Algorithm Design Assignment 8

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Problem 8.6. Given an instance of monotone SAT problem (i.e. all the terms are positive), prove that to find whether there is a case with at most k true variables is NPC.

Solution. We use vertex-cover problem, i.e., can we find a set with at most k vertices, so that each edge must have (at least) one endpoint in this set? This is NPC as we all know.

For any vertex-cover problem with graph $G = (V, E)$, we can construct a monotone SAT problem in polynomial time, i.e., the vertex-cover problem can be reduced to monotone SAT problem. For each $v_i \in V$, we have a variable x_i in monotone SAT problem, and for each $(v_i, v_j) \in E$, we have a clause $c_k = (v_i \vee v_j)$. At last, we have a monotone SAT problem with $c_1 \wedge ... \wedge c_n$. To construct this is obviously in polynomial time, in fact $O(V + E)$.

Then we prove that the monotone SAT problem in this case is "yes" if and only if the original vertex cover problem is "yes".

1. If we can find a vertex-cover set $S \subseteq V$ with $|S| \leq k$, then for any $v_i \in S$, we let x_i be true and other be false. Since we know for each edge, there must be one endpoint in S, we can know each clause is true because there must be one variable which is true. Therefore the monotone SAT problem is true.

2. If we can find a set of variables S' with $|S'| \leq k$ so that the monotone SAT problem is true, then we construct a vertex set S with all v_i if $x_i \in S'$. Since we know each clause must be true, we can know that each edge must have one endpoint in S. Therefore the vertex cover problem is "yes".

Notice monotone SAT problem is in NP, because we can examine a result in polynomial time. In this way, we prove that monotone SAT problem is NPC.

Problem 8.24. Show this problem is NPC: Given constant a, b, a bipartite graph $G = (V, E)$ and $V = X + Y$, where X, Y denote the left and right part, can we find a $(a, b)-$skeleton in G, i.e. an edge set E' so that at most a vertices in X and at least b vertices in Y are incident to E'?

Solution. We call this problem $(a, b)-$skeleton problem and use set cover problem to solve it: Given m sets $S_1, ..., S_m$ and union U, where $|U| = n$, can we find at most k sets to form the union? This is NPC as we all know.

Given any case of set cover problem, we construct a $(a, b)-$skeleton problem. Let each node in set X denote a set S_i and each node in set Y denote an element in U. So we know $|X| = m, |Y| = n$. We construct a bipartite graph $G = (V, E)$. Here, $V = X + Y$ with X on the left and Y on the right. There is an edge between $s_i \in X$ and $u_i \in Y$ if and only if $u_i \in S_i$, where u_i is an element in union U. To construct this is obviously in polynomial time, in fact $O(mn)$.

We will prove that, in this case, the vertex cover problem is "yes" if and only if the $(k, n)-$skeleton problem is "yes". Here k is the bound in vertex cover problem and n is the size of U.

1. If the vertex cover problem is "yes", then we can find some sets (no more than k) forming the union. So the corresponding nodes (We call the set of these nodes X') in X (also no more than k) will have access to all the nodes in Y. So we can find a $(k, n)-$skeleton which includes just all the edges connecting X' and Y.

2. If we can find a $(k, n)-$skeleton, then assuming the set of the nodes in X incident to the skeleton is X', then we only need to use all the sets S_i corresponding to nodes in X' to form the union U, and thus solve the set cover problem. And the set cover problem uses no more than k sets.

Notice that $(a, b)-$skeleton problem is in NP since we can examine a given result in polynomial time. Therefore we prove that $(a, b)-$skeleton problem is NPC.