MATH 333 Discrete Mathematics
Chapter 2 Set and Relation

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Set Introduction

Complex Number $\mathbb{C} = \{1 + i, 1.5, \sqrt{2}, 0, ...\}$

Real Number $\mathbb{R} = \{1.56, \pi, 12, ...\}$

Rational Number $\mathbb{Q} = \{0.1111...., -1, ...\}$

Integer $\mathbb{Z} = \{-1, 0, 1, ...\}$

Natural Number $\mathbb{N} = \{1, 2, 3, ...\}$

Note: In some textbooks, $0 \in \mathbb{N}$ and we use $\mathbb{N}_+$ or $\mathbb{N}^*$ to denote positive integers. But in others, $0 \notin \mathbb{N}$. In depends on the definition.

The relationship

$$\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{Z} \subset \mathbb{C}$$

Set Introduction

A set can include every kind of elements.

A fruit set: $S = \{\text{apple, orange, banana}\}$

A set of set: $S = \{\{1, 2\}, \{1, 3\}\}$

A set with 0 element is empty/null set: $S = \{\} = \emptyset$

A set with 1 element is singleton set: $S = \{1\}$

A set can be either finite, or infinite

$$S_1 = \{1, 2, 3\}, \quad S_2 = \{x | \sqrt{x} \in \mathbb{N}\}$$

Set Introduction

There is no order for elements in a set:

$$S_1 = \{a, b, c\} = \{c, b, a\} = S_2$$

There are no same elements in a set:

$$S \neq \{a, b, c, a, b, c\}$$

Given a set, we will always know whether an arbitrary element belongs to it:

$$1 \in \mathbb{N}, \quad 1.5 \notin \mathbb{N}$$

Set Introduction

There are three ways to describe a set.

1. List all the elements (always finite and small)

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

2. Natural language

$$S = \text{all the prime numbers}$$

3. Use notation

$$S = \{x \in \mathbb{R} | x > 2\}$$

Set Introduction

If all the elements in S_1 and S_2 are the same, then they are equal

$$S_1 = \{1, 2, 3\} = S_2$$

If all the elements in S_1 belong to S_2, then S_1 is the subset of S_2

$$S_1 = \{1, 2\} \subseteq S_2 = \{1, 2, 3\}$$

If all the elements in S_1 belong to S_2, and there exists element in S_2 that does not belong to S_1, then S_1 is the proper subset of S_2

$$S_1 = \{1, 2\} \subset S_2 = \{1, 2, 3\}$$

Notice if $S_1 = S_2$, then they are the subset of each other.

Set Introduction

Power set of S: a set in which elements are subsets of S.

Let $S = \{1, 2, 3\}$, then the power set is

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

We can find that, if $|S| = x$, then

$$|P(S)| = 2^x$$

We have the following rule:

$$P(A) \cup P(B) \subseteq P(A \cup B)$$

Set Operator

1. Union

$$S_1 \cup S_2 = \{x|x \in S_1 \text{ or } x \in S_2\}$$

2. Intersection

$$S_1 \cap S_2 = \{x|x \in S_1 \text{ and } x \in S_2\}$$

Obviously we have

$$S_1 \cap S_2 \subseteq S_1, S_2 \subseteq S_1 \cup S_2$$

Let $A = \{1, 2, 4, 8\}$ and $B = \{x \in \mathbb{N} | 1 \leq x \leq 5\}$. Then we have

$$A \cup B = \{1, 2, 3, 4, 5, 8\}$$
$$A \cap B = \{1, 2, 4\}$$

Set Operator

3. Commutative

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

4. Associative

$$A \cup B \cup C = A \cup (B \cup C)$$
$$A \cap B \cap C = A \cap (B \cap C)$$

5. Distributive

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Operator

Exercise: Prove $S \cup (S \cap T) = S$

Hint: The left is a subset of right, and the right is a subset of left.

Set Operator

Exercise: Prove $S \cup (S \cap T) = S$

Proof.

According to distributive property, we have

$$S \cup (S \cap T) = (S \cup S) \cap (S \cup T) = S \cap (S \cup T) \subseteq S$$

Also we have

$$S \subseteq S \cup (S \cap T)$$

6. Difference

$$S_1 - S_2 = S_1 \backslash S_2 = \{x | x \in S_1 \text{ and } x \notin S_2\}$$

7. Symmetric Difference

$$S_1 \oplus S_2 = (S_1 - S_2) \cup (S_2 - S_1)$$

In fact, we have

$$S_1 \oplus S_2 = S_1 + S_2 - S_1 \cap S_2$$

8. Complement. Suppose U is the universe.

$$\overline{S} = U - S$$

Set Operator

Exercise: Prove $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

Hint: use element belonging

Set Operator

Exercise: Prove $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

Proof.

Let $a \in \overline{(A \cup B)}$. We know a neither belongs to A, nor belongs to B. So it must belong to $\overline{A}$, and also belong to $\overline{B}$. Therefore $a \in \overline{A} \cap \overline{B}$.

Let $a \in \overline{A} \cap \overline{B}$. We know a must belong to $\overline{A}$ and $\overline{B}$. Therefore a cannot belong to A or B. So $a \notin (A \cup B)$. This means that $a \in \overline{(A \cup B)}$.

In the same way we can also prove $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Set Operator

For the union of several sets, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Generally, we have

$$|A \cup B \cup C \cup ...| = (|A| + ...) - (|AB| + ...) + (|ABC| + ...) - ...$$

Relation

Cartesian Product of sets A and B

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Here, (a, b) is called ordered pair. We have

$$(a_1, b_1) = (a_2, b_2) \iff a_1 = a_2 \text{ and } b_1 = b_2$$

The Euclidean space is

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$$

Besides, we have

$$\emptyset \times S = S \times \emptyset = \emptyset$$

Relation

Long Product

$$A \times B \times C = \{(a, b, c) | a \in A, b \in B, c \in C\}$$

Associative

$$A \times B \times C = A \times (B \times C)$$

Cartesian product does not obey commutative property. Usually, we have

$$A \times B \neq B \times A \iff A \neq B$$

Distributive

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Relation

Relation is not necessarily Cartesian product. For example,

$$A = \{a, b, c\}, \quad B = \{1, 2, 3\}, \quad R = \{(a, 1), (b, 2), (c, 3)\}$$

This time R is still a relation between A and B. But we have

$$R \subseteq A \times B$$

Inverse Relation

$$R^{-1} = \{(a, b) | (b, a) \in R\}$$

So the example above is

$$R^{-1} = \{(1, a), (2, b), (3, c)\}$$

Relation

Suppose R is the relation between X and Y.

Domain

$$\mathrm{dom}(R) = \{x | (x, y) \in R\}$$

Range

$$\mathrm{rng}(R) = \{y | (x, y) \in R\}$$

For any set $Z \subseteq X$, we write

$$R(Z) = \{y | z \in Z, (z, y) \in R\}$$

For any set $W \subseteq Y$, we write

$$R^{-1}(W) = \{x | w \in W, (x, w) \in R\}$$

Function

Let f be a relation between X and Y.

If for any $x \in X$, there is at most one $y \in Y$ so that $(x, y) \in f$, then we say f is a function.

Here, y is image and x is pre-image.

According to domain and range, we have

$$\mathrm{dom}(f) = X, \quad \mathrm{rng}(f) = Y$$

Function

One to one / Injective: There is no two $x_1, x_2 \in X$, so that $f(x_1) = f(x_2)$. Or we can say any $y \in Y$ is used at most once.

Onto / Surjective: For any $y \in Y$, we can find at least one $x \in X$ so that $y = f(x)$. Or we can say any $y \in Y$ is used.

Both / Bijective: For any $y \in Y$, we can find just one $x \in X$ so that $y = f(x)$.

Even if $Y \subset X$, we can have a bijection:

$$X = \{x|0 \leq x \leq 1\}, \quad Y = \{y|0 \leq y \leq 0.5\}, \quad y = f(x) = 0.5x$$

Function

Composition: Let $y = f(x), z = g(y)$, then we have

$$g \circ f = g(f(x))$$

1. If f is one-by-one, g is one-by-one, then $g \circ f$ is one-by-one.
2. If f is onto, g is onto, then $g \circ f$ is onto.
3. If f is bijective, g is bijective, then $g \circ f$ is bijective.
4. Associative
$$h \circ g \circ f = h \circ (g \circ f)$$

Special Relation

Suppose R is a relation on A.

1. Reflexive: if $a \in A$, then $(a, a) \in R$

2. Symmetric: if $(a, b) \in R$, then $(b, a) \in R$

3. transitive: if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

4. Closure: R' is a reflexive / symmetric / transitive closure of R if R' is the smallest relation so that $R \subseteq R'$ and R' is reflexive / symmetric / transitive.

Exercise: Let $A = \{1, 2, 3, 4\}$, determine whether the following relation matrix is reflexive, symmetric and transitive? If not, find its closure.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Special Relation

Exercise: Let $A = \{1, 2, 3, 4\}$, determine whether the following relation matrix is reflexive, symmetric and transitive? If not, find its closure.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer: reflexive, symmetric and transitive.

Special Relation

Exercise: Let $S = \{1, 2, 3, 4\}$, $A = S \times S$. Let R be a relation on A. $((a, b), (c, d)) \in R$ if $ad = bc$. Determine whether the following relation matrix is reflexive, symmetric and transitive? If not, find its closure.

Special Relation

Exercise: Let $S = \{1, 2, 3, 4\}$, $A = S \times S$. Let R be a relation on A. $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Determine whether the following relation matrix is reflexive, symmetric and transitive? If not, find its closure.

Answer: reflexive, symmetric and transitive.

1. If $(a, b) = (c, d)$, then $a = c, b = d$, so $ad = ab = bc$. Thus reflexive.

2. If $((a, b), (c, d)) \in R$, then $ad = bc$, then $((c, d), (a, b)) \in R$. Thus symmetric.

3. If $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, then we have $ad = bc$ and $cf = de$. Therefore $acdf = bcfe$, i.e. $af = be$. So We have $((a, b), (e, f)) \in R$. Thus transitive.

Special Relation

Irreflexive: Any $(a, a) \notin R$.

Asymmetric: If $(a, b) \in R$, then $(b, a) \notin R$.

Notice that, asymmetric is not "not symmetric", and irreflexive is not "not reflexive".

If R is transitive and Irreflexive, then it is asymmetric.

Proof.

Suppose $(a, b) \in R$ and $(b, a) \in R$. Then according to transitive, we have $(a, a) \in R$, which is contradicted with irreflexive. Therefore it is asymmetric.

Special Relation

If R is a relation from A to B, then $\overline{R} = A \times B - R$.

Exercise: Let $A = B = \{1, 2, 3\}$. Let $R = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$ and $S = \{(2, 1), (3, 1), (3, 2), (3, 3)\}$.

Calculate: 1. $\overline{R}$; 2. $R \cap S$; 3. $R \cup S$; 4. S^{-1}.

Special Relation

If R is a relation from A to B, then $\overline{R} = A \times B - R$.

Exercise: Let $A = B = \{1, 2, 3\}$. Let $R = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$ and $S = \{(2, 1), (3, 1), (3, 2), (3, 3)\}$.

Calculate: 1. $\overline{R}$; 2. $R \cap S$; 3. $R \cup S$; 4. S^{-1}.

1. $\overline{R} = \{(1, 3), (2, 1), (2, 2), (3, 2), (3, 3)\}$

2. $R \cap S = \{(3, 1)\}$

3. $R \cup S = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)\}$

4. $S^{-1} = \{(1, 2), (1, 3), (2, 3), (3, 3)\}$

Special Relation

Exercise: Let $A = \{1, 2, 3\}$. Given two relations on A, $R = \{(1, 1), (1, 2), (2, 1), (3, 1)\}$ and $S = \{(1, 3), (2, 2), (3, 1), (3, 3)\}$.

Calculate: 1. $R \circ R$; 2. $S \circ S$; 3.$R \circ S$; 4. $S \circ R$.

Special Relation

Exercise: Let $A = \{1, 2, 3\}$. Given two relations on A,
$R = \{(1, 1), (1, 2), (2, 1), (3, 1)\}$ and $S = \{(1, 3), (2, 2), (3, 1), (3, 3)\}$.

Calculate: 1. $R \circ R$; 2. $S \circ S$; 3.$R \circ S$; 4. $S \circ R$.

1. $R \circ R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2)\}$

2. $S \circ S = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$

3. $R \circ S = \{(1, 1), (2, 1), (3, 1)\}$

4. $S \circ R = \{(1, 3), (1, 2), (2, 3), (3, 3)\}$

Special Relation

Partial Order: A relation which is reflexive, asymmetric and transitive.

Example 1: The relation with $\leq$ in $\mathbb{R}$.

1. For any a, we have $a \leq a$.
2. For any $a \leq b$ with $a \neq b$, we can never have $b \leq a$.
3. For any $a \leq b$ and $b \leq c$, we have $a \leq c$.

Special Relation

Example 2: The division with no remain in $\mathbb{R}^*$.

1. For any x, we have $x\%x = 0$
2. For any $x\%y = 0$ and $x \neq y$, we can never have $y\%x = 0$
3. For any $x\%y = 0$ and $y\%z = 0$, we have $x\%z = 0$.

What about other examples?

Exercise 1

Let $A = \{(1, 2, 3, 4)\}$ and $R = \{(2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2)\}$.

Calculate: 1. The reflexive closure of R; 2. The symmetric closure of R; 3. The transitive closure R.

Exercise 1

Let $A = \{(1, 2, 3, 4)\}$ and $R = \{(2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2)\}$.

Calculate: 1. The reflexive closure of R; 2. The symmetric closure of R; 3. The transitive closure R.

1. $r(R) = \{(1, 1), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 4)\}$

2. $s(R) = \{(1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (4, 2)\}$

3. $t(R) = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$

Exercise 2

Let R, S be symmetric relations on A. Are $R + S, R - S$ symmetric?

Exercise 2

Let R, S be symmetric relations on A. Are $R + S, R \oplus S$ symmetric?

1. Yes.

If $r \in R + S$, then $r \in R$ or $r \in S$. Therefore $r^{-1} \in R$ or $r^{-1} \in S$. So $r^{-1} \in R + S$.

2. Yes.

If $r \in R - S$, then $r \in R$ and $r \notin S$. Therefore $r^{-1} \in R$ and $r^{-1} \notin S$. So $r^{-1} \in R - S$.

If $r \in S - R$, then $r \in S$ and $r \notin R$. Therefore $r^{-1} \in S$ and $r^{-1} \notin R$. So $r^{-1} \in S - R$.

Exercise 3

Draw pictures with sets A, B, C that describes:

1. $A + B - A \cup C$

2. $A + B + C - A \cap B - A \cap C$

3. $A + B - C$

4. $A \cup C - A \cap B \cap C$

5. $A \cup B - A \cap C - A \cap C$

Exercise 4

For each case, calculate $A_1 \cup A_2 ...$ and $A_1 \cap A_2 ...$

1. $A_i = \{n | n \in \mathbb{N}_+, n \geq i\}$
2. $A_i = \{0, i\}$
3. $A_i = \{x | x \in \mathbb{R}, 0 < x < i\}$

Exercise 4

For each case, calculate $A_1 \cup A_2 ...$ and $A_1 \cap A_2 ...$

1. $A_i = \{n | n \in \mathbb{N}_+, n \geq i\}$

Answer: $A_1 \cup A_2 ... = \mathbb{N}_+, \quad A_1 \cap A_2 ... = \emptyset$

2. $A_i = \{0, i\}$

Answer: $A_1 \cup A_2 ... = \mathbb{N}, \quad A_1 \cap A_2 ... = \{0\}$

3. $A_i = \{x | x \in \mathbb{R}, 0 < x < i\}$

Answer: $A_1 \cup A_2 ... = \mathbb{R}_+, \quad A_1 \cap A_2 ... = \{x | x \in \mathbb{R}, 0 < x < 1\}$

Exercise 5

Let $B = \{\{1,2\},1,2,3\}$. List the power set $P(B)$.

Exercise 5

Let $B = \{\{1, 2\}, 1, 2, 3\}$. List the power set $P(B)$.

16 elements.

Exercise 6

Draw pictures to denote the sets on the $x-y$ plane.

1. $A = \{(x,y)|x \in [0,1], y \in [0,1]\}$

2. $B = \{(x,y)|x,y \in \mathbb{R}, x,y < 1\}$

3. $C = \{(x,y)|x \in \mathbb{R}, y \in \mathbb{Z}\}$

4. $D = \{(x,x+y)|x \in \mathbb{R}, y \in \mathbb{Z}\}$

5. $E = \{(x,|x+1|)|x \in \mathbb{R}\}$

Exercise 7

1. $f(x) = x^3 + 1, g(x) = x - 1, f \circ g =?, g \circ f =?$

2. $f(x) = |x| - 1, g(x) = |x|, f \circ g =?, g \circ f =?$

Exercise 7

1. $f(x) = x^3 + 1, g(x) = x - 1, f \circ g = ?, g \circ f = ?$

Answer: $f \circ g(x) = (x-1)^3 + 1 = x^3 - 3x^2 + 3x, g \circ f(x) = x^3$

2. $f(x) = |x| - 1, g(x) = |x|, f \circ g = ?, g \circ f = ?$

Answer:

$$f \circ g(x) = |x| - 1, g \circ f(x) = ||x| - 1| = \begin{cases} |x| - 1, x \leq -1 \text{ or } x \geq 1 \\ 1 - |x|, -1 < x < 1 \end{cases}$$

Exercise 8

Given $|A| = 7, |B| = 4$, what is
$|A + B|, |A - B|, |B - A|, |A \cap B|, |A \oplus B|$?

Exercise 8

Given $|A| = 7, |B| = 4$, what is
$|A+B|, |A-B|, |B-A|, |A \cap B|, |A \oplus B|$?

Answer.

1. $|A+B| \in [7, 11]$
2. $|A-B| \in [3, 7]$
3. $|B-A| \in [0, 4]$
4. $|A \cap B| \in [0, 4]$
5. $|A \oplus B| \in [3, 11]$

Exercise 9

How many subsets of $S = \{a, b, c, d\}$ include c but does not include d? List them.

Exercise 9

How many subsets of $S = \{a, b, c, d\}$ include c but does not include d? List them.

Answer: 4.

1. $\{c\}$

2. $\{a, c\}$

3. $\{b, c\}$

4. $\{a, b, c\}$

Exercise 10

Draw picture and write matrix of the following relation based on $S = \{1, 2, 3, 4, 5, 6\}$

$$R = \{(1,1),(1,3),(2,2),(2,4),(2,5),(3,1),$$
$$(3,5),(3,6),(4,5),(5,5),(5,6),(6,2),(6,4)\}$$

Exercise 10

Draw picture and write matrix of the following relation based on $S = \{1, 2, 3, 4, 5, 6\}$

$$R = \{(1,1),(1,3),(2,2),(2,4),(2,5),(3,1),\\ (3,5),(3,6),(4,5),(5,5),(5,6),(6,2),(6,4)\}$$

Solution.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$