MA302 Homework 3

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1. Solution

Considering prove by contradiction, assume $d(A, B) = 0$, thus for every $\varepsilon_n > 0$, there exist $x_n \in A,\ y_n \in B$ such that $0 \leq d(x_n, y_n) < \varepsilon$. As A is compact, thus also sequential compact, there exist subsequence $\{x_{n_k}\} \to x \in A$. Now considering $d(x, y_{n_k}) \leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) \to 0$, thus $\{y_{n_k} \to x\}$, and B is closed, which implies $x \in B$. Thus $A \cap B \neq \emptyset$.

2. Solution

Considering

$$U_1 := \bigcap_{i=1}^{\infty} A_i, \quad A_i := \{x \in X : d(x, A) < 1/i\}$$

also, we define U_2. And claim that $U_1 \cap U_2 = \emptyset$. Prove it by contradiction.

Firstly, check any point x in U_1, here must be a sequence $\{a_n\}$ in A such that $\{a_n\} \to x$ as $\{d(A_i, A)\} \to 0$ and $x \in U_1$, which is the intersection of A_i. Also, we conclude that there exist $\{b_n\} \to y$ for every $y \in U_2$.

Then, assume $U_1 \cap U_2 \neq \emptyset$, which means $\exists x \in U_1 \cap U_2$, then for arbitrary $\varepsilon > 0$, as $\{a_n\} \to x, \{b_n\} \to x$,

$$d(a_n, b_n) \leq d(a_n, x) + d(b_n, x) < \varepsilon/2 + \varepsilon/2$$

Thus, $d(A, B) = 0$, which is contradict to our hypothesis.

3. Solution

To begin with, define

$$diam(A) = \sup_{x,y \in A} d(x, y)$$

1. Firstly, T is continuous map.

   $$\forall \varepsilon > 0, let\ \delta = \varepsilon,\ \forall d(x, y) < \delta, d(Tx, Ty) < d(x, y) < \delta = \varepsilon$$

   Thus, T is continuous, which means $T^n(Y) := T \cdots T(Y)$ is compact set, also closed.

2. The intersection of family $\bigcap_{i=1}^{\infty} A_i$ is non-empty, where $A_i := T^i(Y)$.

   The closed family A has the property that any intersection of finite number subsets has non-empty result. Because it's clearly that

   $$A_{n+1} \subset A_n \subset \cdots \subset A \subset Y$$

   Thus, all the finite intersetion of $\{A_i\}_{i=1}^{n}$ is non-empty, according to Exercise 2.11, we know that the intersection is non-empty.

3. The existence and uniqueness of the limit point.

   Consider $diam(T(Y)) \leq (1-\varepsilon)diam(Y), \varepsilon > 0$, thus $diam(A^n) < diam(A^m)$ for every $n > m$. Define $x_n = T^n(x),\ y^n = T^n(y)$, it's clearly that

   $$d(x_n, y_n) \to 0$$

   as $diam(A_n) \leq (1-\varepsilon)^n \to 0,\ (n \to \infty)$.

   As all the sequence has a convergent subsequence, assume the limitation of some subsequence $\{x_{n_k}\}$ is a. Thus

   $$a \in \bigcap_{i=1}^{\infty} A_i \implies a \in T^n(Y),\ \forall n$$

   And all the convergent subsequence of $\{y_n\}$ must converges to the same point a, otherwise suppose the limit is b. Then

   $$d(x_{n_k}, y_{m_j}) < diam(A_{\min n_k, m_j}) \to 0$$

   which is contradict to $d(a,b) > 0$ as $d(a,b) \leq d(a, x_{n_k}) + d(x_{n_k}, y_{m_j}) + d(y_{m_j}, b) \to 0$.

   Thus there exist an unique point in the intersection.

4. The limit point is the fixed point.

   Prove it by contradiction. Here claim that $Ta = a$, otherwise, $Ta = b,\ a \neq b$, as $d(A_n) \geq d(a,b) > 0$, which is contradict to $diam(A_n) \to 0$.

Above all has shown that the map must has an unique fixed point.