Nash Equilibrium in Hotelling's Game with Weighted Cost Function on a Line Segment

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April 8th, 2022

Model

An infinite number of customers are uniformly distributed on a line segment. Some facilities are also distributed on this segment.

Each customer chooses a facility. The cost is the ratio of distance and function $f(n')$ where n' is the number of facilities at that coordinate. The simplest is $f(x) = x$. Also we consider $f(x) = x^p$, where $p > 0$.

The utility of a facility is the ratio of number of customers choosing this coordinate and the number of facilities at this coordinate. A case is Nash Equilibrium if no facility can increase its utility by moving to another candidate.

Figure 1: Example of 4 facilities on a unit line segment

Example of Nash Equilibrium

Figure 2 is always Nash equilibrium for all $p \geq 1$.
But figure 3 is always Nash equilibrium only for
$1 \leq p \leq \log_3(2+\sqrt{5}) \approx 1.314$ (If $n \to \infty$, then $p \leq 2$).
If $p < 1$ or all the facilities choose different coordinate, then there is no Nash equilibrium.

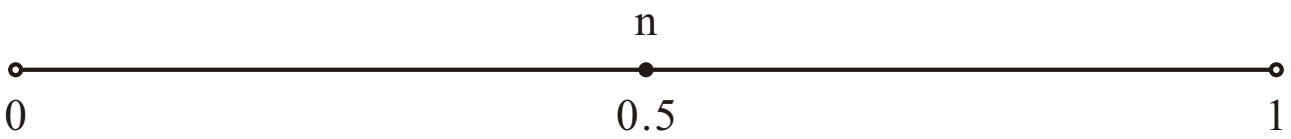

Figure 2: All facilities on the midpoint

Figure 3: Half facilities on 0.25 and the other half on 0.75

Bubble

Definition

Bubble: *If there exists a facility stack with discontinuous control regions on the line segment, then it is called a bubble.*

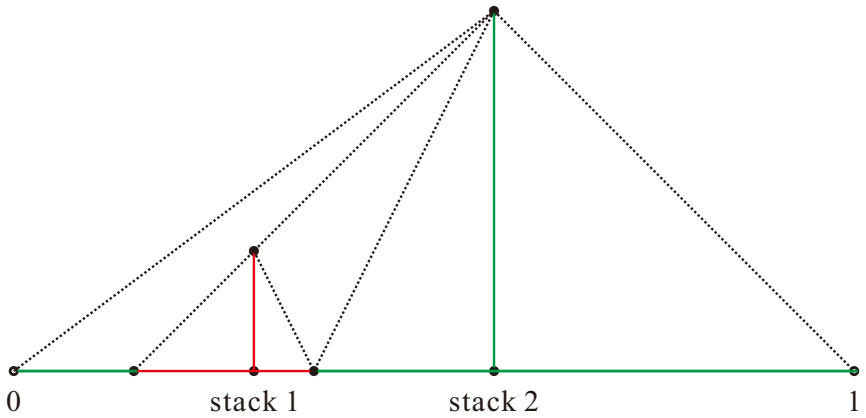

Figure 4: Example of a bubble, the red region is controlled by stack 1 and green region by stack 2

Existence of Bubble

Lemma

If $p = 1$, there exists bubble with Nash equilibrium.

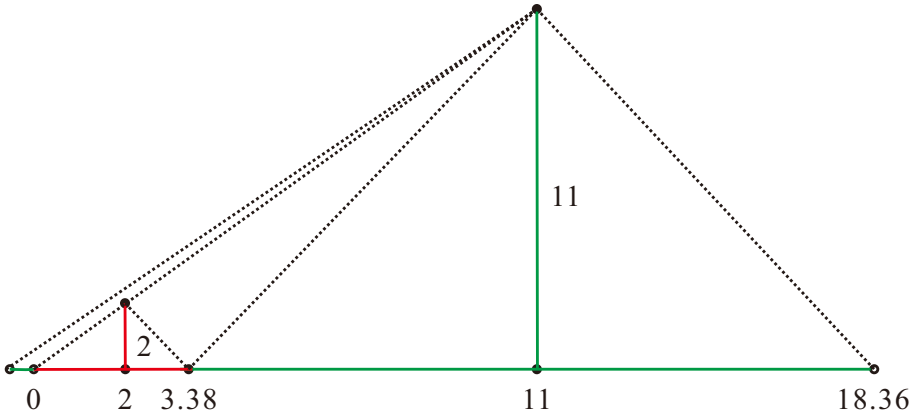

Figure 5: Example of a bubble with Nash equilibrium, the bubble is on the leftmost with size very tiny, such as 0.01

Conclusions about Bubble for 2-Stack

Suppose that the numbers of the two stacks are n_1, n_2 $(n_1 \leq n_2)$ and the length of the line segment is 1. Then we have the following results.

Lemma

If $p \geq 2$, then there is no bubble with Nash equilibrium.

Lemma

When $p = 1$, the largest size of the bubble with Nash equilibrium is about 0.02 with $n_1 = 2, n_2 \in [20, 25]$.

Lemma

When $p = 1$, if $n \to \infty$, then the largest size of the bubble with Nash equilibrium is about $1/(2n)$, where $n = n_1 + n_2$ and $2 \leq n_1 \leq O(\sqrt{n})$.