2012 JMO P3

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Problem

Let a, b, c be positive real numbers. Prove that $\frac{a^3+3b^3}{5a+b} + \frac{b^3+3c^3}{5b+c} + \frac{c^3+3a^3}{5c+a} \geq \frac{2}{3}(a^2+b^2+c^2)$.

Solution

Notice that

$$\frac{a^3+3b^3}{5a+b}+\frac{b^3+3c^3}{5b+c}+\frac{c^3+3a^3}{5c+a}=\sum_{\text{cyc}}\frac{a^4}{5a^2+ab}+3\sum_{\text{cyc}}\frac{b^4}{5ab+b^2}.$$

Now applying Titu's on both expressions, we get

$$\begin{aligned}\sum_{\text{cyc}}\frac{a^4}{5a^2+ab} &\ge \frac{(a^2+b^2+c^2)^2}{5a^2+5b^2+5c^2+ab+bc+ca}\\ &\ge \frac{(a^2+b^2+c^2)^2}{6(a^2+b^2+c^2)}\\ &= \frac{a^2+b^2+c^2}{6}\end{aligned}$$

and

$$\begin{aligned}3\sum_{\text{cyc}}\frac{b^4}{5ab+b^2} &\ge 3\frac{(a^2+b^2+c^2)^2}{5ab+5bc+5ca+a^2+b^2+c^2}\\ &\ge 3\frac{(a^2+b^2+c^2)^2}{6(a^2+b^2+c^2)}\\ &= \frac{a^2+b^2+c^2}{2}\end{aligned}$$

Adding these two gives us the desired result. ■