MA302 Homework 2

WANG Duolei, SID:12012727

wangdl2020@mail.sustech.edu.cn

2.7 Solution

Assume $A \subset X$ is an open set, $\forall x \in A$, $\exists\ B(x,r)$ with $r > 0$ such that $B(x,r) \subset A$, as every point in A is an interior. Because every open ball $B(x_i, r_i)$ is containned in A, we have $\bigcup_{i\in I} B(x_i, r_i) \subset A$. And for every $x_i \in A$, $x \subset B(x_i, r_i) \implies A \subset \bigcup_{i\in I} B(x_i, r_i)$. Above all shows that $A = \bigcup_{i\in I} B(x_i, r_i)$.

2.9 Solution

Assume $E \subseteq X,\ F \subseteq Y$ are the corresponding densen countable subset, then $E \times F$ is also countable. We need to check whether it's dense. Considering $(x_0, t_0) \in X \times Y$ is an arbitrary point, then there exists $\{x_n\} \to x_0,\ \{y_n\} \to y_0$ as the subspaces are dense. Considering the d_p is a metric, which has been proved in 2.2. Thus

$$d_p((x_0, y_0), (x_n, y_m)) \le d_p((x_0, y_0), (x_n, y_0)) + d_p((x_n, y_0), (x_n, y_m))$$

For every $\varepsilon > 0$, there exists $N_x \in \mathbb{N}$ such that

$$d_p((x_0, y_0), (x_n, y_0)) = d_X(x_0, x_n) < \varepsilon/2$$

as $\{x_n\} \to x_0$ in X and $d_Y(y_0, y_0) = 0$. And also, for every given $\varepsilon > 0$, there exists $N_y \in \mathbb{N}$, such that

$$d_p((x_n, y_0), (x_n, y_m)) = d_Y(y_0, y_m) < \varepsilon/2$$

Thus, for every $N \ge \max\{N_x, N_y\}$, we have

$$d_p((x_0, y_0), (x_n, y_m)) \le d_p((x_0, y_0), (x_n, y_0)) + d_p((x_n, y_0), (x_n, y_m)) < \varepsilon$$

2.10 Solution

Fistly, F_α is closed, which means $\bigcap_{\alpha\in A} F_\alpha$ is also closed. We prove by contradiction. Considering the intersection is empty, thus its complement has the property

$$\bigcup_{\alpha\in A} F_\alpha^c = X$$

As X is compact, we can find a finite subcover such that

$$\bigcup_{i=1}^{n} F_{\alpha_i} = X,\ \alpha_i \in A$$

It means $\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$, which is contradict to our hypothesis.

2.11 Solution

According to the definition of $F_j,\ j \in \mathbb{N}^+$, we can conclude that $\bigcap_{i=1}^{n} F_i = F_{i_0} \neq \emptyset, i_0 := \min\{1, 2, \ldots, n\}$. Thus the collection of F has the same hypothesis with 2.10, and also X is compact. Thus $\bigcap_{j \in \mathbb{N}^+} F_j = \bigcap_{j=1}^{\infty} F_j \neq \emptyset$.

2.12 Solution

Assume $x = \sup(S)$, then $\forall s \in S,\ x \geq s$ and $\forall 1/n > 0,\ \exists x_n \in S,\ x_n > x - 1/n$. Thus $d(x_n, x) < 1/n$, $\{x_n\} \to x$. And S is closed, every limit point must lies in S, which means $x = \sup(S) \in S$.

2.13 Solution

Considering X is compact and f is continuous, for every compact subset $U \subseteq X$, we have $f(u)$ is also compact. And in metric space (X, d), compact implies closed. Thus $f(U) = (f^{-1})^{-1}(U)$ is closed for every closed subset $U \in X$, which means f^{-1} is continuous.

2.14 Solution

1. Firstly, we prove that any compact metric space is separable.

   For every $n \in \mathbb{N}^+$, considering $X \subseteq \bigcup_{x \in X} B_x(x, 1/n)$, which is also an open cover, thus there exists an finite subcover $X \subseteq \bigcup_{i=1}^{k(n)} B_{x_i}(x, 1/n)$. Considering the countable subset

   $$\begin{gathered} x_{1,1}, x_{1,2}, \ldots, x_{1,k(1)} \\ x_{2,1}, x_{2,2}, \ldots, x_{2,k(2)} \\ \cdots \\ x_{n,2}, x_{2,2}, \ldots, x_{n,k(n)} \\ \cdots \end{gathered}$$

   It's clearly that there exist $x_{n,i} \in X$ such that $d(x, x_{n,i}) < 1/n,\ \forall x \in X, \forall i \in \{1, 2, \ldots, k(n)\}$. Thus choose a point in every row, such that the sequence has points $x_{n,i}$ satisfying $d(x, x_{n,i}) < 1/n,\ \forall x \in X, \forall i \in \{1, 2, \ldots, k(n)\}$. The subsequece has limit point x. Thus, the countable set is dense in X. Thus X is separable.

2. Considering the construction of the countable subset, and choose a sequence converges to x, we have

   $$d(x, x_k) < \varepsilon,\ \forall k \geq N = floor(\frac{1}{\varepsilon}) + 1$$

   Thus there exist $M(\varepsilon) = N + 1$, such that $d(x, x_M) < \varepsilon$.