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Problem

Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.

Solution

Claim 1. If k is an odd positive integer and p is a prime, then we have $a^p + 1 \mid a^{kp} + 1$.

Proof. Notice that

$$\begin{aligned}
&(a^p+1)(a^{(k-1)p} - a^{(k-2)p} + a^{(k-3)p} - \cdots + 1) \\
&\quad = (a^{kp} - a^{(k-1)p} + a^{(k-2)p} - \cdots + a^p) + (a^{(k-1)p} - a^{(k-2)p} + a^{(k-3)p} - \cdots + 1) \\
&\quad = a^{kp} + 1
\end{aligned}$$

□

Then because of Claim 1, we know that for all $1 \leq i \leq n$ we have $2^{p_i}+1 \mid 2^{p_1p_2\cdots p_n}+1$. Let S be the divisors of the number $p_1p_2\ldots p_n$ in **sorted** order meaning that $S_1 < S_2 < \ldots$. It is easy to see that there are 2^n elements in S. Let S_1 denote the first element of the set. Also let i, j be some positive integers such that $1 \leq i \leq 2^n$ and $j < i$. Now, call a prime "new" if it divides $2^{S_i} + 1$ and none of the numbers $2^{S_j} + 1$ divide this "new" prime. Now considering $S_1 = 1$, we have $2^{S_1} + 1 = 3$ which is a "new" prime. Now for all $1 < k \leq 2^n$ we consider the number $2^{S_k} + 1$. By Zsigmondy's it must have a "new" prime divisor since the set S is sorted. Thus we can conlude that there are 2^{2^n} divisors of $2^{p_1p_2\cdots p_n} + 1$. Since 2^{2^n} is obviously greater than or equal to 4^n, we are done.