2011 USAJMO P4

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November 26, 2021

Problem

A word is defined as any finite string of letters. A word is a palindrome if it reads the same backwards and forwards. Let a sequence of words $W_0, W_1, W_2, ...$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \geq 2$, W_n is the word formed by writing W_{n-2} followed by W_{n-1}. Prove that for any $n \geq 1$, the word formed by writing $W_1, W_2, W_3, ..., W_n$ in succession is a palindrome.

Solution

Because we want to prove it for any integer $n \geq 1$ we think of using induction! Define S_n to be the concatenation of $W_1, W_2, W_3, \ldots, W_n$. Now the base cases are trivial, so lets move onto the inductive step. We assume that S_{n-1} is a palindrome and now we have the inductive step. We have

$$\begin{aligned} S_k &= S_{k-1}W_k \\ &= S_{k-3}W_{k-2}W_{k-1}W_{k-2}W_{k-1} \end{aligned}$$

Now because we have S_{k-1} is a palindrome, reversing $S_{k-3}W_{k-2}W_{k-1}$ will lead to S_k being a palindrome.