2000 IMO P2 / 2000 ISL A1

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Problem

Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Solution

Let $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$. Thus we have

$$\left(\frac{x+z-y}{y}\right)\left(\frac{x+y-z}{z}\right)\left(\frac{y+z-x}{x}\right) \leq 1 \implies (x+z-y)(x+y-z)(y+z-x) \leq xyz.$$

Now expanding LHS we get

$$x^3 + y^3 + z^3 - x^2y - x^2z - y^2z - y^2x - z^2y - z^2x + 3xyz \geq 0$$

which is true by Schur's.