MA302 Homework 5

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4.7 Solution

1. Necessary.

   Considering

   $$x(t) = x_0 + \int_0^t f(x(s))ds,\ t \in [0, T]$$

   Derivate the equation with variable t, one can get

   $$\dot{x}(t) = f(x(t)) \implies \dot{x} = f(x)$$

   And $x(t) = x_0$ is also cleary.

2. Sufficient.

   Considering integrate the equation, and use the initial condition, one can derive the formula

   $$x(t) = x_0 + \int_0^t f(x(s))ds$$

4.8 Solution

According the conclusion in 4.7, one can get that the uniqueness of the solution is equivalent to the uniqueness of the integral from.

Considering an operator I by

$$I(x) = x_0 + \int_0^t f(x(s))ds$$

One can notice that

$$\begin{aligned}
|I(x_1) - I(x_2)| &= |\int_0^t f(x_1(s))ds + \int_0^t f(x_2(s))ds| \\
&\le \int_0^t |f(x_1(s)) - f(x_2(s))|ds \\
&\le L \int_0^t |x_1 - x_2|ds \\
&\le L \int_0^t \|x_1 - x_2\|_\infty
\end{aligned}$$

thus

$$\|I(x_1) - I(x_2)\|_\infty \le LT\|x_1 - x_2\|_\infty$$

When $LT < 1$, the operator I is a contraction, thus has an unique fixed point. And the unique solution in any subinterval of $[0, T]$ has a condition $LT' \leq LT < 1$, which also means the unique solution exist. And use the initial condition x_0, one can derive that the solution in any interval $[0, t]$ has the same formula, thus in the interval $[a, b] \subseteq [0, T]$.

5.4 Solution

Considering x is a point in $X - Y$, then there exist (y_i) such that $d(x, Y) = \lim_{i\to\infty} \|x - y_i\|$. As Y is a subspace thus closed, and there exist some subsequece such that (y_{i_k}) converges to a point $y \in Y$. Considering the sequence $\|x - y_{i_k}\|$, as $\|\cdot\|$ is a continuous function, is also converges. Thus $\|x - y_{i_k}\| \to \|x - y\| = dist(x, Y)$, with $y \in Y$.

5.6 Solution

Considering Y is a proper subspace of X. There exist $x \in X - Y$, which means that $dist(x, Y) = \varepsilon > 0$. Otherwise, if $dist(x, Y) = 0$, there exist $(y_n) \in Y$ such that $\|y_n - x\| \to 0$, what's more, existence of $(y_{n_k}) \to x$ implies x is a limit point of Y thus $x \in Y$, which is contradict to the hypothesis. Thus

$$\forall\ r > 0, dist(\frac{r}{\varepsilon}x, Y) = \frac{r}{\varepsilon}dist(x, Y) = r$$

5.7 Solution

Considering $(e_i)_{i=1}^{\infty}$ is a Hamel Basis of X, define $X_n = Span((e_i)_{i=1}^{n})$. Considering y_i by $y_i \in X_i$ and choose $dist(y_i, X_{i-1}) = 3^{-i}$.

Thus $(y_i)_{i=1}^{n}$ is Cauchy but can't has a limitation in any X_n as

$$d(y_{n+k+1}, X_n) \geq 3{-}n - \sum_{i=1}^{k} 3^{-(n+i)} \geq 3^{-n} - \sum_{i=1}^{\infty} 3^{-(n+i)} = \frac{1}{2}3^{-n} > 0$$

Thus we got a contradiction.