2007 USAMO P1

Lin Liu

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Problem

Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \leq a_k \leq k - 1$ for which $a_1 + a_2 + ... + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, ...$ Prove that for any n the sequence $a_1, a_2, ...$ eventually becomes constant.

Solution

Consider $b_i = \frac{a_1+a_2+\ldots+a_i}{i}$. Then notice that b_i is decreasing because

$$b_{i+1} = \frac{a_1 + a_2 + \ldots + a_{i+1}}{i+1} > \frac{a_1 + a_2 + \ldots + i}{i} = b_i + 1$$

Also notice that $b_i > 0$ for all $i > 0$ which means that b will become a constant eventually which means that a will also become a constant eventually.