2018 ISL A1

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Problem

Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

Solution

Let $P(x,y)$ denote the given assertion. Also notice

$$\begin{aligned} P\left(\frac{1}{f(1)},1\right) &\implies f\left(\frac{1}{f(1)}\right)=1 \\ P\left(x,\frac{1}{f(1)}\right) &\implies f(x)^2=f(x^2) \end{aligned}$$

Thus

$$f(y)=\frac{f((xf(y))^2)}{f(x)^2}=\frac{f(xf(y))^2}{f(x)^2}=\left(\frac{xf(y)}{f(x)}\right)^2.$$

This means that $f(y)$ is the square of a rational number. Now let $f(y)=g(y)^2$. Repeating the same process, we find that $g(y)$ is also the square of a rational number. If we continue this process indefinitely we find that $f(y)=1$ is the only possible solution.