2017 ISL A1

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September 22, 2021

Problem

Let $a_1, a_2, \dots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

Solution

Notice that $a_i + x = (x+1) + \underbrace{1 + 1 + \cdots + 1}_{a_i - 1} \geq a_i(x+1)^{\frac{1}{a_i}}$ Multiplying each we get

$$(x+a_1)(x+a_2)\dots(x+a_n) \geq M(x+1)^k$$

Notice the equality case of the above AM-GM is when $x+1=1 \implies x=0$ which cannot happen because it must be a positive root and 0 is not positive.