2011 USAJMO P2

Lin Liu

October 1, 2021

Problem

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Prove that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

Solution

Let $x = a + b, y = b + c, z = a + c$. It follows that

$$a = \frac{x+z-y}{2}, b = \frac{x+y-z}{2}, c = \frac{y+z-x}{2}$$

Then plugging these numbers into the original equation we have

$$\sum_{\text{cyc}} \frac{x^2 - y^2 - z^2 + 2yz + 4}{4x^2} = \sum_{\text{cyc}} \frac{4 - y^2 - z^2 + x^2 + 2yz}{4x^2} \geq 3$$

Notice that $4 - y^2 - z^2 \geq x^2$ so we have

$$\sum_{\text{cyc}} \frac{2x^2 + 2yz}{4x^2} = \frac{3}{2} + \frac{1}{2} \sum_{\text{cyc}} \frac{yz}{x^2} \geq 3$$

Now we have

$$\frac{yz}{x^2} + \frac{xz}{y^2} + \frac{xy}{z^2} \geq 3\sqrt[3]{1} = 3$$

by AM-GM and we are done.