2012 ISL N2

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Problem

Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Solution

Taking $\pmod x$ we get that $x \mid 4024$. Notice that if $x \geq 503$ we arrive at a contradiction since $503^3 > 2012$ and $z^3 \geq xyz$ and $y^3 > 2$. Thus $x = 1, 2, 4, 8$. If $x \geq 4$ then $\nu_2(x^3(y^3+z^3)) \geq 6$ while $\nu_2(2012(xyz+2)) \leq 3$. Thus $x = 1, 2$.

Claim 1. $y + z \equiv 0 \pmod{503}$

Proof. We have

$$\begin{aligned} y^3 &\equiv -z^3 \pmod{503} \\ y^{502} &\equiv z^{502} \pmod{503} \\ y^{501} = (y^3)^{167} \equiv (-z^3)^{167} &= -z^{501} \pmod{503} \end{aligned}$$

Now multiplying both sides by y in the last congruence we have $y^{502} \equiv -z^{501} \cdot y \pmod{503}$ but because of the second congruence we have $y + z \equiv 0 \pmod{503}$. □

Case 1. $x = 1$
We have $(y+z)(y^2 - yz + z^2) = 2012(yz+2)$. Let $y + z = 503k$. Then since y, z have the same parity, k must be even. Then

$$k((y-z)^2 + yz) = 4yz + 8 \implies k(y-z)^2 + yz(k-4) = 8.$$

Since $yz \geq y + z - 1 \implies yz \geq 502$ we must have $k \leq 4$. This means that $k = 2, 4$.

Case 1.1 $k = 2$
We then have $y + z = 1006$. Thus $2(y-z)^2 - 2yz = 8 \implies (y+z)^2 - 5yz = 4$ which is a contradiction since $1006^2 - 4 \equiv 2 \pmod 5$.

Case 1.2 $k = 4$
We then have $y + z = 2012$. Then $y^2 - 2yz + z^2 = 2 \implies (y-z)^2 = 2$ which is a contradiction.

Case 2. $x = 2$
We get $y^3 + z^3 = 503(yz+1)$. Let $y + z = 503k$ and we get $k(y^2 - yz + z^2) = yz + 1$. Then $k(y-z)^2 + yz(k-1) = 1$. This means that $k \leq 1$ or that $k = 1$. Then we get $y - z = -1$ and then we find the solution $(2, 251, 252)$.

Thus our only solution is the ordered triplet $(2, 251, 252)$. ■