2016 IMO P4

Lin Liu

December 19, 2021

Problem

A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

Solution

Notice that

$$\begin{aligned} k \equiv 2 \pmod 7 &\implies 7 \mid P(k), P(k+2) \\ k \equiv 1 \pmod 3 &\implies 3 \mid P(k), P(k+3) \\ k \equiv 7 \pmod{19} &\implies 19 \mid P(k), P(k+4) \end{aligned}$$

It is easy to see that $b \leq 5$ won't work. Now to prove that $b = 6$ can be achieved we can establish the congruence

$$\begin{aligned} a &\equiv 7 \pmod{19} \\ a &\equiv 1 \pmod 7 \\ a &\equiv 2 \pmod 3 \end{aligned}$$

solve gives us $a \equiv 197 \pmod{399}$ and we are done. ■