2002 ISL A1

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Problem

Find all functions f from the reals to the reals such that

$$f\left(f(x)+y\right)=2x+f\left(f(y)-x\right)$$

for all real x, y.

Solution

We claim that the only solution to this problem is $f(\alpha) = \alpha + c$ where c is any constant. It is easy to see that this function does indeed satisfy the condition.

Let $P(x, y)$ denote the assertion.

Claim 1. f is surjective.

Proof.

$$P(x, -f(x)) \implies f(0) = 2x + f(f(-f(x)) - x).$$

Then plugging in $x = \frac{f(0)-x}{2}$ we get

$$x = f\left(f\left(-f\left(\frac{f(0)-x}{2}\right)\right) - \frac{f(0)-x}{2}\right)$$

which also implies f is surjective. $\square$

Because Claim 1, there exists some u such that $f(u) = 0$. Then

$$P(u, x) \implies f(y) = 2u + f(f(y) - u).$$

Again because Claim 1, for all real α there must exist some y such that $f(y) - u = \alpha$ so we get $f(\alpha) = \alpha + u - 2u = \alpha - u$.