1985 IMO P1

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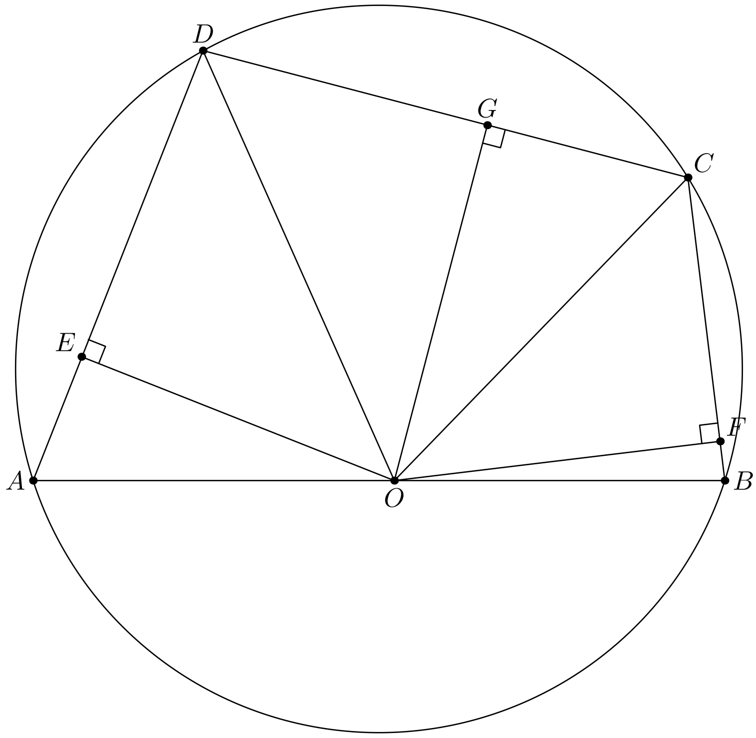
Problem

A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Solution



O is the center of the circle that is tangent to sides AD, DC, and BC. Since O is the center of that circle, we know that $OE = OG = OF$. Furthermore, since the three sides are tangent to the circle, $\angle DEO = \angle CGO = \angle CFO = 90^\circ$.

Claim 1. $\sin(x)\sin(2x-90) + \cos(x)\cos(2x-90) = \sin(x)$.

Proof. Because

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$$

we have

$$\begin{aligned}&\sin(x)\sin(2x-90) + \cos(x)\cos(2x-90)\\ &\qquad = \frac{1}{2}(\cos(x-90) - \cos(3x-90)) + \frac{1}{2}(\cos(3x-90) + \cos(x-90))\\ &\qquad = \cos(x-90)\\ &\qquad = \sin(x)\end{aligned}$$

□

Now let $OE = OG = OF = x$ and let $\angle EDO = \angle GDO = \alpha$ and let $\angle GCO = \angle CFO = \beta$. Now we have

$$AD = x\left(\tan(2\beta-90) + \frac{1}{\tan(\alpha)}\right)$$
$$BC = x\left(\tan(2\alpha-90) + \frac{1}{\tan(\beta)}\right)$$
$$AB = x\left(\frac{1}{\cos(2\beta-90)} + \frac{1}{\cos(2\alpha-90)}\right)$$

So it remains to prove

$$\tan(2\beta - 90) + \frac{1}{\tan(\beta)} + \tan(2\alpha - 90) + \frac{1}{\tan(\alpha)} = \frac{1}{\cos(2\beta - 90)} + \frac{1}{\cos(2\alpha - 90)}.$$

Because $\tan(x) = \frac{\sin(x)}{\cos(x)}$ we have

$$\frac{\sin(2\beta - 90)}{\cos(2\beta - 90)} + \frac{\cos(\beta)}{\sin(\beta)} + \frac{\sin(2\alpha - 90)}{\cos(2\alpha - 90)} + \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{1}{\cos(2\beta - 90)} + \frac{1}{\cos(2\alpha - 90)}.$$

So if we prove that

$$\frac{\sin(2x - 90)}{\cos(2x - 90)} + \frac{\cos(x)}{\sin(x)}$$

we would be done. Now notice

$$\frac{\sin(2x - 90)}{\cos(2x - 90)} + \frac{\cos(x)}{\sin(x)} = \frac{\sin(2x - 90)\sin(x) + \cos(x)\cos(2x - 90)}{\sin(x)\cos(2x - 90)} = \frac{1}{\cos(2x - 90)}$$
$$\implies \sin(2x - 90)\sin(x) + \cos(x)\cos(2x - 90) = \sin(x)$$

which is true because of Claim 1. ■