2019 ISL G1

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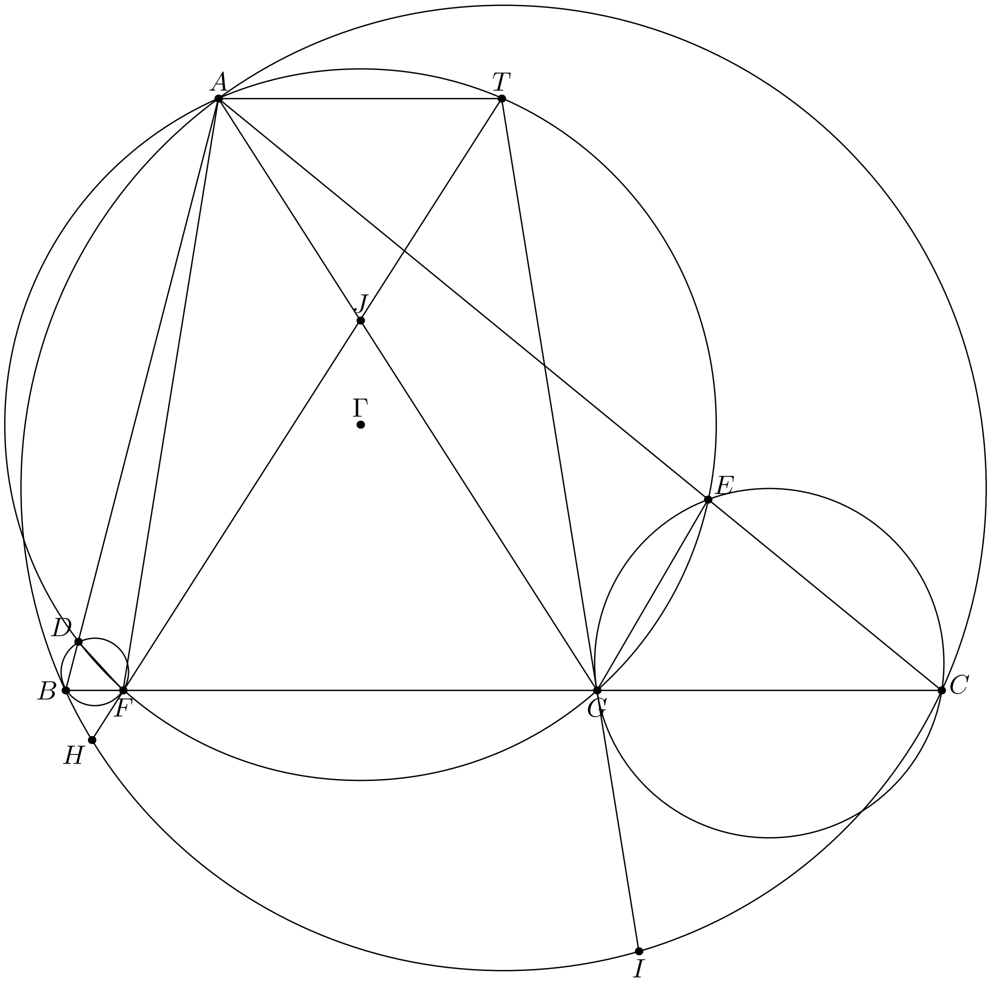
Problem

Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

Solution



Claim 1. $\angle TFG = \angle AGF$

Proof. Let $\angle TFG = \theta$. Thus we have

$$\angle TFG = \angle HFB = \angle BDF = 180 - \angle ADF = \theta \implies \angle ADF = 180 - \theta$$

Now consider the cyclic quadrilateral $ADFG$, this means that $\angle AGF = \theta$. □

Claim 2. $ATGF$ is a cyclic quadrilateral.

Proof. It suffices to prove that $\angle AFT = \angle AGT$. Let $\angle FAG = \alpha$, $\angle AJF = \beta$, and $\angle AGT = x$. We have $\angle JFA = 180 - \alpha - \beta$. Then we have

$$\angle TGF = \angle IGC = \angle GEC = 180 - \angle AEG = \theta + x \implies \angle AEG = 180 - \theta - x$$

Now consider the cyclic quadrilateral $AFGE$. Thus we have

$$\angle AEG + \angle AFG = (180 - \theta - x) + (180 - \alpha - \beta + \theta) = 180$$
$$\angle AGT = x = 180 - \alpha - \beta = \angle AFT$$

□

Claim 3. $ATGF$ is an isosceles trapezoid.

Proof. Obvious. $\square$

Because all trapezoids have thier bases parallel, we are done.