1991 ISL N4

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Problem

Find all triples (x, y, z) such that $x, y, z \in \mathbb{Z}^+$ and that

$$3^x + 4^y = 5^z.$$

Solution

$$\begin{aligned}(-1)^x \equiv 1^y \pmod 4 &\implies 2 \mid x\\ 1^y \equiv 2^z \pmod 3 &\implies 2 \mid z\end{aligned}$$

Let $x = 2a, z = 2c$ and thus we have

$$2^{2y} = 5^{2c} - 3^{2a} = (5^c - 3^a)(5^c + 3^a).$$

Notice that $5^c \equiv 1 \pmod 4$ and that $3^a \equiv \pm 1 \pmod 4$. This means that exactly 1 of $5^c - 3^a, 5^c + 3^a$ is divisible by 4. Now this means that one of them is equal to 2.

Case 1: $5^c + 3^a = 2$
This means that $a = c = 0$ which is a contradiction since $x, y, z > 0$.

Case 2: $5^c - 3^a = 2$
It is easy to see that $(a, c) = (1, 1)$ is a solution. We claim that this is the only solution. For the rest of the solution we assume that $a, c > 0$.
Taking mod 5 we get

$$-3^a \equiv 2 \pmod 5 \implies 3^a \equiv 3 \pmod 5$$

The cycle is $(3, 4, 2, 1)$ which means that $a \equiv 1 \pmod 4$. Now let $a = 4\beta + 1$.

Taking mod 16 we get

$$5^c - 3^{4\beta} \cdot 3 \equiv 2 \pmod{16} \implies 5^c \equiv 5 \pmod{16}$$

The cycle is $5, 9, 13, 1$ which means that

$$c \equiv 1 \pmod 4 \qquad (2)$$

Let $c = 4\alpha + 1$.

Taking mod 13 we get

$$5^{4\alpha} \cdot 5 - 3^a \equiv 2 \pmod{13} \implies 3^a \equiv 3 \pmod{13}$$

The cycle is $3, 9, 1$ which means that

$$a \equiv 1 \pmod 3 \qquad (3)$$

Combining (1) and (3) we get

$$a \equiv 1 \pmod{12} \qquad (4)$$

Let $a = 12\lambda + 1$.

Taking mod 7 we get

$$5^c - 3^{12\lambda} \cdot 3 \equiv 2 \pmod 7 \implies 5^c \equiv 5 \pmod 7$$

The cycle is $5, 4, 6, 2, 3, 1$ which means that

$$c \equiv 1 \pmod 6 \qquad (5)$$

Combining this with (2) we get

$$c \equiv 1 \pmod{12} \qquad (6)$$

Finally taking mod 9 we get

$$5^c \equiv 2 \pmod 9$$

Note that since we assumed that $a, c > 0$ this means that from (4) we have $\lambda > 0$ which means that $3^{12\lambda+1} \equiv 0 \pmod 9$. The cycle is $5, 7, 8, 4, 2, 1$ which means that $c \equiv 5 \pmod 6$ which is a contradiction with (5).
Since $(a, c) = (1, 1)$ is the only solution to the above diophantine equation, we get that $x = z = 2$. Thus the only solution is $\boxed{(2, 2, 2)}$.