Linear Maps between Normed Spaces

Definition 11.1 A linear map $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is *bounded* if there exists a constant M such that

$$\|Tx\|_Y \leq M\|x\|_X \qquad \text{for all} \qquad x \in X. \tag{11.1}$$

Lemma 11.2 *If X is a finite-dimensional vector space, then any linear map $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is bounded.*

Lemma 11.3 *A linear map $T\colon X \to Y$ is continuous if and only if it is bounded.*

Proof Suppose that T is bounded; then for some $M > 0$

$$\|Tx_n - Tx\|_Y = \|T(x_n - x)\|_Y \leq M\|x_n - x\|_X,$$

and so T is continuous. Now suppose that T is continuous; then in particular it is continuous at zero, and so, taking $\varepsilon = 1$ in the definition of continuity, there exists a $\delta > 0$ such that

$$\|Tx\| \leq 1 \qquad \text{for all} \qquad \|x\| \leq \delta.$$

It follows that for $z \neq 0$

$$\|Tz\| = \left\|T\left(\frac{\|z\|}{\delta}\frac{\delta z}{\|z\|}\right)\right\| = \frac{\|z\|}{\delta}\left\|T\left(\frac{\delta z}{\|z\|}\right)\right\| \leq \frac{1}{\delta}\|z\|,$$

and so T is bounded. □

Corollary *The null space $\mathcal{N}(T)$ of a bounded operator $T : X \to Y$ is closed.*

Let X be a normed linear space. A linear map from X to 𝕂 is called a linear functional. The space of all bounded linear functionals on X is denoted by X* and called the dual of X.

Theorem *A linear functional on the normed space X is bounded if and only if its null space is closed.*

Proof. (Necessity.) This is a special case of Corollary above.

(Sufficiency.) Let f be a linear functional on X such that $\mathcal{N}(f)$ is a closed subspace of X. Suppose that f is unbounded. Then for every $n \in \mathbf{N}$, there exists $x_n \in X$ such that $\|x_n\| = 1$ and $|f(x_n)| > n$. Note that $\mathcal{N}(f) \neq X$. Let $x \in X \setminus \mathcal{N}(f)$, so $f(x) \neq 0$, and define

$$y_n = x - \frac{f(x)}{f(x_n)} x_n \qquad n \in \mathbf{N}.$$

For every $n \in \mathbf{N}$ we have

$$f(y_n) = f(x) - \frac{f(x)}{f(x_n)} f(x_n) = 0.$$

Hence $y_n \in \mathcal{N}(f)$ for all $n \in \mathbf{N}$. Furthermore,

$$\|y_n - x\| = \left\| \frac{f(x)}{f(x_n)} x_n \right\| = \frac{|f(x)|}{|f(x_n)|} \|x\| < \frac{|f(x)| \|x\|}{n} \longrightarrow 0,$$

as $n \longrightarrow \infty$. Therefore, the sequence (y_n) converges to x. Because $\mathcal{N}(f)$ is closed, we have $x \in \mathcal{N}(f)$, a contradiction.

The space of all bounded linear maps from X into Y is denoted by $B(X, Y)$; we write $B(X)$ for the space $B(X, X)$ of all bounded linear maps from X into itself.

Definition 11.4 The norm in $B(X, Y)$ or *operator norm* of a linear map $T : X \to Y$ is the smallest value of M such that (11.1) holds,

$$\|T\|_{B(X,Y)} := \inf\{M : \|Tx\|_Y \leq M\|x\|_X \text{ for all } x \in X\}. \tag{11.2}$$

Lemma 11.5 *As defined in (11.2)* $\|\cdot\|_{B(X,Y)}$ *is a norm on* $B(X, Y)$. *The norm in* $B(X, Y)$ *is also given by*

$$\|T\|_{B(X,Y)} = \sup_{\|x\|_X=1} \|Tx\|_Y.$$

$$T \in B(X,Y),\ S \in B(Y,Z) \quad \Rightarrow \quad S \circ T \in B(X,Z)$$

If $T : X \to Y$, then in order to find $\|T\|$ one can try the following: first show that

$$\|Tx\|_Y \leq M\|x\|_X \tag{11.7}$$

for some $M > 0$, i.e. show that T is bounded. It follows that $\|T\| \leq M$ (since $\|T\|$ is the infimum of all M such that (11.7) holds). Then, in order to show that in fact $\|T\| = M$, find an example of a particular $z \in X$ such that

$$\|Tz\|_Y = M\|z\|_X.$$

This shows from the definition that $\|T\| \geq M$ and hence that in fact $\|T\| = M$.

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces. Then a linear operator $A : X \to Y$ is continuous if and only if the following implication holds

$$(*) \quad x_n \in X,\ \|x_n\|_X \to 0 \implies (\|Ax_n\|_Y) \text{ is a bounded sequence.}$$

Proof

Obviously, if A is continuous, then $(*)$ holds true. For the converse implication, it suffices to prove that there exists an $r > 0$ such that $A\big(B_X(0,r)\big) = \{Ax;\ x \in X,\ \|x\|_X < r\}$ is bounded in $(Y, \|\cdot\|_Y)$. Assume the contrary, i.e., for all $n \in \mathbb{N}$, the set $A\big(B_X(0,1/n)\big)$ is unbounded. This means there is a sequence (x_n) in X such that $\|x_n\|_X < 1/n$, $\|Ax_n\|_Y > n$, for all $n \in \mathbb{N}$, which contradicts $(*)$.

Example 11.7 Consider the right- and left- shift operators $\mathfrak{s}_r : \ell^2 \to \ell^2$ and $\mathfrak{s}_l : \ell^2 \to \ell^2$, given by

$$\mathfrak{s}_r(\boldsymbol{x}) = (0, x_1, x_2, \ldots) \qquad \text{and} \qquad \mathfrak{s}_l(\boldsymbol{x}) = (x_2, x_3, x_4, \ldots).$$

Both operators are linear with $\|\mathfrak{s}_r\| = \|\mathfrak{s}_l\| = 1$.

Proof It is clear that the operators are linear. We have

$$\|\mathfrak{s}_r(\boldsymbol{x})\|_{\ell^2}^2 = \sum_{i=1}^{\infty} |x_i|^2 = \|\boldsymbol{x}\|_{\ell^2}^2,$$

so that $\|\mathfrak{s}_r\| = 1$, and

$$\|\mathfrak{s}_l(\boldsymbol{x})\|_{\ell^2}^2 = \sum_{i=2}^{\infty} |x_i|^2 \leq \|\boldsymbol{x}\|_{\ell^2}^2,$$

so that $\|\mathfrak{s}_l\| \leq 1$. However, if we choose an $\boldsymbol{x}$ with

$$\boldsymbol{x} = (0, x_2, x_3, \ldots)$$

then we have

$$\|\mathfrak{s}_l(\boldsymbol{x})\|_{\ell^2}^2 = \sum_{j=2}^{\infty} |x_j|^2 = \|\boldsymbol{x}\|_{\ell^2}^2,$$

and so we must have $\|\mathfrak{s}_l\| = 1$. $\square$

Example 11.9 Consider the map from $X = C([a,b])$ to $\mathbb{R}$ given by

$$Tf = \int_a^b \phi(x) f(x)\, \mathrm{d}x,$$

where $\phi \in C([a,b])$. Then T is linear with $\|T\|_{B(X;\mathbb{R})} = \|\phi\|_{L^1}$.

Proof Linearity is clear. For the upper bound we have

$$|Tf| \le \int_a^b \|f\|_\infty |\phi(x)|\, dx = \|f\|_\infty \|\phi\|_{L^1}.$$

Setting

$$f_\varepsilon(x) = \frac{\phi(x)}{|\phi(x)| + \varepsilon}$$

for $\varepsilon > 0$. Since ϕ attains its maximum on $[a,b]$, we have

$$\frac{\|\phi\|_\infty}{\|\phi\|_\infty + \varepsilon} \le \|f_\varepsilon\|_\infty \le 1.$$

Then

$$\begin{aligned}\int_a^b |\phi(x)|\, \mathrm{d}x - \int_a^b \phi(x) f_\varepsilon(x)\, \mathrm{d}x &= \int_a^b |\phi(x)| - \frac{|\phi(x)|^2}{|\phi(x)| + \varepsilon}\, \mathrm{d}x \\ &= \int_a^b \frac{\varepsilon|\phi(x)|}{|\phi(x)| + \varepsilon} \\ &\le (b-a)\varepsilon.\end{aligned}$$

This shows that

$$\begin{aligned}\left|\int_a^b \phi(x) f_\varepsilon(x)\, \mathrm{d}x\right| &\ge \big(\|\phi\|_{L^1} - (b-a)\varepsilon\big) \\ &\ge \big(\|\phi\|_{L^1} - (b-a)\varepsilon\big)\, \|f_\varepsilon\|_\infty;\end{aligned}$$

letting $\varepsilon \to 0$ guarantees that $\|T\| \ge \|\phi\|_{L^1}$, yielding the required equality.

Definition A sequence of vectors (e_n) in a normed space X is said to be a *Schauder basis* (or *basis*) for X if for every $x \in X$, there is a unique sequence of scalars (λ_n) such that the series $\sum_{k=1}^{\infty} \lambda_k e_k$ converges to x, that is,

$$x = \sum_{k=1}^{\infty} \lambda_k e_k.$$

Let c_{00} be the set of sequences (x_n) such that $x_n = 0$ for all but finitely many $n \in \mathbf{N}$.

Example Let $T : c_{00} \to c_{00}$ be a linear operator defined by

$$T : (x_1, x_2, \ldots, x_n, \ldots) \mapsto (x_1, 2x_2, \ldots, nx_n, \ldots).$$

The null space of T is the trivial subspace $\{0\}$, which is closed. For the vectors in the standard basis (e_n) of c_{00}, we have

$$\|e_n\| = 1 \qquad \text{and} \qquad \|Te_n\| = n, \qquad \text{for all } n \in \mathbf{N}.$$

Hence T is an unbounded operator.

Example Let X be the normed space of the polynomials on $[0, 1]$ with the sup-norm. A *differentiation* operator T is defined on X by

$$Tx(t) = \frac{d}{dt} x(t), \qquad \text{for } x(t) \in X.$$

For $x_n(t) = t^n$, $n > 1$, we have $Tx_n(t) = nt^{n-1}$. It is clear that $\|x_n\| = 1$ and $\|Tx_n\| = n$. It follows that T is an unbounded operator.

Example For $d = (d_n) \in \ell_\infty$, the *diagonal operator* $T_d : \ell_p \to \ell_p$ is defined by

$$T_d : (x_1, \ldots, x_n, \ldots) \mapsto (d_1 x_1, \ldots, d_n x_n, \ldots).$$

For $x = (x_n) \in \ell_p$, we have

$$\|T_d x\|_p = \left(\sum_{k=1}^{\infty} |d_k x_k|^p \right)^{1/p} \leq \sup_{k \in \mathbf{N}} |d_k| \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} = \|d\|_\infty \|x\|_p.$$

Hence, T_d is well defined and $\|T_d\| \leq \|d\|_\infty$. Let (e_n) be the standard basis in ℓ_p Clearly, $\|T_d e_n\|_p = |d_n|$ for $n \in \mathbf{N}$. We have

$$\|T_d\| = \sup_{\|x\|=1} \|T_d x\|_p \geq \sup_{n \in \mathbf{N}} \|T_d e_n\| = \|d\|_\infty.$$

It follows that $\|T_d\| = \|d\|_\infty$.

Example On the space $C[0,1]$, the *operator of indefinite integration* is defined by

$$(Tx)(t) = \int_0^t x(u)\,du, \qquad 0 \le t \le 1,\ x \in C[0,1].$$

The integral on the right-hand side is a continuous function of the upper limit. Hence T indeed maps the space $C[0,1]$ into itself. For $x \in C[0,1]$, we have

$$\begin{aligned}\|Tx\| &= \sup_{t\in[0,1]} \left| \int_0^t x(u)\,du \right| \le \sup_{t\in[0,1]} \int_0^t |x(u)|\,du \le \int_0^1 |x(u)|\,du \\ &\le \sup_{u\in[0,1]} |x(u)| = \|x\|.\end{aligned}$$

For the constant function $x = 1$ on $[0,1]$, we have

$$\|Tx\| = \sup_{t\in[0,1]} \left| \int_0^t 1\,du \right| = \sup_{t\in[0,1]} t = 1 = \|x\|.$$

It follows that $\|T\| = 1$.

Theorem 11.11 *If X is a normed space and Y is a Banach space, then $B(X, Y)$ is a Banach space.*

Proof Given any Cauchy sequence (T_n) in $B(X, Y)$ we need to show that $T_n \to T$ for some $T \in B(X, Y)$. Since (T_n) is Cauchy, given $\varepsilon > 0$ there exists an N_ε such that

$$\|T_n - T_m\|_{B(X,Y)} \leq \varepsilon \qquad \text{for all} \quad n, m \geq N_\varepsilon. \tag{11.9}$$

We now show that for every fixed $x \in X$ the sequence $(T_n x)$ is Cauchy in Y. This follows since

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\|_{B(X,Y)} \|x\|_X, \tag{11.10}$$

and (T_n) is Cauchy in $B(X, Y)$. Since Y is complete, it follows that

$$T_n x \to z$$

for some $z \in Y$, which depends on x. We can therefore define a mapping $T : X \to Y$ by setting $Tx = z$.

Now that we have identified our expected limit we need to make sure that $T \in B(X, Y)$ and that $T_n \to T$ in $B(X, Y)$.

First, T is linear since for any $x, y \in X$, $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x + \beta y) = \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y \\ &= \alpha T x + \beta T y. \end{aligned}$$

To show that T is bounded take $n, m \geq N_\varepsilon$ (from (11.9)) in (11.10), and let $m \to \infty$. Since $T_m x \to Tx$ this limiting process shows that

$$\|T_n x - Tx\|_Y \leq \varepsilon \|x\|_X. \tag{11.11}$$

Since (11.11) holds for every x, it follows that

$$\|T_n - T\|_{B(X,Y)} \leq \varepsilon \qquad \text{for } n \geq N_\varepsilon. \tag{11.12}$$

In particular, $T_{N_\varepsilon} - T \in B(X, Y)$, and since $B(X, Y)$ is a vector space and we have $T_{N_\varepsilon} \in B(X, Y)$, it follows that $T \in B(X, Y)$. Finally, (11.12) also shows that $T_n \to T$ in $B(X, Y)$. $\square$

Given a linear map $T: X \to Y$, we define

$\mathrm{Ker}(T) := \{x \in X : Tx = 0\}$ $\quad$ $\mathrm{Range}(T) := \{y \in Y : y = Tx \text{ for some } x \in X\}$.

Lemma 11.12 *If $T \in B(X, Y)$, then* Ker T *is a closed linear subspace of X.*

While $T \in B(X, Y)$ implies that $\mathrm{Ker}(T)$ is closed, the same is not true for the range of T: it need not be closed. Indeed, consider the map from ℓ^2 into itself given by

$$Tx = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots\right). \tag{11.13}$$

Since

$$\|Tx\|_{\ell^2}^2 = \sum_{j=1}^{\infty} \frac{1}{j^2}|x_j|^2 \le \sum_{j=1}^{\infty} |x_j|^2 = \|x\|_{\ell^2}^2,$$

we have $\|T\| \le 1$ and T is bounded. Now consider $y^{(n)} \in \mathrm{Range}(T)$, where

$$y^{(n)} = T(\underbrace{1, 1, 1, \dots, 1}_{\text{first } n \text{ terms}}, 0 \dots) = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, \dots\right).$$

We have $y^{(n)} \to y$, where y is the element of ℓ^2 with $y_j = j^{-1}$ (observe that $y \in \ell^2$ since $\sum_{j=1}^{\infty} j^{-2} < \infty$). However, there is no $x \in \ell^2$ such that $T(x) = y$: the only candidate is $x = (1, 1, 1, \dots)$, but this is not in ℓ^2 since its ℓ^2 norm is not finite, so $y \notin \mathrm{Range}(T)$.

Definition 11.13 An operator $T \in B(X,Y)$ is *invertible* if there exists an $S \in B(Y,X)$ such that[2] $ST = I_X$ and $TS = I_Y$, and then $T^{-1} = S$ is the inverse of T.

If X and Y are both Banach spaces and T : X ⟶ Y is a continuous bijection, then T^{-1} is bounded.

Lemma 11.14 *Suppose that X and Y are both normed spaces. Then for any $T \in B(X,Y)$, the following are equivalent:*

(i) *T is invertible;*
(ii) *T is a bijection and $T^{-1} \in B(Y,X)$;*
(iii) *T is onto and for some $c > 0$*

$$\|Tx\|_Y \geq c\|x\|_X \qquad \text{for every } x \in X. \tag{11.14}$$

Corollary 11.15 *If X is finite-dimensional, then a linear operator $T : X \to X$ is invertible if and only if* $\mathrm{Ker}(T) = \{0\}$.

Lemma 11.16 *If X and Y are Banach spaces and $T \in B(X, Y)$ is invertible, then so is $T + S$ for any $S \in B(X, Y)$ with $\|S\|\|T^{-1}\| < 1$. Consequently, the subset of $B(X, Y)$ consisting of invertible operators is open.*

Proof Suppose that $T \in B(X, Y)$ is invertible; then by (i)⇒(iii) we know that T is onto and that

$$\|Tx\|_Y \geq \frac{1}{\|T^{-1}\|}\|x\|_X.$$

We will show that for any $S \in B(X, Y)$ with $\|S\|\|T^{-1}\| = \alpha < 1$, $T + S$ is invertible.

First we show that $T + S$ is onto: given $y \in Y$, we want to ensure that there is an $x \in X$ such that

$$(T + S)x = y.$$

Consider the map $\mathcal{J}: X \to X$ defined by setting

$$x \mapsto \mathcal{J}(x) := T^{-1}(y - Sx).$$

Then

$$\begin{aligned}
\|\mathcal{J}(x) - \mathcal{J}(x')\|_X &= \left\|T^{-1}(y - Sx) - T^{-1}(y - Sx')\right\|_X \\
&= \left\|T^{-1}S(x - x')\right\|_X \\
&\leq \|T^{-1}\|\|S\|\|x - x'\|_X \\
&= \alpha\|x - x'\|_X,
\end{aligned}$$

where $\alpha < 1$ by assumption. Since X is a Banach space, we can use the Contraction Mapping Theorem to ensure that there is a unique $x \in X$ such that $x = \mathcal{J}(x)$, i.e. such that $x = T^{-1}(y - Sx)$. Applying T to both sides guarantees that $y = (T + S)x$ and so $T + S$ is onto.

Note that since

$$\|S\|\|T^{-1}\| < 1$$

we have

$$\frac{1}{\|T^{-1}\|} - \|S\| = c > 0.$$

Therefore

$$\begin{aligned}
\|(T + S)x\|_Y &\geq \|Tx\|_Y - \|Sx\|_Y \\
&\geq \frac{1}{\|T^{-1}\|}\|x\|_X - \|S\|\|x\|_X = c\|x\|_X.
\end{aligned}$$

Now using (iii)⇒(i) we deduce that $(T + S)$ is invertible. □

If $\{T_n\}$ and $\{S_n\}$ are sequences in $B(X)$ such that $\lim_{n\to\infty} T_n = T$ and $\lim_{n\to\infty} S_n = S$ then $\lim_{n\to\infty} S_nT_n = ST$.

Proof

As $\{T_n\}$ is convergent it is bounded so there exists $K > 0$ such that $\|T_n\| \leq K$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that when $n \geq N_1$

$$\|S_n - S\| < \frac{\epsilon}{2K},$$

and $N_2 \in \mathbb{N}$ such that when $n \geq N_2$

$$\|T_n - T\| < \frac{\epsilon}{2(\|S\| + 1)}.$$

As

$$\|S_nT_n - ST\| \leq \|S_nT_n - ST_n\| + \|ST_n - ST\| \leq K\|S_n - S\| + \|S\|\|T_n - T\|,$$

when $n \geq \max(N_1, N_2)$ we have

$$\|S_nT_n - ST\| \leq K\|S_n - S\| + \|S\|\|T_n - T\| < \epsilon.$$

Thus $\lim_{n\to\infty} S_nT_n = ST$. $\square$

Let X be a Banach space. If $T \in B(X)$ is an operator with $\|T\| < 1$ then $I - T$ is invertible and the inverse is given by

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Proof

Because X is a Banach space, so is $B(X)$. Since $\|T\| < 1$ the series $\sum_{n=0}^{\infty} \|T\|^n$ converges and hence, as $\|T^n\| \leq \|T\|^n$ for all $n \in \mathbb{N}$, the series $\sum_{n=0}^{\infty} \|T^n\|$ also converges. Therefore $\sum_{n=0}^{\infty} T^n$ converges . Let $S = \sum_{n=0}^{\infty} T^n$ and let $S_k = \sum_{n=0}^{k} T^n$. Then the sequence $\{S_k\}$ converges to S in $B(X)$. Now

$$\|(I - T)S_k - I\| = \|I - T^{k+1} - I\| = \| - T^{k+1}\| \leq \|T\|^{k+1}.$$

Since $\|T\| < 1$ we deduce that $\lim_{k\to\infty} (I - T)S_k = I$. Therefore,

$$(I - T)S = (I - T) \lim_{k\to\infty} S_k = \lim_{k\to\infty} (I - T)S_k = I,$$

Similarly, $S(I - T) = I$ so $I - T$ is invertible and $(I - T)^{-1} = S$.

Lemma 11.17 *If $T \in B(X, Y)$ and $S \in B(Y, Z)$ are invertible, then so is $ST \in B(X, Z)$, and $(ST)^{-1} = T^{-1}S^{-1}$.*

Proposition 11.18 *If $\{T_1, \ldots, T_n\}$ are commuting operators in $B(X)$, then*

$$T_1 \cdots T_n$$

is invertible if and only if every T_j, $j = 1, \ldots, n$, is invertible.

Proof One direction follows from Lemma 11.17 and induction. For the other direction, suppose that $\mathcal{T} = T_1 \cdots T_n$ is invertible; since T_1 commutes with $\mathcal{T}$ it also commutes with $\mathcal{T}^{-1}$, and so

$$T_1[\mathcal{T}^{-1}T_2 \cdots T_n] = \mathcal{T}^{-1}T_1T_2 \cdots T_n = \mathcal{T}^{-1}\mathcal{T} = I.$$

Since $\{T_1, \ldots, T_n\}$ commute we have

$$[\mathcal{T}^{-1}T_2 \cdots T_n]T_1 = \mathcal{T}^{-1}\mathcal{T} = I$$

as well. All the operators are bounded, so T_1 is invertible.

For other values of j we can use the fact that the $\{T_j\}$ commute to reorder the factors of $\mathcal{T}$ so that the first is T_j. We can now apply the same argument to show that T_j is invertible. $\square$