Answer for 3.4

$$\hat{l} = \hat{r} \times \hat{p} = \varepsilon_{ijk}\hat{i}r_j p_k \Rightarrow \hat{l}_i = \varepsilon_{ijk} r_j p_k$$

$$\begin{aligned}
\left[\hat{l_x}, \hat{l_y}\right] =& \left[r_y p_z - r_z p_y, r_z p_x - r_x p_z\right] \\
=& \left[r_y p_z, r_z p_x\right] - \left[r_y p_z, r_x p_z\right] - \left[r_z p_y, r_z p_x\right] + \left[r_z p_y, r_x p_z\right] \\
=& \left[r_y, r_z p_x\right] p_z + r_y \left[p_z, r_z p_x\right] - \left[r_y, r_x p_z\right] p_z - r_y \left[p_z, r_x p_z\right] \\
& - \left[r_z, r_z p_x\right] p_y - r_z \left[p_y, r_z p_x\right] + \left[r_z, r_x p_z\right] p_y + r_z \left[p_y, r_x p_z\right] \\
=& 0 + r_y p_x(-i\hbar) - 0 - 0 - 0 - 0 + 0 + r_x p_y(i\hbar) + 0 \\
=& i\hbar l_z
\end{aligned}$$

同理有:

$$\left[\hat{l_y}, \hat{l_z}\right] = i\hbar\hat{l_x} \qquad \left[\hat{l_z}, \hat{l_x}\right] = i\hbar\hat{l_x}$$

Answer for 3.6

对于激发态电子组态为 (1s, 2p) 的氦原子, 有

$$l_1 = 0, \quad l_2 = 1, \quad S_1 = S_2 = \frac{1}{2}$$

所以:

$$L = 1, \quad S = 0\,\text{or}\,1, \quad J = 0, 1, 2$$

该组态所有可能的原子能级为: $0P_1$, ${}^3P_0$, ${}^3P_1$, ${}^3P_2$

Answer for 3.7

(a)

对超精细结构有:

$$\Delta E_{HFS} = A(J)\frac{\hbar^2}{2}\left[F(F+1) - J(J+1) - I(I+1)\right]$$

对于不同的 F, 其能级差为

$$\Delta E_{HFS} - \Delta E'_{HFS} = A(J)\frac{\hbar^2}{2}\left[F(F+1) - F'(F'+1)\right]$$

有 $F = F' + 1$, 所以:

$$\Delta E_{HFS} - \Delta E'_{HFS} = A(J)\frac{\hbar^2}{2}\left[F(F+1) - (F-1)F\right] = A(J)\hbar^2 F$$

能级差正是正比于 F

(b)

自旋-轨道角动量相互作用导致的能量偏移也同样符合类似的关系:

$$\Delta E_{so} = C'\left[J(J+1) - L(L+1) - S(S+1)\right] \Rightarrow \Delta E_{soJ} - \Delta E_{soJ-1} = 2C'J$$

(c)

(i)

对于纳的 $3p^2P_{3/2}$ 能级, 其电子角动量 J 为 $\frac{3}{2}$, 包括核自旋的总角动量 $\hat{F}=\hat{J}+\hat{I}$, 其中 $\hat{I}$ 为核自旋角动量. 由于包括四条超精细能级, 所以该总角动量 $\hat{F}$ 有四个值, 若 $I=\frac{1}{2}$, 则 F 仅能为 1 和 2, 仅有两条超精细能级, 不符合情况. 仅当 $I\geqslant\frac{3}{2}$ 时, 才有可能有四条超精细能级.

(ii)

有:

$$\begin{aligned}
\Delta E_{HFS1}-\Delta E_{HFS2}&=A(J)\hbar^2F\Rightarrow 60\text{MHz}\\
\Delta E_{HFS2}-\Delta E_{HFS3}&=A(J)\hbar^2(F-1)\Rightarrow 36\text{MHz}\\
\Delta E_{HFS3}-\Delta E_{HFS4}&=A(J)\hbar^2(F-2)\Rightarrow 17\text{MHz}
\end{aligned}$$

可以大致推测 $F=3$, 也即 $I=\frac{3}{2}$

Answer for 3.12

第一个 PBS 两个输出的正交偏振的光强为:

$$I_{1v}=\frac{I_0}{2},\qquad I_{1h}=\frac{I_0}{2}$$

第二个 PBS 两个输出的正交偏振的光强为:

$$\begin{aligned}
I_{2v}&=I_{1v}\cos^2\theta=\frac{I_0}{2}\cos^2\theta\\
I_{2h}&=I_{1v}\sin^2\theta=\frac{I_0}{2}\sin^2\theta
\end{aligned}$$

所以光子在第二个 PBS 垂直偏振光输出口的单光子计数器记录的概率为 $\frac{1}{2}\cos^2\theta$, 在第二个 PBS 水平偏振光输出口的单光子计数器记录的概率为 $\frac{1}{2}\sin^2\theta$

Answer for Lodon 4.2

$$\begin{aligned}
\left[\hat{a},(\hat{a}^\dagger)^2\right]&=\left[\hat{a},\hat{a}^\dagger\right]\hat{a}^\dagger+\hat{a}^\dagger\left[\hat{a},\hat{a}^\dagger\right]\\
&=2\hat{a}^\dagger
\end{aligned}$$

$$\begin{aligned}
\left[(\hat{a})^2,\hat{a}^\dagger\right]&=\left[\hat{a},\hat{a}^\dagger\right]\hat{a}+\hat{a}\left[\hat{a},\hat{a}^\dagger\right]\\
&=2\hat{a}
\end{aligned}$$

$$\begin{aligned}
\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] &= \left[\hat{a}, \hat{a}^\dagger\right] \left(\hat{a}^\dagger\right)^{n-1} + \hat{a}^\dagger \left[\hat{a}, \left(\hat{a}^\dagger\right)^{n-1}\right] \\
&= \left(\hat{a}^\dagger\right)^{n-1} + \hat{a}^\dagger \left[\hat{a}, \left(\hat{a}^\dagger\right)^{n-1}\right] \\
&= \left(\hat{a}^\dagger\right)^{n-1} + \hat{a}^\dagger \left[\left(\hat{a}^\dagger\right)^{n-2} + \hat{a}^\dagger \left[\hat{a}, \left(\hat{a}^\dagger\right)^{n-2}\right]\right] \\
&= 2\left(\hat{a}^\dagger\right)^{n-1} + \left(\hat{a}^\dagger\right)^2 \left[\hat{a}, \left(\hat{a}^\dagger\right)^{n-2}\right] \\
&= \cdots \\
&= (n-2)\left(\hat{a}^\dagger\right)^{n-2} + \left(\hat{a}^\dagger\right)^{n-2} \left[\hat{a}, \left(\hat{a}^\dagger\right)^2\right] \\
&= (n-1)\left(\hat{a}^\dagger\right)^{n-1} + \left(\hat{a}^\dagger\right)^{n-1} \left[\hat{a}, \hat{a}^\dagger\right] \\
&= n\left(\hat{a}^\dagger\right)^{n-1}
\end{aligned}$$

$$\begin{aligned}
\left[(\hat{a})^n, \hat{a}^\dagger\right] &= \left[\hat{a}, \hat{a}^\dagger\right] (\hat{a})^{n-1} + \hat{a} \left[(\hat{a})^{n-1}, \hat{a}^\dagger\right] \\
&= (\hat{a})^{n-1} + \hat{a} \left[(\hat{a})^{n-1}, \hat{a}^\dagger\right] \\
&= \left(\hat{a}^\dagger\right)^{n-1} + \hat{a}^\dagger \left[(\hat{a})^{n-2} + \hat{a} \left[(\hat{a})^{n-2}, \hat{a}^\dagger\right]\right] \\
&= 2(\hat{a})^{n-1} + (\hat{a})^2 \left[(\hat{a})^{n-2}, \hat{a}^\dagger\right] \\
&= \cdots \\
&= (n-2)(\hat{a})^{n-2} + (\hat{a})^{n-2} \left[(\hat{a})^2, \hat{a}^\dagger\right] \\
&= (n-1)(\hat{a})^{n-1} + (\hat{a})^{n-1} \left[\hat{a}, \hat{a}^\dagger\right] \\
&= n(\hat{a})^{n-1}
\end{aligned}$$

$$\begin{aligned}
\left[\hat{a}, \exp\left(\beta\hat{a}^\dagger\right)\right] &= \sum_n \left[\hat{a}, \frac{\beta^n}{n!}\hat{a}^\dagger\right] \\
&= \sum_n \frac{\beta^n}{n!} n \left(\hat{a}^\dagger\right)^{n-1} \\
&= \beta \sum_n \frac{\beta^{n-1}}{(n-1)!} \left(\hat{a}^\dagger\right)^{n-1} \\
&= \beta \exp\left(\beta\hat{a}^\dagger\right)
\end{aligned}$$

Answer for Lodon 4.3

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \Rightarrow \frac{\hat{a}^\dagger}{\sqrt{n+1}} |n\rangle = |n+1\rangle$$

$$
\begin{aligned}
|n\rangle =& \frac{\hat{a}^\dagger}{\sqrt{n}} |n-1\rangle \\
=& \frac{\left(\hat{a}^\dagger\right)^2}{\sqrt{n(n-1)}} |n-2\rangle \\
=& \cdots \\
=& \frac{\left(\hat{a}^\dagger\right)^{n-1}}{\sqrt{n(n-1)\cdots 3\cdot 2}} |1\rangle \\
=& \frac{\left(\hat{a}^\dagger\right)^{n}}{\sqrt{n!}} |1\rangle \\
=& \hat{N}(n) |0\rangle
\end{aligned}
$$

Answer for cmmutative of $\hat{X}_1$ and $\hat{X}_2$

有施瓦茨不等式:

$$
\begin{aligned}
\langle \hat{A}^2\rangle \langle \hat{B}^2\rangle \geqslant & \left|\langle AB\rangle\right|^2 \\
=& \left|\langle \frac{AB+BA}{2} + \frac{AB-BA}{2}\rangle\right|^2 \\
=& \left|\langle \frac{AB+BA}{2}\rangle\right|^2 + \left|\langle \frac{AB-BA}{2}\rangle\right|^2 \\
=& \frac{1}{4}\left(\left|\langle\{A,B\}\right|\right|^2 + \left|\langle[A,B]\rangle\right|^2\right) \\
\geqslant & \frac{1}{4}\left|\langle[A,B]\rangle\right|^2
\end{aligned}
$$

有:

$$
\begin{aligned}
\left[\Delta\hat{X}_1, \Delta\hat{X}_2\right] =& \left[\hat{X}_1 - \langle\hat{X}_1\rangle, \hat{X}_2 - \langle\hat{X}_2\rangle\right] \\
=& \left[\hat{X}_1, \hat{X}_2\right] \\
=& \frac{i}{2}
\end{aligned}
$$

便可得到目标不确定关系:

$$
\left(\Delta\hat{X}_1\right)^2 \cdot \left(\Delta\hat{X}_2\right)^2 \geqslant \frac{1}{16} \Rightarrow \Delta\hat{X}_1 \cdot \Delta\hat{X}_2 \geqslant \frac{1}{4}
$$