Functional Analysis Homework

WANG Duolei, SID:12012727

wangdl2020@mail.sustech.edu.cn

19.1 Solution

Considering the Riesz Representation Theorem, we can derive a vector $u_\phi \in U$ such that

$$(x, u_\phi) = \phi(x), \forall x \in U$$

Also, the $u_\phi \in U$, which means we can define a map $f_\phi \in H^*$ by

$$f_\phi(x) = (x, u_\phi), \forall x \in H$$

One can check that the f_ϕ clearly satisfies the property that $f(x) = \phi(x), \forall x \in U$.

Now we check the Dual Norm of f_ϕ. And clearly

$$f(x) = (x, u_\phi)_H \leq \|x\|\|u_\phi\|$$

has the same norm as ϕ. And the inverse inequality

$$\sup_{\|x\|=1} f(x) = \|u_\phi\|_H = \|u_\phi\|_U$$

also holds.

Thus $\|f\|_{H^*} = \phi_{U^*}$.

19.2 Solution

As the $u_\phi \in U$ is unique, the definition of f is clearly unique. Also, one can check it by assuming there exists another $\bar{f}$ such that $\bar{f} = f$. Which also means

$$(x, u_f) = (x, u_{\bar{f}}), \forall x \in H$$

And the H is a Hilbert Space, which means

$$(x, u_f - u_{\bar{f}}) = 0, \forall x \in H$$

This is equal to $u_f - u_{\bar{f}} = 0$, and also $u_f = u_{\bar{f}}$.

19.3 Solution

For any $x \in \bar{U}$, there exists a sequence $(x_n) \in U$ such that $x_n \to x$. Define

$$\phi(x) = \lim_{n\to\infty} \hat{\phi}(x_n)$$

It's clearly that the ϕ is just the continuous extension. And the linear also holds in the process of limitation.

As $|\hat{\phi}(x)| \leq M\|x\|$, and the norm is continuous as a map. Thus the inequality holds, which means

$$|\phi(x)| \leq M\|x\|$$

19.4 Solution

(i) Considering that

$$p(0) = p(k0) = |k|p(0), \forall k \in \mathbb{K}$$

Thus, $(1-|k|)p(0) = 0$. Choose the k to be a number with the norm not be one. One can get the result.

(ii) One can notice that

$$p(x) = p(y + x - y) \le p(y) + p(x - y)$$

and

$$p(y) = p(x - y + y) \le p(x - y) + p(y)$$

Thus,

$$p(x) - p(y) \le p(x - y),\ p(y) - p(y) \le p(y - x) \implies |p(x) - p(y)| \le p(x - y)$$

(iii) Choose the y to be zero in the (ii), one can get the result $|p(x)| \le p(x)$. And $p(x) \le |p(x)|$ implies that $p(x) = |p(x)|$, which also means $p(x) \ge 0$.

(iv) Noted the set by A. Assume $x, y \in A$. Thus,

$$0 \le p(x + y) \le p(x) + p(y) = 0 \implies p(x + y) = 0 \implies x + y \in A$$

And $p(kx) = |k|p(x) = 0$, which also implies $kx \in A$.

Thus the subset A also be a subspace.