# 2002 USAMO P4

Lin Liu

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## Problem

Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers $x$ and $y$.

## Solution

Let $P(x, y)$ denote the assertion.

$$P(0,0) \implies f(0) = 0$$
$$P(x,0) \implies f(x^2) = xf(x)$$

Thus the original condition turns into

$$f(x^2 - y^2) = f(x^2) - f(y^2).$$

Now we have

$$P(\sqrt{a+b}, \sqrt{b}) \implies f(a) = f(a+b) - f(b) \implies f(a+b) = f(a) + f(b)$$

which means that $f$ is additive. Then we have

$$f((x+1)^2) = (x+1)f(x+1)$$
$$f(x^2) + f(x) + f(x) + f(1) = xf(x) + f(x) + xf(1) + f(1)$$
$$f(x) = xf(1)$$

and we are done. ■