MATH 333 Discrete Mathematics
Chapter 4 Formal Proof

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Definition

Direct Proof for $p \to q$.

1. Ordinary Proof: given p, we prove q.

2. Contradiction Proof: given p, suppose $\neg q$, then we get into a contradiction.

3. Contraposition Proof: given $\neg q$, prove $\neg p$.

Example: Prove only 2 is even prime number.

Definition

Induction Proof.

1. Ordinary Induction: prove $n = 1$, then suppose $n = k - 1$ holds and prove $n = k$.

2. Strong Induction: prove $n = 1$, then suppose $n = 1, ..., k - 1$ holds and prove $n = k$.

3. Multi-dimensional Induction: for 2−dimensional case, prove $m = 1, \forall n$ and $\forall m, n = 1$ both hold, then suppose $m = i, n = j$ holds and prove $m = i + 1, n = j$ and $m = i, n = j + 1$ hold.

Exercise

Prove $\sqrt{5} + \sqrt{22} < \sqrt{48}$.

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Proof.

$$
\begin{aligned}
\sqrt{5} + \sqrt{22} < \sqrt{48} &\iff \left(\sqrt{5} + \sqrt{22}\right)^2 < \left(\sqrt{48}\right)^2 \\
&\iff 27 + 2\sqrt{110} < 48 \\
&\iff 2\sqrt{110} < 21 \\
&\iff 440 < 441
\end{aligned}
$$

Exercise

Prove: For $\forall n \in \mathbb{Z}$, $n^2 + 3n + 4$ is even, i.e., $n^2 + 3n + 4 = 2k$ where $k \in \mathbb{Z}$.

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Prove: For $\forall n \in \mathbb{Z}$, $n^2 + 3n + 4$ is even, i.e., there $\exists k \in \mathbb{Z}$, so that $n^2 + 3n + 4 = 2k$.

Proof.

1. If n is odd, then n^2 is odd, $3n$ is odd, so $n^2 + 3n + 4$ is even.

2. If n is even, then n^2 is even, $3n$ is even, so $n^2 + 3n + 4$ is even.

Exercise

Prove: For $\forall a, b, c \in \mathbb{Z}$, if $a^2 + b^2 = c^2$, then at least one of a, b is even.

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Proof.

Suppose a, b are both odd, so a^2, b^2 are odd. Therefore c^2 is even, and c is even.

Then let $a = 2k_1 + 1$ and $b = 2k_2 + 1$. So

$$c^2 = (2k_1 + 1)^2 + (2k_2 + 1)^2 = 4(k_1^2 + k_2^2 + k_1 + k_2) + 2$$

Since c is an even integer, $c^2 \bmod 4 = c^2 \ \% \ 4 = 0$. However, we get into a contradiction. Therefore the original proposition holds.

Exercise

Prove: if $2^n - 1$ is prime, then n is prime.

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Proof.

Suppose n is not prime, then we prove $2^n - 1$ is not prime.

Let $n = p \cdot q$ where $p, q > 1$ are positive integers. Then we have

$$\begin{aligned} 2^n - 1 &= 2^{p \cdot q} - 1 \\ &= (2^p)^q - 1 \\ &= (2^p - 1) \cdot \left((2^p)^q + (2^p)^{q-1} + ... + 1\right) \end{aligned}$$

So $2^n - 1$ is not prime. The original proposition holds.

Exercise

Prove: For any odd number $n = 2k + 1 > 0$, there $\exists a, b \in \mathbb{N}$ so that $n = a^2 - b^2$. Here, we let $0 \in \mathbb{N}$.

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Proof.

Notice $1^2 - 0^2 = 1, 2^2 - 1^2 = 3, 3^2 - 2^2 = 5$, we can recall this

$$(k+1)^2 - k^2 = 2k + 1$$

So given arbitrary $n = 2k + 1$, we let $a = k + 1, b = k$.

Exercise

Prove: For $n \geq 2$, none of $(n! + 2, n! + 3, ..., n! + n)$ is prime. Here, we have $n \in \mathbb{N}$.

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Proof.

For $\forall 2 \leq k \leq n$, we have

$$\begin{aligned} n! + k &= (k-1)! \cdot k \cdot (k+1) \cdot ... \cdot n + k \\ &= k \cdot [(k-1)! \cdot (k+1) \cdot (k+2) \cdot ... \cdot n + 1] \end{aligned}$$

This is not prime.

Exercise

Prove: For $\forall n \in \mathbb{N}$, the following cannot be simplified.

$$\frac{21n+4}{14n+3}$$

For example, $2/4$ can be simplified to $1/2$.
Hint: If a/b cannot be simplified, then $(a-b)/b$ cannot be simplified, either.

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Hint: If a/b cannot be simplified, then $(a-b)/b$ cannot be simplified, either.

Proof.

$$\frac{21n+4}{14n+3} = 1 + \frac{7n+1}{14n+3} = 1 + \frac{7n+1}{2(7n+1)+1}$$

Obviously, k and $2k+1$ are always co-prime. So the proposition holds.

Exercise

Prove

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

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Proof.

When $n = 1$, it holds obviously. Suppose it holds for $n = k$, so we have

$$
\begin{aligned}
&1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6} \\
\Longleftrightarrow &1^2 + 2^2 + ... + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
\Longleftrightarrow &1^2 + 2^2 + ... + k^2 + (k+1)^2 = \frac{(k+1)([2k^2+k] + 6[k+1])}{6} \\
\Longleftrightarrow &1^2 + 2^2 + ... + k^2 + (k+1)^2 = \frac{(k+1)(2k^2+7k+6)}{6} \\
\Longleftrightarrow &1^2 + 2^2 + ... + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}
$$

Exercise

Prove

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

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Proof.

For $n = 1$, we have $1 \geq 1$. Suppose $n = k$ holds, then

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} \geq \sqrt{k}$$

Therefore we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

So we only need to prove

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$$

Exercise

We have

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1} \iff \sqrt{k(k+1)} + 1 \geq k + 1$$
$$\iff \sqrt{k(k+1)} \geq k$$
$$\iff k(k+1) \geq k^2$$

We know $n = k + 1$ obviously holds, so the original proposition holds.

Exercise

Prove

$$1^3 + 2^3 + 3^3 + ... + n^3 \geq \frac{1}{4}n^4$$

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Prove

$$1^3 + 2^3 + 3^3 + ... + n^3 \geq \frac{1}{4}n^4$$

Proof.

For $n = 1$, we have $1 \geq 1$. Suppose $n = k$ holds, then

$$1^3 + 2^3 + 3^3 + ... + k^3 \geq \frac{1}{4}k^4$$

Therefore we have

$$1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 \geq \frac{1}{4}k^4 + (k+1)^3$$

We only need to prove

$$\frac{1}{4}k^4 + (k+1)^3 \geq \frac{1}{4}(k+1)^4$$

Exercise

We have

$$\frac{1}{4}k^4 + (k+1)^3 \geq \frac{1}{4}(k+1)^4 \iff k^4 + 4(k+1)^3 \geq (k+1)^4$$
$$\iff k^4 + 4(k^3 + 3k^2 + 3k + 1) \geq (k+1)^4$$
$$\iff k^4 + 4k^3 + 12k^2 + 12k + 4 \geq (k+1)^4$$

We know

$$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$$

Therefore

$$k^4 + 4k^3 + 12k^2 + 12k + 4 \geq k^4 + 4k^3 + 6k^2 + 4k + 1$$

So $n = k + 1$ holds.

Exercise

Prove

$$\sum_{i=1}^{n} i \cdot (i+1) \cdot (i+2) = \frac{1}{4} n(n+1)(n+2)(n+3)$$

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$$\sum_{i=1}^{n} i \cdot (i+1) \cdot (i+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

Proof.

For $n = 1$, we have $1 \cdot 2 \cdot 3 = 1 \cdot 2 \cdot 3 \cdot 4/4$. Suppose $n = k$ holds, then we have

$$\sum_{i=1}^{k} i \cdot (i+1) \cdot (i+2) = \frac{1}{4}k(k+1)(k+2)(k+3)$$

So we have

$$\begin{aligned}\sum_{i=1}^{k+1} i \cdot (i+1) \cdot (i+2) &= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3) \\ &= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)\end{aligned}$$

Exercise

Prove $n! \geq n^n \cdot e^{-n}$. Hint:

$$\left(1+\frac{1}{n}\right)^n \leq e$$

Exercise

Prove $n! \geq n^n \cdot e^{-n}$. Hint:

$$\left(1 + \frac{1}{n}\right)^n \leq e$$

Proof.

For $n = 1$, we have $1 \geq 1 \cdot e^{-1}$. Suppose $n = k$ holds, then we have

$$k! \geq k^k \cdot e^{-k}$$

So for $n = k + 1$, we have

$$(k+1)! = k! \cdot (k+1) \geq k^k \cdot e^{-k} \cdot (k+1)$$

We only need to prove

$$k^k \cdot e^{-k} \cdot (k+1) \geq (k+1)^{k+1} \cdot e^{-(k+1)}$$

Exercise

We have

$$\begin{aligned}\frac{k^k\cdot(k+1)}{e^k}\geq\frac{(k+1)^{k+1}}{e^{k+1}} &\iff e\cdot k^k\cdot(k+1)\geq(k+1)^{k+1}\\ &\iff e\cdot k^k\geq(k+1)^k\\ &\iff e\geq\frac{(k+1)^k}{k^k}\\ &\iff e\geq\left(\frac{k+1}{k}\right)^k\\ &\iff e\geq\left(1+\frac{1}{k}\right)^k\end{aligned}$$

So, $n=k+1$ holds.