other different-order poles, above expression can be modified accordingly.), where

1. B_r's can be derived by long division of the numerator by the denominator,
2. $A_k = \left(1 - d_k z^{-1}\right) X(z) \left.\right|_{z=d_k}$,
3. $$\begin{aligned} C_m &= \frac{1}{(s-m)!\left(-d_{N-s+1}\right)^{s-m}} \\ &\quad \cdot \left.\left\{ \frac{d^{s-m}}{dw^{s-m}} \left[\left(1 - d_{N-s+1} w\right)^s X\left(w^{-1}\right) \right] \right\}\right|_{w=1/d_{N-s+1}}. \end{aligned}$$

- Ex: Find $x[n]$ if $X(z)$ is given by

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \text{ for } |z| > 1.$$

Now,

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}.$$

Fig. 3.11

We have $M = N = 2$ and two poles of order one, thus

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

where

1. B_0 is derived

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & z^{-2} - 3z^{-1} + 2 \\ \hline & \quad 5z^{-1} \quad -1 \end{array}$$

$\Rightarrow B_0 = 2$

2.

$$\begin{aligned} A_1 &= \left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} \\ &= \frac{1+2z^{-1}+z^{-2}}{1-z^{-1}}\Big|_{z=\frac{1}{2}} \\ &= -9 \end{aligned}$$

$$\begin{aligned} A_2 &= \left(1-z^{-1}\right)X(z)\Big|_{z=1} \\ &= \frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}}\Big|_{z=1} \\ &= 8. \end{aligned}$$

That is,

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}} \qquad |z| > 1.$$

By inspection with Table 3.1,

$$\begin{aligned} 2\delta[n] &\overset{\mathcal{Z}}{\longleftrightarrow} 2 \qquad \text{all } z \\ \left(\frac{1}{2}\right)^n u[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2} \\ u[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}} \qquad |z| > 1. \end{aligned}$$

So,

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

3.3.3 Power Series Expansion

- If we can expand $X(z)$ in terms of power series (Laurent series)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

then $x[n]$ can be obtained as the coefficient of z^{-n}.

- Ex:

$$\begin{aligned}
X(z) &= z^2\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right) \qquad |z|>0 \\
&= z^2-\frac{1}{2}z-1+\frac{1}{2}z^{-1}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & x[n]=\begin{cases} 1, & n=-2 \\ -\frac{1}{2}, & n=-1 \\ -1, & n=0 \\ \frac{1}{2}, & n=1 \\ 0, & \text{otherwise} \end{cases} \\
\Rightarrow \quad & x[n]=\delta[n+2]-\frac{1}{2}\delta[n+1]-\delta[n]+\frac{1}{2}\delta[n-1].
\end{aligned}$$

- Ex:

  $$X(z)=\log\left(1+az^{-1}\right) \qquad |z|>|a|$$

  Since

  $$\log(1+x)=\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}x^n \text{ for } |x|<1$$

  we have

  $$\begin{aligned}
  X(z) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}a^n z^{-n}}{n} \\
  \Rightarrow \quad x[n] &= (-1)^{n+1}\frac{a^n}{n}u[n-1].
  \end{aligned}$$

- Ex: Consider

  $$X(z)=\frac{1}{1-az^{-1}}, \quad |z|>|a|.$$

  Since

  $$\frac{1}{1-x}=1+x+x^2+... \text{ for } |x|<1$$

  we have

  $$X(z)=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n \Rightarrow x[n]=a^n u[n].$$

- Ex: Consider

  $$X(z)=\frac{1}{1-az^{-1}}, \quad |z|<|a|.$$

Now

$$X(z) = \frac{a^{-1}z}{a^{-1}z - 1} = -a^{-1}z\left(\frac{1}{-a^{-1}z + 1}\right).$$

Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$

we have

$$\begin{aligned}
X(z) &= -a^{-1}z\sum_{n=0}^{\infty} a^{-n}z^n = -a^{-1}z\sum_{n=-\infty}^{0} a^n z^{-n} \\
&= -\sum_{n=-\infty}^{0} a^{n-1}z^{-(n-1)} \\
&= \sum_{n=-\infty}^{-1} (-a^n)\, z^{-n} \\
\Rightarrow x[n] &= -a^n u[-n-1].
\end{aligned}$$

3.3.4 Cauchy Residue Theorem

- From Cauchy Residue Theorem,

$$\begin{aligned}
x[n] &= \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz \\
&= \sum \left(\text{residues of } X(z)\, z^{n-1} \text{ at the poles inside } C\right)
\end{aligned}$$

- If $X(z)$ is a rational function of z, then $X(z)\, z^{n-1}$ has the form of

$$X(z)\, z^{n-1} = \frac{\Psi(z)}{(z-d_0)^s}$$

where $X(z)\, z^{n-1}$ has a pole of order s at $z = d_0$ and $\Psi(z)$ has no poles nor zeros at $z = d_0$. Then,

$$Res\left[X(z)\, z^{n-1} \text{ at } z = d_0\right] = \frac{1}{(s-1)!}\left[\frac{d^{s-1}\Psi(z)}{dz^{s-1}}\right]\Bigg|_{z=d_0}.$$

If $s = 1$,

$$Res\left[X(z)\, z^{n-1} \text{ at } z = d_0\right] = \Psi(d_0).$$

- Ex: Consider

$$
\begin{aligned}
X(z) &= \frac{1}{1-az^{-1}}, \quad |z| > |a| \\
\Rightarrow \quad X(z) z^{n-1} &= \frac{z^n}{z-a}.
\end{aligned}
$$

1. For $n \geq 0$, there is only one pole of order one at $z = a$ and

$$Res\left[X(z) z^{n-1} \text{ at } z = a\right] = a^n, \; n \geq 0.$$

2. For $n < 0$, there is $|n|$-th order pole at $z = 0$ and first order pole at $z = a$, and

$$
\begin{aligned}
Res\left[X(z) z^{n-1} \text{ at } z = a\right] &= a^n \\
Res\left[X(z) z^{n-1} \text{ at } z = 0\right] &= \frac{1}{(|n|-1)!} \frac{d^{|n|-1} (z-a)^{-1}}{dz^{|n|-1}} \Bigg|_{z=0} \\
&= (-1)^{|n|-1} \frac{(|n|-1)!}{(|n|-1)!} (z-a)^{-|n|} \Bigg|_{z=0} \\
&= (-1)(-1)^{|n|} (-a)^{-|n|} \\
&= -\left(a^{-1}\right)^{|n|} \\
&= -a^n.
\end{aligned}
$$

$\Rightarrow x[n] = a^n - a^n = 0$ for $n < 0$.

3. Therefore, from cases 1 and 2,

$$x[n] = a^n u[n].$$

3.4 Z-Transform Properties

- We shall adopt the notations

$$
\begin{aligned}
x[n] &\overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad \text{ROC} = R_x \\
x_1[n] &\overset{\mathcal{Z}}{\longleftrightarrow} X_1(z), \qquad \text{ROC} = R_{x_1} \\
x_2[n] &\overset{\mathcal{Z}}{\longleftrightarrow} X_2(z), \qquad \text{ROC} = R_{x_2}
\end{aligned}
$$

and

$$
\begin{aligned}
X(z) &= \mathcal{Z}\{x[n]\} \\
x[n] &= \mathcal{Z}^{-1}\{X(z)\}.
\end{aligned}
$$

- Properties:
  1. Linearity:

$$a x_1[n] + b x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} a X_1(z) + b X_2(z), \qquad \text{ROC} = R_{x_1} \cap R_{x_2}.$$

  2. Time-Shifting:

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z), \qquad \text{ROC} = R_x \text{ (may exclude } z=0 \text{ or } |z|=\infty\text{)}.$$

Pf:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n} &= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} \qquad (m = n - n_0) \\ &= z^{-n_0} X(z). \end{aligned}$$

(ROC should be reconsidered at $z = 0$ and $|z| = \infty$.) QED

Ex: Let

$$X(z) = \frac{1}{z - \frac{1}{4}}, \;\; |z| > \frac{1}{4}.$$

First, from partial fraction expansion,

$$\begin{aligned} X(z) &= \frac{z^{-1}}{1 - \frac{1}{4} z^{-1}}, \;\; |z| > \frac{1}{4} \\ &= -4 + \frac{4}{1 - \frac{1}{4} z^{-1}}. \end{aligned}$$

Since

$$\begin{aligned} &\delta[n] \overset{\mathcal{Z}}{\longleftrightarrow} 1, \;\; \forall z \\ &a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - a z^{-1}}, \;\; |z| > |a| \end{aligned}$$

$$\Rightarrow x[n] = -4\delta[n] + 4\left(\frac{1}{4}\right)^n u[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1].$$

Alternatively, since

$$\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{4} z^{-1}}, \;\; |z| > \frac{1}{4}$$

we have

$$\left(\frac{1}{4}\right)^{n-1} u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{1 - \frac{1}{4} z^{-1}}$$

from the time-shifting property.

3. Multiplication by z_0^n:

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{z_0}\right), \qquad \text{ROC} = |z_0| R_x.$$

Pf:

$$\begin{aligned} X_{new}(z) &= \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{z_0}\right)^{-n} \\ &= X\left(\frac{z}{z_0}\right) \qquad \frac{z}{z_0} \in R_x. \qquad \text{QED} \end{aligned}$$

Ex: Find $X(z)$ of $x[n] = r^n \cos(\omega_0 n) u[n]$.
Now,

$$x[n] = \frac{1}{2}\left(re^{j\omega_0}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_0}\right)^n u[n].$$

Since,

$$\begin{aligned} &u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \ \ |z| > 1 \\ &\left(re^{j\omega_0}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\left(\frac{z}{re^{j\omega_0}}\right)^{-1}}, \ \ |z| > \left|re^{j\omega_0}\right| = r \\ &\left(re^{-j\omega_0}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\left(\frac{z}{re^{-j\omega_0}}\right)^{-1}}, \ \ |z| > r \end{aligned}$$

$$\begin{aligned} \Rightarrow \ \ X(z) &= \frac{1}{2}\left[\frac{1}{1-re^{j\omega_0}z^{-1}} + \frac{1}{1-re^{-j\omega_0}z^{-1}}\right], \ \ |z| > r \\ &= \frac{1-r\cos\omega_0 z^{-1}}{1-2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \ \ |z| > r. \end{aligned}$$

4. Differentiation of $X(z)$:

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x \text{ (may exclude } z=0 \text{ or } |z|=\infty\text{)}.$$

Pf:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} nx[n] z^{-n} &= z\sum_{n=-\infty}^{\infty} nx[n] z^{-n-1} \\ &= -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= -z\frac{dX(z)}{dz} \qquad \text{QED} \end{aligned}$$

Ex: Find $X(z)$ of $na^n u[n]$.
Since

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$\begin{aligned}
\Rightarrow \quad na^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} & -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right), \quad |z| > |a| \\
&= \frac{z\left(az^{-2}\right)}{\left(1-az^{-1}\right)^2}, \quad |z| > |a| \\
&= \frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z| > |a|.
\end{aligned}$$

($|z| = \infty$ can be included in the ROC)

5. Conjugation of a Complex Sequence:

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC} = R_x.$$

Pf:

$$\begin{aligned}
\sum_n x^*[n] z^{-n} &= \left(\sum_n x[n]\left(z^{-n}\right)^*\right)^* \\
&= \left(\sum_n x[n]\left(z^*\right)^{-n}\right)^* \\
&= X^*(z^*), \ z^* \in R_x \Rightarrow z \in R_x. \quad \text{QED}
\end{aligned}$$

6. Time Reversal:

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{1}{z}\right), \ z^{-1} \in R_x.$$

Pf:

$$\sum_n x[-n] z^{-n} \underset{n'=-n}{=} \sum_{n'} x\left[n'\right]\left(z^{-1}\right)^{-n'} = X\left(z^{-1}\right), \ z^{-1} \in R_x. \quad \text{QED}$$

Ex: Find z-transform of $a^{-n}u[-n]$.
Since

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$\Rightarrow a^{-n}u[-n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az}, \quad |z| < \left|a^{-1}\right|.$$

7. Convolution of Sequences:

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) X_2(z), \text{ ROC} = R_{x_1} \cap R_{x_2}.$$

Pf:

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (x_1[n] * x_2[n]) z^{-n} \\
=& \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] z^{-(n-k)} z^{-k} \\
=& \left(\sum_{m=-\infty}^{\infty} x_2[m] z^{-m} \right) \left(\sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \right) \ (m = n-k) \\
=& X_1(z) X_2(z), \ z \in R_{x_1} \text{ and } z \in R_{x_2}. \quad \text{QED}
\end{aligned}$$

Ex:

$$x_1[n] = a^n u[n], \ |a| < 1 \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) = \frac{1}{1 - az^{-1}}, \ |z| > |a|$$

$$x_2[n] = u[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z) = \frac{1}{1 - z^{-1}}, \ |z| > 1$$

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \underbrace{\frac{1}{1 - z^{-1}} \frac{1}{1 - az^{-1}}}_{Y(z)}, \ |z| > 1$$

By partial fraction expansion,

$$Y(z) = \frac{1}{1-a} \left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}} \right), \ |z| > 1$$

$$\Rightarrow y[n] = \frac{1}{1-a} (u[n] - a \cdot a^n u[n]) = \frac{1 - a^{n+1}}{1-a} u[n].$$

8. Initial Value Theorem:
   If $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z).$$

Pf:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$\Rightarrow \lim_{z \to \infty} X(z) = \sum_{n=1}^{\infty} x[n] \lim_{z \to \infty} z^{-n} + x[0] = x[0]. \quad \text{QED}$$

- See Table 3.2 for a list of properties.

3.5 Complex Convolution Theorem (Supplemental)

- Question: What is the z-transform of $x_1[n]\, x_2[n]$?

  Answer: Let

  $$w[n] = x_1[n]\, x_2[n].$$

  Then

  $$W(z) = \sum_{n=-\infty}^{\infty} x_1[n]\, x_2[n]\, z^{-n}.$$

  From contour-integral definition of inverse z-transform,

  $$x_2[n] = \frac{1}{2\pi j} \oint_{C_2} X_2(v)\, v^{n-1} dv$$

  $$\begin{aligned} \Rightarrow \quad W(z) &= \frac{1}{2\pi j} \sum_{n=-\infty}^{\infty} x_1[n] \oint_{C_2} X_2(v) \left(\frac{z}{v}\right)^{-n} v^{-1} dv \\ &= \frac{1}{2\pi j} \oint_{C_2} \underbrace{\left[\sum_{n=-\infty}^{\infty} x_1[n] \left(\frac{z}{v}\right)^{-n}\right]}_{X_1\left(\frac{z}{v}\right)} X_2(v)\, v^{-1} dv. \end{aligned}$$

  If we choose C_2 as a closed contour in the overlap of ROCs of $X_1\left(\frac{z}{v}\right)$ and $X_2(v)$, then

  $$\boxed{W(z) = \frac{1}{2\pi j} \oint_{C_2} X_1\left(\frac{z}{v}\right) X_2(v)\, v^{-1} dv.}$$

  Alternatively, if we choose C_1 as a closed contour in the overlap of ROCs of $X_1(v)$ and $X_2\left(\frac{z}{v}\right)$, then

  $$\boxed{W(z) = \frac{1}{2\pi j} \oint_{C_1} X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1} dv.}$$

- Question: What is the ROC for $W(z)$?

  Answer: Let

  $$\begin{aligned} R_{x_1} &: r_{R_1} < |z| < r_{L_1} \\ R_{x_2} &: r_{R_2} < |z| < r_{L_2} \end{aligned}$$

  $\Rightarrow$

1. C_2 is determined by

$$\left.\begin{array}{l} r_{R_2} < |v| < r_{L_2} \\ r_{R_1} < \left|\frac{z}{v}\right| < r_{L_1} \end{array}\right\} \Rightarrow r_{R_1} r_{R_2} < |z| < r_{L_1} r_{L_2}.$$

2. C_1 is determined by

$$\left.\begin{array}{l} r_{R_1} < |v| < r_{L_1} \\ r_{R_2} < \left|\frac{z}{v}\right| < r_{L_2} \end{array}\right\} \Rightarrow r_{R_1} r_{R_2} < |z| < r_{L_1} r_{L_2}.$$

$\Rightarrow$ The ROC of $W(z)$ should contain $r_{R_1} r_{R_2} < |z| < r_{L_1} r_{L_2}$, denoted by R_w which may be different from $R_{x_1} \cap R_{x_2}$ (since $X_1(v) X_2\left(\frac{z}{v}\right) v^{-1}$ may have different "poles" for z).

- Note: As $v = e^{j\theta}$, $z = e^{j\omega}$, $C_1 = C_2 =$ unit circle, we have

$$W\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1\left(e^{j\theta}\right) X_2\left(e^{j(\omega-\theta)}\right) d\theta$$

which is exactly the windowing theorem in the Fourier transform.

- Ex: Given $x_1[n] = a^n u[n]$ and $x_2[n] = b^n u[n]$ where $|a| < 1$ and $|b| < 1$, find $w[n] = x_1[n] x_2[n]$.

Now,

$$\begin{aligned} X_1(z) &= \frac{1}{1 - az^{-1}}, \quad |z| > |a| \\ X_2(z) &= \frac{1}{1 - bz^{-1}}, \quad |z| > |b| \end{aligned}$$

$$\Rightarrow W(z) = \frac{1}{2\pi j} \oint_{C_2} \underbrace{\frac{-z/a}{v - z/a}}_{X_1\left(\frac{z}{v}\right)} \underbrace{\frac{1}{v - b}}_{X_2(v) v^{-1}} dv$$

where C_2 is chosen as

Fig. 44-F1

with $\frac{|z|}{|a|} > |b|$, i.e., $|z| > |ab|$. Now, from Cauchy Residue Theorem,

$$Res\left[\frac{-z/a}{v - z/a} \frac{1}{v - b} \text{ at } v = b\right] = \frac{-z/a}{b - z/a}$$

$$\begin{aligned} \Rightarrow \quad & W(z) = \frac{1}{1 - abz^{-1}}, \quad |z| > |ab| \\ \Rightarrow \quad & w[n] = (ab)^n u[n] \text{ as expected!} \end{aligned}$$

Note: In this example R_w is different from $R_{x_1} \cap R_{x_2}$.

3.6 Parseval's Relation (Supplemental)

- For two complex sequences $x_1[n]$ and $x_2[n]$, Parseval's Relation states that

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$$

where $\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n]$ is assumed to exist and C is in the overlap of ROCs of $X_1(v)$ and $X_2^*\left(\frac{1}{v^*}\right)$.

Pf: Let $y[n] \triangleq x_1[n]\, x_2^*[n]$. From the Complex Convolution Theorem,

$$Y(z) = \frac{1}{2\pi j} \oint_C X_1(v)\, U\left(\frac{z}{v}\right) v^{-1} dv$$

where $U(z) = \mathcal{Z}\{x_2^*[n]\}$ and C is in the overlap of ROC's of $X_1(v)$ and $U\left(\frac{z}{v}\right)$. Now, from conjugation of a complex sequence, ($x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$, ROC $= R_x$),

$$\begin{aligned} & U(z) = X_2^*(z^*), \text{ ROC} = R_{x_2} \\ \Longrightarrow\ & Y(z) = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{z^*}{v^*}\right) v^{-1} dv. \end{aligned}$$

Furthermore,

$$Y(z)\big|_{z=1} = \sum_{n=-\infty}^{\infty} y[n]$$

yields the relation

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv. \quad \text{QED}$$

- Notes:

1. When $v = e^{j\omega}$, we have the Parseval's Relation in terms of the Fourier Transform,

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1\left(e^{j\omega}\right) X_2^*\left(e^{j\omega}\right) d\omega$$

which is difficult, in general, to evaluate.

2. Parseval's relation in terms of z-transform is easy to evaluate by use of the Cauchy Residue Theorem.

3. When $x_1[n] = x_2[n] = x[n]$ is real,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \text{energy of } x[n] = \frac{1}{2\pi j}\oint_C X(v)\, X^*\left(\frac{1}{v^*}\right) v^{-1} dv$$

where $C \in ROC_x \cap (ROC_x)^{-1}$, with $ROC_x = \{z | r_R < |z| < r_L\}$ and $(ROC_x)^{-1} = \{z | r_L^{-1} < |z| < r_R^{-1}\}$.

- Ex: Find the energy of real sequence $x[n]$ with

$$X(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})}, \quad \begin{array}{l} |a| < 1, |b| < 1, a \neq b \\ |z| > \max\{|a|, |b|\} \end{array}.$$

Now,

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi j}\oint_C \frac{1}{(1-av^{-1})(1-bv^{-1})} \frac{1}{(1-av)(1-bv)} v^{-1} dv$$

with C being unit circle. This can be simplified to

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi j}\oint_C \frac{v}{(v-a)(v-b)(1-av)(1-bv)} dv.$$

Fig. 45-B1

1.

$$\begin{aligned} & Res\left[\frac{v}{(v-a)(v-b)(1-av)(1-bv)} \text{ at } v = a\right] \\ = & \frac{a}{(a-b)(1-a^2)(1-ab)} \end{aligned}$$

2.

$$\begin{aligned} & Res\left[\frac{v}{(v-a)(v-b)(1-av)(1-bv)} \text{ at } v = b\right] \\ = & \frac{b}{(b-a)(1-ab)(1-b^2)} \end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x^2[n] &= \frac{1}{(a-b)(1-ab)}\left[\frac{a}{1-a^2}-\frac{b}{1-b^2}\right] \\
&= \frac{(a-b)+ab(a-b)}{(a-b)(1-ab)(1-a^2)(1-b^2)} \\
&= \frac{1+ab}{(1-ab)(1-a^2)(1-b^2)}.
\end{aligned}$$

3.7 z-Transforms and LTI Systems

- For an LTI system with input $x[n]$, output $y[n]$, and impulse response $h[n]$,

$$\begin{aligned}
& y[n] = x[n] * h[n] \\
\Longleftrightarrow\ & Y(z) = X(z)\,H(z)\,,\ \text{ROC} = R_y = R_x \cap R_h
\end{aligned}$$

where $X(z) = \mathcal{Z}\{x[n]\}$ for $z \in R_x$, $Y(z) = \mathcal{Z}\{y[n]\}$ for $z \in R_y$, and $H(z) = \mathcal{Z}\{h[n]\}$ for $z \in R_h$, by convolution property. Here, $H(z)$ is called the *system function* of the LTI system with impulse response $h[n]$.

- Ex: Let $h[n] = a^n u[n]$ with $|a| < 1$ and $x[n] = Au[n]$. Now, the system function is

$$H(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

and the z-transform of the input is

$$X(z) = \frac{A}{1-z^{-1}}, \quad |z| > 1.$$

Thus, the z-transform of the output is

$$\begin{aligned}
Y(z) &= \frac{A}{(1-z^{-1})(1-az^{-1})}, \quad |z| > 1 \\
&= \frac{Az^2}{(z-1)(z-a)} \\
&= \frac{A}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right).
\end{aligned}$$

Fig. 3.12

Taking the inverse z-transform of $Y(z)$ yields

$$y[n] = \frac{A}{1-a}(1-a^{n+1})u[n].$$

Alternatively, we can evaluate $y[n]$ by convolution sum as

$$\begin{aligned} y[n] &= Au[n] * a^n u[n] \\ &= \sum_{k=-\infty}^{\infty} Au[k]a^{n-k}u[n-k] \\ &= A\sum_{k=0}^{n} a^{n-k}u[n] \\ &= \frac{A}{1-a}(1-a^{n+1})u[n]. \end{aligned}$$

Note that the z-transform is useful for describing the input-output relation of an LTI system.

- Consider the causal LTI system described by a linear constant-coefficient difference equation

$$y[n] = -\sum_{k=1}^{N}(\frac{a_k}{a_0})y[n-k] + \sum_{k=0}^{M}(\frac{b_k}{a_0})x[n-k]$$

with $a_0 \neq 0$. Let input $x[n]$ and output $y[n]$ be both causal. Taking z-transform, the equation becomes after applying linearity and time-shift properties

$$\begin{aligned} Y(z) &= -\sum_{k=1}^{N}(\frac{a_k}{a_0})z^{-k}Y(z) + \sum_{k=0}^{M}(\frac{b_k}{a_0})z^{-k}X(z) \\ &\Longrightarrow Y(z) = H(z)X(z) \end{aligned}$$

with the system function of the causal LTI system given by

$$H(z) = \frac{\sum_{k=0}^{M}(\frac{b_k}{a_0})z^{-k}}{\sum_{k=0}^{N}(\frac{a_k}{a_0})z^{-k}}$$

for $|z| > r_R$ (since $h[n]$ is causal).

Notes:

1. r_R is the magnitude of the pole of $H(z)$ farthest from the origin.
2. If $r_R < 1$, all poles of $H(z)$ locate inside the unit circle. Thus, the system is stable and the frequency response $H(e^{j\omega})$ of the LTI system exists.

Recall that an LTI system is stable iff its impulse response is absolutely summable, i.e., $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$. Now, if the ROC of $H(z)$ contains the unit circle, then $h[n]$ is absolutely summable and this guarantees the system stability.

- Ex: First-Order Causal LTI System

  Consider $y[n] = ay[n-1] + x[n]$. By inspection, the system function is given by

$$
\begin{aligned}
& H(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a| \\
\Leftrightarrow \quad & h[n] = a^n u[n].
\end{aligned}
$$

3.8 The Unilateral z-Transform

- Defn: The unilateral z-transform of $x[n]$ is defined and denoted by

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n] \, z^{-n}.$$

- Note:

  1. Conventional z-transform is bilateral!
  2. If $x[n] = 0$ for $n < 0$, i.e., the sequence is causal, unilateral and bilateral z-transforms will be identical; and share the *same* properties.
  3. $\mathcal{X}(z)$ has a ROC with the form $|z| > r_R$. Thus, if $\mathcal{X}(z)$ is a rational function, r_R is defined by the pole with the largest magnitude.

- Ex: $x[n] = \delta[n]$

  $\Rightarrow$ For unilateral z-transform,

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} \delta[n] \, z^{-n} = 1 \qquad \forall z$$

  $\Rightarrow$ For bilateral z-transform,

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] \, z^{-n} = 1 \qquad \forall z.$$

- Ex: $x[n] = \delta[n+1]$

  $\Rightarrow$ For unilateral z-transform,

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} \delta[n+1] z^{-n} = 0 \qquad \forall z$$

  $\Rightarrow$ For bilateral z-transform,

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z \qquad |z| < \infty$$

- Note: For $x[n] \neq 0$, $n < 0$, i.e., any noncausal sequence, $X(z)$ and $\mathcal{X}(z)$ are different.

- Further Notes:

  1. The time-shift properties for unilateral and bilateral z-transforms are different.

     (a) For bilateral z-transform,

$$\begin{aligned} x[n] &\overset{\mathcal{Z}}{\longleftrightarrow} X(z) \\ x[n-m] &\overset{\mathcal{Z}}{\longleftrightarrow} z^{-m} X(z) \end{aligned}$$

     (b) For unilateral z-transform,
     let $y[n] = x[n-m]$, $m > 0$.

$$\begin{aligned} \mathcal{Y}(z) &= \sum_{n=0}^{\infty} y[n] z^{-n} \\ &= \sum_{n=0}^{\infty} x[n-m] z^{-n} \\ &= \sum_{\substack{n'=-m \\ (n'=n-m)}}^{\infty} x\left[n'\right] z^{-n'-m} \\ &= x[-m] + x[-m+1] z^{-1} + x[-m+2] z^{-2} + \\ &\quad \ldots + x[-1] z^{1-m} + \left(\sum_{n'=0}^{\infty} x\left[n'\right] z^{-n'} \right) z^{-m} \\ &= \underbrace{\sum_{k=1}^{m} x[-m+(k-1)] z^{-k+1}}_{\text{different from that of bilateral } z\text{-transform}} + z^{-m} \mathcal{X}(z) \end{aligned}$$

2. The linearities for both are, however, the same.

- Ex: Consider a system with (1) I/O relationship

$$y[n] - \frac{1}{2}y[n-1] = x[n] \qquad \circledast$$

(2) $x[n] = 1$, $n \geq 0$, and (3) $y[-1] = 1$. We want to know the behavior of $y[n]$ for $n \geq 0$.

Now, applying unilateral transform to both sides of $\circledast$ yields

$$\mathcal{Y}(z) - \frac{1}{2}\left(\underbrace{y[-1]}_{\text{initial output}} + z^{-1}\mathcal{Y}(z)\right) = \mathcal{X}(z).$$

Since

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} \qquad |z| > 1$$

we have

$$\begin{aligned}
\mathcal{Y}(z) &= \frac{1}{1-\frac{1}{2}z^{-1}}\left(\frac{1}{2}y[-1] + \frac{1}{1-z^{-1}}\right) \qquad |z| > 1 \\
&= \frac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}} + \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \\
&= \frac{1}{2}\frac{1}{1-\frac{1}{2}z^{-1}} \underbrace{- \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-z^{-1}}}_{\text{partial fraction expansion}} \\
&= \frac{2}{1-z^{-1}} - \frac{1}{2}\frac{1}{1-\frac{1}{2}z^{-1}}.
\end{aligned}$$

Now,

$$\frac{2}{1-z^{-1}} \underset{\text{inverse unilateral } z\text{-transform}}{\longrightarrow} 2u[n]$$

$$\frac{1}{2}\frac{1}{1-\frac{1}{2}z^{-1}} \underset{\text{inverse unilateral } z\text{-transform}}{\longrightarrow} \frac{1}{2}\left(\frac{1}{2}\right)^{n} u[n]$$

$$\Rightarrow y[n] = \left(2 - \left(\frac{1}{2}\right)^{n+1}\right) u[n].$$

- Note: Unilateral z-transform is good for analyzing systems described by LCCD equations with *nonzero initial* conditions (i.e., noncausal output sequence).