2011 USAMO P1

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Problem

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

Solution

Let $x = a + b, y = b + c, z = a + c$. It follows that

$$a = \frac{x + z - y}{2}, b = \frac{x + y - z}{2}, c = \frac{y + z - x}{2}.$$

Then plugging these numbers into the original equation we have

$$\sum_{\text{cyc}} \frac{x^2 - y^2 - z^2 + 2yz + 4}{4x^2} = \sum_{\text{cyc}} \frac{4 - y^2 - z^2 + x^2 + 2yz}{4x^2} \geq 3.$$

Notice that $4 - y^2 - z^2 \geq x^2$ so we have

$$\sum_{\text{cyc}} \frac{2x^2 + 2yz}{4x^2} = \frac{3}{2} + \frac{1}{2} \sum_{\text{cyc}} \frac{yz}{x^2} \geq 3.$$

Now we have

$$\frac{yz}{x^2} + \frac{xz}{y^2} + \frac{xy}{z^2} \geq 3\sqrt[3]{1} = 3$$

by AM-GM and we are done.