

# Induction

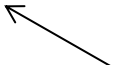
Mathematical induction is a technique for showing that a statement  $P(n)$  is true for all natural numbers  $n$ , or for some infinite subset of the natural numbers (e.g. all positive even integers).

A proof by induction has the following outline:

**Claim:**  $P(n)$  is true for all positive integers  $n$ .

**Proof:** We'll use induction on  $n$ .  induction variable

**Base:** We need to show that  $P(1)$  is true.

**Induction:** Suppose that  $P(n)$  is true for  $n = 1, 2, \dots, k-1$ .  
We need to show that  $P(k)$  is true. 

inductive hypothesis

# Simple Example

**Claim**      *For any positive integer  $n$ ,  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ .*

**Proof:** We will show that  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$  for any positive integer  $n$ , using induction on  $n$ .

**Base:** We need to show that the formula holds for  $n = 1$ .  $\sum_{i=1}^1 i = 1$ . And also  $\frac{1 \cdot 2}{2} = 1$ . So the two are equal for  $n = 1$ .

**Induction:** Suppose that  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$  for  $n = 1, 2, \dots, k-1$ . We need to show that  $\sum_{i=1}^k i = \frac{k(k+1)}{2}$ .

By the definition of summation notation,  $\sum_{i=1}^k i = (\sum_{i=1}^{k-1} i) + k$

Our inductive hypothesis states that at  $n = k-1$ ,  $\sum_{i=1}^{k-1} i = (\frac{(k-1)k}{2})$ .

Combining these two formulas, we get that  $\sum_{i=1}^k i = (\frac{(k-1)k}{2}) + k$ .

But  $(\frac{(k-1)k}{2}) + k = (\frac{(k-1)k}{2}) + \frac{2k}{2} = (\frac{(k-1+2)k}{2}) = \frac{k(k+1)}{2}$ .

So, combining these equations, we get that  $\sum_{i=1}^k i = \frac{k(k+1)}{2}$  which is what we needed to show.

# Why is the induction legit?

## Domino Theory (intuitively ):

- Imagine an infinite line of dominoes.
- The base step pushes the first one over.
- The inductive step claims that one domino falling down will push over the next domino in the line.
- So dominos will start to fall from the beginning all the way down the line.
- This process continues forever, because the line is infinitely long.
- However, if you focus on any specific domino, it falls after some specific finite delay.

## Another example

**Claim**     *For any natural number  $n$ ,  $n^3 - n$  is divisible by 3.*

Proof: By induction on  $n$ .

**Base:** Let  $n = 0$ . Then  $n^3 - n = 0^3 - 0 = 0$  which is divisible by 3.

**Induction:** Suppose that  $n^3 - n$  is divisible by 3, for  $n = 0, 1, \dots, k$ .

We need to show that  $(k + 1)^3 - (k + 1)$  is divisible by 3.

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3(k^2 + k)$$

From the inductive hypothesis,  $(k^3 - k)$  is divisible by 3. And  $3(k^2 + k)$  is divisible by 3 since  $(k^2 + k)$  is an integer. So their sum is divisible by 3. That is  $(k + 1)^3 - (k + 1)$  is divisible by 3.

## Variation in notation

Certain details of the induction outline vary, depending on the individual preferences of the author and the specific claim being proved.

- Some folks prefer to assume the statement is true for  $k$  and prove it's true for  $k + 1$ .
- Other assume it's true for  $k - 1$  and prove it's true for  $k$ .
- For a specific problems, sometimes one or the other choice yields a slightly simpler proofs.
- Folks differ as to whether the notation  $n = 0, 1, \dots, k$  implies that  $k$  is necessarily at least 0, at least 1, or at least 2.