

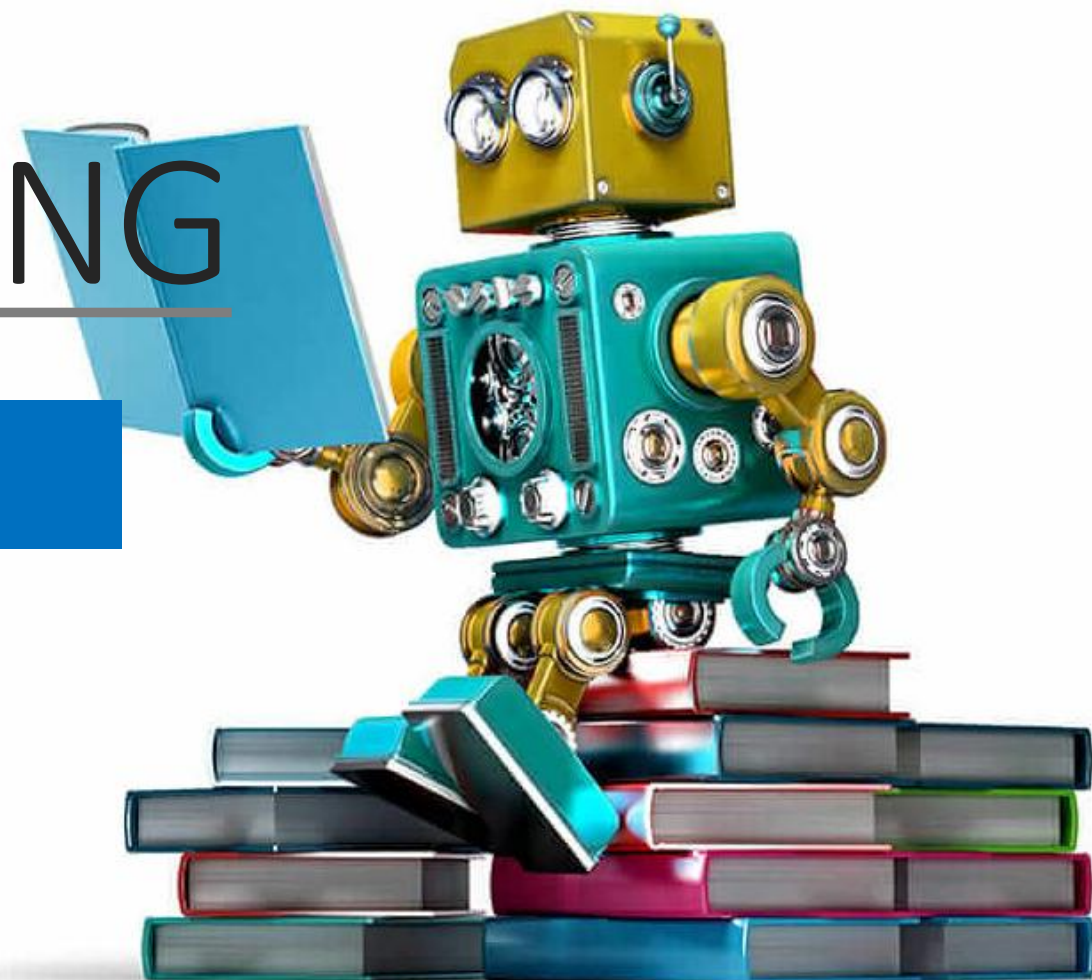
# MACHINE LEARNING

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## LAB3 Bayesian Learning

贾艳红 Jana

Email: [jiayh@mail.sustech.edu.cn](mailto:jiayh@mail.sustech.edu.cn)



# OBJECTIVES

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01

**Understand the Naive Bayes algorithm**

02

**Learn how to implement the Naive Bayes Classifier in python**





# Outline



- Background and Probability Basics
- Probabilistic Classification Principle
  - Probabilistic discriminative models
  - Generative models and their application to classification
  - MAP and converting generative into discriminative
- Naïve Bayes – an generative model
  - Principle and Algorithms (discrete vs. continuous)
  - Example: Play Tennis
- Zero Conditional Probability and Treatment
- Summary



# Background



- There are three methodologies:
  - ✓ **Model a classification rule directly**  
Examples: k-NN, linear classifier, SVM, neural nets, ..
  - ✓ **Model the probability of class memberships given input data**  
Examples: logistic regression, probabilistic neural nets (softmax),...
  - ✓ **Make a probabilistic model of data within each class**  
Examples: naive Bayes, model-based ....
- Important ML taxonomy for learning models
  - probabilistic models vs non-probabilistic models**
  - discriminative models vs generative models**



# Background



- Based on the taxonomy, we can see different the essence of learning models (classifiers) more clearly.

	Probabilistic	Non-Probabilistic
Discriminative	<ul style="list-style-type: none"><li>Logistic Regression</li><li>Probabilistic neural nets</li><li>.....</li></ul>	<ul style="list-style-type: none"><li>K-nn</li><li>Linear classifier</li><li>SVM</li><li>Neural networks</li><li>.....</li></ul>
Generative	<ul style="list-style-type: none"><li>Naïve Bayes</li><li>Model-based (e.g., GMM)</li><li>.....</li></ul>	N.A. (?)



# Probability Basics



## ➤ Prior, conditional and joint probability for random variables

- Prior probability:  $P(x)$
- Conditional probability:  $P(x_1|x_2), P(x_2|x_1)$
- Joint probability:  $\mathbf{x}=(x_1, x_2), P(\mathbf{x})=P(x_1, x_2)$
- Relationship:  $P(x_1, x_2)=P(x_2|x_1)P(x_1)=P(x_1|x_2)P(x_2)$
- Independence:  $P(x_2|x_1)=P(x_2), P(x_1|x_2)=P(x_1), P(x_1, x_2)=P(x_1)P(x_2)$

## ➤ Bayesian Rule

$$P(c|\mathbf{x}) = \frac{P(\mathbf{x}|c)P(c)}{P(\mathbf{x})}$$

Discriminative

$$Posterior = \frac{Likelihood \times Prior}{Evidence}$$

Generative



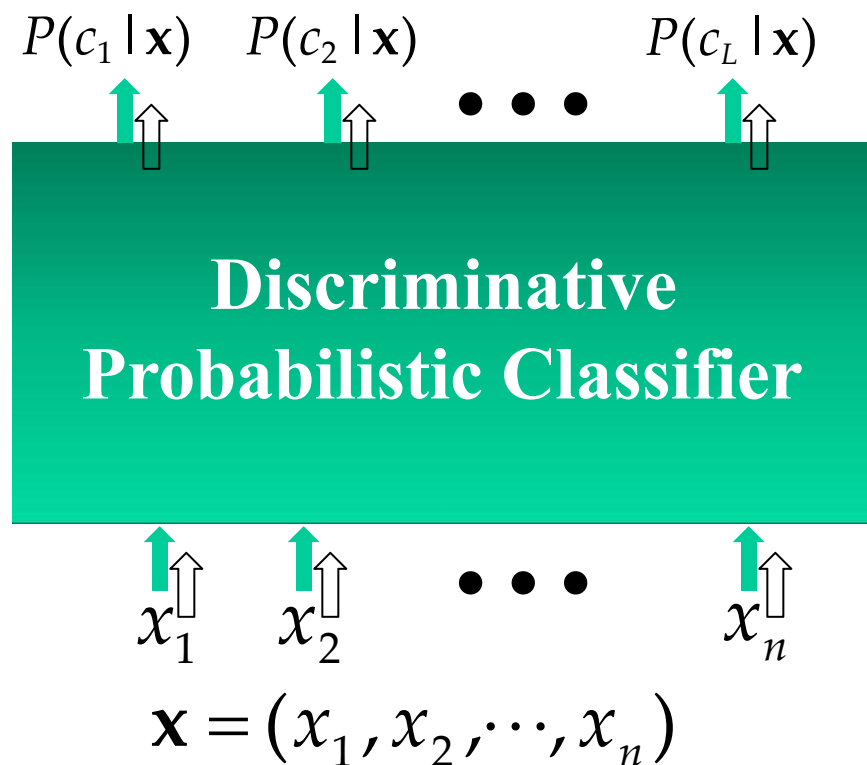
# Probabilistic Classification Principle



➤ Establishing a probabilistic model for classification

– **Discriminative model**

$$P(c | \mathbf{x}) \quad c = c_1, \dots, c_L, \mathbf{x} = (x_1, \dots, x_n)$$



- To train a discriminative classifier regardless its probabilistic or non-probabilistic nature, **all training examples of different classes must be jointly used to build up a single discriminative classifier.**
- **Output  $L$  probabilities for  $L$  class labels in a probabilistic classifier** while a single label is achieved by a non-probabilistic classifier.



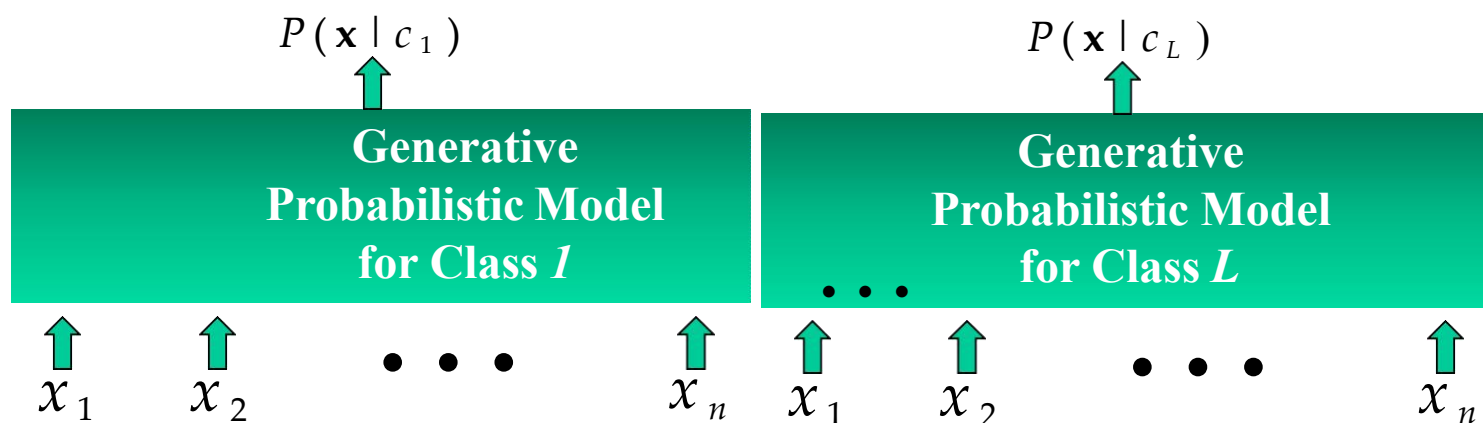
# Probabilistic Classification Principle



➤ Establishing a probabilistic model for classification (cont.)

– **Generative model (must be probabilistic)**

$$P(\mathbf{x} | c) \quad c = c_1, \dots, c_L, \mathbf{x} = (x_1, \dots, x_n)$$



$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

- $L$  probabilistic models have to be trained independently
- **Each is trained on only the examples of the same label**
- **Output  $L$  probabilities for a given input with  $L$  models**
- “Generative” means that such a model produces data subject to the distribution via sampling.





# Probabilistic Classification Principle



- **M**aximum **A** **P**osterior (**MAP**) classification rule
  - For an input  $\mathbf{x}$ , find the largest one from  $L$  probabilities output by a discriminative probabilistic classifier  $P(c_1 | \mathbf{x}), \dots, P(c_L | \mathbf{x})$ .
  - Assign  $\mathbf{x}$  to label  $c^*$  if  $P(c^* | \mathbf{x})$  is the largest.
- Generative classification with the MAP rule
  - Apply Bayesian rule to convert them into posterior probabilities

$$P(c_i | \mathbf{x}) = \frac{P(\mathbf{x} | c_i)P(c_i)}{P(\mathbf{x})} \propto P(\mathbf{x} | c_i)P(c_i)$$

for  $i = 1, 2, \dots, L$

Common factor for  
all  $L$  probabilities

- Then apply the MAP rule to assign a label



# Naive Bayes



## ➤ Bayes classification

$$P(c | \mathbf{x}) \propto P(\mathbf{x} | c)P(c) = P(x_1, \dots, x_n | c)P(c) \text{ for } c = c_1, \dots, c_L.$$

Difficulty: learning the joint probability  $P(x_1, \dots, x_n | c)$  is infeasible!

## ➤ Naïve Bayes classification

- Assume **all input features are class conditionally independent!**

$$P(x_1, x_2, \dots, x_n | c) = P(x_1 | x_2, \dots, x_n, c)P(x_2, \dots, x_n | c)$$

Applying the  
independence  
assumption

$$= P(x_1 | c)P(x_2, \dots, x_n | c)$$

$$= P(x_1 | c)P(x_2 | c) \cdots P(x_n | c)$$

- Apply the MAP classification rule: assign  $\mathbf{x}' = (a_1, a_2, \dots, a_n)$  to  $c^*$  if

$$\underbrace{[P(a_1 | c^*) \cdots P(a_n | c^*)]P(c^*)}_{\text{estimate of } P(a_1, \dots, a_n | c^*)} > \underbrace{[P(a_1 | c) \cdots P(a_n | c)]P(c)}_{\text{estimate of } P(a_1, \dots, a_n | c)}, \quad c \neq c^*, c = c_1, \dots, c_L$$

estimate of  $P(a_1, \dots, a_n | c^*)$

estimate of  $P(a_1, \dots, a_n | c)$



# Naive Bayes



- Algorithm: Discrete-Valued Features

- Learning Phase: Given a training set  $S$  of  $F$  features and  $L$  classes,

For each target value of  $c_i$  ( $c_i = c_1, \dots, c_L$ )

$\hat{P}(c_i) \leftarrow$  estimate  $P(c_i)$  with examples in  $S$ ;

For every feature value  $x_{jk}$  of each feature  $x_j$  ( $j = 1, \dots, F; k = 1, \dots, N_j$ )

$\hat{P}(x_j = x_{jk} | c_i) \leftarrow$  estimate  $P(x_{jk} | c_i)$  with examples in  $S$ ;

Output:  $F * L$  conditional probabilistic (generative) models

- Test Phase: Given an unknown instance  $\mathbf{x}' = (a'_1, \dots, a'_m)$

“Look up tables” to assign the label  $c^*$  to  $\mathbf{x}'$  if

$$[\hat{P}(a'_1 | c^*) \cdots \hat{P}(a'_n | c^*)] \hat{P}(c^*) > [\hat{P}(a'_1 | c) \cdots \hat{P}(a'_n | c)] \hat{P}(c), \quad c_i \neq c^*, c_i = c_1, \dots, c_L$$



# Example-Play Tennis

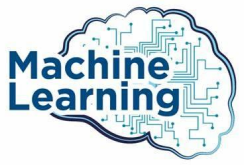


## *PlayTennis: training examples*

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No



# Example-Play Tennis



- Learning Phase

Outlook	Play=Yes	Play=No
<i>Sunny</i>	2/9	3/5
<i>Overcast</i>	4/9	0/5
<i>Rain</i>	3/9	2/5

Temperature	Play=Yes	Play=No
<i>Hot</i>	2/9	2/5
<i>Mild</i>	4/9	2/5
<i>Cool</i>	3/9	1/5

Humidity	Play=Yes	Play=No
<i>High</i>	3/9	4/5
<i>Normal</i>	6/9	1/5

Wind	Play=Yes	Play=No
<i>Strong</i>	3/9	3/5
<i>Weak</i>	6/9	2/5

$$P(\text{Play}=\text{Yes}) = 9/14 \quad P(\text{Play}=\text{No}) = 5/14$$



# Example-Play Tennis



- Test Phase

- Given a new instance, predict its label

$\mathbf{x}' = (\text{Outlook}=\textit{Sunny}, \text{Temperature}=\textit{Cool}, \text{Humidity}=\textit{High}, \text{Wind}=\textit{Strong})$

- Look up tables achieved in the learning phrase

$$P(\text{Outlook}=\textit{Sunny}|\text{Play}=\textit{Yes}) = 2/9$$

$$P(\text{Outlook}=\textit{Sunny}|\text{Play}=\textit{No}) = 3/5$$

$$P(\text{Temperature}=\textit{Cool}|\text{Play}=\textit{Yes}) = 3/9$$

$$P(\text{Temperature}=\textit{Cool}|\text{Play}=\textit{No}) = 1/5$$

$$P(\text{Humidity}=\textit{High}|\text{Play}=\textit{Yes}) = 3/9$$

$$P(\text{Humidity}=\textit{High}|\text{Play}=\textit{No}) = 4/5$$

$$P(\text{Wind}=\textit{Strong}|\text{Play}=\textit{Yes}) = 3/9$$

$$P(\text{Wind}=\textit{Strong}|\text{Play}=\textit{No}) = 3/5$$

$$P(\text{Play}=\textit{Yes}) = 9/14$$

$$P(\text{Play}=\textit{No}) = 5/14$$

- Decision making with the MAP rule

$$P(\text{Yes}|\mathbf{x}') \approx [P(\textit{Sunny}|\textit{Yes})P(\textit{Cool}|\textit{Yes})P(\textit{High}|\textit{Yes})P(\textit{Strong}|\textit{Yes})]P(\text{Play}=\textit{Yes}) = 0.0053$$

$$P(\text{No}|\mathbf{x}') \approx [P(\textit{Sunny}|\textit{No})P(\textit{Cool}|\textit{No})P(\textit{High}|\textit{No})P(\textit{Strong}|\textit{No})]P(\text{Play}=\textit{No}) = 0.0206$$

Given the fact  $P(\text{Yes}|\mathbf{x}') < P(\text{No}|\mathbf{x}')$ , we label  $\mathbf{x}'$  to be “No”.



# Naive Bayes



- Algorithm: Continuous-valued Features
  - Numberless values taken by a continuous-valued feature
  - Conditional probability often modeled with the normal distribution

$$\hat{P}(x_j | c_i) = \frac{1}{\sqrt{2\pi}\sigma_{ji}} \exp\left(-\frac{(x_j - \mu_{ji})^2}{2\sigma_{ji}^2}\right)$$

$\mu_{ji}$  : mean (average) of feature values  $x_j$  of examples for which  $c = c_i$

$\sigma_{ji}$  : standard deviation of feature values  $x_j$  of examples for which  $c = c_i$

- **Learning Phase:** for  $\mathbf{X} = (X_1, \dots, X_F)$ ,  $C = c_1, \dots, c_L$   
Output: FxL normal distributions and  $P(C = c_i)$   $i = 1, \dots, L$
- **Test Phase:** Given an unknown instance  $\mathbf{X}' = (a'_1, \dots, a'_F)$ 
  - Instead of looking-up tables, calculate conditional probabilities with all the normal distributions achieved in the learning phase
  - Apply the MAP rule to assign a label (the same as done for the discrete case)



# Naive Bayes



- Example: Continuous-valued Features

- Temperature is naturally of continuous value.

**Yes:** 25.2, 19.3, 18.5, 21.7, 20.1, 24.3, 22.8, 23.1, 19.8

**No:** 27.3, 30.1, 17.4, 29.5, 15.1

- Estimate mean and variance for each class

$$\mu = \frac{1}{N} \sum_{n=1}^N x_n, \quad \sigma^2 = \frac{1}{N} \sum_{n=1}^N (x_n - \mu)^2$$

$$\mu_{Yes} = 21.64, \sigma_{Yes} = 2.35$$

$$\mu_{No} = 23.88, \sigma_{No} = 7.09$$

- **Learning Phase:** output two Gaussian models for  $P(\text{temp} | C)$

$$\hat{P}(x | Yes) = \frac{1}{2.35\sqrt{2\pi}} \exp\left(-\frac{(x - 21.64)^2}{2 \times 2.35^2}\right) = \frac{1}{2.35\sqrt{2\pi}} \exp\left(-\frac{(x - 21.64)^2}{11.09}\right)$$

$$\hat{P}(x | No) = \frac{1}{7.09\sqrt{2\pi}} \exp\left(-\frac{(x - 23.88)^2}{2 \times 7.09^2}\right) = \frac{1}{7.09\sqrt{2\pi}} \exp\left(-\frac{(x - 23.88)^2}{50.25}\right)$$





# Zero conditional probability



- If no example contains the feature value
  - In this circumstance, we face a zero conditional probability problem during test

$$\hat{P}(x_1 | c_i) \cdots \hat{P}(a_{jk} | c_i) \cdots \hat{P}(x_n | c_i) = 0 \text{ for } x_j = a_{jk}, \hat{P}(a_{jk} | c_i) = 0$$

- For a remedy, class conditional probabilities re-estimated with

$$\hat{P}(a_{jk} | c_i) = \frac{n_c + mp}{n + m} \quad \text{(m-estimate)}$$

$n_c$  : number of training examples for which  $x_j = a_{jk}$  and  $c = c_i$   
 $n$  : number of training examples for which  $c = c_i$

$p$  : prior estimate (usually,  $p = 1/t$  for  $t$  possible values of  $x_j$ )

$m$  : weight to prior (number of "virtual" examples,  $m \geq 1$ )



# Zero conditional probability



- Example:  $P(\text{outlook}=\text{overcast}|\text{no})=0$  in the play-tennis dataset
  - Adding  $m$  “virtual” examples
    - In this dataset, training examples for the “no” class is 5.
    - We can only add  $m=1$  “virtual” example in our m-estimate remedy.
  - The “outlook” feature can takes only 3 values. So  $p=1/3$ .
  - Re-estimate  $P(\text{outlook}|\text{no})$  with the m-estimate

$$P(\text{overcast}|\text{no}) = \frac{0+1*\left(\frac{1}{3}\right)}{5+1} = \frac{1}{6}$$

$$P(\text{sunny}|\text{no}) = \frac{3+1*\left(\frac{1}{3}\right)}{5+1} = \frac{5}{6} \quad P(\text{rain}|\text{no}) = \frac{2+1*\left(\frac{1}{3}\right)}{5+1} = \frac{7}{6}$$

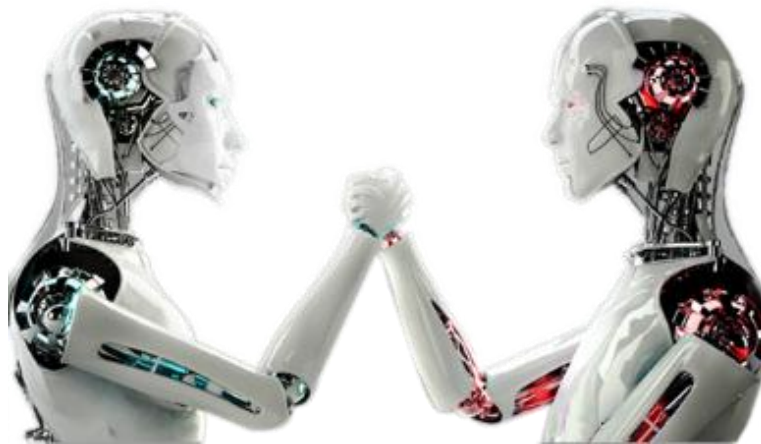


# Summary



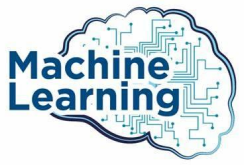
- Probabilistic Classification Principle
  - Discriminative vs. Generative models: learning  $P(c|x)$  vs.  $P(x|c)$
  - Generative models for classification: MAP and Bayesian rule
- Naïve Bayes: the **conditional independence** assumption
  - Training and test are very efficient.
  - Two different data types lead to two different learning algorithms.
  - Working well sometimes for data violating the assumption!
- Naïve Bayes: a popular **generative** model for classification
  - Performance competitive to most of state-of-the-art classifiers even in presence of violating independence assumption
  - Many successful applications, e.g., spam mail filtering
  - A good candidate of a base learner in ensemble learning
  - Apart from classification, naïve Bayes can do more...

# Lab Task





# Lab Task



1. Complete the exercises and questions in the lab03\_Bayes.pdf
2. Submit it to bb.

# Thanks

贾艳红 Jana

Email: [jiayh@mail.sustech.edu.cn](mailto:jiayh@mail.sustech.edu.cn)

