Normed Linear Spaces

Definition Let X be a vector space over the field $\mathbb{K}$. A norm on X is a function $\|\cdot\| : X \to [0, \infty)$, with the properties

i) $\|\lambda x\| = |\lambda|\|x\|$;

ii) $\|x + y\| \leq \|x\| + \|y\|$;

iii) $\|x\| = 0 \Leftrightarrow x = 0$.

for all $\lambda \in \mathbb{K}$ and $x, y \in X$. The pair $(X, \|\cdot\|)$ is said to be a normed space.

A function $p : X \to [0, \infty)$ is said to be a seminorm on X if

(a) $p(\lambda x) = |\lambda| p(x)$,

(b) $p(x + y) \leq p(x) + p(y)$.

Every normed space is also a metric space with respect to the induced metric $d(x, y) := \|x - y\|$.

Convergence in a normed space is defined by the metric d, that is, $x_n \to x$ in $(X, \|\cdot\|)$ if $\|x_n - x\| \to 0$ as $n \to \infty$.

A Banach space is a normed space that is complete w.r.t. the induced metric.

Definition 1.1 A subset K of a vector space V is convex if whenever $x, y \in K$ the line segment joining x and y, lies in K, i.e, for every $\lambda \in [0,1]$ we have $\lambda x + (1-\lambda)y \in K$.

Lemma 1.1 In any normed space $(X, \|\cdot\|)$, $\mathbb{B}_X := \{x \in X, \|x\| \leq 1\}$ and $B_X := \{x \in X, \|x\| < 1\}$ are convex.

Lemma 1.2 Suppose that $N : X \to [0, \infty)$ satisfies
(i) $N(x) = 0$ if and only if $x = 0$;
(ii) $N(\lambda x) = |\lambda| N(x)$ for every $\lambda \in \mathbb{K}, x \in X$
and, in addition, that the set $B := \{x : N(x) \leq 1\}$ is convex. Then N satisfies

$$N(x + y) \leq N(x) + N(y) \tag{1.1}$$

and so N defines a norm on X.

Proof We can assume that $N(x) > 0, N(y) > 0$. In this case $x/N(x) \in B$ and $y/N(y) \in B$ so using the convexity of B we have

$$\frac{N(x)}{N(x) + N(y)} \frac{x}{N(x)} + \frac{N(y)}{N(x) + N(y)} \frac{y}{N(y)} \in B.$$

Thus

$$\frac{x + y}{N(x) + N(y)} \in B,$$

which implies that

$$1 \geq N\left(\frac{x + y}{N(x) + N(y)}\right) = \frac{N(x + y)}{N(x) + N(y)} \Rightarrow N(x + y) \leq N(x) + N(y).$$

$\square$

$f : [a, b] \to \mathbb{R}$, is convex if whenever $x, y \in [a, b]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \; \forall \lambda \in (0, 1). \quad (1.2)$$

If $f \in C^2(a, b) \cap C^1([a, b])$ and $f''(x) \geq 0 \; \forall x \in (a, b)$, then f is convex.

Note that $s \to |s|^p$ is convex for all $1 \leq p < \infty$ and that $s \to e^s$ is convex.

Examples i) $(\mathbb{K}^n, \|\cdot\|)$ with

$$\|x\| = \|x\|_{l^2} = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \quad x \in \mathbb{K}^n. \tag{1.3}$$

ii) $(\mathbb{K}^n, \|\cdot\|_{l^p})$, $1 \le p < \infty$, where

$$\|x\|_{l^p} = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \quad x \in \mathbb{K}^n. \tag{1.4}$$

iii) $(\mathbb{K}^n, \|\cdot\|_{l^\infty})$ with

$$\|x\|_{l^\infty} = \max_{j=1,\cdots,n} |x_j|. \tag{1.5}$$

Lemma 1.3 (Minkowski's inequality in $\mathbb{K}^n$) For all $1 \le p \le \infty$, if $x, y \in \mathbb{K}^n$, then

$$\|x + y\|_{l^p} \le \|x\|_{l^p} + \|y\|_{l^p}. \tag{1.6}$$

Proof. The case $p = \infty$ is obvious. Suppose that $1 \le p < \infty$. By Lemma 1.2, we need only to show that

$$B = \{x \in \mathbb{K}^n : \|x\|_{l^p} \le 1\} = \{x \in \mathbb{K}^n : \|x\|_{l^p}^p \le 1\}$$

is convex. Note that $\forall 1 \le p < \infty$, $t \to |t|^p$ is convex. If $x, y \in \mathbf{B}$, $\lambda \in (0,1)$, then

$$\|\lambda x + (1 - \lambda)y\|_{l^p}^p = \sum_{i=1}^n |\lambda x_i + (1 - \lambda)y_i|^p$$

$$\le \sum_{i=1}^n (\lambda |x_i| + (1 - \lambda)|y_i|)^p \le \sum_{i=1}^n (\lambda |x_i|^p + (1 - \lambda)|y_i|^p) \le 1,$$

and so $\lambda x + (1 - \lambda)y \in B$ and B is convex. $\qquad\square$

Lemma 1.4 Let $\dim(V) < \infty$ and $\{e_j\}_{j=1}^n$ be a basis of V. Then

$$\|x\|_E = \left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2}, \text{ for } x = \sum_{j=1}^n \alpha_j e_j \tag{1.7}$$

is a norm on V.

Proof. We check the triangle inequality. Let
$x = \sum_{j=1}^n \alpha_j e_j, y = \sum_{j=1}^n \beta_j e_j \in V$; then

$$\|x + y\|_E = \|\alpha + \beta\|_{l^2} \leq \|\alpha\|_{l^2} + \|\beta\|_{l^2} = \|x\|_E + \|y\|_E,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$. $\qquad\square$

Example iv) The sequence space $l^p(\mathbb{K}), 1 \le p < \infty$, consists of all $\mathbb{K}$- valued sequences $x = (x_j)_{j=1}^\infty$ such that the l^p norm is finite, where

$$\|x\|_{l^p} = \left(\sum_{j=1}^\infty |x_j|^p \right)^{1/p} ; \qquad (1.8)$$

and $l^\infty(\mathbb{K})$ is the space of bounded sequences equipped with the norm

$$\|x\|_\infty = \sup_{j \in \mathbb{N}} |x_j|. \qquad (1.9)$$

Lemma 1.5 (Minkowski's inequality in $l^p(\mathbb{K})$) For all $1 \leq p \leq \infty$, if $x, y \in l^p(\mathbb{K})$, then $x + y \in l^p(\mathbb{K})$ and

$$\|x + y\|_{l^p} \leq \|x\|_{l^p} + \|y\|_{l^p}. \tag{1.10}$$

Proof. We consider the case $1 \leq p < \infty$ and use (1.6) to get

$$\left(\sum_{j=1}^{n} |x_j + y_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^{n} |y_j|^p \right)^{1/p}$$

$$\leq \|x\|_{l^p} + \|y\|_{l^p}.$$

Taking the limit as $n \to \infty$ one deduces (1.10). □

Example v) On the Space C([0,1]), the collection of continuous functions on [0, 1], we define

$$\|f\| = \max_{x\in[0,1]} |f(x)|.$$

$(C([0,1]), \|\cdot\|)$ is a normed space.

Example vi) If X is a metric space consider the space $C_b(X;\mathbb{K})$ of all bounded continuous functions from X into $\mathbb{K}$. Then

$$\|x\|_\infty := \sup_{x\in X} |f(x)|$$

defines a norm on $C_b(X;\mathbb{K})$. If X is compact, then $C_b(X;\mathbb{K}) = C(X;\mathbb{K})$, and this norm is the same as the maximum norm

$$\|x\|_\infty := \max_{x\in X} |f(x)|.$$

Example vii) On the Space C([0,1]) for any $1 \leq p < \infty$,

$$\|f\|_p := \left(\int_0^1 |f(x)|^p dx \right)^{1/p}$$

defines a norm.

• If $x, y \in X$, then

$$|\|x\| - \|y\|| \leq \|x - y\|;$$

in particular, the map $x \to \|x\|$ is continuous from $(X, \|\cdot\|)$ into $\mathbb{R}$.

Definition Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on a vector space X are equivalent - we write $\| \cdot \|_1 \sim \| \cdot \|_2$ -if there exist constants $0 < c_1 \leq c_2$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1 \quad \text{for all } x \in X. \qquad (1.11)$$

• The above notion of 'equivalence' is an equivalence relation.

• Two equivalent norms on X induce the same topology on X.

• If $\| \cdot \|_1 \sim \| \cdot \|_2$ then

$$\|x_n - x\|_1 \to 0 \Leftrightarrow \|x_n - x\|_2 \to 0.$$

• If $(x_n), (y_n) \subset X$ with $x_n \to x, y_n \to y$ and if $(a_n) \subset \mathbb{K}$ with $a_n \to a$, then $\|x_n\| \to \|x\|$; $x_n + y_n \to x + y$, and $a_n x_n \to ax$.

• If $1 \leq p \leq q \leq \infty$, then $l^p(\mathbb{K}) \subset l^q(\mathbb{K})$ and

$$\|x\|_{l^q} \leq \|x\|_{l^p}, \ \forall x \in l^p(\mathbb{K}). \qquad (1.12)$$

Proof Suppose that $x \in \ell^p$ with $\|x\|_{\ell^p} = 1$. Then in particular, $|x_j| \leq 1$ for every j, in which case, since $q \geq p$, it follows that $|x_j|^q \leq |x_j|^p$. Therefore

$$\|x\|_{\ell^q}^q = \sum_{j=1}^{\infty} |x_j|^q \leq \sum_{j=1}^{\infty} |x_j|^p = 1,$$

i.e. $\|x\|_{\ell^q} \leq 1$. Given a general non-zero $x \in \ell^p$, consider $y := x/\|x\|_{\ell^p}$; then $\|y\|_{\ell^p} = 1$ and so

$$\|y\|_{\ell^q} = \left\| \frac{x}{\|x\|_{\ell^p}} \right\|_{\ell^q} = \frac{\|x\|_{\ell^q}}{\|x\|_{\ell^p}} \leq 1.$$

- Let $1 \leq p < q$. For any $C > 0$, we do not have

$$\|x\|_{l^p} \leq C\|x\|_{l^q}, \ \text{for every} \ x \in l^1(\mathbb{K}). \tag{1.13}$$

In fact, define $x^{(n)} \in l^1$ by

$$x_j^{(n)} = \left\{ \begin{array}{ll} j^{-1/p}, & j = 1, \cdots, n \\ 0 & j = n+1, \cdots. \end{array} \right.$$

Then $\|x^{(n)}\|_{l^q} < \sum_{j=1}^{\infty} j^{-q/p} < \infty \ \ \forall n$ but $\|x^{(n)}\|_{l^p} = \sum_{j=1}^{n} j^{-1}$ is unbounded, so (1.13) cannot hold for any $C > 0$.

Definition $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isomorphic, write $X \simeq Y$, if $\exists$ a bijective linear map $T: X \to Y$ and $c_1 > 0, c_2 > 0$ such that

$$c_1 \|x\|_X \leq \|T(x)\|_Y \leq c_2 \|x\|_X \ \forall x \in X; \qquad (1.14)$$

and are isometrically isomorphic if in addition T preserves the norm,

$$\|T(x)\|_Y = \|x\|_X \ \forall x \in X,$$

T is then called an isometric isomorphism from X onto Y and write $X \equiv Y$.

• If X is an n-dim. vector space. Then $(X, \|\cdot\|_E) \equiv (\mathbb{K}^n, \|\cdot\|_{l^2})$.

Theorem 1.1 If $\dim(X) = n < \infty$, then any two norms on X are equivalent.

Proof. Let $\{u_j\}_{j=1}^n$ be a base of X and define

$$\left\| \sum_{j=1}^n x_j u_j \right\|_E = \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}.$$

$\|\cdot\|_E$ is a norm on X. It suffices to show that any norm $\|\cdot\|$ on X is equivalent to $\|\cdot\|_E$. We have

$$\left\|\sum_{j=1}^n x_j u_j\right\| \le \beta \left\|\sum_{j=1}^n x_j u_j\right\|_E, \qquad (1.15)$$

where $\beta = \left(\sum_{j=1}^n \|u_j\|^2\right)^{1/2}$. Note that
$f : (\mathbb{K}^n, \|\cdot\|) \to (X, \|\cdot\|_E)$, $f(x_1, \cdots, x_n) = \sum_{j=1}^n x_j u_j$ is an isometry. Thus,
$B := \{u \in X : \|u\|_E = 1\} = \{\sum_{j=1}^n x_j u_j : \sum_{j=1}^n |x_j|^2 = 1\}$ is a compact subset of $(X, \|\cdot\|_E)$. By (1.14), $\|\cdot\|$ is continuous on $(X, \|\cdot\|_E)$. Hence $\exists v_0 \in B$ such that $\|v_0\| \le \|u\|$, $\forall u \in B$ and so $\|v_0\|\|w\|_E \le \|w\|$, $\forall w \in X$. $\qquad \square$

- If S is a nonempty subset of a linear space X, we denote by Span(S) the collection of all finite linear combinations of elements of S, i.e.,

$$\text{Span}\,S = \left\{ \sum_{i=1}^{n} x_i u_i, \ x_i \in \mathbb{K}, n \in \mathbb{N}, u_i \in S, i = 1, ..., n \right\}.$$

- $\text{Span}(S)$ is a linear subspace of X, called the linear subspace spanned by S and call $\overline{\text{Span(S)}}$ the closed linear span of S. If $X = \text{Span}(S)$ we say that S spans X.

- A set $E \subset X$ is linearly independent if any finite collection of elements of E is linearly independent.

- A Hamel basis for a vector space X is any linearly independent spanning set.

- Every vector space has a Hamel basis.

Lemma If X is a normed space, the following are equivalent:

(i) X is separable (i.e. X contains a countable dense subset);

(ii) The unit sphere in X, $S_X := \{x \in X : \|x\| = 1\}$, is separable;

(iii) X contains a countable set $\{x_j\}_{j=1}^\infty$ s.t. $\overline{\mathrm{Span}\{x_j\}_{j=1}^\infty} = X$.

Proof. ii)$\Rightarrow$ iii): Let $\{x_j\}_{j=1}^\infty$ be a dense subset of S_X. If $0 \neq x \in X$ and $\epsilon > 0$, then from $x/\|x\| \in S_X$, $\exists x_k$ s.t.

$$\|x_k - x/\|x\|\| < \epsilon/\|x\|, \ i.e., \|x - x_k\|x\|\| < \epsilon,$$

and since $x_k\|x\| \in \overline{\mathrm{Span}\{x_j\}_{j=1}^\infty}$ this gives (iii).

iii)$\Rightarrow$ i): The collection of finite linear combinations of the $\{x_j\}$ with rational coefficients is countable. This countable collection is dense in X. $\qquad \square$

• If $(X, \|\cdot\|)$ is a Banach space and Y is a linear subspace of X then $(Y, \|\cdot\|)$ is a Banach space if and only Y is closed.

Lemma For $1 \leq p < \infty$ the space $l^p(\mathbb{K})$ is separable, but $l^\infty(\mathbb{K})$ is not separable.

Proof If $1 \leq p < \infty$, then the linear span of $\{e^{(j)}\}_{j=1}^{\infty}$, where $e_i^{(j)} = \delta_{ij}$ is dense in $l^p(\mathbb{K})$: given any $x \in l^p(\mathbb{K})$ and any $\epsilon > 0$ $\exists N$ s.t.

$$\sum_{j=n+1}^{\infty} |x_j|^p < \epsilon \ \text{ for every } n \geq N$$

and so

$$\left\| x - \sum_{j=1}^{n} x_j e^{(j)} \right\|_{l^p} = \left(\sum_{j=n+1}^{\infty} |x_j|^p \right)^{1/p} < \epsilon.$$

In the space $l^\infty(\mathbb{K})$, consider the uncountable set

$$S := \{x \in l^\infty : x_j = 0 \text{ or } 1 \text{ for each } j \in \mathbb{N}\}.$$

Any two distinct elements x and y in S satisfy

$$\|x - y\|_{l^\infty} = 1$$

since they must differ by 1 in at least one term. Any dense set A of $l^\infty(\mathbb{K})$ must contain an uncountable number of elements: since A is dense, for every $x \in S$ there must be some $x' \in A$ such that $\|x' - x\|_{l^\infty} < 1/3$. But if x, y are distinct elements of S, then $x' \neq y'$ since

$$\|x' - y'\|_{l^\infty} > 1/3$$

by triangular inequality. Thus A contains an uncountable number of elements.

Definition A normed space $(X, \|\cdot\|)$ is complete if every Cauchy sequence in X converges in X (to a limit that lies in X). A Banach space is a complete normed space.

Steps to show completeness:

(i) use the definition of what it means for a sequence to be Cauchy to identify a possible limit;

(ii) show that the original sequence converges to this 'possible limit' in the appropriate norm;

(iii) check that the 'limit' lies in the correct space.

Theorem For each $1 \leq p \leq \infty$ the space $l^p(\mathbb{K})$ is complete w.r.t. its standard norm.

Proof We consider the case $1 \leq p < \infty$ and let $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots) \subset l^p(\mathbb{K})$ be Cauchy. Then $\forall \epsilon > 0, \exists N_\epsilon$ s.t.

$$\|x^{(n)} - x^{(m)}\|_{l^p}^p = \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}| < \epsilon^p \ \forall n, m \geq N_\epsilon. \quad (1.16)$$

Thus $(x_j^{(n)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{K}$ for every j and so for each $j \in \mathbb{N}$

$$x_j^{(n)} \to a_j \ \text{ as } \ n \to \infty$$

for some $a_j \in \mathbb{K}$. Set $a = (a_1, a_2, \cdots)$; then (1.16) implies that

$$\|x^{(n)} - a\|_{l^p}^p = \sum_{j=1}^{\infty} |x_j^{(n)} - a_j| < \epsilon^p \ \forall n \geq N_\epsilon,$$

and so $x^{(n)} - a \in l^p$ provided that $n \geq N_\epsilon$. This implies that $a \in l^p$ and $x^{(n)} \to a$ in $l^p(\mathbb{K})$.

Theorem Let X be a metric space and let $\mathbb{F}_b(X : \mathbb{K})$ be the collection of all functions $f : X \to \mathbb{K}$ that are bounded, i.e. $\sup_{x \in X} |f(x)| < \infty$. Then $\mathbb{F}_b(X : \mathbb{K})$ is complete with the supremum norm

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

Proof Note that

$$\|f_n - f\|_\infty \to 0 \Leftrightarrow f_n \to f \text{ uniformly}.$$

If (f_n) is a Cauchy sequence in $\mathbb{F}_b(X : \mathbb{K})$, then given any $\epsilon > 0$, $\exists N$ s.t.

$$\|f_n - f_m\|_\infty = \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon \ \ \forall m, n \ge N. \quad (1.17)$$

Thus for each $x \in X$, $(f_n(x))$ is a Cauchy sequence in $\mathbb{K}$ and so we can set

$$f(x) = \lim_{k \to \infty} f_k(x).$$

From (1.17) we have

$$|f_n(x) - f_m(x)| < \epsilon \ \ \forall m, n \geq N,$$

where N does not depend on x. Letting $m \to \infty$ we obtain

$$|f_n(x) - f(x)| < \epsilon \ \ \forall n \geq N,$$

where N does not depend on x again. It follows that

$$\sup_{x \in X} |f_n(x) - f(x)| < \epsilon \ \ \forall n \geq N \tag{1.18}$$

and so $f_n \to f$ uniformly on X. Since f_N is bounded and we have from (1.18) that $\|f_N - f\|_\infty < \epsilon$. Thus f is bounded. $\qquad \square$

Corollary If (X, d) is any metric space, then $C_b(X : \mathbb{K})$, the space of all bounded continuous functions from X to $\mathbb{K}$, is complete when equipped w.r.t. the supremum norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

Proof Note that $C_b(X : \mathbb{K})$ is a linear subspace of $\mathbb{F}_b(X : \mathbb{K})$. It then suffices to show that $C_b(X : \mathbb{K})$ is a closed subspace. Let $(f_k) \in C_b(X : \mathbb{K})$ and $\|f_k - f\|_\infty \to 0$. Fix $x \in X$. Given $\epsilon > 0$, $\exists N$ such that $\|f_n - f\|_\infty < \epsilon/3$ for every $n \geq N$. Since $f_N \in C_b(X : \mathbb{K})$, $\exists \delta > 0$ s.t. $d(x, y) < \delta \Rightarrow |f_N(y) - f_N(x)| < \epsilon/3$. Thus if $d(x, y) < \delta$, then

$$
\begin{aligned}
|f(y) - f(x)| \leq\ & |f(y) - f_N(x)| + |f_N(y) - f_N(x)| \\
& + |f_N(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}
$$

which shows that f is continuous at x. Since this holds for any $x \in X$, we conclude that $f \in C_b(X : \mathbb{K})$. $\qquad \square$

Theorem The space $C([a,b])$ of all continuously differentiable functions on $[a,b]$ is complete with the C^1 norm

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty.$$

Proof Let $(f_n) \in C([a,b])$ be Cauchy in C^1 norm. $\forall \epsilon > 0, \exists N$ s.t.

$$\|f_n - f_m\|_\infty + \|f'_n - f'_m\|_\infty < \epsilon \quad n, m \geq N. \qquad (1.19)$$

Thus (f_n) and (f'_n) are both Cauchy sequences in $C([a,b])$ with the supremum norm and so $\exists f, g \in C([a,b])$ s.t. such that $\|f_n - f\|_\infty \to 0, \|f'_n - g\|_\infty \to 0$. **Claim**: $g = f'$. In fact, from

$$\left| \int_a^x f'_n(t)dt - \int_a^x g(t)dt \right| \leq \int_a^x |f'_n(t) - g(t)|dt \leq (b-a)\|f'_n - g\|_\infty,$$

we have by taking $n \to \infty$ in $f_n(x) = f_n(a) + \int_a^x f'_n(t)dt$ that $f(x) = f(a) + \int_a^x g(t)dt$ and so $f' = g$. Hence $f \in C'([a,b])$. Taking $m \to \infty$ in (1.19), we have $\|f_n - f\|_\infty + \|f'_n - f'\|_\infty < \epsilon \ \forall n \geq N$ which shows that $f_n \to f$ in $C^1([a,b])$. $\qquad \square$

The space $C^k([a,b])$ of all k times continuously differentiable functions on $[a,b]$ with the C^k norm

$$\|f\|_{C^k} = \sum_{j=0}^{k} \|f^{(j)}\|_\infty, \text{ where } f^{(j)} = \frac{d^j f}{dx^j}.$$

is complete.

If $\{x_n\}$ is a sequence in a normed linear space V, the series $\sum_{n=1}^{\infty} x_n$ is said to converge to x if $\sum_{n=1}^{N} x_n \to x$ as $N \to \infty$, and it is called absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\| < \infty$

Lemma A normed linear space V is complete iff every absolutely convergent series in V converges.

Proof. If V is complete and $\sum_{n=1}^{\infty} ||x_n|| < \infty$, let $S_N = \sum_{n=1}^{N} x_n$. Then for $N > M$, we have

$$||S_N - S_M|| \leq \sum_{n=M+1}^{N} ||x_n|| \to 0, \ as \ M, N \to \infty.$$

so the sequence $\{S_N\}$ is Cauchy and hence convergent. Conversely, let $\{x_n\}$ be a Cauchy sequence. We can choose $n_1 < n_2 < \cdots$ such that $||x_n - x_m|| < 2^{-j}$ for $m, n \geq n_j$. Let $y_1 = x_{n_1}, \ y_j = x_{n_j} - x_{n_{j-1}}$ for $j > 1$. Then $\sum_{j=1}^{k} y_j = x_{n_k}$, and

$$\sum_{j=1}^{\infty} ||y_j|| \leq ||y_1|| + \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty,$$

so $\lim_{k \to \infty} x_{n_k} = \sum_{j=1}^{\infty} y_j$ exists. But since $\{x_n\}$ is Cauchy, we know that $\{x_n\}$ converges to the same limit as $\{x_{n_k}\}$. $\square$

• If $(X, \|\cdot\|_X) \simeq (Y, \|\cdot\|_Y)$, then $(X, \|\cdot\|_X)$ is complete iff $(Y, \|\cdot\|_Y)$ is complete.

Proof Let $T : X \to Y$ be an isomorphism between X and Y with

$$c_1\|x\|_X \le \|Tx\|_Y \le c_2\|x\|_X, \ \forall x \in X$$

for some $c_1 > 0, c_2 > 0$. Assume that Y is complete. Let (x_n) be a Cauchy sequence in X; then (Tx_n) is Cauchy in Y and so $Tx_n \to y$ for some $y \in Y$. Let $x = T^{-1}y$; then

$$\|x_n - x\|_X \le \frac{1}{c_1}\|Tx_n - y\|_Y \to 0.$$

Hence, X is complete. $\qquad\square$

• If X is a normed space and $\|\cdot\|_1)$ and $\|\cdot\|_2)$ are two equivalent norms on X, then $(X, \|\cdot\|_1)$ is complete $\Leftrightarrow$ $(X, \|\cdot\|_2)$ is complete.

• If $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are complete, then $X \times Y$ is complete for the norm $\|(x, y)\|_1 := \|x\|_X + \|y\|_Y$ and also for the equivalent norm $\|(x, y)\|_2 := \left(\|x\|_X^2 + \|y\|_Y^2 \right)^{1/2}$.

Proof If (x_n, y_n) is a Cauchy sequence in $X \times Y$ using the norm $\| \cdot \|_1$, then from

$$\|(x_n, y_n) - (x_m, y_m)\|_1 = \|x_n - x_m\|_X + \|y_n - y_m\|_y$$

we know that (x_n) is Cauchy in $(X, \| \cdot \|_X)$ and (y_n) is Cauchy in $(Y, \| \cdot \|_Y)$. Thus $\exists x \in X, y \in Y$ s.t. $x_n \to x$ in $(X, \| \cdot \|_X)$, $y_n \to y$ in $(Y, \| \cdot \|_Y)$. Since

$$\|(x_n, y_n) - (x, y)\|_1 = \|x_n - x\|_X + \|y_n - y\|,$$

it follows that $(x_n, y_n) \to (x, y)$ in $(X \times Y, \| \cdot \|_1)$, and so this space is complete. $\qquad \square$

Riesz's Lemma Let $(X, \| \cdot \|)$ be a normed space and Y a proper closed subspace of X. Then $\exists x \in X$ with $\|x\| = 1$ such that $\|x - y\| > 1/2$ for every $y \in Y$.

Proof Let $x_0 \in X \setminus Y$ and set $d = d(x_0, Y) := \inf_{y \in Y} \|x_0 - y\|$. If $d = d(x_0, Y) = 0$, then $\exists (y_n) \in Y$ s.t. $\|y_n - y_0\| \to 0$, i.e. $y_n \to x_0$ and then since Y is closed, we would have $x_0 \in Y$. Thus, $d > 0$.

Choose $y_0 \in Y$ s.t. $d \leq \|x_0 - y_0\| \leq 2d$ and set $x = \frac{x_0 - y_0}{\|x_0 - y_0\|}$; then $\|x\| = 1$, and for any $y \in Y$ we have

$$
\begin{aligned}
\|x - y\| &= \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - y \right\| \\
&= \frac{1}{\|x_0 - y_0\|} \left\| x_0 - [y_0 + y\|x_0 - y_0\|] \right\| \\
&\geq \frac{d}{\|x_0 - y_0\|} \geq \frac{d}{2d} = \frac{1}{2}.
\end{aligned}
$$

$\square$

• A finite-dimensional subspace of a normed linear space X is complete and so is closed.

Theorem A normed space X is finite-dimensional iff its closed unit ball is compact.

Proof Let $\dim X = \infty$. Take any $x_1 \in X$ with $\|x_1\| = 1$. Then $\mathrm{span}\{x_1\}$ is a proper closed linear subspace of X, so $\exists x_2 \in X$ with $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq 1/2$. Now $\mathrm{Span}\{x_1, x_2\}$ is a proper closed linear subspace of X so $\exists x_3 \in X$ with $\|x_3\| = 1$ and $\|x_3 - x_2\| \geq 1/2$ and $\|x_3 - x_1\| \geq 1/2$.

One can continue inductively to obtain a sequence (x_n) with $\|x_n\| = 1$ and $\|x_i - x_j\| \geq 1/2$ whenever $i \neq j$. No subsequence of the (x_n) can be Cauchy, so no subsequence can converge, from which it follows that the closed unit ball in X is not compact. $\square$

- A set is precompact if its closure is compact.

Lemma A subset A of a complete normed space $(X, \|\cdot\|)$ is precompact iff any sequence in A has a Cauchy subsequence.

Proof
$\Rightarrow$: Let A be precompact and that $(x_n) \in A$. Since $(x_n) \in \overline{A}$ and $\overline{A}$ is compact, (x_n) has a convergent subsequence, and any convergent sequence is Cauchy.

$\Leftarrow$: Suppose that any sequence in A has a Cauchy subsequence. Take a sequence $(y_n) \in \overline{A}$; then $\exists (x_n) \in A$ such that $\|x_n - y_n\| < 1/n$. Let x_{n_k} be a Cauchy subsequence of (x_n); then (y_{n_k}) is Cauchy too, and so converges to a limit y, which is contained in $\overline{A}$ since this set is closed. This shows that $\overline{A}$ is compact. $\qquad \square$

Arzelá-Ascoli Theorem If X is a compact metric space then
$A \subset C(X : \mathbb{R})$ is precompact $\Leftrightarrow A$ is bounded ($\exists R > 0$ s.t.
$\|f\|_\infty \leq R \ \forall f \in A$) and equicontinuous, i.e. $\forall \epsilon > 0 \ \exists \delta > 0$ s.t.

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon \text{ for every } f \in A, x, y \in X.$$

Proof $\Leftarrow$: Let $(f_j) \in A$ and take a countable set (x_n) as below:
• (Exercise 2.14) In any compact metric space $\exists$ a countable
subset $(x_n)_{n=1}^\infty$ s.t. $\forall \epsilon > 0 \ \exists M(\epsilon)$ s.t. for every $x \in X$ we have

$$d(x_j, x) < \epsilon \text{ for some } 1 < j < M(\epsilon).$$

Since (f_j) is bounded, we can use a 'diagonal argument' to find a
subsequence (which we relabel) such that $f_j(x_k)$ converges for
every k. The idea is to repeatedly extract subsequences to ensure
that $f_j(x_k)$ converges for more and more of the (x_k).

(f_j) is bounded $\Rightarrow f_j(x_1)$ is bounded in $\mathbb{K} \Rightarrow \exists (f_{1,j}) \subset (f_j)$ s.t. $f_{1,j}(x_1)$ converges. Since $(f_{1,j}(x_2))$ is bounded in $\mathbb{K}$, $\exists (f_{2,j}) \subset (f_{1,j})$ s.t. $f_{2,j}(x_2)$ converges. Since $(f_{2,j}(x_1)) \subset (f_{1,j}(x_1))$, $(f_{2,j}(x_1))$ still converges.

We continue, extracting subsequences of subsequences, so that

$$f_{n,j}(x_i) \text{ converges for all } i = 1, ..., n.$$

Consider the sequence $(f_{m,m})_{m=1}^{\infty}$. This is a subsequence of (f_j), and a subsequence of $(f_{n,j})$ once $m \geq n$. We have $(f_{m,m}(x_k))_{m=k}^{\infty} \subset (f_{k,j}(x_k))_{j=1}^{\infty}$. Thus

$$f_{m,m}(x_k) \text{ converges for all } k \in \mathbb{N}.$$

Set $g_m = f_{m,m}$. Let's show that (g_m) is Cauchy in the supremum norm.

Given $\epsilon > 0$, since $(g_n) \subset (f_j)$ is equicontinuous $\exists \delta > 0$ s.t.

$$d(x, y) < \delta \Rightarrow |g_n(x) - g_n(y)| < \epsilon/3 \ \ \forall \ n, \ x, y.$$

$\exists M(\delta)$ s.t. $\forall x \in X$, $\exists \ x_i$ with $1 \leq i \leq M$ s.t. $d(x, x_i) < \delta$. Since $g_n(x_i)$ converges for every i, $\exists N$ s.t. if $n, m \geq N$, then

$$|g_n(x_i) - g_m(x_i)| < \epsilon/3, \ 1 \leq i \leq M$$

Thus, if $n, m > N$, we have for any x, by taking $i \in \{1, \cdots, M\}$ with $d(x, x_i) < \delta$ that

$$\begin{aligned} |g_n(x) - g_m(x)| &\leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| \\ &+ |g_m(x_i) - g_m(x)| < \epsilon, \end{aligned}$$

which shows that if $n, m \geq N$, then $\|g_n - g_m\|_\infty \leq \epsilon$ and so (g_n) is Cauchy.

$\Rightarrow$: The boundedness of A follows from the fact that $\overline{A}$ is compact.

Since A is precompact, it is totally bounded. Thus, $\forall\, \epsilon > 0$, $\exists\{f_1, ..., f_n\}$ s.t. $\forall f \in A$, we have

$$\|f - f_i\|_\infty < \epsilon/3 \quad \text{for some } i \in \{1, ..., n\}.$$

Since the f_j are all continuous functions on the compact set X they are uniformly continuous, so $\exists \delta > 0$ s.t. $\forall i = 1, \cdots, n$

$$d(x,y) < \delta \Rightarrow |f_i(x) - f_i(y)| < \epsilon/3.$$

For any $f \in A$ choose j such that $\|f - f_j\|_\infty < \epsilon/3$. It follows that if $d(x,y) < \delta$ then

$$
\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\
&\leq \|f - f_j\|_\infty + |f_j(x) - f_j(y)| + \|f_j - f\|_\infty < \epsilon,
\end{aligned}
$$

so A is equicontinuous. $\qquad\square$