MATH 333 Discrete Mathematics
Chapter 9 Probability Theory

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Probability

Event: The coin is head up.

Probability Space: All the events. For example, the coin is head / tail up.

The probability of an event e

$$Pr(e) = P(e) \in [0, 1]$$

Notice, $P = 0$ does not mean it never happens. For example, select arbitrary point in interval $[0, 1]$, then the probability to choose 0 is 0. But it may happen!

Probability

If we want to describe event A or B happen, then we write $A \cup B$. If we want to describe event A and B happen, then we write $A \cap B$.

If A and B are independent, i.e., no relation each other, then

$$P(A \cap B) = P(A) \cdot P(B)$$

and

$$\begin{aligned} P(A \cup B) &= 1 - P(\overline{A \cup B}) \\ &= 1 - P(\overline{A} \cap \overline{B}) \\ &= 1 - P(\overline{A}) \cdot P(\overline{B}) \\ &= 1 - (1 - P(A)) \cdot (1 - P(B)) \\ &= P(A) + P(B) - P(A) \cdot P(B) \end{aligned}$$

Here, $\overline{A}$ is the opposite of A, like the definition in set.

Probability

Frequency: We do experiments and get the results about ratio of event.

Probability: Suppose the time of experiments is infinite and what is the frequency?

The probability calculation is the same as set.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We can simplify $P(A \cap B)$ with $P(AB)$.

Also we have

$$P(\overline{A \cup B}) = P(\overline{A} \cap \overline{B})$$
$$P(\overline{A \cap B}) = P(\overline{A} \cup \overline{B})$$

Probability

Classic Probability: We have n events A_i with probability equal to $1/n$ each.

Example: Coin's head and tail; dice with 1, 2, 3, 4, 5, 6 face.

All the events are independent and cannot happen at the same time. We have

$$P(A_1 \cup ... \cup A_k) = \frac{k}{n}$$

Probability

Problem: We throw a coin with head and tail 3 times. What's the probability of: 1. Exactly one head and two tails; 2. At least two heads; 3. At least one head.

Probability

Problem: We throw a coin with head and tail 3 times. What's the probability of: 1. Exactly one head and two tails; 2. At least two heads; 3. At least one head.

Answer.

1. There are $2^3 = 8$ results. 2 heads and 1 tail mean: HHT, HTH, THH. So 3/8. In fact, we can also write

$$C_3^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right) = \frac{3}{8}$$

Let A_i denote event: there are i times of heads. We have

$$A_0 = C_3^0 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$A_1 = C_3^1 \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

$$A_2 = C_3^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right) = \frac{3}{8}$$

$$A_3 = C_3^3 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Therefore:

2. The answer is $A_2 + A_3 = 0.5$.
3. The answer is $A_1 + A_2 + A_3 = 1 - A_0 = 7/8$.

In this way, in classical probability model, suppose there are independent n events with equal probability and we repeat the experiment k times. Then the probability of event A_i appearing j times is

$$A_{i,j} = C_k^j \cdot \left(\frac{1}{n}\right)^j \cdot \left(1 - \frac{1}{n}\right)^{k-j}$$

Consider a simple case. Suppose event A has probability p. Then if we repeat the experiment k times, then the probability of A appearing i times is:

$$A_i = C_k^i \cdot p^i \cdot (1-p)^{k-i}$$

In fact, this is the same as binomial.

Probability

We have 4 red balls and 2 white balls. Calculate the probability of selecting two balls with putting it back after each operation.

1. The two are both white.

2. One is white and the other is red. We do not care about the order.

3. At least one is white.

Probability

We have 4 red balls and 2 white balls. Calculate the probability of selecting two balls with putting it back after each operation.

1. The two are both white.

$$P = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

2. One is white and the other is red. We do not care about the order.

$$P = C_2^1 \cdot \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right) = \frac{4}{9}$$

3. At least one is white.

$$P = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$$

Probability

What if we select a ball without putting it back? So let's change the problem.

We have 4 red balls and 2 white balls. Calculate the probability of selecting two balls *without* putting it back after each operation.

1. The two are both white.

2. One is white and the other is red. We do not care about the order.

3. At least one is white.

Probability

We have 4 red balls and 2 white balls. Calculate the probability of selecting two balls *without* putting it back after each operation.

1. The two are both white.

Notice the probability should be written as $P(\text{two white}) = P(\text{the first is white, and the second is white based on the first is white})$.

$$P = \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{15}$$

This is called conditioned probability. We use $B|A$ to denote B based on A happening, and we have

$$P(AB) = P(A) \cdot P(B|A)$$

If A, B are independent, then $P(B|A) = P(B)$.

Probability

2. One is white and the other is red. We do not care about the order.

Maybe red + white, or white + red.

$$P = \frac{4}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{4}{5} = \frac{8}{15}$$

3. At least one is white.

We can use 1 minus the probability of two red.

$$P = 1 - \frac{4}{6} \cdot \frac{3}{5} = \frac{3}{5}$$

Probability

Problem: There are n students ($n < 365$). Assume there are 365 days a year. What's the probability that at least two students have the same birthday?

Write the formula and test it with some values by calculator.

Probability

Problem: There are n students ($n < 365$). Assume there are 365 days a year. What's the probability that at least two students have the same birthday?

Answer: the formula is as follows

$$P = 1 - \frac{365 \cdot 364 \cdot ... \cdot (365 - n + 1)}{365^n} = 1 - \frac{A_{365}^n}{365^n}$$

This is because there are 365^n cases totally. We calculate the probability that no students have the same birthday. The first student has 365 choices, then the second 364 ......

If $n = 20, P = 0.4$; if $n = 30, P = 0.7$; if $n = 100, P = 0.9999997$. So if we have 100 students in a class, this is nearly guaranteed.

Probability

Problem: There are n balls with k red. We select $d \leq k$ balls without putting it back. Then what is the probability that we get $i \leq d$ ball?

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Answer.

$$P = \frac{C_k^i \cdot C_{n-k}^{d-i}}{C_n^d}$$

1. There are C_n^d cases if we select d from n balls and do not care about the order.

2. There are C_k^i cases if we select i from k red balls and do not care about the order.

3. There are C_{n-k}^{d-i} cases if we select $d-i$ from $n-k$ non-red balls and do not care about the order.

Probability

There are a red and b white balls. $k \leq a + b$ people select a ball one by one without putting it back. What is the probability that person i selects a white ball?

PS: If we put it back, then obviously it is

$$\frac{b}{a+b}$$

So case without putting it back is challenging.

Probability

There are a red and b white balls. $k \leq a + b$ people select a ball one by one without putting it back. What is the probability that person i selects a white ball?

Answer. There are totally A_{a+b}^{k} different cases because they select the ball one by one with orders. So how many cases are there for person i selects a white ball? The answer is

$$b \cdot A_{a+b-1}^{k-1}$$

Therefore

$$P = \frac{b \cdot A_{a+b-1}^{k-1}}{A_{a+b}^{k}} = b \cdot \frac{(a+b-1) \cdot \ldots \cdot (a+b-k+1)}{(a+b) \cdot \ldots \cdot (a+b-k+1)} = \frac{b}{a+b}$$

It is the same as putting it back!

Probability

Select a number randomly from 1 to 2000. What is the probability that it can neither be divided by 8 nor be divided by 6?

Select a number randomly from 1 to 2000. What is the probability that it can neither be divided by 8 nor be divided by 6?

Answer: Let A denote it can be divided by 8, B denote it can be divided by 6. So AB is it can be divided by $\operatorname{lcm}(6,8) = 24$.

$$
\begin{aligned}
P(\overline{AB}) &= 1 - P(A \cup B) \\
&= 1 - P(A) - P(B) + P(AB) \\
&= \frac{2000 - 250 - 333 + 83}{2000} \\
&= \frac{1500}{2000} \\
&= \frac{3}{4}
\end{aligned}
$$

Probability

We have 12 boys and 3 girls (like the ratio in CS department) and we will divide them into three groups with each group 5 people.

1. What is the probability that each group has one girl?

2. What is the probability that the three girls are in the same group?

Probability

We have 12 boys and 3 girls (like the ratio in CS department) and we will divide them into three groups with each group 5 people.

1. What is the probability that each group has one girl?

$$P = \frac{3! \cdot 12!}{4! \cdot 4! \cdot 4!} \Big/ \frac{15!}{5! \cdot 5! \cdot 5!} = \frac{25}{91}$$

2. What is the probability that the three girls are in the same group?

$$P = \frac{3 \cdot 12!}{2! \cdot 5! \cdot 5!} \Big/ \frac{15!}{5! \cdot 5! \cdot 5!} = \frac{6}{91}$$

Probability

Suppose a restaurant serves 12 people last week (none of the person comes in group). All of them go to the restaurant in Tuesday and Thursday. Can we believe that the restaurant does not open in some days?

Suppose a restaurant serves 12 people last week (none of the person comes in group). All of them go to the restaurant in Tuesday and Thursday. Can we believe that the restaurant does not open in some days?

Answer: Suppose the restaurant opens all week long. Then this probability is

$$P = \left(\frac{2}{7}\right)^{12} = 0.0000003$$

So we can believe it closes in some days.

Bayes

Condition Probability: $P(B|A)$ means with A happening, what is the probability of B?

We have the following rule

$$P(B_1 \cup B_2|A) = P(B_1|A) + P(B_2|A) - P(B_1B_2|A)$$

If B_1 and B_2 are independent, then

$$P(B_1 \cup B_2|A) = P(B_1|A) + P(B_2|A)$$

If A and B are independent, then

$$P(B|A) = P(B)$$

Bayes

Notice a basic formula

$$P(B|A) = \frac{P(AB)}{P(A)}$$

Problem: We have 3 red and 1 white balls. We select two balls one by one without putting it back. What is the probability that the second is white based on the first is red?

Bayes

Notice a basic formula

$$P(B|A) = \frac{P(AB)}{P(A)}$$

Problem: We have 3 red and 1 white balls. We select two balls one by one without putting it back. What is the probability that the second is white based on the first is red?

Answer:

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{3/12}{3/4} = \frac{1}{3}$$

Or we can get directly 1/3 by counting examples.

Bayes

According to the formula

$$P(B|A) = \frac{P(AB)}{P(A)}$$

We have

$$P(AB) = P(A) \cdot P(B|A)$$

In this way we have

$$P(ABC) = P(A) \cdot P(B|A) \cdot P(C|AB)$$

Bayes

Problem: We have a red and b white balls. Each time we select a ball and observe its color called d. Then we put it back and add c balls with color d. Repeat it 4 times. What is the probability that we have red in 1, 2 and white in 3, 4?

Bayes

Problem: We have a red and b white balls. Each time we select a ball and observe its color called d. Then we put it back and add c balls with color d. Repeat it 4 times. What is the probability that we have red in 1, 2 and white in 3, 4?

Answer: Let A_i denote we get red in i.

$$P(A_1A_2\overline{A_3A_4}) = P(A_1)\cdot P(A_2|A_1)\cdot P(\overline{A_3}|A_1A_2)\cdot P(\overline{A_4}|A_1A_2\overline{A_3})$$
$$= \frac{a}{a+b}\cdot\frac{a+c}{a+b+c}\cdot\frac{b}{a+b+2c}\cdot\frac{b+c}{a+b+3c}$$

Bayes

Problem: We have an egg. First time we drop it, it breaks with probability $1/2$. Second time (of course this means it does not break in the first time) $7/10$, and third time $9/10$. What is the probability that we drop it three times and it is still safe?

Bayes

Problem: We have an egg. First time we drop it, it breaks with probability 1/2. Second time (of course this means it does not break in the first time) 7/10, and third time 9/10. What is the probability that we drop it three times and it is still safe?

Answer: Let A_i denote it breaks in i time. Then we have

$$\begin{aligned}
P(\overline{A_1 A_2 A_3}) &= P(\overline{A_1}) \cdot P(\overline{A_2} | \overline{A_1}) \cdot P(\overline{A_3} | \overline{A_1 A_2}) \\
&= \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{7}{10}\right) \cdot \left(1 - \frac{9}{10}\right) \\
&= \frac{3}{200}
\end{aligned}$$

Bayes

Suppose $B_1 \cup B_2 \cup ... \cup B_n$ include all the events and $\forall i, j, B_i \cap B_j = \emptyset$, then we have

$$P(A) = P(A|B_1) + P(A|B_2) + ... + P(A|B_n)$$

This is called law of total probability. Another is a more important formula

$$P(B_i|A) = \frac{P(AB_i)}{P(A)} = \frac{P(A|B_i) \cdot P(B_i)}{P(A|B_1) + P(A|B_2) + ... + P(A|B_n)}$$

This is called Bayes formula.

Bayes

Problem: It is reported that about 0.1% of the people have lung cancer. About 20% of the people are smokers. The probability of a smoker having lung cancer is 0.4%. What is the probability of a non-smoker having lung cancer?

Bayes

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Answer: Let A denote smoker and C denote cancer. We have

$$P(C) = P(C|A) \cdot P(A) + P(C|\overline{A}) \cdot P(\overline{A})$$

Therefore

$$0.001 = 0.004 \cdot 0.2 + P \cdot 0.8$$

We have $P = 0.00025 = 0.25\%$.

Bayes

Problem: Medical tests are the primary means of detecting cancer. Suppose we test some people who suspect they have cancer (so the cancer ratio is much larger than ordinary people). It is reported that, if a person has cancer, then its probability with test result being positive is 95%. Also if a person does not have cancer, then its probability with test result being negative is 95%. We have known that about 0.5% of these people have cancer. So what is the probability of a positive testing person has cancer?

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Answer: Let A denote positive and C denote cancer. We know $P(A|C) = 0.95$ and $P(\overline{A}|\overline{C}) = 0.95$. Also $P(C) = 0.005$. Therefore we know

$$P(\overline{C}) = 1 - P(C) = 0.995. \qquad P(A|\overline{C}) = 1 - P(\overline{A}|\overline{C}) = 0.05$$

Bayes

In this way, what we want is $P(C|A)$, so we have

$$\begin{aligned} P(C|A) &= \frac{P(A|C) \cdot P(C)}{P(A|C) \cdot P(C) + P(A|\overline{C}) \cdot P(\overline{C})} \\ &= \frac{0.95 \cdot 0.005}{0.95 \cdot 0.005 + 0.05 \cdot 0.995} \\ &= 0.087 \quad (8.7\%) \end{aligned}$$

It's amazing that positive test only means that you have 8.7% of the probability to have the cancer.

Independent

A, B are independent if and only if $P(AB) = P(A) \cdot P(B)$. And also we know $A, \overline{B}$ are independent, $\overline{A}, \overline{B}$ are independent...

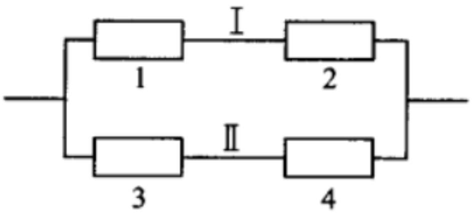
A, B, C are independent if and only if

$$
\begin{aligned}
P(AB) &= P(A) \cdot P(B) \\
P(AC) &= P(A) \cdot P(C) \\
P(BC) &= P(B) \cdot P(C) \\
P(ABC) &= P(A) \cdot P(B) \cdot P(C)
\end{aligned}
$$

Notice we must guarantee all the four equations hold.

Independent

Problem: The probability that each electronic component works well is p_i. What is the probability that the system works well?



Independent

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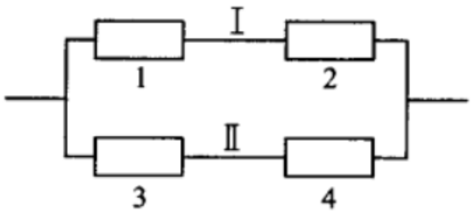


Answer: Let A_i denote that component i works well. We have

$$
\begin{aligned}
P(A) = P(A_1A_2 \cup A_3A_4) &= P(A_1A_2) + P(A_3A_4) - P(A_1A_2A_3A_4) \\
&= p_1p_2 + p_3p_4 - p_1p_2p_3p_4
\end{aligned}
$$

Independent

A and B play games. If one wins 2 games of the three, then he/she wins. Suppose the probability A win is p and B win is $1 - p$. What is the probability that A wins the whole game?

Independent

A and B play games. If one wins 2 games of the three, then he/she wins. Suppose the probability A win is p and B win is $1 - p$. What is the probability that A wins the whole game?

Answer: Three cases: AA, ABA, BAA. So we have

$$P = p^2 + 2p^2(1 - p) = p^2(3 - 2p)$$

Next question: What if we change win 2/3 into win 3/5?

Independent

Problem: What if we change win 2/3 into win 3/5?

Answer: three main cases: $AAA, 3A + B, 3A + 2B$. Notice the last game must be won by A. Therefore

$$P = p^3 + C_3^1 \cdot p^3(1-p) + C_4^2 \cdot p^3(1-p)^2$$

Next question: Which rule is better for A?

Independent

Problem: Which rule is better for A?

Answer.

$$P_1 = p^2 + 2p^2(1-p) = p^2(3-2p)$$
$$P_2 = p^3 + C_3^1 \cdot p^3(1-p) + C_4^2 \cdot p^3(1-p)^2$$

Therefore

$$\begin{aligned} P_2 - P_1 &= p^2(6p^3 - 15p^2 + 12p - 3) \\ &= p^2(p-1)^2(2p-1) \end{aligned}$$

It depends on the relation between p and 0.5.

Random Variable

We can write discrete random variable with tables.

For example:

X	0	1	2	3	4	5
$P(X)$	0.1	0.2	0.1	0.2	0.2	0.2

Then we introduce some special discrete random variables.

Random Variable

$0-1$ Distribution.

X	0	1
$P(X)$	$1-p$	p

So $P(X=1)=p, P(0)=1-p$.

Random Variable

Bernoulli (binomial) distribution for $0 \le X \le N$:

$$P(X) = C_N^X \cdot p^X \cdot (1-p)^{N-X}$$

For example, if $N = 4, p = 0.5$, we have

X	0	1	2	3	4
$P(X)$	$1/16$	$1/4$	$3/8$	$1/4$	$1/16$

Random Variable

Problem: A monkey picked up a gun and shot 400 times. Each time it has 0.02 probability to hit the target apple. What's the probability that it hit at least 2 times?

Random Variable

Problem: A monkey picked up a gun and shot 400 times. Each time it has 0.02 probability to hit the target apple. What's the probability that it hit at least 2 times?

Answer:

$$P(A_k) = P(X = k) = C_{400}^k \cdot 0.02^k \cdot 0.8^{400-k}$$

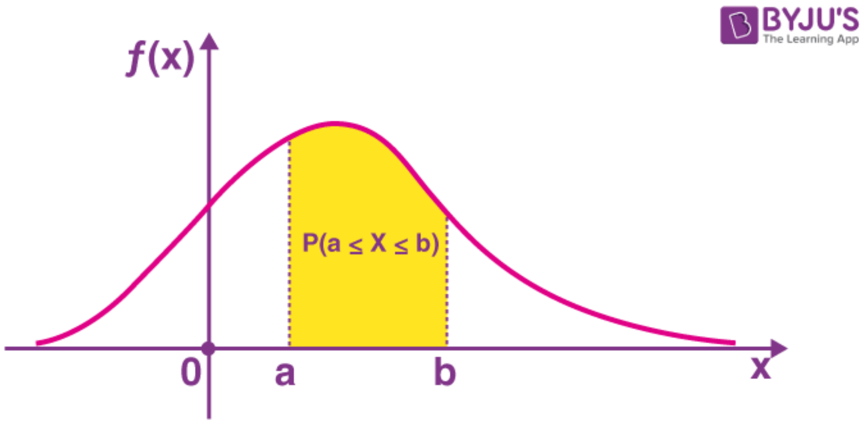
Therefore

$$P(A_{\geq 2}) = 1 - P(A_0) - P(A_1) = 1 - 0.98^{400} - 400 \cdot 0.02 \cdot 0.98^{399} = 0.9972$$

Random Variable

Continuous random variable cannot be denoted by tables. But we can write functions called probability distribution function (PDF).

The integral of the probability distribution function is the cumulative distribution function (CDF).



Random Variable

Obviously we know

$$P(x_1 < x \leq x_2) = P(x \leq x_2) - P(x \leq x_1).$$

Let $F(x_0) = P(x \leq x_0)$ be the CDF function.

Naturally we have

$$F(\infty) = 1, \quad F(-\infty) = 0$$

Let $f(x) = F'(x)$ be the PDF function. Or we can say

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

We introduce some distribution.

Uniform distribution.

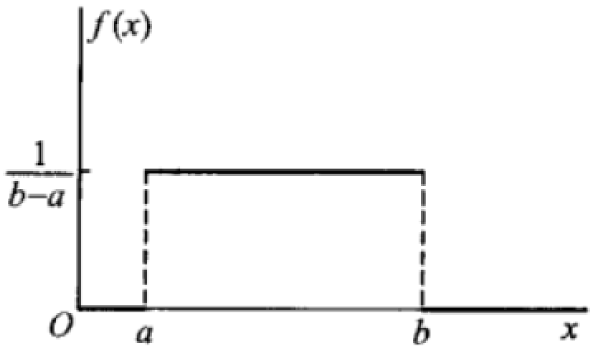
$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & x < a \text{ or } x > b \end{cases}$$

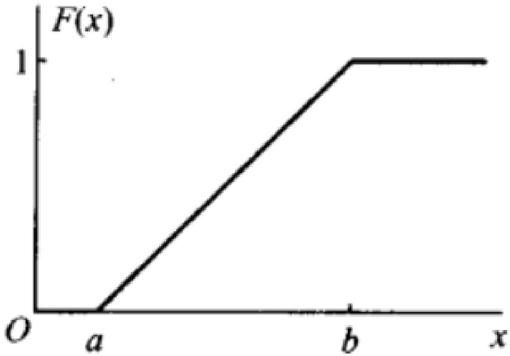
We have

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

Random Variable

Uniform distribution.





Random Variable

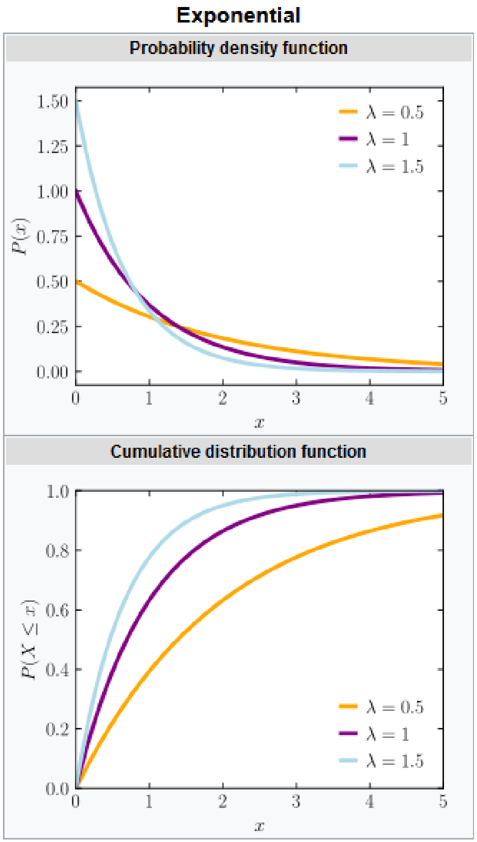
Exponential distribution.

$$f(x) = \begin{cases} 0, & x < 0 \\ \dfrac{e^{-x/\theta}}{\theta}, & x \geq 0 \end{cases}$$

We have

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/\theta}, & x \geq 1 \end{cases}$$

Random Variable



Random Variable

Given CDF as follows

$$F(x) = \begin{cases} 0, & x < 1 \\ \ln(x), & 1 \leq x \leq e \\ 1, & x > e \end{cases}$$

Calculate: 1. $P(X < 2)$. 2. $P(0 < X \leq 3)$. 3. $f(x)$

Random Variable

Given CDF as follows

$$F(x) = \begin{cases} 0, & x < 1 \\ \ln(x), & 1 \leq x \leq e \\ 1, & x > e \end{cases}$$

1. $P(X < 2) = F(2) = \ln(2)$.

2. $P(0 < X \leq 3) = 1$.

3. $$f(x) = \begin{cases} 0, & x < 1 \\ \dfrac{1}{x}, & 1 \leq x \leq e \\ 0, & x > e \end{cases}$$

Expectation

For discrete random variables $x_1, x_2, ..., x_n$ with probability p_i, the expectation is

$$E(x) = \sum_{i=1}^{n} x_i \cdot p_i$$

For continuous random variables with probability $f(x)$, the expectation is

$$E(x) = \int_{\infty}^{\infty} x \cdot f(x) dx$$

Expectation

The expectation of $0-1$ distribution is p, which is obvious.

The expectation of binomial distribution is

$$E(x)=\sum_{i=0}^{n} C_n^i \cdot p^i \cdot (1-p)^i = np$$

This is because, binomial distribution is $0-1$ distribution $\times n$.
Therefore $E = np$.

We have $E(a+b) = E(a) + E(b)$.

Expectation

The expectation of uniform distribution is $0.5 \cdot (a + b)$, which is obvious.

The expectation of exponential distribution is

$$
\begin{aligned}
E(x) &= \int_0^\infty x \cdot \frac{e^{-x/\theta}}{\theta} dx \\
&= \theta \int_0^\infty \frac{x}{\theta} e^{-x/\theta} d\left(\frac{x}{\theta}\right) \\
&= \theta \left(-e^{-x/\theta} - \frac{x}{\theta} e^{-x/\theta}\right)_0^\infty \\
&= \theta
\end{aligned}
$$

Variance

Variance of discrete random variables is

$$D(x) = \frac{1}{n}\sum_{i=1}^{n}(x_i - E(x))^2 p_i = E((x - E(x))^2)$$

Variance of continuous random variables is

$$D(x) = \int_{-\infty}^{\infty}(x - E(x))^2 f(x)dx$$

Besides, we have

$$\begin{aligned} D(x) &= E((x - E(x))^2) = E(x^2) - 2E(x) \cdot E(x) + E(x)^2 \\ &= E(x^2) - E(x)^2 \end{aligned}$$

Notice $E(c) = c$.

Variance

The variance of $0-1$ distribution is $p(1-p)$.

The variance of binomial distribution is $np(1-p)$.

The variance of uniform distribution is

$$D(x) = \frac{(b-a)^2}{12}$$

The variance of exponential distribution is

$$D(x) = \theta^2$$

Can you prove them?

Variance

1. Uniform

$$D(X) = E(X^2) - [E(X)]^2$$
$$= \int_a^b x^2 \frac{1}{b-a} \mathrm{d}x - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

2. Exponential

$$E(X) = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x = \int_0^{\infty} x \frac{1}{\theta} \mathrm{e}^{-x/\theta} \mathrm{d}x$$
$$= -x\mathrm{e}^{-x/\theta} \Big|_0^{\infty} + \int_0^{\infty} \mathrm{e}^{-x/\theta} \mathrm{d}x = \theta,$$
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \mathrm{d}x = \int_0^{\infty} x^2 \frac{1}{\theta} \mathrm{e}^{-x/\theta} \mathrm{d}x$$
$$= -x^2 \mathrm{e}^{-x/\theta} \Big|_0^{\infty} + \int_0^{\infty} 2x \mathrm{e}^{-x/\theta} \mathrm{d}x = 2\theta^2,$$
$$D(X) = E(X^2) - [E(X)]^2 = 2\theta^2 - \theta^2 = \theta^2.$$