2004 ISL A5

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Problem

If a, b, c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

Solution

Let

$$x = \sqrt[3]{\frac{1}{a} + 6b}, y = \sqrt[3]{\frac{1}{b} + 6c}, z = \sqrt[3]{\frac{1}{c} + 6a}.$$

Then by CS, we have

$$(1+1+1)(1+1+1)(x^3+y^3+z^3) \geq (x+y+z)^3.$$

Thus we have

$$\begin{aligned}
\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} &\leq \sqrt[3]{9\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 6a + 6b + 6c\right)} \\
&= \sqrt[3]{9\left(\frac{ab+bc+ca+6(abc)(a+b+c)}{abc}\right)} \\
&= \sqrt[3]{9\left(\frac{1+6(abc)(a+b+c)}{abc}\right)}
\end{aligned}$$

Claim 1. $6(abc)(a+b+c) \leq 2(ab+bc+ca)^2$

Proof. Dividing by 2 on both sides and expanding it remains to prove

$$a^2bc + ab^2 + abc^2 \leq a^2b^2 + b^2c^2 + c^2a^2.$$

If we let $x = ab, y = bc, z = ca$ then we get

$$xy + yz + zx \leq x^2 + y^2 + z^2$$

which is obvious by rearrangement inequality. □

Because Claim 1, we have

$$\begin{aligned}
\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} &\leq \sqrt[3]{9\left(\frac{1+2(ab+bc+ca)^2}{abc}\right)} \\
&= \sqrt[3]{9\left(\frac{3}{abc}\right)} \\
&= \frac{3}{\sqrt[3]{abc}} \\
&= \frac{3\sqrt[3]{a^2b^2c^2}}{abc} \\
&\leq \frac{ab+bc+ca}{abc} \\
&= \frac{1}{abc}
\end{aligned}$$