Answer for 9.1

对于受迫运动的微分方程:

$$m_e \frac{d^2x}{dt^2} + m_e\gamma\frac{dx}{dt} + m_e\omega_0^2 x = F_0 \cos\omega t$$

有解:

$$x(t) = A_0 e^{-\gamma t/2} \cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}t + \varphi_0\right) + A\cos(\omega t + \varphi) \tag{1}$$

其中, 对于振幅 A 有如下近似:

$$A = \frac{F_0}{m_e\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}} \tag{2}$$

对其求导可获得最大值位于 $\omega = \omega_0$ 处:

$$\frac{dA}{d\omega} = 0 \Rightarrow \omega_m = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} \approx \omega_0,\ A_m = \frac{F_0}{m_e\gamma\omega_0} \qquad (\text{since } \omega_0 \gg \gamma) \tag{3}$$

代入 $A = \frac{A_m}{2}$:

$$\begin{aligned}
A = \frac{A_m}{2} \Rightarrow \frac{F_0}{2m_e\sqrt{\frac{\gamma^4}{4} + \gamma^2\left(\omega_0^2 - \frac{\gamma^2}{2}\right)}} &= \frac{F_0}{m_e\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \\
4\gamma^2\omega_0^2 - \gamma^4 &= \omega_0^4 + \omega^4 - 2\omega_0^2\omega^2 + \gamma^2\omega^2 \\
\left(\omega^2 - \omega_m^2\right)^2 &= 3\gamma^2\left(\omega_m^2 + \frac{\gamma^2}{4}\right) \\
\omega^2 &= \omega_m^2 \pm \gamma\sqrt{3\left(\omega_m^2 + \frac{\gamma^2}{4}\right)} \\
\omega^2 &\approx \omega_m^2 \pm \gamma\omega_m + \frac{\gamma^2}{4} \\
\omega^2 &= \left(\omega_m \pm \frac{\gamma}{2}\right)^2 \\
\omega_\pm &= \omega_m \pm \frac{\gamma}{2}
\end{aligned} \tag{4}$$

求得半高宽 $\text{FWHM} = \omega_+ - \omega_- = \gamma$

Answer for 9.5

(a)

有旋波近似下 $c_1(t)$ 和 $c_2(t)$ 的微分方程:

$$\dot{c}_1(t) = \frac{i}{2}\Omega_R e^{i\delta\omega t} c_2(t) \tag{5}$$

$$\dot{c}_2(t) = \frac{i}{2}\Omega_R e^{-i\delta\omega t} c_1(t) \tag{6}$$

对 $c_2(t)$ 的微分方程两边求导:

$$\begin{aligned}
\ddot{c}_2(t) =& \frac{d}{dt}\left(\frac{i}{2}\Omega_R e^{-i\delta\omega t} c_1(t)\right) \\
=& \frac{i}{2}\Omega_R \left(-i\delta\omega e^{-i\delta\omega t} c_1(t) + e^{-i\delta\omega t}\dot{c}_1(t)\right) \\
=& \frac{i}{2}\Omega_R \left[-i\delta\omega e^{-i\delta\omega t}\left(-\frac{2ie^{i\delta\omega t}}{\Omega_R}\dot{c}_2(t)\right) + e^{-i\delta\omega t}\left(\frac{i}{2}\Omega_R e^{i\delta\omega t} c_2(t)\right)\right] \\
=& - i\delta\omega\dot{c}_2(t) - \frac{\Omega^2}{4} c_2(t)
\end{aligned} \tag{7}$$

将 (5) 代入 (7) 得到目标:

$$\ddot{c}_2(t) + i\delta\omega\dot{c}_2(t) + \frac{\Omega_R^2}{4} c_2(t) = 0$$

(b)

代入试探解求得目标:

$$\begin{aligned}
\frac{d^2 c_2(t)}{dt^2} + i\delta\omega\frac{dc_2(t)}{dt} + \frac{\Omega_R^2}{4} c_2(t) =& 0 \\
\frac{d^2}{dt^2}\left(Ce^{-i\zeta t}\right) + i\delta\omega\frac{d}{dt}\left(Ce^{-i\zeta t}\right) + \frac{\Omega_R^2}{4} Ce^{-i\zeta t} =& 0 \\
\left(-i\zeta\right)\frac{d}{dt}\left(e^{-i\zeta t}\right) + i\delta\omega\left(-i\zeta\right)e^{-i\zeta t} + \frac{\Omega_R^2}{4} e^{-i\zeta t} =& 0 \\
\left(-i\zeta\right)^2 e^{-i\zeta t} + \left(\delta\omega\zeta + \frac{\Omega_R^2}{4}\right) e^{-i\zeta t} =& 0 \\
-\zeta^2 + \delta\omega\zeta + \frac{\Omega_R^2}{4} =& 0 \\
\zeta^2 - \delta\omega\zeta + \frac{1}{2}\delta\omega =& \frac{1}{4}\delta\omega + \frac{\Omega_R^2}{4} \\
\left(\zeta - \frac{1}{2}\delta\omega\right)^2 =& \frac{1}{4}\delta\omega + \frac{\Omega_R^2}{4} \\
\zeta_\pm =& \frac{1}{2}\delta\omega \pm \frac{1}{2}\left(\delta\omega + \Omega_R^2\right)^{\frac{1}{2}}
\end{aligned} \tag{8}$$

(c)

使用 $c_2(0) = 0$ 的初始条件获得 C_+ 与 C_- 的关系:

$$\begin{aligned}
c_2(0) = 0 &\Rightarrow C_+ + C_- = 0 \\
&\Rightarrow C_+ = -C_- \\
&\Rightarrow c_2(t) = C_+\left(e^{-i\zeta_+ t} - e^{-i\zeta_- t}\right)
\end{aligned} \tag{9}$$

再使用 $c_1(0) = 1$ 的初始条件获得 C_+ 与 C_- 的值:

$$\begin{aligned}
c_1(0) = 1 &\Rightarrow \dot{c}_2(0) = \frac{i}{2}\Omega_R \\
&\Rightarrow C_+\left(-i\zeta_+ + i\zeta_-\right) = \frac{i}{2}\Omega_R \\
&\Rightarrow C_+ = \frac{\Omega_R}{-\Omega + (-\Omega)} = -\frac{\Omega_R}{2\Omega} \\
&\Rightarrow c_2(t) = -\frac{\Omega_R}{2\Omega}\left(e^{-i\zeta_+ t} - e^{-i\zeta_- t}\right)
\end{aligned} \tag{10}$$

最后代入 C_+ 与 C_- 得到结果:

$$
\begin{aligned}
|c_2(t)|^2 =&\frac{\Omega_R^2}{4\Omega^2}\left(e^{-i\zeta_+ t}-e^{-i\zeta_- t}\right)\left(e^{i\zeta_+ t}-e^{i\zeta_- t}\right)\\
=&\frac{\Omega_R^2}{4\Omega^2}\left(-e^{-i(\zeta_+-\zeta_-)t}-e^{i(\zeta_+-\zeta_-)t}+2\right)\\
=&\frac{\Omega_R^2}{4\Omega^2}\left(-e^{-i\Omega t}-e^{i\Omega t}+2\right)\\
=&\frac{\Omega_R^2}{4\Omega^2}\left\{-\left[\cos\left(\Omega t\right)-i\sin\left(\Omega t\right)\right]-\left[\cos\left(\Omega t\right)+i\sin\left(\Omega t\right)\right]+2\right\}\\
=&\frac{\Omega_R^2}{4\Omega^2}\left[-2\cos\left(\Omega t\right)+2\right]\\
=&\frac{\Omega_R^2}{\Omega^2}\sin^2\left(\Omega t/2\right)
\end{aligned}
\tag{11}
$$

Answer for 9.9

对于极坐标, 并且在布洛赫球上，恒有 $r=1$, 知 x, y, z 与极坐标关系:

$$
\begin{cases}
x &= \sin\theta\cos\varphi\\
y &= \sin\theta\sin\varphi\\
z &= \cos\theta
\end{cases}
\tag{12}
$$

通过变换, 得到 c_1,c_2 与极坐标关系:

$$
\begin{aligned}
1-z =&1-\cos\theta\\
1-|c_2|^2+|c_1|^2 =&2\sin^2\left(\frac{\theta}{2}\right)\\
|c_1|^2 =&\sin^2\left(\frac{\theta}{2}\right)\\
c_1 =&e^{i\alpha}\sin\left(\frac{\theta}{2}\right)
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
x+iy =&\sin\theta\left(\cos\varphi+i\sin\varphi\right)\\
2\mathrm{Re}\left\langle c_1c_2\right\rangle+2i\mathrm{Im}\left\langle c_1c_2\right\rangle =&\sin\theta e^{i\varphi}\\
c_1c_2 =&e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\\
c_2 =&e^{i\alpha}e^{i(\varphi-2\alpha)}\cos\left(\frac{\theta}{2}\right)
\end{aligned}
\tag{14}
$$

由于 xy 轴的方向是任意选择的, 所以 φ 也可平移变换:

$$
\varphi' = \varphi - 2\alpha
\tag{15}
$$

丢弃全局相位 $e^{i\alpha}$, 得到目标关系式:

$$
\begin{cases}
c_1 &= \sin\left(\frac{\theta}{2}\right)\\
c_2 &= e^{i\varphi'}\cos\left(\frac{\theta}{2}\right)
\end{cases}
\tag{16}
$$

Answer for 9.11

(a)

$$
x=\frac{2\sqrt{2}}{3}\qquad y=0\qquad z=\frac{1}{3}
$$

(b)

$$x = 0 \qquad y = \frac{2\sqrt{2}}{3} \qquad z = \frac{1}{3}$$

(c)

$$x = \frac{\sqrt{2}}{2} \qquad y = \frac{\sqrt{2}}{2} \qquad z = 0$$

Answer for 9.14

$$\Theta(t) = |\frac{\mu_{12}}{\hbar}\int_t^0 \mathcal{E}_0(\tau)d\tau|$$

默认一开始绕 y 轴转并且初态为 1 态位于布洛赫球的南极

(a)

$$\Theta_1 = \frac{\pi}{4} \qquad \Theta_2 = \frac{3\pi}{4} \qquad \phi = 0$$

即先绕 y 轴转 $\frac{\pi}{4}$, 再继续绕 y 轴转 $\frac{3\pi}{4}$, 得到:

$$|\psi\rangle = |2\rangle$$

(a)

$$\Theta_1 = \frac{\pi}{2} \qquad \Theta_2 = \pi \qquad \phi = \frac{\pi}{2}$$

即先绕 y 轴转 $\frac{\pi}{2}$, 再继续绕 $-x$ 轴转 π, 得到:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$$

(c)

$$\Theta_1 = \frac{\pi}{2} \qquad \Theta_2 = \pi \qquad \phi = \frac{\pi}{4}$$

即先绕 y 轴转 $\frac{\pi}{2}$, 再继续绕与 y 轴和 $-x$ 轴各成 $45°$ 的轴转 π, 得到:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}|2\rangle$$