2013 IMO P4

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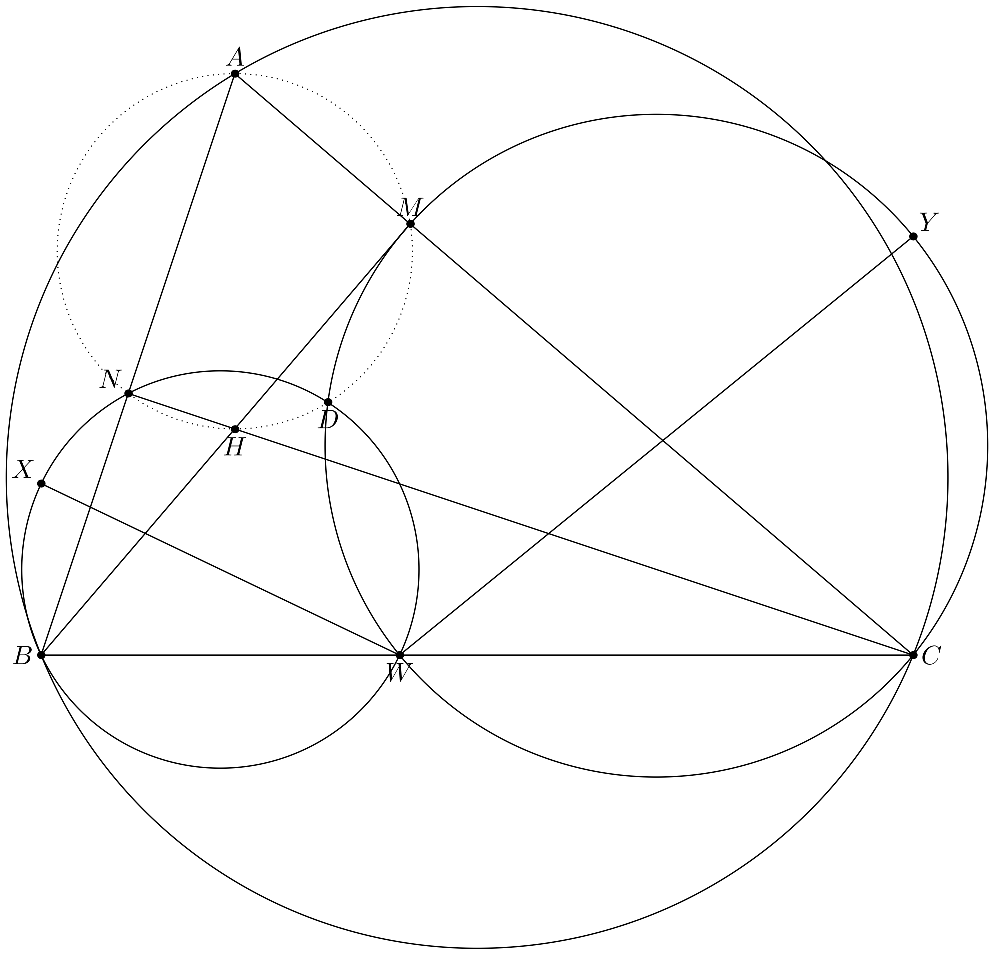
Problem

Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Solution



Claim 1. $(AMN), (BNW), (CMW)$ concur at a point D.

Proof. Notice that D is the miquel point of the triangle, so we are done. □

Claim 2. $AMDHN$ is a cyclic pentagon.

Proof. Notice that $AMHN$ is cyclic because $\measuredangle ANH = \measuredangle AMH = 90°$. □

Claim 3. A, D, W are collinear.

Proof. Notice that

$$\measuredangle MDA = \measuredangle MNA = \measuredangle ACB = 180 - \measuredangle WDM$$

□

Claim 4. X, H, Y are collinear.

Proof. We know that $\measuredangle WDY = 90°$ and $\measuredangle ADH = 90°$ and because Claim 3, we know that H, D, Y are collinear. Analogously, we know that X, H, D are collinear. Thus X, H, Y are collinear. □