MATH 333 Discrete Mathematics
Chapter 1 Mathematics Basis

Jianan Lin

linj21@rpi.edu

May 24, 2022

Number

1. What's the prime factorization of 252?
Answer: $252 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$

2. What's the calculation rules for exponents?
Answer: $a^{x_1+x_2} = a^{x_1} \cdot a^{x_2}$ and $a^{x_1-x_2} = a^{x_1}/a^{x_2}$

3. What's the calculation rules for logarithm?
Answer: $\log_a (x_1 \cdot x_2) = \log_a (x_1) + \log_a (x_2)$

4. What's the summation of arithmetic progression $a_i = a + (i-1)d$?

$$\sum_{i=1}^{n} (a + (i-1)d) = \frac{n(2a + (n-1)d)}{2}$$

5. What's the summation of proportional sequence $a_i = a \cdot q^{i-1}$?

$$\sum_{i=1}^{n} \left(a \cdot q^{i-1}\right) = \frac{a(1-q^n)}{1-q}$$

Notice that $(1 + x + x^2 + ... + x^n)(1 - x) = 1 - x^{n+1}$

6. What is the factorial of a number?

$$n! = \prod_{i=1}^{n} i = n \cdot (n-1) \cdot ...... \cdot 1$$

Notice that $0! = 1$

Number

7. What's the greatest common divisor (gcd) of 72 and 120?
Answer: $\gcd(72, 120) = 24$
Notice that if $a > b$ and $a\%b \neq 0$, then we have
$\gcd(a, b) = \gcd(a - b, b) = \gcd(a\%b, b)$

8. What's the least least common multiple (lcm) of 72 and 120?
Answer: $\text{lcm}(72, 120) = 360$
Notice we have $\gcd(a, b) \cdot \text{lcm}(a, b) = |a \cdot b|$

9. What's the approximate sum of harmonic series?

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln(n)$$

Function

1. Exponential function

$$f(x) = a^x$$

2. Logarithmic function

$$f(x) = \log_a(x)$$

3. Polynomial function

$$f(x) = a_0 + a_1 x + ... + a_n x^n$$

4. Trigonometric functions

$$f(x) = \sin(x), \cos(x), \tan(x), \cot(x)$$

5. Inverse trigonometric functions

$$f(x) = \sin^{-1}(x) = \arcsin(x), \arccos(x), \arctan(x), \operatorname{arccot}(x)$$

Derivative

1. Exponential function

$$f'(x) = a^x \ln(a)$$

2. Logarithmic function

$$f'(x) = \frac{1}{x \ln(a)}$$

3. Polynomial function

$$f'(x) = a_1 + 2 \cdot a_2 x + ... + n \cdot a_n x^{n-1}$$

4. Trigonometric functions

$$f'(x) = \cos(x), -\sin(x), 1 + \tan^2(x)$$

5. Inverse trigonometric functions

$$f'(x) = \frac{1}{\sqrt{1-x^2}}, -\frac{1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}$$

Derivative

6. Plus and Minus

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

7. Multiplication

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

8. Division

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

9. Chain Rule (Let $u = g(x)$ and $y = f(u)$)

$$(f \circ g(x))' = f'(u) \cdot g'(x)$$

Integral

1. Indefinite Integral

$$\int f(x)dx = F(x) + C$$

2. Definite Integral

$$\int_a^b f(x)dx = F(b) - F(a)$$

Function Approximation

1. Exponential function $e^x \geq 1 + x$

$$e^x \approx 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

2. Logarithmic function $\ln(x+1) \leq x$

$$\ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

3. Trigonometric function $\sin(x) \leq x$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Determinant

1. 2×2 Determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

2. 3×3 Determinant

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

3. The same way for $n \times n$ determinant

Matrix

1. Plus and Minus

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{bmatrix}$$

2. Multiplication (Suppose A is $m \times p$, B is $p \times n$, and $C = AB$)

$$c_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj}$$

3. Inverse Matrix

$$AA^{-1} = I$$

Set

1. Element and Set

$$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}$$

2. Union

$$U = A + B = A \cup B = \{1, 2, 3, 4, 5\}$$

3. Intersection

$$A \cap B = \{3\}$$

4. Subtraction

$$A - B = A \backslash B = \{1, 2\}$$

5. Complement

$$\overline{A} = U - A = \{4, 5\}$$

Set

6. Belong

$$1 \in A, \quad 5 \notin A$$

7. Subset

$$\{1,2\} \subset A, \quad A \not\subset A$$

8. Subset Equal

$$\{1,2\} \subseteq A, \quad A \subseteq A$$

9. Cardinality

$$|A| = 3, \quad |\emptyset| = 0$$

10. Equation

$$|A \cup B| + |A \cap B| = |A| + |B|$$

Set

Complex Number $\mathbb{C} = \{1 + i, 1.5, \sqrt{2}, 0, ...\}$

Real Number $\mathbb{R} = \{1.56, \pi, 12, ...\}$

Rational Number $\mathbb{Q} = \{0.1111...., -1, ...\}$

Natural Number $\mathbb{N} = \{1, 2, 3, ...\}$

Note: In some textbooks, $0 \in \mathbb{N}$ and we use $\mathbb{N}_+$ or $\mathbb{N}^*$ to denote positive integers. But in others, $0 \notin \mathbb{N}$. In depends on the definition.

The relationship

$$\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Algebra

1. Square Difference Formula

$$a^2 - b^2 = (a+b)(a-b)$$

2. Perfect Square Formula

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

3. General Formula

$$(a+p)(a+q) = a^2 + (p+q) \cdot a + pq$$

4. Cube Formula

$$(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^2 \pm b^3$$

Complex Number

We let $z = x + y \cdot i$, where x is called the real part and y is called the imaginary part (Notice that $i = \sqrt{-1}$). Define $z_1 = x_1 + y_1 \cdot i$ and $z_2 = x_2 + y_2 \cdot i$

1. Plus and Minus

$$z_1 \pm z_2 = (x_1 \pm x_2) + (y_1 \pm y_2) \cdot i$$

2. Multiplication

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) \cdot i$$

3. Division

$$\frac{z_1}{z_2} = \frac{(x_1 + y_1 \cdot i)(x_2 - y_2 \cdot i)}{(x_2 + y_2 \cdot i)(x_2 - y_2 \cdot i)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \cdot i$$

Complex Number

Euler's Formula

$$e^{ia} = \cos(a) + \sin(a) \cdot i$$

In this way, we can write a complex number in the form of

$$z = Ae^{ia} = A\cos(a) + A\sin(a) \cdot i$$

In a 2-dimensional flat, let x coordinate be the real part and y coordinate be the imaginary part. We draw a vector from $(0, 0)$ to $(A\cos(a), A\sin(a))$. Then this denote the complex number z.

Here, A is the length of the vector, and a is the angle between this vector and x axis.

Exercise 1

Problem: Prove $\sqrt{2}$ is irrational.

Hint: if x is a rational number, then there are two co-prime numbers p, q so that $x = p/q$.

Exercise 1

Problem: Prove $\sqrt{2}$ is irrational.

Proof.

Suppose $\sqrt{2}$ is rational, so there exists two coprime positive integers p, q (i.e. $\gcd(p, q) = 1$) so that $p/q = \sqrt{2}$. So we have

$$\frac{p^2}{q^2} = 2 \quad \Longrightarrow \quad p^2 = 2q^2$$

However, since $\gcd(p, q) = 1$, we also have $\gcd(p^2, q^2) = 1$ (this is obvious). Therefore we have a contradiction. If we want both equations hold, we have to let $q = 1$, but p will never be an integer.

In this way prove $\sqrt{2}$ is irrational. In fact, in the same way, we can prove that for any prime number x, $\sqrt{x}$ is irrational. $\square$

Exercise 2

Problem: There are n trucks. Each truck can carry at most c fuels. We allow a truck to give its fuel to another. For simplicity, we suppose 1 unit of fuel can let the truck move 1km. What's the largest distance that a truck can move? Here, we only consider the truck which moves the farthest.

Hint: consider $n = 2$, then $n = 3$, last for general n.

Exercise 2

Problem: There are n trucks. Each truck can carry at most c fuels. We allow a truck to give its fuel to another. For simplicity, we suppose 1 unit of fuel can let the truck move 1km. What's the largest distance that a truck can move? Here, we only consider the truck which moves the farthest.

Solution.

Firstly we consider there are two trucks. Obviously, the best strategy is that, we let the two trucks move some distance x, and let one truck give all its fuel to the other. Then the second truck goes on until the fuel is used up.

If $x \leq 0.5c$, then the second truck will receive $\min(c-x, x) = x$ units of fuels because it uses x. So this truck can move $x + c \leq 1.5c$.

Exercise 2

If $x \geq 0.5c$, then the second truck will receive $\min(c-x, x) = c-x$ units of fuels because the first truck has only $c-x$ left. So this truck can move $2c - x \leq 1.5c$.

Therefore if $n = 2$, we have $\max(d) = 1.5c$.

Next we consider general n. In the same way, we know, it is the best strategy to let one truck give all its remained fuel to others after moving some distance. We also know that this distance x should satisfy

$$x + (n-1) \cdot x = c \quad \Longrightarrow \quad x = \frac{c}{n}$$

This is because, when $x = c/n$, the truck just gives all its fuel to others. It uses c/n and gives $c(n-1)/n$.

Exercise 2

Then we can reconsider this question and write it as follows: we have $n-1$ trucks and they start at coordinate c/n. What's the largest distance?

So this time after moving x distance, we should also let a truck give all its fuel to others. In the same way, we have

$$x + (n-2) \cdot x = c \quad \Longrightarrow \quad x = \frac{c}{n-1}$$

Therefore we iterate this until $n = 1$, we have

$$\max(d) = \frac{c}{n} + \frac{c}{n-1} + \ldots + c = c\sum_{i=1}^{n} \frac{1}{i} = cH_n$$

Here, H_n is called Harmonic series. The difference between H_n and $\log(n)$ is a constant ≈ 0.577.

Exercise 3

Given $\sin(z) = 2$, calculate complex number z.

Hint: Use Euler's formula

Exercise 3

Given $\sin(z) = 2$, calculate complex number z.

Solution.

According to Euler's formula, we have

$$
\begin{aligned}
e^{iz} &= \cos(z) + \sin(z) \cdot i \\
e^{-iz} &= \cos(z) - \sin(z) \cdot i
\end{aligned}
$$

Therefore let equation 1 minus equation 2, and we have

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = 2$$

So

$$e^{2iz} - 4ie^{iz} - 1 = 0$$

Exercise 3

We can regard this as an quadratic equation:

$$\left(e^{iz}\right)^2 - 4i \cdot e^{iz} - 1 = 0$$

Solving this, we have

$$e^{iz} = \frac{4i \pm \sqrt{-16+4}}{2} = 2i \pm \sqrt{-3} = \left(2 \pm \sqrt{3}\right) \cdot i$$

So we have

$$z = \frac{\ln\left(\left(2 \pm \sqrt{3}\right) \cdot i\right)}{i} = \frac{\ln(2+\sqrt{3}) + \ln(i)}{i}$$

Last we calculate $\ln(i)$. Notice that according to Euler's formula, we have

$$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \cdot i = i$$

Exercise 3

In this way we know that

$$\ln(i) = \ln\left(e^{i\frac{\pi}{2}}\right) = \frac{\pi}{2} \cdot i$$

Therefore we have the solution

$$z = \frac{\pi}{2} - \ln\left(2 \pm \sqrt{3}\right) \cdot i = \frac{\pi}{2} \pm \ln\left(2 + \sqrt{3}\right) \cdot i$$

Using calculator, we have

$$z \approx 1.5708 \pm 1.31696i$$

Exercise 4

Problem: There are 99 machines. At least 50 of them are good, and others are bad. You can use a machine called A to test another. If A is good, then it will always tell you the correct answer. If A is bad, then it will always tell you the false answer. How many times do we need to guarantee that we can clarify a good machine? Consider the best and worst luck respectively.

Hint: If machine A tests B, when will they belong to the same type?

Exercise 4

Problem: There are 99 machines. At least 50 of them are good, and others are bad. You can use a machine called A to test another. If A is good, then it will always tell you the correct answer. If A is bad, then it will always tell you the false answer. How many times do we need to guarantee that we can clarify a good machine? Consider the best and worst luck respectively.

Solution.

For any two machines, let A test B. If A says B is good, then either they are both good, or they are both bad. So they are in the same kinds. In the same way, if A says B is bad, then they in different kinds.

We label the machines from index 1 to 99. We start from 1 and let it test 2. We provide two groups X and Y, and put 1 into X. If 1 reports 2 is good, then put 2 into X, otherwise put 2 into Y.

Exercise 4

In the same way, we always let i test $i+1$ and decide which group we put $i+1$ into. If machines from 1 to 50 are all in X, then we ensure that they are good. This is the best luck.

For the worst luck, after we use 97 to test 98, there are 49 in X and 49 in Y. Therefore we have to repeat it for the 98-th times. Therefore the answer is: the best luck is 49 and the worst luck is 98.

If we modify 99 to some odd number $2k+1$, then the answer will be: for the best luck, at least k times and for the worst luck, at least $2k$ times.