Exercises in Functional Analysis (week 3)

1. Let (X, d) be a metric space. Let A be a compact subset and B be a closed subset of X such that $A \cap B = \emptyset$. Show that

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y) > 0.$$

2. Let A, B be two subsets of a metric space (X, d) with

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y) > 0.$$

Show that there are two open subsets U_1 and U_2 of X such that $U_1 \supset A$, $U_2 \supset B$ and $U_1 \cap U_2 = \emptyset$.

3. Let (X, d) be a metric space and Y be a compact subset of X. Assume that a map $T : Y \to Y$ satisfies $d(Tx, Ty) < d(x, y)$, $\forall x, y \in Y, x \neq y$. Show that T has a unique fixed point.