Question 6.1

Prove:

$$
\begin{aligned}
\langle\alpha,\xi|\,\hat{a}\,|\alpha,\xi\rangle &= \alpha \\
\langle\alpha,\xi|\,\hat{a}^2\,|\alpha,\xi\rangle &= \alpha^2 - e^{i\theta}\sinh r\cosh r \\
\langle\alpha,\xi|\,\hat{a}^\dagger\hat{a}\,|\alpha,\xi\rangle &= |\alpha|^2 + \sinh^2 r
\end{aligned}
$$

Tips:

$$
\begin{aligned}
\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) &= \hat{a} + \alpha \\
\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) &= \hat{a}^\dagger + \alpha^*
\end{aligned}
$$

Answer

$$
\begin{aligned}
\langle\alpha,\xi|\,\hat{a}\,|\alpha,\xi\rangle &= \langle 0|\,\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\,\hat{S}^\dagger(\xi)\left(\hat{a}+\alpha\right)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\left(\hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r\right)|0\rangle + \alpha \\
&= \alpha
\end{aligned}
$$

$$
\begin{aligned}
\langle\alpha,\xi|\,\hat{a}^2\,|\alpha,\xi\rangle &= \langle 0|\,\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}^2\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\,\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\,\hat{S}^\dagger(\xi)\left(\hat{a}+\alpha\right)^2\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\,\hat{S}^\dagger(\xi)\left(\hat{a}^2+2\alpha\hat{a}+\alpha^2\right)\hat{S}(\xi)\,|0\rangle \\
&= \alpha^2 + \langle 0|\,\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)\,|0\rangle \\
&= \alpha^2 + \langle 0|\left(\hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r\right)^2|0\rangle \\
&= \alpha^2 + \langle 0|\left[\hat{a}^2\cosh^2 r - \left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)e^{i\theta}\cosh r\sinh r + \hat{a}^{\dagger 2}e^{2i\theta}\sinh^2 r\right]|0\rangle \\
&= \alpha^2 - e^{i\theta}\cosh r\sinh r
\end{aligned}
$$

$$
\begin{aligned}
\langle\alpha,\xi|\,\hat{a}^\dagger\hat{a}\,|\alpha,\xi\rangle &= \langle 0|\,\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{a}\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\,\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\,\hat{S}^\dagger(\xi)\left(\hat{a}^\dagger+\alpha^*\right)\left(\hat{a}+\alpha\right)\hat{S}(\xi)\,|0\rangle \\
&= \langle 0|\,\hat{S}^\dagger(\xi)\left(\hat{a}^\dagger\hat{a}+\alpha\hat{a}^\dagger+\alpha^*\hat{a}+|\alpha|^2\right)\hat{S}(\xi)\,|0\rangle \\
&= |\alpha|^2 + \langle 0|\,\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi)\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)\,|0\rangle \\
&= |\alpha|^2 + \langle 0|\left(\hat{a}^\dagger\cosh r - \hat{a}e^{-i\theta}\sinh r\right)\left(\hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r\right)|0\rangle \\
&= |\alpha|^2 + \langle 0|\left[\hat{a}^\dagger\hat{a}\cosh^2 r - \left(\hat{a}^\dagger\hat{a}^\dagger e^{i\theta} + \hat{a}\hat{a}e^{-i\theta}\right)\cosh r\sinh r + \hat{a}\hat{a}^\dagger\sinh^2 r\right]|0\rangle \\
&= |\alpha|^2 + \sinh^2 r
\end{aligned}
$$

Question 6.2

Prove:

$$\left(\mu\hat{a}+\nu\hat{a}^{\dagger}\right)|\alpha,\xi\rangle=\left(\alpha\cosh r+\alpha^{*}e^{i\theta}\sinh r\right)|\alpha,\xi\rangle\equiv r\,|\alpha,\xi\rangle$$

Answer

$$\begin{aligned}
\left(\mu\hat{a}+\nu\hat{a}^{\dagger}\right)|\alpha,\xi\rangle &= \left(\mu\hat{a}+\nu\hat{a}^{\dagger}\right)\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle\\
&=\hat{D}(\alpha)\hat{D}^{\dagger}(\alpha)\left(\mu\hat{a}+\nu\hat{a}^{\dagger}\right)\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle\\
&=\hat{D}(\alpha)\left[\mu\left(\hat{a}+\alpha\right)+\nu\left(\hat{a}^{\dagger}+\alpha^{*}\right)\right]\hat{S}(\xi)\,|0\rangle\\
&=\left(\mu\alpha+\nu\alpha^{*}\right)\hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle+\hat{D}(\alpha)\hat{S}(\xi)\hat{S}^{\dagger}(\xi)\left(\mu\hat{a}+\nu\hat{a}^{\dagger}\right)\hat{S}(\xi)\,|0\rangle\\
&=\left(\mu\alpha+\nu\alpha^{*}\right)|\alpha,\xi\rangle+\hat{D}(\alpha)\hat{S}(\xi)\left[\mu\left(\hat{a}\mu-\hat{a}^{\dagger}\nu\right)+\nu\left(\hat{a}^{\dagger}\mu-\hat{a}e^{-i\theta}\sinh r\right)\right]|0\rangle\\
&=\left(\mu\alpha+\nu\alpha^{*}\right)|\alpha,\xi\rangle+\hat{D}(\alpha)\hat{S}(\xi)\left(\mu^{2}-\nu e^{-i\theta\sinh r}\right)\hat{a}\,|0\rangle\\
&=\left(\alpha\cosh r+\alpha^{*}e^{i\theta}\sinh r\right)|\alpha,\xi\rangle\\
&\equiv r\,|\alpha,\xi\rangle
\end{aligned}$$

Question 6.3

Prove:

$$P_n=|\langle n|\alpha,\xi\rangle|^2=\frac{\left(\frac{1}{2}\tanh r\right)^n}{n!\cosh r}\exp\left[-|\alpha|^2-\frac{1}{2}\left(\alpha^{*2}e^{i\theta}+\alpha^2e^{-i\theta}\right)\tanh r\right]|H_n\left[r\left(e^{i\theta}\sinh 2r\right)^{-\frac{1}{2}}\right]|^2$$

Answer

对于压缩态, 使用数态展开为:

$$|\alpha,\xi\rangle=\frac{1}{\sqrt{\cosh r}}\exp\left(-\frac{1}{2}|\alpha|^2-\frac{1}{2}\alpha^{*2}e^{i\theta}\tanh r\right)\times\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{n/2}}{\sqrt{n!}}H_n\left[\gamma\left(e^{i\theta}\sinh 2r\right)^{-1/2}\right]|n\rangle$$

那么在场中发现 n 个光子的概率即为:

$$\begin{aligned}
P_n =&|\langle n|\alpha,\xi\rangle|^2\\
=&\langle n|\alpha,\xi\rangle\langle\alpha,\xi|n\rangle\\
=&\frac{1}{\cosh r}\exp\left[-|\alpha|^2-\frac{1}{2}\left(\alpha^{*2}e^{i\theta}+\alpha^2e^{-i\theta}\right)\tanh r\right]\left|\frac{\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{n/2}}{\sqrt{n!}}H_n\left[\gamma\left(e^{i\theta}\sinh 2r\right)^{-1/2}\right]\right|^2\\
=&\frac{\left(\frac{1}{2}\tanh r\right)^n}{n!\cosh r}\exp\left[-|\alpha|^2-\frac{1}{2}\left(\alpha^{*2}e^{i\theta}+\alpha^2e^{-i\theta}\right)\tanh r\right]\left|H_n\left[\gamma\left(e^{i\theta}\sinh 2r\right)^{-1/2}\right]\right|^2
\end{aligned}$$

Answer for 7.11

对于极强正交幅压缩的光, 最有可能满足振幅压缩的情况为椭圆的短轴定位相干态的相矢量方向. 而由于振幅的不确定度极小, 导致在椭圆面积仍满足不确定度关系的前提下, 要求椭圆的长轴极长. 也就导致椭圆无法完全落在振幅不确定度的范围, 长轴的两极会落在范围外导致不再是振幅压缩光.
强振幅压缩的光在相位不确定度上极大, 其不确定度区域需要落在相平面以原点为圆心的圆环内, 也就导致其不确定区域需要适当弯曲为” 香蕉状”.

Answer for 7.15

光照强度和电场振幅有如下关系

$$I = \frac{1}{2}c\epsilon_0 n|\mathcal{E}_p|^2 \Rightarrow Re(\mathcal{E}_p) = \sqrt{\frac{2I}{c\epsilon_0 n}}$$

将其代入衰减因子中即可得到期望的正交压缩比 η:

$$\eta = 1 - \exp(-\gamma L) = 1 - \exp\left(-\frac{\omega\chi^{(2)}\mathcal{E}_p}{2nc}L\right) = 1 - \exp\left(-\frac{2\pi\chi^{(2)}L\sqrt{\frac{2I}{c\epsilon_0 n}}}{2n\lambda}\right) \approx 18\%$$