MA302 Homework 4

WANG Duolei, SID:12012727

wangdl2020@mail.sustech.edu.cn

3.1 Solution

- Translation invariant.

  Considering

$$\begin{aligned} d(x+z, y+z) &= \|x+z-(y+z)\| \\ &= \|x-y\| \\ &= d(x,y) \end{aligned}$$

- Homogeneous.

  Considering

$$\begin{aligned} d(tx, ty) &= \|tx-ty\| \\ &= |t|\|x-y\| \\ &= |t|d(x,y) \end{aligned}$$

3.5 Solution

1. The inequality.

   Considering the left inequality, it's clearly as m^p is actually a part of the right side.

   Considering that $|x_j|^p \le m,\ \forall j$, the right side is clearly holds.

2. The limitation.

   Considering

$$\begin{aligned} (m^p)^{1/p} \le (\sum_{j=1}^{n} |x_j|^p)^{1/p} \le (nm^p)^{1/p} \\ \implies \|x\|_{l^\infty} \le \|x\|_{l^p} \le n^{1/p}\|x\|_{l^\infty} \end{aligned}$$

   And $\lim_{p\to\infty} n^{1/p} = 1,\ \forall n \ge 1$. Thus $\|x\|_{l^\infty} = \lim_{p\to\infty} \|x\|_{l^p}$

3.6Solution

Firstly, considering that in a finite dimension vector space, the property

$$\|\cdot\|_{l^\infty} < \|\cdot\|_{l^p} \leq n^{1/p}\|\cdot\|_{l^\infty}$$

holds.

And for arbitrary vector space, $\mathbf{x} \in l^1$ implies it can minus some remainder to be approximate by finite dimension vector with arbitrary precise. Note the vector as x

Thus

$$\|\mathbf{x}\|_{l^\infty} - \varepsilon \leq \|x\|_{l^\infty} \leq \|x\|_{l^p} \leq \|x\|_{l^\infty} + \varepsilon \leq \|\mathbf{x}\|_{l^\infty}$$

and

$$\|\mathbf{x}\|_{l^p} - \varepsilon \leq \|x\|_{l^p} \leq \|x\|_{l^\infty} + \varepsilon \leq \|\mathbf{x}\|_{l^\infty} + \varepsilon$$

Considering the conclusion of 3.5

$$\|x\|_{l^\infty} \leq \|x\|_{l^p}$$

which implies $\|\mathbf{x}\|_{l^\infty} - \varepsilon \leq \|\mathbf{x}\|_{l^p}$. Thus

$$\|\mathbf{x}\|_{l^\infty} - \varepsilon \leq \|\mathbf{x}\|_{l^p} \leq \|\mathbf{x}\|_{l^\infty} + 2\varepsilon$$

Thus the limitation holds.

3.7 Solution

Notice that $p/q > 1$. And the series

$$\sum_{i=1}^{\infty} \frac{1}{[n(\ln(n))^2]^k}$$

is converge for $k \geq 1$ and diverges for $k < 1$. Thus construct $x_n = a_n$, where a_n is just the term of the series above.

3.8 Solution

As $U \subseteq X$ is a open normed vector space. There exist $\varepsilon > 0$, $s.t.$ $B(0, \varepsilon) \subseteq U$. Considering the vector space is closed onto the add, scalar-multiply, thus $kB(0, \varepsilon) \subset U$. Considering

$$\forall x \in X,\ x \in \frac{2\|x\|}{\varepsilon} B(0, \varepsilon) \cap X$$

Thus $x \in U$, which implies $X \subset U$. Thus $U = X$.

3.9 Solution

Firstly, $\bar{U}$ is closed. We need to check whether it's linear subspace. Considering two sequences converges in $\bar{U}$, $(x_n) \to x$, $(y_n) \to y$. Then $(x_n + y_n) \to x + y$, $(kx_n) \to kx$ is clearly as the sequence space of $\bar{U}$ is subspace of $\bar{U}$, thus it's a linear subspace.

3.11 Solution

Firstly,

$$\|f_n - f\|_{L^p}^p = \int_I |f_n(x) - f(x)|^p dx \leq \int_I \|f_n - f\|_\infty^p dx = \|f_n - f\|_\infty^p$$

thus, $\|f_n - f\|_\infty \to 0 \implies \|f_n - f\|_\infty^p \to 0 \implies \|f_n - f\|_{L^p}^p \to 0 \implies \|f_n - f\|_{L^p} \to 0$.
Secondly,

$$|f_n(x) - f(x)| \leq \|f_n - f\|_\infty$$

thus it's clearly that $\|f_n - f\|_\infty \to 0 \implies |f_n(x) - f(x)| \to 0$.

3.12 Solution

To prove $c_0(\mathbb{K})$ is separable, we can prove it has a countable set with a dense linear span. Here I claim that the set

$$E := \bigcup_{k=1}^{\infty} E_k,\ E_k^i := (0, \ldots, x^i, \ldots)$$

where the k-th coordinate has nonzero the i-th term of sequence $\{\frac{1}{2^i}\}_{i=1}^{\infty}$ satisfies the requirement.

It can be checked as every null sequence can be approximate in the order of coordinates with the help of binary division. Thus $c_0(\mathbb{K})$ is separable.

3.13 Solution

- Sufficient.

  Considering X is separable, then for every vector $y \in Y$, as T is bijection, $T^{-1}(y) \in X$. Note the countable dense subset as $E = \{x_i\}_{x=1}^{\infty}$. Consider $T(E)$ is countable, we need to check whether it's dense.

  There exist a sequence $(x_i)_{i=1}^{\infty} \subset E$ such that

  $$\|x_i - T^{-1}(y)\|_X \to 0$$

  Then,

  $$c_1\|x_i - T^{-1}(y)\|_X \leq \|T(x_i) - y\|_Y \leq c_2\|x_i - T^{-1}(y)\|_X \implies \|T(x_i) - y\|_Y = 0$$

  Thus the set $T(E) \subset Y$ is the countable dense subset.

- Necessary.

  As the same, and use the truth

  $$c_1'\|y\|_Y \leq \|T^{-1}(y)\|_X \leq c_2'\|y\|_Y$$

  one can prove it as same as the sufficient.

3.16 Solution

- Sufficient.

  Consider $(X, \|\cdot\|)$ is separable, then it has a countable dense subset, note it as $E = \{x_1, x_2, \ldots\}$. Considering

  $$X_i := \mathrm{Span}(x_i)$$

  Then $X = \mathrm{Span}(E) = \bigcup_{i=1}^{\infty} X_i$, which is just the hypothesis.

- Necessary.

  Considering X can be written as the form. Notice that every X_i is finite dimension, we can find a basis, note it as $E_i := \{v_i^1, v_i^2, \dots, v_i^{n_i}\}$.

  Then
  $$X = \bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty} \mathrm{Span}(E_i)$$

  And considering $E := \bigcup_{i=1}^{\infty} E_i$ must be contable, thus

  $$X = \bigcup_{i=1}^{\infty} \mathrm{Span}(E_i) = \mathrm{Span}(E)$$

  the set E is just a countable dense subset.