2008 ISL N2

Lin Liu

December 19, 2021

Problem

Let a_1, a_2, $\ldots$, a_n be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1$, $3a_2$, $\ldots$, $3a_n$.

Solution

WLOG the set of elements is sorted such that a_n is the largest and a_1 is the smallest.

Now notice that if we find some i such that $a_n + a_i \not\equiv 0 \pmod 3$ then we would be done because if there exists some k such that $a_n + a_i \mid 3a_k$ then $3a_k = a_n + a_i$ or $3a_k = 2(a_n + a_i)$ because of a simple bounding argument. Obviously none of these equations have a solution so we are done. So now we must have

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{n-1} \equiv -a_n \pmod 3.$$

Establishing this, we will proceed with a contradiction-construction argument. First choose two numbers a_1, a_n such that $a_n > a_1$ and $a_n \neq 2a_1$. We will designate a_n as the largest in the set and each element that we add in the set must be smaller than a_n.

Case 1. $2a_1 > a_n$
Define $f(x) = \frac{x+a_n}{3}$. Since we are trying to achieve a contradiction, we must add $f(a_1)$ to the set. If we added $2f(a_1)$ to the set then $2f(a_1)$ would become the new largest which we don't want to happen. Then we continue and add $f(f(a_1))$ the the set in order to satisfy the condition. Notice that this process is decreasing meaning that

$$f(a_1) > f(f(a_1)) > f(f(f(a_1))) > \dots.$$

Note that in each step of the process we will assume that it will remain an integer. Thus we will eventually end at $\lfloor \frac{a_n}{2} \rfloor + 1$.

Case 1.1. a_n is even
The process will end at $\frac{a_n}{2} + 1$ and it is easy to see that $a_n + \frac{a_n}{2} + 1 \equiv 1 \pmod 3$.

Case 1.2. a_n is odd
It will end at $\frac{a_n-1}{2} + 1 = \frac{a_n+1}{2}$. It is also easy to see that $a_n + \frac{a_n+1}{2} \equiv 2 \pmod 3$ so we are done.

Case 2. $2a_1 < a_n$.
Like the above case we will define $f(x) = \frac{x+a_n}{3}$. If we decide to use $2f(a_1)$ then this will be a continuation of Case 1 and we would be done. So we must use $f(a_1)$ and $f(f(a_1))$ and so on. Now notice that this process is increasing meaning that

$$f(a_1) < f(f(a_1)) < f(f(f(a_1))) < \dots.$$

Note that in each step of the process we will assume that it will remain an integer.

Case 2.1. a_n is even
This process will eventually end at $\frac{a_n}{2} - 1$ and since $a_n + \frac{a_n}{2} - 1 \equiv 2 \pmod 3$ we are done with this case.

Case 2.2. a_n is odd
This process will end at $\lfloor \frac{a_n}{2} \rfloor = \frac{a_n-1}{2}$ and since $a_n + \frac{a_n-1}{2} \equiv 1 \pmod 3$ we are done.

Combining all of these cases gives us the desired contradiction and we are done. ■