MA302 Homework 11

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13.5 Solution

1. T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

   Firstly, T is invertible, which means there exists $S \in B(K, H)$ s.t. $TS = id_K,\ ST = id_H$. And note the Hilbert Adjoint of S by S^*, we know that $S^* \in B(H, K)$. Claim that $(T^*)^{-1} = S^*$.

   (a) $T^*S^* = id_H$. Firstly, $T^*S^* \in B(H)$ is clearly because of the combination of two bounded linear map is also bounded linear. And

   $$(T^*S^*x, y)_H = (x, (T^*S^*)^*y) = (x, STy)_H = (x, y)_H,\ \forall x \in H, y \in K$$

   thus, $T^*S^* = id_H$.

   (b) $S^*T^* = id_K$. One can prove this like (a).

   Thus, $(T^*)^{-1} = S^* = (T^{-1})^*$.

2. Consider $T \in B(H)$ is self-adjoint. According to 1's conclusion, we know that $T^{-1} = (T^*)^{-1} = (T^{-1})^*$. Thus T^{-1} is also a self-adjoint operator.

14. Solution

(i) Considering that $\{\alpha_i\}_{i=1}^{\infty} \subseteq \sigma_p(D_\alpha)$ is clearly. We consider the inverse. For every $\alpha \in \sigma_p(D_\alpha)$, we have some nonzero x_0 such that

$$D_\alpha x_0 = (\alpha_1 x_1, \alpha_2 x_2, \ldots) = \beta x_0$$

As $x_0 \neq 0$, there must exist some coordinates be nonzero, and consider the nonzero terms as $x_{i_1}, x_{i_2}, \ldots$, we know that

$$\beta = \alpha_{i_1} = \alpha_{i_2} = \ldots$$

thus, $\beta \in \{\alpha_i\}_i^{\infty}$

1. Considering a $\lambda \in \mathbb{C}$ and $\lambda \notin \overline{\sigma_p(D_\alpha)}$, then $|\lambda - a_i| \geq \delta$. Thus consider every coordinates, we know it's $(a_i - \lambda)x_i$, and the norm is also bigger than δ. Thus, the $D_\alpha - \lambda I$ is invertible, which has a bounded inverse. Thus $D_\alpha - \lambda I$ is also invertible. And the spectrum is closed, the two set are same is clear.

14.4 Solution

Considering the Spectral Mapping Theorem, we know that

$$\lambda \in \sigma(T) \implies \lambda^n \in \sigma(T^n)$$

thus $(r_\sigma(T))^n = r_\sigma(T^n)$. And $\sigma(T^n) \subset \{\lambda : \lambda \leq \|T^n\|\}$, which means the upper bound of $r_\sigma(T)$ is $\|T^n\|^{1/n}$. Thus

$$r_\sigma(T) \leq \varliminf_{n\to\infty} \|T^n\|^{1/n}$$

14.5 Solution

Consider this operator is just a special case in 11.7, we know that the bound is $\|T^n\|_{B(X)} \leq \frac{1}{n!}$

And according to 14.4

$$r_\sigma(T) \leq \underset{\liminf}{n \to \infty} = 0$$

Thus, all the eigenvalues must be zero. And we know that T is not invertible, as the 0 is not an eigenvalue, which is because

$$|(T - 0I)f| = 0 \implies Tf = 0 \implies f = 0$$

This also means T is also not surjective.