Induction

Mathematical induction is a technique for showing that a statement $P(n)$ is true for all natural numbers n, or for some infinite subset of the natural numbers (e.g. all positive even integers).

A proof by induction has the following outline:

Claim: P(n) is true for all positive integers n.

Proof: We'll use induction on n. ← induction variable

Base: We need to show that P(1) is true.

Induction: Suppose that P(n) is true for n = 1, 2, . . . , k−1.

We need to show that P(k) is true.

inductive hypothesis

Simple Example

Claim *For any positive integer n,* $\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof: We will show that $\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any positive integer n, using induction on n.

Base: We need to show that the formula holds for $n = 1$. $\Sigma_{i=1}^{1} i = 1$. And also $\frac{1 \cdot 2}{2} = 1$. So the two are equal for $n = 1$.

Induction: Suppose that $\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n = 1, 2, \ldots, k-1$. We need to show that $\Sigma_{i=1}^{k} i = \frac{k(k+1)}{2}$.

By the definition of summation notation, $\Sigma_{i=1}^{k} i = (\Sigma_{i=1}^{k-1} i) + k$

Our inductive hypothesis states that at $n = k-1$, $\Sigma_{i=1}^{k-1} i = (\frac{(k-1)k}{2})$.

Combining these two formulas, we get that $\Sigma_{i=1}^{k} i = (\frac{(k-1)k}{2}) + k$.

But $(\frac{(k-1)k}{2}) + k = (\frac{(k-1)k}{2}) + \frac{2k}{2} = (\frac{(k-1+2)k}{2}) = \frac{k(k+1)}{2}$.

So, combining these equations, we get that $\Sigma_{i=1}^{k} i = \frac{k(k+1)}{2}$ which is what we needed to show.

Why is the induction legit?

Domino Theory (intuitively):

- Imagine an infinite line of dominoes.
- The base step pushes the first one over.
- The inductive step claims that one domino falling down will push over the next domino in the line.
- So dominos will start to fall from the beginning all the way down the line.
- This process continues forever, because the line is infinitely long.
- However, if you focus on any specific domino, it falls after some specific finite delay.

Another example

Claim *For any natural number n, $n^3 - n$ is divisible by 3.*

Proof: By induction on n.

Base: Let $n = 0$. Then $n^3 - n = 0^3 - 0 = 0$ which is divisible by 3.

Induction: Suppose that $n^3 - n$ is divisible by 3, for $n = 0, 1, \ldots, k$. We need to show that $(k+1)^3 - (k+1)$ is divisible by 3.

$$(k+1)^3-(k+1) = (k^3+3k^2+3k+1)-(k+1) = (k^3-k)+3(k^2+k)$$

From the inductive hypothesis, $(k^3 - k)$ is divisible by 3. And $3(k^2 + k)$ is divisible by 3 since $(k^2 + k)$ is an integer. So their sum is divisible by 3. That is $(k+1)^3 - (k+1)$ is divisible by 3.

Variation in notation

Certain details of the induction outline vary, depending on the individual preferences of the author and the specific claim being proved.

- Some folks prefer to assume the statement is true for k and prove it's true for k + 1.
- Other assume it's true for k – 1 and prove it's true for k.
- For a specific problems, sometimes one or the other choice yields a slightly simpler proofs.
- Folks differ as to whether the notation n = 0, 1, . . . , k implies that k is necessarily at least 0, at least 1, or at least 2.