2020 IberoAmerican P2

LIN LIU

October 13, 2021

Problem

Let T_n denotes the least natural such that

$$n \mid 1 + 2 + 3 + \cdots + T_n = \sum_{i=1}^{T_n} i$$

Find all naturals m such that $m \geq T_m$.

Proposed by Nicolás De la Hoz

Solution

We claim that all m works except for when m is a power of two and is greater than 1.

Claim 1. All odd m works.

Proof. We know that $T_m = m$ works, but this might not be the smallest, so we have

$$T_m \leq m \implies m \geq T_m.$$

$\square$

Claim 2. $m = 1$ works.

Proof. Obvious. If you don't understand why this is true, stop reading this solution and go to first grade, please. $\square$

Claim 3. When m is a power of two, it doesn't satify the condition.

Proof. Notice that the condition implies that there exists a positive integer a such that $a(a+1) = 2m$. Now if a is even, then $a+1$ is odd which means $\nu_2(a+1) = 0$. So $\nu_2(a) + \nu_2(a+1) = \nu_2(a)$. Now let $m = 2^b$ and so $\nu_2(2m) = b+1$. Also notice that $\nu_2(a) \leq b$ because $a \leq m$, and hence a contradiction. The case when a is odd, and $a+1$ is even follows the same logic and we are done. $\square$

Claim 4. All m that is even and not a power of two work.

Proof. Let

$$a = 2^{\nu_2(m)+1}$$
$$b = \frac{m}{2^{\nu_2(m)}}$$

Now if $a > b$, let $a \equiv x \pmod b$. Then let y be the modular inverse of $a \mod b$. This means that $ay \equiv 1 \pmod b$ and $0 < y < b$. Because we know $0 < y < b$, we know that $ay < 2m$. Now if $ay \leq m$ then we can have $T_m = ay - 1$ and we would be done. Now if $ay > m$ then we consider the number $2n - ay$. Notice that

$$2n - ay \equiv 1 \pmod b$$
$$2n - ay \equiv 0 \pmod a$$

and so we can have $T_m = 2n - ay - 1$. Using the same proof as above for when $b > a$, we are done. □