2010 USAMO P1

Lin Liu

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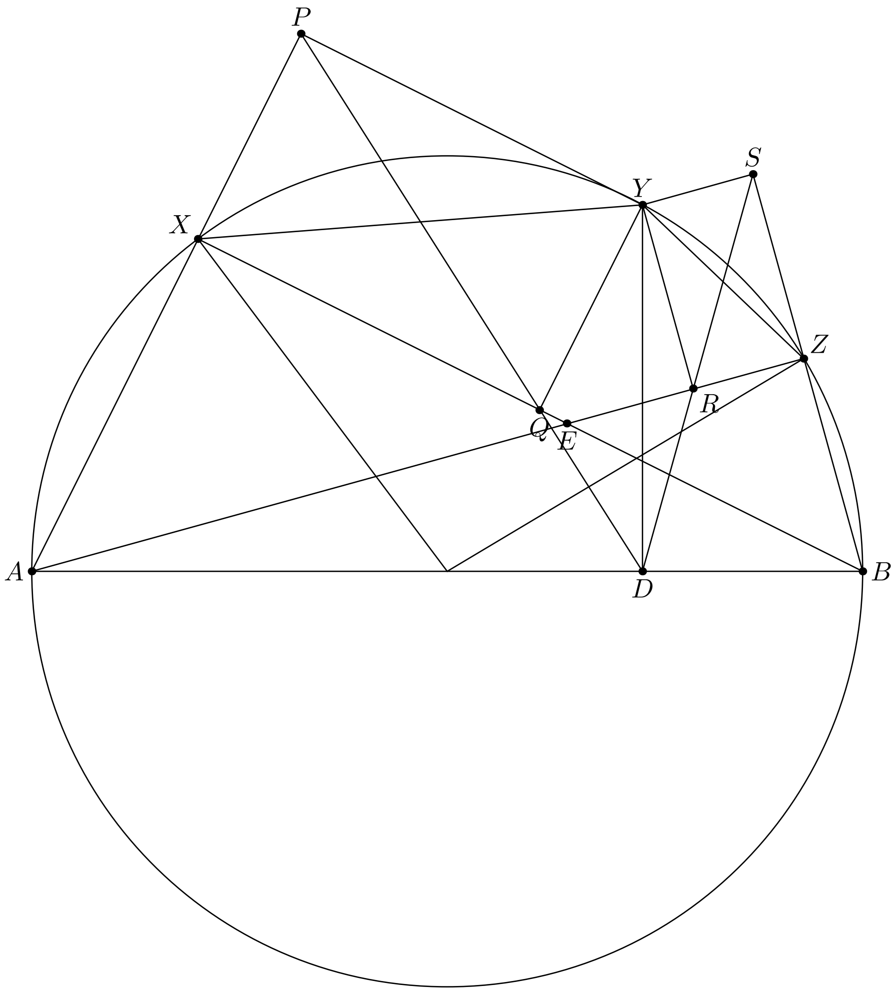
Problem

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Solution



Let D be the foot of the altitude from Y to AB and let $E = \overline{BX} \cap \overline{AZ}$.

Claim 1. $D = \overline{PQ} \cap \overline{SR}$.

Proof. Notice that PQ and SR are simson lines and they must intersect at the foot of the altitude from Y to AB which is the point D. $\square$

Claim 2. $\angle SDP = \angle YQP + \angle SRY - \angle QYR$

Proof. Notice that $\angle DQY = 180 - \angle YQP$ and that $\angle YRD = 180 - \angle SRY$. Also we know that

$$\begin{aligned} &\angle QYR + \angle DQY + \angle RDQ + \angle YRD = 360^\circ \\ \implies &\angle QYR + 180 - \angle YQP + \angle RDQ + 180 - \angle SRY = 360^\circ \\ \implies &\angle RDQ = \angle SDP = \angle YQP + \angle SRY - \angle QYR \end{aligned}$$

$\square$

Claim 3. $\angle ZAP = 90 - \angle QYR$

Proof. Notice the cyclic quadrilateral $YQER$. Now we have

$$\begin{aligned}\angle ZAP = 90 - \angle XEA = 90 - (180 - \angle REQ) &= \angle REQ - 90 \\ &= (180 - \angle QYR) - 90 = 90 - \angle QYR\end{aligned}$$

□

Claim 4. $\angle YQP + \angle SRY = 90°$

Proof. Notice the cyclic quadrilateral $AXYZ$. So we have

$$\angle ZAP = 180 - (\angle XYZ) = 180 - (\angle XYQ + \angle QYR + \angle RYZ)$$
$$90 - \angle QYR = 180 - \angle XYQ - \angle QYR - \angle RYZ \implies \angle XYQ + \angle RYZ = 90°$$

Now since $PYRX$ and $SZRY$ are both rectangles, we have $\angle XYQ = \angle YQP$ and $\angle RYZ = \angle ZRS$ so we are done. □

Claim 5. $\angle PAZ = \angle SDP$

Proof. Because of Claim 2, 3, and 4, we have

$$\angle ZAP = \angle YQP + \angle SRY - \angle QYR = \angle SDP.$$

□

Now we are done because we also know that $\angle PAZ = \frac{\angle XOZ}{2}$