

2019 ISL A1

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Problem

Let \mathbb{Z} be the set of integers. Determine all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that, for all integers a and b ,

$$f(2a) + 2f(b) = f(f(a + b)).$$

Solution

Let $P(x, y)$ denote the assertion. Then

$$P(1, n-1) \implies f(2) + 2f(n-1) = f(f(n))$$

And

$$P(0, n) \implies f(0) + 2f(n) = f(f(n))$$

Then

$$f(2) + 2f(n-1) = f(0) + 2f(n) \implies f(n) - f(n-1) = c$$

This means that f is a linear function. Then we find that $f(x) = 2x + c$ or $f(x) = 0$ are the only possible solutions.