2018 IberoAmerican P2

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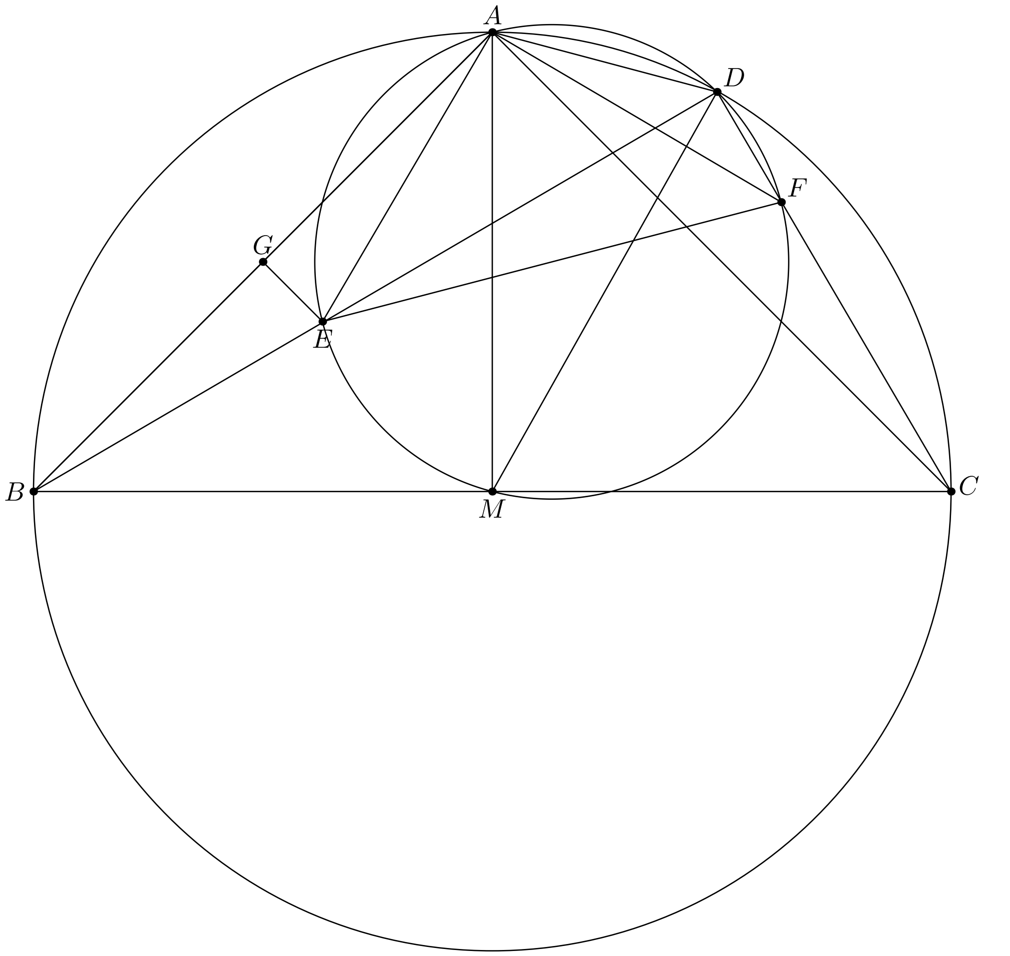
Problem

Let ABC be a triangle such that $\angle BAC = 90^\circ$ and $AB = AC$. Let M be the midpoint of BC. A point $D \neq A$ is chosen on the semicircle with diameter BC that contains A. The circumcircle of triangle DAM cuts lines DB and DC at E and F respectively. Show that $BE = CF$.

Solution



For the sake of my sanity, call $\measuredangle EBG = \alpha$. Also let G be the midpoint of AB. Then we do a bit of angle chasing.

Because $\measuredangle DCA = \alpha$, we have $\measuredangle DMA = \measuredangle DEA = 2\alpha$. Also notice that since $\measuredangle ACB = 45°$, $\measuredangle ADB = \measuredangle ADE = 45°$ as well. Notice $\measuredangle CBD = 45 - \alpha$ and $\measuredangle DCB = 45 + \alpha$.

Claim 1. EF is diameter of (ADM)

Proof. Because $\measuredangle BDC = 90°$, $\measuredangle EDF = 90°$. □

Claim 2. $\measuredangle FED = 45 - 2\alpha$, $\measuredangle DFE = 45 + 2\alpha$

Proof. Because Claim 1, we have $\measuredangle EDF = 90°$ we have

$$\measuredangle ADF = \measuredangle ADE + \measuredangle EDF = 135°.$$

Now considering the cyclic quadrilateral $ADFE$ we know that

$$\measuredangle FEA = \measuredangle DEA + \measuredangle FED = 2\alpha + \measuredangle FED = 45°$$

where the last equality is from $\measuredangle ADF + \measuredangle FEA = 180°$. So we know that $\measuredangle FED = 45 - 2\alpha$. This also shows that $\measuredangle DFE = 45 + 2\alpha$ because of Claim 1. □

Claim 3. $\frac{BC}{EF} = 2\cos(\alpha)$

Proof. Consider $\triangle AEF$. It is a $45-45-90$ triangle and is similar to $\triangle ABC$. Furthermore, we know that

$$\measuredangle AEB = 180 - \measuredangle DEA = 180 - 2\alpha.$$

This means that $\measuredangle BAE = \alpha$ implying $\triangle AEB$ is isosceles. Since G is the midpoint of AB we know that $\measuredangle BGE = 90°$. Then $\frac{BG}{EB} = \cos(\alpha)$ and since

$$\frac{BG}{EB} = \frac{AB}{2AE} = \cos(\alpha) \implies \frac{AB}{AE} = 2\cos(\alpha).$$

Since $\triangle AEF \sim \triangle ABC$ we also know that $\frac{BC}{EF} = 2\cos(\alpha)$. □

Claim 4. $BE = CF$

Proof.

$$\begin{aligned} BD &= BC\sin(45+\alpha) \\ ED &= EF\sin(45+2\alpha) \\ CD &= BC\sin(45-\alpha) \\ FD &= EF\sin(45-2\alpha) \end{aligned}$$

Now

$$\begin{aligned} BE &= BD - ED \\ &= BC\sin(45+\alpha) - EF\sin(45+2\alpha) \\ &= \frac{\sqrt{2}}{2}BC(\sin(\alpha)+\cos(\alpha)) - \frac{\sqrt{2}}{2}EF(\sin(2\alpha)+\cos(2\alpha)) \end{aligned}$$

And

$$\begin{aligned} CF &= CD - FD \\ &= BC\sin(45-\alpha) - EF\sin(45-2\alpha) \\ &= \frac{\sqrt{2}}{2}BC(\cos(\alpha)-\sin(\alpha)) - \frac{\sqrt{2}}{2}EF(\cos(2\alpha)-\sin(2\alpha)) \end{aligned}$$

If we set these two to be equal to each other we get

$$\begin{aligned} BC(\sin(\alpha)+\cos(\alpha)) - EF(\sin(2\alpha)+&\cos(2\alpha)) \\ &= BC(\cos(\alpha)-\sin(\alpha)) - EF(\cos(2\alpha)-\sin(2\alpha)) \\ BC(-2\sin(\alpha)) &= EF(-2\sin(2\alpha)) \\ \frac{BC}{EF} &= 2\cos(\alpha) \end{aligned}$$

which we have already proved in Claim 3 and thus completing our proof. □