2005 USAMO P1

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Problem

Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Solution

We will prove that the only time that n doesn't work is when $n = pq$ where p, q are distinct primes.

It is easy to see that if $n = pq$ where p and q are distinct primes, it is impossible to place p, q, pq in a circle such that p and q are not bordering. If $n = p^k$ it is easy to see that it will work since any divisor of n that is greater than 1 will be a multiple of p. Now we will prove that other n work as well.

Let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ where $p_1, p_2, \dots, p_k$ are distinct primes and is in strictly increasing order. That is, $p_1 < p_2 < \dots < p_k$.

Now we define A_i from $1 \le i \le k$ to be the **strictly increasing** set containing the divisors of n such that the divisor is a multiple of p_i but is not a multiple of p_j with $j < i$. We will now perform two changes on A_1.

- Take the last element of A_1 and move it to the front of the set without changing the order of the other elements.
- Take the element $p_1 \cdot p_2$ and move it to the end of the set without changing the order of the other elements.

For the second operation, it is quite obvious that the number $p_1 \cdot p_2$ does appear in A_1 since it is a multiple of p_1. Now imagine if we strung together all of these sets into a circle so that the last element of some A_j and the first element of A_{j+1} are neighboring. If $j = k$ then the last element of A_k and the first element of A_1 are neighboring. We will claim that this construction works. It is easy to see that the first element of A_1 and the last element of A_k are not coprime because the first element of A_1 is now n. Notice that the last element of A_1 is now $p_1 p_2$ and the first element of A_2 is p_2 so the connection between A_1 and A_2 works. Furthermore, it is easy to see that if $1 < j < k$ then the last element of A_j, $p_j p_{j+1} \dots p_k$ and the first element of A_{j+1} which is p_{j+1} are obviously not coprime so we are done. $\blacksquare$