2019 ISL G1

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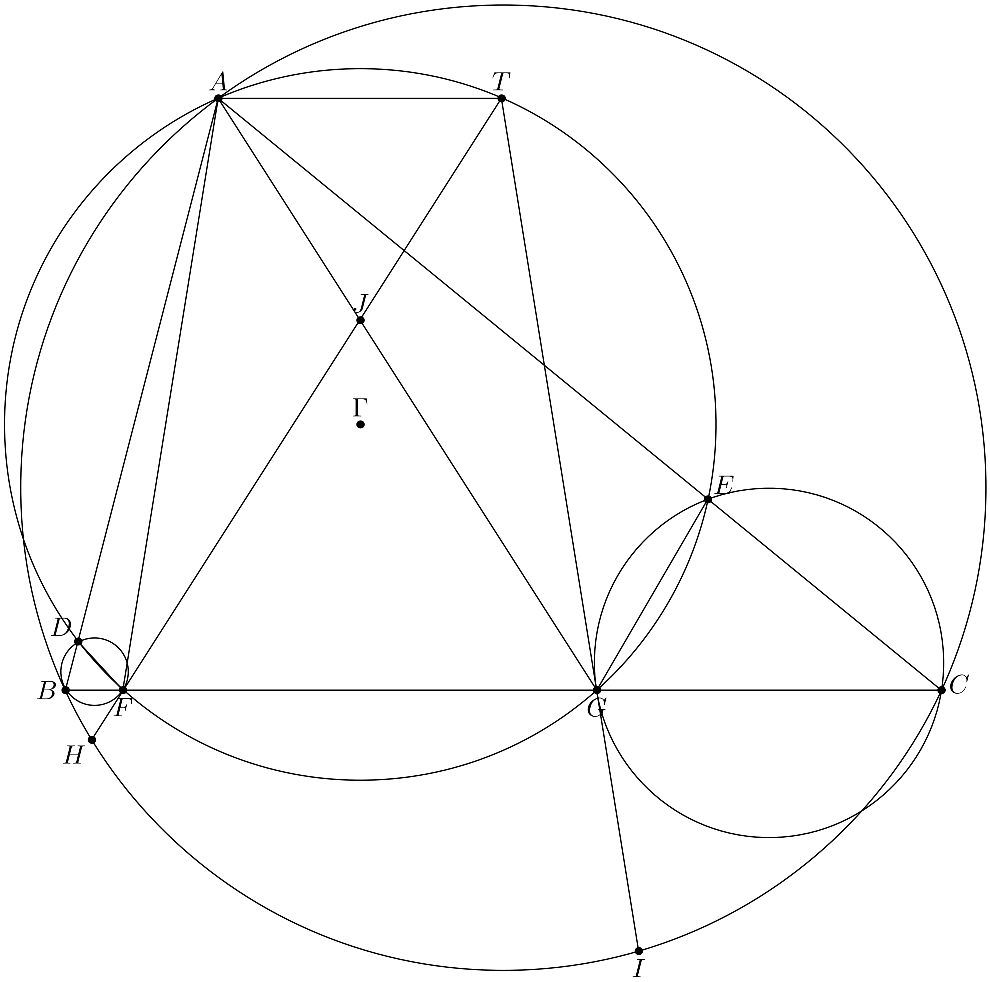
Problem

Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

Solution



Claim 1. $\measuredangle GFT = \measuredangle AGF$

Proof. We have

$$\measuredangle GFT = \measuredangle BFH = \measuredangle BDF = \measuredangle ADF$$

Now consider the cyclic quadrilateral $ADFG$, this means that $\measuredangle GFT = \measuredangle AGF$. $\square$

Claim 2. $ATGF$ is a cyclic quadrilateral.

Proof. It suffices to prove that $\measuredangle TFA = \measuredangle TGA$. Notice that $\measuredangle TGF = \measuredangle TGA + \measuredangle AGF$. Then we have

$$\measuredangle TGF = \measuredangle IGC = \measuredangle GEC = \measuredangle GEA \implies \measuredangle GEA = \measuredangle TGA + \measuredangle AGF$$

Now consider the cyclic quadrilateral $AFGE$. Thus we have

$$\measuredangle GEA = \measuredangle GFA \implies \measuredangle TGA + \measuredangle AGF = \measuredangle GFT + \measuredangle TFA$$

Since we know that $\measuredangle AGF = \measuredangle GFT$ from Claim 1 we get that $\measuredangle TGA = \measuredangle TFA$. $\square$

Claim 3. $ATGF$ is an isosceles trapezoid.

Proof. Obvious. $\square$

Because all trapezoids have their bases parallel, we are done.