1997 IMO P5

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Problem

Find all pairs (a, b) of positive integers that satisfy the equation: $a^{b^2} = b^a$.

Solution

The answer is $(27,3),(16,2),(1,1)$.
It is easy to see that if $a^m = b^n$ and $a > b$ then $a = b^k$.

Case 1: $a > b$
Then $a = b^k$. Substituting we get

$$b^{k^{b^2}} = b^{b^2 k} = b^{b^k} \implies b^2 k = b^k \implies k = b^{k-2}$$

Notice that $1^{k-2} < k < 2^{k-2}$ if $k > 4$ which means that we only need to test out k from $[2,4]$. Testing out each each k we find the only solutions are $(27,3),(16,2)$.

Case 2: $a = b$
This is trivial, so I won't go into detail. In the end we get $(1,1)$.

Case 3: $a < b$
This means that $b = a^k$. Doing the same steps as above we get $k = a^{2k-1}$. Notice that $1^{2k-1} < k < 2^{2k-1}$ if $k > 1$. Now if $k = 1$ then we have Case 2 which we already covered.

Thus we can conlude that these are the only solutions.