MATH 333 Discrete Mathematics
Chapter 8 Counting and Number

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Permutation and Combination

We have some numbers $a_1, a_2, ..., a_n$.

How many orders do we have?

Answer: We have n choices for the first number, then $n-1$, then....
Therefore we have $n! = n \cdot (n-1) \cdot ... \cdot 1$ choices.

In this way we have: n different things have $n!$ different orders.

What about we select m from n? Here we require $m \leq n$.

Permutation and Combination

We have n choices for the first number, then $n-1$, then... until $n-m+1$. Therefore we have

$$A_n^m = P_m^n = n \cdot (n-1) \cdot \ldots \cdot (n-m+1) = \frac{n!}{(n-m)!}$$

Different textbooks use different symbols, so it is recommended to use

$$\frac{n!}{(n-m)!}$$

This is called permutation. The order is important. Notice if $m = n$, then it is $n!$.

Permutation and Combination

What if we do not care the order? For the same question, select m from n, how many choices?

We have A_n^m permutations. And notice the m elements have $m!$ orders. Therefore the choice is

$$C_n^m = \binom{n}{m} = \frac{A_n^m}{m!} = \frac{n!}{m!(n-m)!}$$

This is called combination. Also, different textbooks use different symbols.

Permutation and Combination

Practice: $A_4^2, A_6^2, A_8^4, A_6^6$

Practice: $C_4^2, C_6^2, C_8^4, C_6^6, C_6^0$

Practice: $4!, 5!, 6!$

Can you find some rules for combination?

Pascal / Yanghui's Triangle: The $i-$th row, $j-$th element is C_{i-1}^{j-1}

							1								Row 0
						1		1							Row 1
					1		2		1						Row 2
				1		3		3		1					Row 3
			1		4		6		4		1				Row 4
		1		5		10		10		5		1			Row 5
	1		6		15		20		15		6		1		Row 6
1		7		21		35		35		21		7		1	Row 7

Try some values!

Permutation and Combination

Try to prove the following rules.

Rule 1: $C_n^m = C_n^{n-m}$

Rule 2: $C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$

In fact, the proof does not need the triangle above. We can prove it by definition.

Permutation and Combination

Rule 1. $C_n^m = C_n^{n-m}$

$$C_n^m = \frac{n!}{m!(n-m)!} = C_n^{n-m}$$

Rule 2. $C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$

$$\begin{aligned}
C_{n-1}^{m-1} + C_{n-1}^m &= \frac{(n-1)!}{(n-m)!(m-1)!} + \frac{(n-1)!}{m!(n-m-1)!} \\
&= \frac{(n-1)!}{(m-1)!(n-m-1)!}\left(\frac{1}{m} + \frac{1}{n-m}\right) \\
&= \frac{(n-1)!}{(m-1)!(n-m-1)!} \cdot \frac{n}{m(n-m)} \\
&= \frac{n!}{m!(n-m)!} = C_n^m
\end{aligned}$$

Permutation and Combination

When we want to have some permutation or combination, what if not all the elements are different?

Suppose there are 2 red, 2 yellow, 2 blue. How many orders do we have?

When all the colors are different, we have 6!. Since there are two red, so the order of these two is not important (also 2! orders). We should let 6!/2!. In the same way, we have

$$\text{result} = \frac{6!}{2! \cdot 2! \cdot 2!} = 90$$

Permutation and Combination

In this way we have the following rules. If there are k kinds of elements with $a_1, a_2, ..., a_k$ numbers, then the permutation is

$$\frac{\left(\sum_{i=1}^{k} a_i\right)!}{\prod_{i=1}^{k} a_i!} = \frac{(a_1 + a_2 + ... + a_k)!}{a_1! \cdot a_2! \cdot ... \cdot a_k!}$$

Note: If we want to calculate the permutation on a ring, instead of a line, then we should let result $/n$ where n is the total number selected.

Permutation and Combination

How many solutions are there for the equation below? Here $x_i \geq 0$ is an integer. And we only consider the combination number, i.e. do not care about the order.

$$x_1 + x_2 + ... + x_k = n$$

Permutation and Combination

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$$x_1 + x_2 + ... + x_k = n$$

We draw n balls on a line: oooooooooooooooo....

We should use $k-1$ dividers to divide them into n parts. This becomes a new case: When we add the first divider, we have $n+1$, then $n+2$... until $n+k-1$. Notice all the dividers are the same, we have to divide the result by $(k-1)!$. So

$$\text{result} = \frac{(n+1)(n+2)...(n+k-1)}{(k-1)!} = C^k_{n+k-1}$$

Permutation Combination

Exercise: We have 2 red, 3 yellow, and 4 green balls. How many permutation?

Permutation Combination

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Answer:

$$\frac{9!}{2! \cdot 3! \cdot 4!} = 1260$$

Permutation and Combination

Exercise: We have 10 different balls. We select 6 of them and put them on a *ring*. How many permutation?

$$\frac{A_{10}^{6}}{6} = 840$$

Permutation and Combination

Exercise: We have 6 different color. We want to paint 4 balls. How many combinations?

Hint: Recall $x_1 + x_2 + ... + x_k = n$

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Hint: Recall $x_1 + x_2 + ... + x_k = n$

Answer: It is the same as: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 4$. So

$$C_{4+6-1}^{6-1} = C_9^5 = C_9^4 = 126$$

Binomial

We have the following formula

$$(a+b)^n = \sum_{i=0}^{n} C_n^i a^{n-i} b^i$$

For example,

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Try some others!

Binomial

Compute

$$C_{10}^{2}C_{8}^{4} - C_{10}^{4}C_{6}^{2}$$

Binomial

Compute

$$C_{10}^2 C_8^4 - C_{10}^4 C_6^2$$

Answer.

$$\begin{aligned}
& C_{10}^2 C_8^4 - C_{10}^4 C_6^2 \\
=& \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{2! \cdot 4!} - \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{4! \cdot 2!} \\
=& 0
\end{aligned}$$

Binomial

Compute

$$\sum_{i=0}^{n} C_n^i$$

Binomial

Compute

$$\sum_{i=0}^{n} C_n^i$$

Answer.

$$\begin{aligned}\sum_{i=0}^{n} C_n^i &= \sum_{i=0}^{n} C_n^i \cdot 1^{n-i} \cdot 1^i \\ &= (1+1)^n = 2^n\end{aligned}$$

Binomial

What's the coefficient of x^3 in:
1.$(1+x)^3$;   2.$(3-2x)^6$;   3.$(2x+1)^10-(3-2x)^6$?

Hint: coefficient of x^3 in $4x^3$ is 4.

Binomial

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Hint: coefficient of x^3 in $4x^3$ is 4.

Answer:

1. $C_6^3 = 20.$

2. $C_6^3 \cdot 3^3 \cdot (-2)^3 = 20 \cdot 27 \cdot (-8) = 4320$

3. $C_{10}^3 \cdot 2^3 - C_6^3 \cdot 3^3 \cdot (-2)^3 = 9600 - 4320 = 5280$

Binomial

Prove

$$\sum_{i=k}^{n} C_i^k = C_{n+1}^{k+1}$$

Hint: Recall

$$C_{n+1}^{k+1} = C_n^k + C_n^{k+1}$$

Binomial

Prove

$$\sum_{i=k}^{n} C_i^k = C_{n+1}^{k+1}$$

Answer:

$$\begin{aligned}
C_{n+1}^{k+1} &= C_n^k + C_n^{k+1} \\
&= C_n^k + C_{n-1}^k + C_{n-1}^{k+1} \\
&= C_n^k + C_{n-1}^k + ... + C_{k+1}^k + C_{k+1}^{k+1} \\
&= C_n^k + C_{n-1}^k + ... + C_{k+1}^k + C_k^k \\
&= \sum_{i=k}^{n} C_i^k
\end{aligned}$$

Binomial

Prove

$$\sum_{i=0}^{n} \left(C_n^i\right)^2 = C_{2n}^n$$

Hint: Use $(1+x)^{2n} = (1+x)^n \cdot (1+x)^n$ and consider the coefficient.

Binomial

Prove

$$\sum_{i=0}^{n} \left(C_n^i\right)^2 = C_{2n}^n$$

Hint: Use $(1+x)^{2n} = (1+x)^n \cdot (1+x)^n$ and consider the coefficient.

Answer: The coefficient of x^n in $(1+x)^2$ is C_{2n}^n. The coefficient of x^n in $(1+x)^n \cdot (1+x)^n$ is

$$C_n^0 \cdot C_n^n + C_n^1 \cdot C_n^{n+1} + ... = \sum_{i=0}^{n} \left(C_n^i\right)^2$$

Binomial

Prove

$$\sum_{i=1}^{n} C_n^k (-1)^k = 0$$

Binomial

Prove

$$\sum_{i=0}^{n} C_n^k (-1)^k = 0$$

Answer:

$$\sum_{i=0}^{n} C_n^k (-1)^k = (1-1)^n = 0$$

Number Theory

We know $5/3 = 1......2$. Therefore $5\%3 = 2$.

If $a\%b = 0$, then we can say $a|b$. But this is not easy to understand. So $a\%b = 0$ is recommended.

Greatest Common Divisor (GCD): $\gcd(15, 25) = 5$. It is the largest number x so that $a|x$ and $b|x$. Here we only consider $a, b > 0$.

Least Common Multiple (LCM): $\mathrm{lcm}(15, 25) = 75$. It is the smallest number x so that $x|a$ and $x|b$.

We can find that $\gcd(a, b) \cdot \mathrm{lcm}(a, b) = a \cdot b$. This is easy to prove. Try it.

Prove $\gcd(a, b) \cdot \mathrm{lcm}(a, b) = a \cdot b$.

Proof.

Suppose $\gcd(a, b) = c$ and $a = ck_1$, $b = ck_2$. We know k_1 and k_2 are co-prime otherwise c is not gcd.

Therefore let $d = ab/c = ck_1k_2$. We know $d|a$ and $d|b$.

There is no $x < d$ so that $x|a$ and $x|b$. This is because $d/a = k_2$ and $d/b = k_1$, which are co-prime.

Some rules: $\gcd(ka, kb) = k \cdot \gcd(a, b)$. The same for lcm.

Number Theory

How to get the gcd of a, b? Suppose $a \geq b$.

If $a = b$, then obviously $\gcd(a, b) = a$. If $a|b$, then obviously $\gcd(a, b) = b$. For other cases, we use the algorithm below.

While $a\%b \neq 0$, we let $a \leftarrow a\%b$. If This time $a < b$ then we swap a, b.

Until we meet $a\%b = 0$, this time we return b.

A rule: $\gcd(a, b) = \gcd(a\%b, b)$ for $a > b > 0$.

How to prove the rule? Try it.

Number Theory

1. $\gcd(1250, 2000) = ?$
2. $\gcd(169, 195) = ?$
3. $\operatorname{lcm}(1250, 2000) = ?$
4. $\operatorname{lcm}(169, 195) = ?$

Number Theory

1. $\gcd(1250, 2000) = 250$
2. $\gcd(169, 195) = 13$
3. $\operatorname{lcm}(1250, 2000) = 10000$
4. $\operatorname{lcm}(169, 195) = 2535$

Number Theory

What's the largest digit of $1122334455667788 99^{123456789}$?

Number Theory

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Answer: the last digit is r. Then we have

$$\begin{aligned} r &= 112233445566778899^{123456789} \% 10 \\ &= (10 \cdot 11223344556677889 + 9)^{123456789} \% 10 \\ &= 9^{123456789} \% 10 \end{aligned}$$

Then we consider $9^1 \% 10 = 9, 9^2 \% 10 = 1, 9^3 \% 10 = 9......$ So we know it can only be 1 or 9 according to odd and even. This is obvious and you can try to prove it.

$$\begin{aligned} r &= 9^{123456789} \% 10 \\ &= 9^{(123456788+1)} \% 10 \\ &= 9 \% 10 = 9 \end{aligned}$$

Prove: for $\forall k \geq 1$, $2^k + 1$ and $2^k - 1$ are co-prime. It means, gcd is 1.

Hint: Contradiction Proof.

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Hint: Contradiction Proof.
Proof.

Suppose $c > 1$ so that $c = \gcd(2^k + 1, 2^k - 1)$. Therefore we can assume $2^k + 1 = cm, 2^k - 1 = cn$. So we have

$$(2^k + 1) - (2^k - 1) = 2 = cm - cn = c(m - n)$$

Therefore we have to let $c = 2, m - n = 1$. But notice $2^k + 1$ is odd, so it cannot be divided by 2. We get into a contradiction.

Number Theory

Prove $\mathbb{Z} = \{2x + 3y | x, y \in \mathbb{Z}\}$. It means we can denote all the integers with $2x + 3y$.

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Proof.

There are three kinds of integers according to $n\%3 = 0, 1, 2$. We call them A, B, C set.

Let $x = 0$, it covers A. Let $x = 1$, it covers C. Let $x = 2$, it covers B.

Also you can use $n\%2 = 0, 1$ to prove. Try it.

Number Theory

Prove: $(2^{70} + 3^{70}) \% 13 = 0$.

Recall: $x^3 + 1 = (x+1)(x^2 - x + 1)$. What about we change 3 to other odd number?

Number Theory

Prove: $(2^{70} + 3^{70})\%13 = 0$.

Proof.

We know for any $n = 2k + 1$, we can use factorization for $a^n + b^n$.

$$a^{2k+1} + b^{2k+1} = (a + b) \cdot (a^{2k} - a^{2k-1}b + a^{2k-2}b^2 - ... + b^{2k})$$

Let $a = 4, b = 9, n = 35$, we have

$$4^{35} + 9^{35} = (4 + 9)(4^{34} - 4^{33} \cdot 9 + ... + 9^{34})$$

In this way, we know $(2^{70} + 3^{70})\%13 = 0$.

Number Theory

There are 24 hours per day. Assume it is 15:00 now. What is the time after the following hours?

1. 233. 2. $14 \cdot 233$. 3. 233^{233}

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1. 233. 2. $14 \cdot 233$. 3. 233^{233}

Answer.

1. $233\%24 = 17$. It's 22:00.

2. $14 \cdot 233\%24 = 14 \cdot 17\%24 = 238\%24 = 22$. It's 3:00.

3.
$$\begin{aligned} 233^{233}\%24 &= 17^{233}\%24 = 17 \cdot 289^{116}\%24 \\ &= 17 \cdot (1 + 12 \cdot 24)^{116}\%24 = 17\%24 = 17 \end{aligned}$$

So it's 22:00.