

Proof by Contradiction

Contradiction is a powerful proof technique that can be extremely useful in the right circumstances.

However, contradiction proofs tend to be less convincing and harder to write than direct proofs.

In proof by contradiction, we show that a claim P is true by showing that its negation $\neg P$ leads to a contradiction.

If $\neg P$ leads to a contradiction, then $\neg P$ can't be true, and therefore P must be true.

- A contradiction can be any statement that is well-known to be false or
- a set of statements that are obviously inconsistent with one another. e.g.
 - ✓ n is odd and n is even, or
 - ✓ $x < 2$ and $x > 7$.

Simple Example

Claim There is no largest even integer.

Proof:

- Suppose not. That is, suppose that there were a largest even integer. Let's call it k .
- Since k is even, it has the form $2n$, where n is an integer.
- Consider $k + 2$. $k + 2 = (2n) + 2 = 2(n + 1)$.
- So $k + 2$ is even. But $k + 2$ is larger than k .
- This contradicts our assumption that k was the largest even integer. So our original claim must have been true.

Another Example: $\sqrt{2}$ is irrational

Proof:

Suppose not. That is, suppose that $\sqrt{2}$ were rational.

- Then we can write $\sqrt{2}$ as a fraction $\frac{a}{b}$, where a and b are integers with **no common factors**.
- Since $\sqrt{2} = \frac{a}{b}$, $2 = \frac{a^2}{b^2}$. So $2b^2 = a^2$.
- By the definition of even, this means a^2 is even.
- But then a must be even. So $a = 2n$ for some integer n .
- If $a = 2n$ and $2b^2 = a^2$, then $2b^2 = 4n^2$. So $b^2 = 2n^2$. This means that b^2 is even, so b must be even.
- We now have a contradiction. **a and b** were chosen not to have any common factors. But they are both even, i.e. they **are both divisible by 2**.
- Because assuming that $\sqrt{2}$ was rational led to a contradiction, it must be the case that $\sqrt{2}$ is irrational.

Philosophy

- Proof by contradiction strikes many people as mysterious, because the argument starts with an assumption known to be false.
- The whole proof consists of building up a fantasy world and then knocking it down.
- Although the method is accepted as valid by the vast majority of theoreticians, these proofs are less satisfying than direct proofs which construct the world as we believe it to be.

There is, in fact, a minority but long-standing thread within theoretical mathematics, called “constructive mathematics,” which does not accept this proof method.