2017 IMO P1

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November 21, 2021

Problem

For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

Solution

We claim that a_0 works if and only if $3 \mid a_0$. We proceed with casework.

Case 1. $a_0 \equiv 2 \pmod 3$ This obviously doesn't work since no perfect square is 2 mod 3.

Case 2. $a_0 \equiv 0 \pmod 3$ We proceed with strong induction. Base case is $a_0 = 3$. This works because the cycle repeats $3, 6, 9, 3, \ldots$. Then let $a_0 = 3n$. Let the first square greater than a_0 be k^2. We would like to prove that $k < a_0$ so that we could use our induction to finish. If $k \geq a_0$ then consider the number $(a_0 - 1)^2$. We know that $(a_0 - 1)^2 \geq k^2$ which means $k \leq a_0 - 1$ so we are done because we already covered this because of the induction.

Case 3. $a_0 \equiv 1 \pmod 3$. We will also use strong induction for this case. $a_0 = 4, 7$ are base cases and it is easy to see that these don't work. For the $a_0 = 4$ case we have the sequence $4, 2, 5, 8, 11, \ldots$ which is covered by Case 1. $a_0 = 7$ is similar. Now let $a_0 = 3x + 1$ and k^2 is the smallest square number larger than a_0 such that $k^2 \equiv 1 \pmod 3$. So if $k \equiv 2 \pmod 3$ then we are done. Now we must have $k^2 \equiv 1 \pmod 3$. So now notice $(a_0 - 3)^2 > k^2$ because k^2 is the **smallest** square number which is 1 mod 3. Now this means $k < a_0 - 3$ which we have already covered because of our induction.