MATH 333 Discrete Mathematics
Chapter 3 Logic and Propositions

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Definition

Variables: x, y, z. Their values are true of false.

Today is a sunny day.

I registered for MATH 333 course.

4 is an even number.

I can fly.

Other example that is either true or false.

Definition

Connector

NOT: $\neg p$. If p is true, then return false. If p is false, then return true.

AND: $p \wedge q$. Return true if and only if p is true and q is true.

OR: $p \vee q$. Return true if either p or q is true.

IF...THEN...: $p \to q$. Return false if and only if p is true and q is false.

In fact, $p \to q = \neg p \vee q$.

Definition

All the connectors can be denoted by only $\neg$ and one of $\wedge, \vee$.

How to use two connectors to denote others?

Use $\neg, \wedge$ to denote $\vee$:

$$p \vee q = \neg\left(\neg p \wedge \neg q\right)$$

Use $\neg, \vee$ to denote $\wedge$:

$$p \wedge q = \neg\left(\neg p \vee \neg q\right)$$

All other connectors can be denoted by these three basic connectors.

Definition

Commutative

$$p \wedge q = q \wedge p$$
$$p \vee q = q \vee p$$

Associative

$$(p \wedge q) \wedge r = p \wedge (q \wedge r)$$
$$(p \vee q) \vee r = p \vee (q \vee r)$$

Distributive

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

Definition

Negations

$$\neg(\neg p) = p$$
$$\neg(p \wedge q) = \neg p \vee \neg q$$
$$\neg(p \vee q) = \neg p \wedge \neg p$$

Implication

$$p \rightarrow q = \neg q \rightarrow \neg p$$
$$p \rightarrow q = \neg p \vee q$$

Exclusive Or / XOR, return true if the values of p and q are different.

$$p \oplus q = (p \wedge \neg q) \vee (\neg p \wedge q)$$

Truth Table

Truth table is a convenient way to calculate a sentence.

Index	p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \oplus q$
1	F	F	T	T	F	F	T	F
2	F	T	T	F	F	T	T	T
3	T	F	F	T	F	T	F	T
4	T	T	F	F	T	T	T	F

Truth Table

Exercise: Given the truth table below, write the sentence with $\neg, \wedge, \vee$.

p	q	value
T	T	F
T	F	T
F	T	F
F	F	F

Truth Table

Exercise: Given the truth table below, write the sentence with $\neg, \wedge, \vee$.

p	q	value
T	T	F
T	F	T
F	T	F
F	F	F

Answer: $p \wedge \neg q$. Notice, first calculate $\neg$, then $\wedge$, then $\vee$, last others.

Truth Table

Exercise: Given the truth table below, write the sentence with $\neg, \wedge, \vee$.

p	q	value
T	T	F
T	F	T
F	T	F
F	F	T

Truth Table

Exercise: Given the truth table below, write the sentence with $\neg, \wedge, \vee$.

p	q	value
T	T	F
T	F	T
F	T	F
F	F	T

Answer: $\neg q$

Truth Table

Exercise: Given the truth table below, write the sentence with $\neg, \wedge, \vee$.

p	q	r	value
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

Truth Table

Exercise: Given the truth table below, write the sentence with $\neg, \wedge, \vee$.

p	q	r	value
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

Answer: $\neg p \wedge q \wedge \neg r$

Truth Table

Exercise: Given the truth table below, write the sentence with $\neg, \wedge, \vee$.

p	q	r	value
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Truth Table

Exercise: Given the truth table below, write the sentence with $\neg, \wedge, \vee$.

p	q	r	value
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Answer: $(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r)$

SAT

Conjunctive Normal Form (CNF)

Sentence $S = A_1 \wedge A_2 \wedge ...$

Clause $A_i = a_1 \vee a_2 \vee a_3 ...$, where a_i can be either positive or negative, i.e. p or $\neg p$.

Every sentence can be written in CNF.

Example: $(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r)$

SAT

Exercise: write the following sentence in CNF.

$$\neg((p \vee q) \to r)) \to (p \vee r)$$

SAT

Exercise: write the following sentence in CNF.

$$
\begin{aligned}
&\neg((p \vee q) \to r)) \to (p \vee r) \\
&= \neg(\neg(p \vee q) \vee r) \to (p \vee r) \\
&= (\neg(p \vee q) \vee r) \vee (p \vee r) \\
&= (\neg(p \vee q) \vee r) \vee p \vee r \\
&= \neg(p \vee q) \vee r \vee p \vee r \\
&= \neg(p \vee q) \vee p \vee r \\
&= (\neg p \wedge \neg q) \vee (p \vee r) \\
&= (\neg p \vee p \vee r) \wedge (\neg q \vee p \vee r) \\
&= \text{True} \wedge (p \vee \neg q \vee r) \\
&= (p \vee \neg q \vee r)
\end{aligned}
$$

SAT

Disjunctive Normal Form (DNF)

Sentence $S = A_1 \vee A_2 \vee ...$

Clause $A_i = a_1 \wedge a_2 \wedge a_3 ...$, where a_i can be either positive or negative, i.e. p or $\neg p$.

Every sentence can be written in DNF.

Example: $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$

SAT

Exercise: write the following sentence in DNF.

$$\neg((\neg p \wedge \neg q) \to r)) \to (p \wedge r)$$

SAT

Exercise: write the following sentence in DNF.

$$
\begin{aligned}
&\neg((\neg p \wedge \neg q) \rightarrow r)) \rightarrow (p \wedge r) \\
&= \neg(\neg(\neg p \wedge \neg q) \vee r) \rightarrow (p \wedge r) \\
&= (\neg(\neg p \wedge \neg q) \vee r) \vee (p \wedge r) \\
&= \neg(\neg p \wedge \neg q) \vee r \vee (p \wedge r) \\
&= p \vee q \vee (p \wedge r) \\
&= p \vee q
\end{aligned}
$$

SAT

SAT problem: Given a CNF sentence, choose value for variables to make sentence true.

Example: $(p \vee q) \wedge (p \vee \neg q)$

We can let $p = q = T$ or $p = T, q = F$.

Most common case is 3-SAT problem. Each clause has no more than 3 variables.

Example: $(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r)$

We can let $p = q = r = T$.

3-SAT problem is the first NPC problem.

SAT

Exercise: Given the 3-SAT problem, find values for variables if there is a solution. Otherwise answer no.

$$(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r)$$

SAT

Exercise: Given the 3-SAT problem, find values for variables if there is a solution. Otherwise answer no.

$$(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r)$$

Answer

1. $p = r = T, q = F$

2. $p = q = F, r = T$

3. $p = r = F, q = T$

SAT

About 3-SAT: It is a NP complete problem. Therefore if someone finds a polynomial algorithm with parameter n (variables) and m (clauses), then $P = NP$.

There is an approximation algorithm in polynomial time. It can satisfies at least 7/8 of the clauses.

Predicate Logic

Let c denote a car, and $P(c)$ denote "car c has 4 wheels".

Any car has 4 wheels.

$$\forall c : P(c)$$

There exists some car that has more than or less than 4 wheels.

$$\exists c : \neg P(c)$$

These two claims are opposite. Also, the following two are opposite:

$$\forall c : \neg P(c) \qquad \exists c : P(c)$$

Proposition

There are four kinds of proposition.

Statement If p , then q .

Converse If q , then p .

Inverse If not p , then not q .

Contrapositive If not q , then not p .

The following equations hold.

$$p \to q \Longleftrightarrow \neg q \to \neg p$$
$$p \to \neg q \Longleftrightarrow q \to \neg p$$

Exercise 1

Write the sentences in "if ... then ..." form.

1. All roads lead to Rome.
2. You will pass this exam only if you study hard.
3. A natural number cannot be an odd prime unless it is larger than 2.
4. Attending class is necessary for being familiar with the teacher.
5. I can not fly unless I am on the plane.

Exercise 1

Write the sentences in "if ... then ..." form.

1. All roads lead to Rome.

Answer: If x is a road, then x leads to Rome.

2. You will pass this exam only if you study hard.

Answer: If you pass the exam, then you study hard.

There will not be such natural language problem in the exam.

Exercise 1

3. A natural number cannot be an odd prime unless it is larger than 2.

Answer: If a natural number is an odd prime, then it is larger than 2.

4. Attending class is necessary for being familiar with the teacher.

Answer: If I'm familiar with the teacher, I attend the class.

5. I can not fly unless I am on the plane.

Answer: If I can fly, then I am on the plane.

Exercise 2

Let p denote "I am talented" and q denote "I can fly". Then write the following sentences.

1. I am talented and I can fly.
2. Neither am I talented nor can I fly.
3. I am not talented, but I can fly.
4. I can fly or I am talented, but not both.
5. I am either not talented, or I can not fly.

Exercise 2

Let p denote "I am talented" and q denote "I can fly". Then write the following sentences.

1. I am talented and I can fly.

Answer: $p \wedge q$

2. Neither am I talented nor can I fly.

Answer: $\neg(p \vee q) = \neg p \wedge \neg q$

Exercise 2

3. I am not talented, but I can fly.

Answer: $\neg p \wedge q$

4. I can fly or I am talented, but not both.

Answer: $(p \wedge \neg q) \vee (\neg p \wedge q)$

5. I am either not talented, or I can not fly.
Answer: $\neg(p \wedge q) = \neg p \vee \neg q$

Exercise 3

Write the following sentences in predicates.

1. All balls are red.

2. Some ball is not red.

3. No ball is red.

Exercise 3

Let $P(x)$ denote "x is a ball" and $Q(x)$ denote "x is red".

1. All balls are red.

Answer: $\forall x : P(x) \to Q(x)$

2. Some ball is not red.

Answer: $\exists x : P(x) \wedge \neg Q(x)$

3. No ball is red.

Answer: $\forall x : P(x) \to \neg Q(x)$