Nash Equilibrium in Hotelling's Game with Weighted Cost Function on a Line Segment

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Model

An infinite number of customers are uniformly distributed on a line segment. Some facilities are also distributed on this segment.

Each customer chooses a facility with lowest cost.

The cost is the ratio of distance and function $f(n')$ where n' is the number of facilities at that coordinate.

The simplest is $f(x) = x$.

Also we consider $f(x) = x^p$, where $p > 0$.

The utility of a facility is the ratio of number of customers choosing this coordinate and the number of facilities at this coordinate.

A case is Nash Equilibrium if no facility can increase its utility by moving to another candidate.

Some Examples and Nash Equilibrium

Figure 1: Example of 4 facilities on a unit line segment.

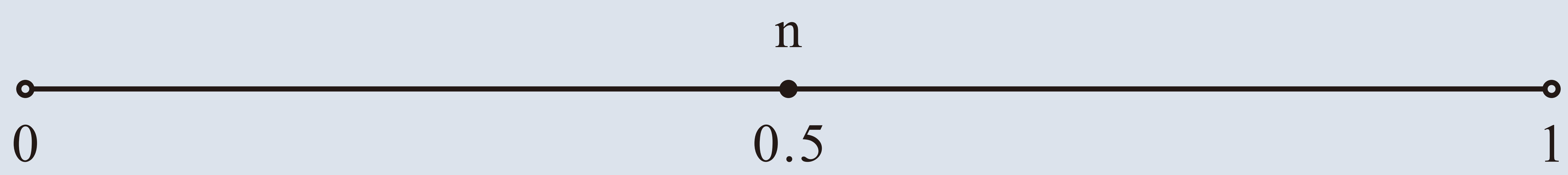

Figure 2: All facilities on the midpoint. This is always Nash equilibrium for all $p, n \geq 1$.

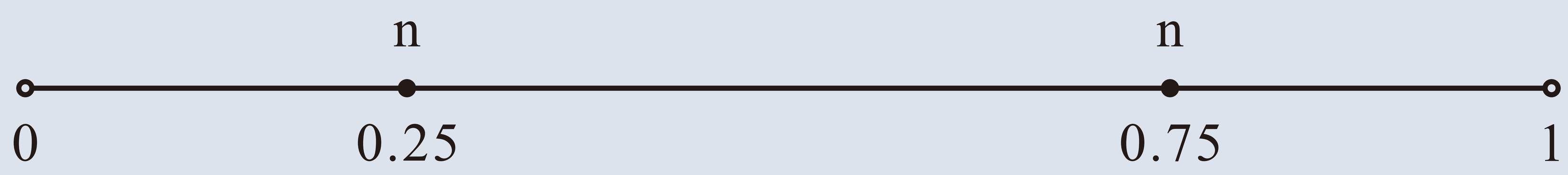

Figure 3: Half facilities on 0.25 and the other half on 0.75. This is always Nash equilibrium only for $1 \leq p \leq \log_3(2+\sqrt{5}) \approx 1.314$ and $n \geq 2$. (If $n \to \infty$, then $1 \leq p \leq 2$).

Bubble

Definition 1 Bubble*: If there exists a facility stack with discontinuous control regions on the line segment, then it is called a bubble.*

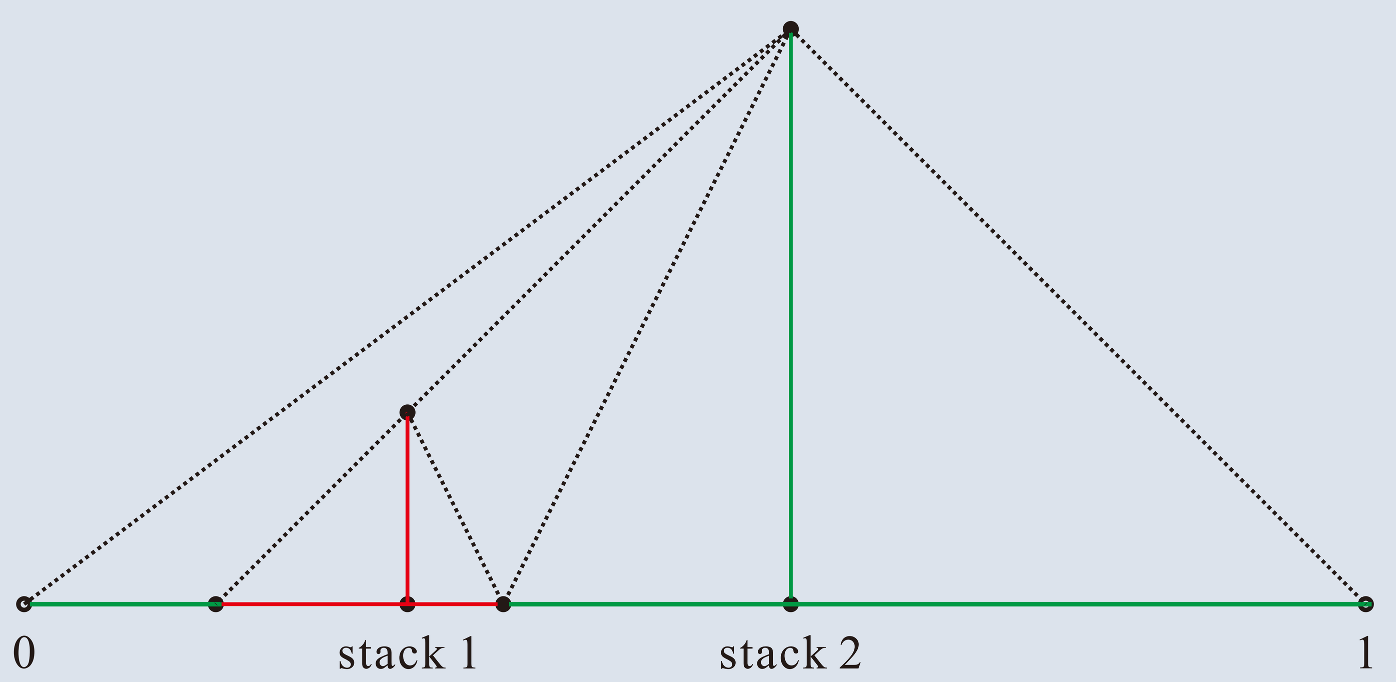

Figure 4: Example of a bubble, the red region is controlled by stack 1 and green region by stack 2.

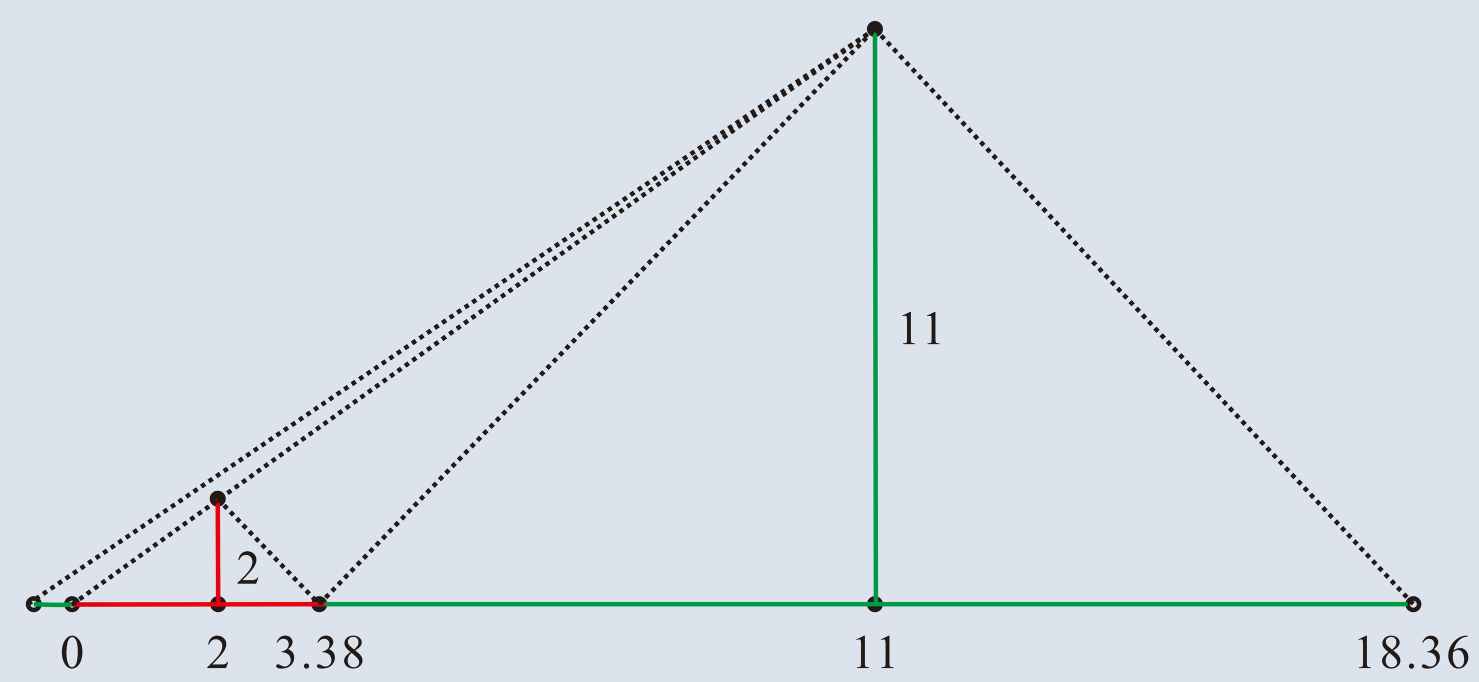

Figure 5: Example of Nash equilibrium with bubble, the bubble is on the leftmost with size very tiny, such as 0.01.

Why do we care about "bubble"?

We focus on Price of Anarchy (POA). This is the ratio between the total cost in Nash equilibrium and optimal cost globally. But the existence of bubble makes it difficult. So it's important to find the largest size of bubble to get an approximate POA by deleting the bubble.

Meaning of triangles above: Use similar triangles to calculate the control area of different stacks.

Results

Three main results for "bubble". Here we require that there should be only two stacks, i.e., there should be only two points where facility exists.

Lemma 1 *If $p \geq 2$, then there is no bubble with Nash equilibrium.*

Lemma 2 *When $p = 1$, the largest size of the bubble with Nash equilibrium is about 2% of the whole length with $n_1 = 2, n_2 \in [20, 25]$.*

Lemma 3 *When $p = 1$, if $n \to \infty$, then the largest size of the bubble with Nash equilibrium is about $1/(2n)$, where $n = n_1 + n_2$ and $2 \leq n_1 \leq O(\sqrt{n})$.*

Some main results for Nash Equilibrium and POA.

Lemma 4 *When $p < 1$ or all the facilities choose different coordinate, then there is no Nash equilibrium.*

Lemma 5 *The Price of Anarchy is no more than 2.*

Lemma 6 *Locating all the facilities into one point in interval $[s, 1-s](0 \leq s \leq 0.5)$ is always Nash equilibrium if and only if $(n-1)^p \geq 2n - 2s - 1$.*