Solution to Functional Analysis

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Considering the following two cases

1. The case $p = \infty$.

   - Positive definite.
     It's clearly because that every metric in X_i itself is positive define, and there are only absolute value, power function used in all the definition
   - Symmetric. Trivial.
   - Traingle Inequality.
     Considering that

$$\begin{aligned}
&\max_i d_i(x_i, y_i) \\
\leq &\max_i \{d_i(x_i, z_i) + d_i(z_i, y_i)\} \\
\leq &\max_i \{d_i(x_i, z_i)\}
\end{aligned}$$

2. The case $1 \leq p < \infty$.

   - Positive definite. It's clearly.
   - Symmetric is trivial.
   - Traingle Inequality.
     Considering that

$$\begin{aligned}
d(x, y) &= [d_1(x_1, y_1)^p + d_2(x_2, y_2)^p + \cdots + d_n(x_n, y_n)^p]^{1/p} \\
&\leq [(d_1(x_1, z_1) + d_1(y_1, z_1))^p + (d_2(x_2, z_2) + d_2(y_2, z_2))^{1/p} \cdots + (d_n(x_n, z_n) + d_n(y_n, z_n))^p]^{1/p} \\
&\leq [d_1(x_1, z_1) + d_2(x_2, z_2) + \cdots + d_n(x_n, z_n)]^{1/p} + [d_1(x_1, y_1) + d_2(x_2, y_2) + \cdots + d_n(x_n, y_n)]^{1/p} \\
&= d(x, z) + d(y, z)
\end{aligned}$$

   The last inequality can be proved by the hint with the htlp of mathematical induction.

2.3 Solution

- Positive definite. Considering that

$$\hat{d} = 1 - \frac{1}{1 + d} \geq 0$$

  and $\hat{d} = 0 \iff d = 0 \iff x = y$.

- Symmetric. Trivial.
- Traingle Inequality.

  Note that the function

  $$f(x) = 1 - 1/(1+x),\ x \in \mathbb{R}^+$$

  is monotonic increasing and convex. And $\hat{d}(x,y) = f(d(x,y))$

  Thus,

  $$\begin{aligned} & d(x,y) \le d(x,z) + d(y,z) \\ \implies & f(d(x,y)) \le f(d(x,z) + d(y,z)) \le f(d(x,z)) + f(d(y,z)) \\ \implies & \hat{d}(x,y) \le \hat{d}(x,z) + \hat{d}(y,z) \end{aligned}$$

  which means the traingle inequality holds.

2.4 Solution

Note that here the metric induced by a same increasing convex in 2.3, and the image of f is $\mathbb{R}^+$ exactly. Thus, the metric defined in 2.4 is uniformly converges by comparasion law of series.

(1) Sufficient.

For every given i, we have

$$d_i(x_i, y_i) \le 2^i d(x,y)$$

Thus, when $d(x,y)$ converges to 0, $d_i(x_i, y_i)$ converges to 0, which means that $(x^{(n)}) \to y^{(j)}$.

(2) Necessary.

Considering that

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} d(x_i, y_i) \le \sum_{i=1}^{\infty} d_i(x_i, y_i)$$

Thus, $(x)_n$ must be convergent when ${(x)^{(n)}}_n$ converges.

2.5 Solution

(1) Finite union $\bigcup_{i=1}^N F_i$ is closed i.f.f $\bigcap_{i=1}^N F_i^c$ is open. Considering an arbitrary point in the finite intersection of complement, as every complement set is open, there exist $\{B(x,r_1), \dots, B(x,r_N)\}$ such that $B(x,r_i) \subset F_i^c$. Considering the intersection of $B(x,r_i)$ is also open. Thus

$$B(x,r_0) \subset \bigcap_{i=1}^N F_i^c,\ r_0 := \min\{r_1, \dots, r_N\}$$

Thus, x is also a interior. Thus, it's open, which also means the finite union of closed set is closed.

(2) Finite intersection. Also considering the complement $\bigcup_{i=1}^N F_i^c$, it's clearly that the countable infinite union of open set is union, thus the finite must also, which also means the finite intersection is closed.

2.6 Solution

2.7 Solution

Assume $A \subset X$ is an open set, $\forall x \in A$, $\exists\ B(x,r)$ with $r > 0$ such that $B(x,r) \subset A$, as every point in A is an interior. Because every open ball $B(x_i, r_i)$ is containned in A, we have $\bigcup_{i \in I} B(x_i, r_i) \subset A$. And for every $x_i \in A$, $x \subset B(x_i, r_i) \implies A \subset \bigcup_{i \in I} B(x_i, r_i)$. Above all shows that $A = \bigcup_{i \in I} B(x_i, r_i)$.

2.8 Solution

2.9 Solution

Assume $E \subseteq X,\ F \subseteq Y$ are the corresponding densen countable subset, then $E \times F$ is also countable. We need to check whether it's dense. Considering $(x_0, t_0) \in X \times Y$ is an arbitrary point, then there exists $\{x_n\} \to x_0,\ \{y_n\} \to y_0$ as the subspaces are dense. Considering the d_p is a metric, which has been proved in 2.2. Thus

$$d_p((x_0, y_0), (x_n, y_m)) \le d_p((x_0, y_0), (x_n, y_0)) + d_p((x_n, y_0), (x_n, y_m))$$

For every $\varepsilon > 0$, there exists $N_x \in \mathbb{N}$ such that

$$d_p((x_0, y_0), (x_n, y_0)) = d_X(x_0, x_n) < \varepsilon/2$$

as $\{x_n\} \to x_0$ in X and $d_Y(y_0, y_0) = 0$. And also, for every given $\varepsilon > 0$, there exists $N_y \in \mathbb{N}$, such that

$$d_p((x_n, y_0), (x_n, y_m)) = d_Y(y_0, y_m) < \varepsilon/2$$

Thus, for every $N \ge \max\{N_x, N_y\}$, we have

$$d_p((x_0, y_0), (x_n, y_m)) \le d_p((x_0, y_0), (x_n, y_0)) + d_p((x_n, y_0), (x_n, y_m)) < \varepsilon$$

2.10 Solution

Fistly, F_α is closed, which means $\bigcap_{\alpha \in A} F_\alpha$ is also closed. We prove by contradiction. Considering the intersection is empty, thus its complement has the property

$$\bigcup_{\alpha \in A} F_\alpha^c = X$$

As X is compact, we can find a finite subcover such that

$$\bigcup_{i=1}^{n} F_{\alpha_i} = X,\ \ \alpha_i \in A$$

It means $\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$, which is contradict to our hypothesis.

2.11 Solution

According to the definition of $F_j,\ j \in \mathbb{N}^+$, we can conclude that $\bigcap_{i=1}^{n} F_i = F_{i_0} \neq \emptyset, i_0 := \min\{1, 2, \ldots, n\}$. Thus the collection of F has the same hypothesis with 2.10, and also X is compact. Thus $\bigcap_{j \in \mathbb{N}^+} F_j = \bigcap_{j=1}^{\infty} F_j \neq \emptyset$.

2.12 Solution

Assume $x = \sup(S)$, then $\forall s \in S,\ x \ge s$ and $\forall 1/n > 0,\ \exists x_n \in S,\ x_n > x - 1/n$. Thus $d(x_n, x) < 1/n$, $\{x_n\} \to x$. And S is closed, every limit point must lies in S, which means $x = \sup(S) \in S$.

2.13 Solution

Considering X is compact and f is continuous, for every compact subset $U \subseteq X$, we have $f(u)$ is also compact. And in metric space (X, d), compact implies closed. Thus $f(U) = \left(f^{-1}\right)^{-1}(U)$ is closed for every closed subset $U \in X$, which means f^{-1} is continuous.

2.14 Solution

1. Firstly, we prove that any compact metric space is separable.

   For every $n \in \mathbb{N}^+$, considering $X \subseteq \bigcup_{x \in X} B_x(x, 1/n)$, which is also an open cover, thus there exists an finite subcover $X \subseteq \bigcup_{i=1}^{k(n)} B_{x_i}(x, 1/n)$. Considering the countable subset

$$
\begin{aligned}
&x_{1,1}, x_{1,2}, \dots, x_{1,k(1)} \\
&x_{2,1}, x_{2,2}, \dots, x_{2,k(2)} \\
&\dots \\
&x_{n,2}, x_{2,2}, \dots, x_{n,k(n)} \\
&\dots
\end{aligned}
$$

   It's clearly that there exist $x_{n,i} \in X$ such that $d(x, x_{n,i}) < 1/n,\ \forall x \in X, \forall i \in \{1, 2, \dots, k(n)\}$. Thus choose a point in every row, such that the sequence has points $x_{n,i}$ satisfying $d(x, x_{n,i}) < 1/n,\ \forall x \in X, \forall i \in \{1, 2, \dots, k(n)\}$. The subsequece has limit point x. Thus, the countable set is dense in X. Thus X is separable.

2. Considering the construction of the countable subset, and choose a sequence converges to x, we have

$$
d(x, x_k) < \varepsilon,\ \forall k \geq N = floor(\frac{1}{\varepsilon}) + 1
$$

   Thus there exist $M(\varepsilon) = N + 1$, such that $d(x, x_M) < \varepsilon$.

3 Norms and Normed Spaces

3.1 Solution

- Translation invariant.

  Considering

$$
\begin{aligned}
d(x+z, y+z) &= \|x + z - (y+z)\| \\
&= \|x - y\| \\
&= d(x, y)
\end{aligned}
$$

- Homogeneous.

  Considering

$$
\begin{aligned}
d(tx, ty) &= \|tx - ty\| \\
&= |t|\|x - y\| \\
&= |t|d(x, y)
\end{aligned}
$$

3.2

3.3

3.4

3.5 Solution

1. The inequality.

Considering the left inequality, it's clearly as m^p is actually a part of the right side.

Considering that $|x_j|^p \leq m, \ \forall j$, the right side is clearly holds.

2. The limitation.

   Considering

$$(m^p)^{1/p} \leq (\sum_{j=1}^{n} |x_j|^p)^{1/p} \leq (nm^p)^{1/p}$$
$$\implies \|x\|_{l^\infty} \leq \|x\|_{l^p} \leq n^{1/p}\|x\|_{l^\infty}$$

   And $\lim_{p\to\infty} n^{1/p} = 1, \ \forall n \geq 1$. Thus $\|x\|_{l^\infty} = \lim_{p\to\infty} \|x\|_{l^p}$

3.6 Solution

Firstly, considering that in a finite dimension vector space, the property

$$\|\cdot\|_{l^\infty} < \|\cdot\|_{l^p} \leq n^{1/p}\|\cdot\|_{l^\infty}$$

holds.

And for arbitrary vector space, $\mathbf{x} \in l^1$ implies it can minus some remainder to be approximate by finite dimension vector with arbitrary precise. Note the vector as x

Thus

$$\|\mathbf{x}\|_{l^\infty} - \varepsilon \leq \|x\|_{l^\infty} \leq \|x\|_{l^p} \leq \|x\|_{l^\infty} + \varepsilon \leq \|\mathbf{x}\|_{l^\infty}$$

and

$$\|\mathbf{x}\|_{l^p} - \varepsilon \leq \|x\|_{l^p} \leq \|x\|_{l^\infty} + \varepsilon \leq \|\mathbf{x}\|_{l^\infty} + \varepsilon$$

Considering the conclusion of 3.5

$$\|x\|_{l^\infty} \leq \|x\|_{l^p}$$

which implies $\|\mathbf{x}\|_{l^\infty} - \varepsilon \leq \|\mathbf{x}\|_{l^p}$. Thus

$$\|\mathbf{x}\|_{l^\infty} - \varepsilon \leq \|\mathbf{x}\|_{l^p} \leq \|\mathbf{x}\|_{l^\infty} + 2\varepsilon$$

Thus the limitation holds.

3.7 Solution

Notice that $p/q > 1$. And the series

$$\sum_{i=1}^{\infty} \frac{1}{[n(\ln(n))^2]^k}$$

is converge for $k \geq 1$ and diverges for $k < 1$. Thus construct $x_n = a_n$, where a_n is just the term of the series above.

3.8 Solution

As $U \subseteq X$ is a open normed vector space. There exist $\varepsilon > 0, \ s.t. \ B(0, \varepsilon) \subseteq U$. Considering the vector space is closed onto the add, scalar-multiply, thus $kB(0, \varepsilon) \subset U$. Considering

$$\forall x \in X, \ x \in \frac{2\|x\|}{\varepsilon} B(0, \varepsilon) \cap X$$

Thus $x \in U$, which implies $X \subset U$. Thus $U = X$.

3.9 Solution

Firstly, $\bar{U}$ is closed. We need to check whether it's linear subspace. Considering two sequences converges in $\bar{U}$, $(x_n) \to x,\ (y_n) \to y$. Then $(x_n + y_n) \to x + y,\ (kx_n) \to kx$ is clearly as the sequence space of $\bar{U}$ is subspace of $\bar{U}$, thus it's a linear subspace.

3.10

3.11 Solution

Firstly,

$$\|f_n - f\|_{L^p}^p = \int_I |f_n(x) - f(x)|^p dx \le \int_I \|f_n - f\|_\infty^p dx = \|f_n - f\|_\infty^p$$

thus, $\|f_n - f\|_\infty \to 0 \implies \|f_n - f\|_\infty^p \to 0 \implies \|f_n - f\|_{L^p}^p \to 0 \implies \|f_n - f\|_{L^p} \to 0.$

Secondly,

$$|f_n(x) - f(x)| \le \|f_n - f\|_\infty$$

thus it's clearly that $\|f_n - f\|_\infty \to 0 \implies |f_n(x) - f(x)| \to 0.$

3.12 Solution

To prove $c_0(\mathbb{K})$ is separable, we can prove it has a countable set with a dense linear span. Here I claim that the set

$$E := \bigcup_{k=1}^{\infty} E_k,\ E_k^i := (0, \dots, x^i, \dots)$$

where the k-th coordinate has nonzero the i-th term of sequence $\{\frac{1}{2^i}\}_{i=1}^{\infty}$ satisfies the requirement.

It can be checked as every null sequence can be approximate in the order of coordinates with the help of binary division. Thus $c_0(\mathbb{K})$ is separable.

3.13 Solution

- Sufficient.

  Considering X is separable, then for every vector $y \in Y$, as T is bijection, $T^{-1}(y) \in X$. Note the countable dense subset as $E = \{x_i\}_{x=1}^{\infty}$. Consider $T(E)$ is countable, we need to check whether it's dense.

  There exist a sequence $(x_i)_{i=1}^{\infty} \subset E$ such that

  $$\|x_i - T^{-1}(y)\|_X \to 0$$

  Then,

  $$c_1\|x_i - T^{-1}(y)\|_X \le \|T(x_i) - y\|_Y \le c_2\|x_i - T^{-1}(y)\|_X \implies \|T(x_i) - y\|_Y = 0$$

  Thus the set $T(E) \subset Y$ is the countable dense subset.

- Necessary.

  As the same, and use the truth

  $$c_1'\|y\|_Y \le \|T^{-1}(y)\|_X \le c_2'\|y\|_Y$$

  one can prove it as same as the sufficient.

3.14

3.15

3.16 Solution

- Sufficient.

  Consider $(X, \|\cdot\|)$ is separable, then it has a countable dense subset, note it as $E = \{x_1, x_2, \ldots\}$. Considering

  $$X_i := \mathrm{Span}(x_i)$$

  Then $X = \mathrm{Span}(E) = \bigcup_{i=1}^{\infty} X_i$, which is just the hypothesis.

- Necessary.

  Considering X can be written as the form. Notice that every X_i is finite dimension, we can find a basis, note it as $E_i := \{v_i^1, v_i^2, \ldots, v_i^{n_i}\}$.

  Then

  $$X = \bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty} \mathrm{Span}(E_i)$$

  And considering $E := \bigcup_{i=1}^{\infty} E_i$ must be contable, thus

  $$X = \bigcup_{i=1}^{\infty} \mathrm{Span}(E_i) = \mathrm{Span}(E)$$

  the set E is just a countable dense subset.

4 Complete Normed Spaces

4.7 Solution

1. Necessary.

   Considering

   $$x(t) = x_0 + \int_0^t f(x(s))ds, \ t \in [0, T]$$

   Derivate the equation with variable t, one can get

   $$\dot{x}(t) = f(x(t)) \implies \dot{x} = f(x)$$

   And $x(t) = x_0$ is also cleary.

2. Sufficient.

   Considering integrate the equation, and use the initial condition, one can derive the formula

   $$x(t) = x_0 + \int_0^t f(x(s))ds$$

4.8 Solution

According the conclusion in 4.7, one can get that the uniqueness of the solution is equivalent to the uniqueness of the integral from.

Considering an operator I by

$$I(x) = x_0 + \int_0^t f(x(s))ds$$

One can notice that

$$\begin{aligned}|I(x_1) - I(x_2)| &= |\int_0^t f(x_1(s))ds + \int_0^t f(x_2(s))ds| \\ &\le \int_0^t |f(x_1(s)) - f(x_2(s))|ds \\ &\le L\int_0^t |x_1 - x_2|ds \\ &\le L\int_0^t \|x_1 - x_2\|_\infty\end{aligned}$$

thus

$$\|I(x_1) - I(x_2)\|_\infty \le LT\|x_1 - x_2\|_\infty$$

When $LT < 1$, the operator I is a contraction, thus has an unique fixed point. And the unique solution in any subinterval of $[0, T]$ has a condition $LT' \le LT < 1$, which also means the unique solution exist. And use the initial condition x_0, one can derive that the solution in any interval $[0, t]$ has the same formula, thus in the interval $[a, b] \subseteq [0, T]$.

5 Finite Dimension Normed Spaces

5.4 Solution

Considering x is a point in $X - Y$, then there exist (y_i) such that $d(x, Y) = \lim_{i\to\infty} \|x - y_i\|$. As Y is a subspace thus closed, and there exist some subsequece such that (y_{i_k}) converges to a point $y \in Y$. Considering the sequence $\|x - y_{i_k}\|$, as $\|\cdot\|$ is a continuous function, is also converges. Thus $\|x - y_{i_k}\| \to \|x - y\| = dist(x, Y)$, with $y \in Y$.

5.5 Solution

1. Considering

$$dist(ax, Y) = \inf_{y\in Y} \|ax - y\| = |a| \inf_{y\in Y} \|x - \frac{y}{a}\|$$

   And for every $y \in Y$, $y/a \in Y$ as Y is a subspace, thus the two inferior is equivalent.

2. Considering

$$dist(x + w, Y) = \inf_{y\in Y} \|x + w - y\| = \inf_{y\in Y} \|x - (y - w)\|$$

   Given the w, for every $y \in Y$, $y - w \in Y$. As same as above, the two inferior is equivalent.

5.6 Solution

Considering Y is a proper subspace of X. There exist $x \in X - Y$, which means that $dist(x, Y) = \varepsilon > 0$. Otherwise, if $dist(x, Y) = 0$, there exist $(y_n) \in Y$ such that $\|y_n - x\| \to 0$, what's more, existence of $(y_{n_k}) \to x$ implies x is a limit point of Y thus $x \in Y$, which is contradict to the hypothesis. Thus

$$\forall\ r > 0, dist(\frac{r}{\varepsilon}x, Y) = \frac{r}{\varepsilon}dist(x, Y) = r$$

5.7 Solution

Considering $(e_i)_{i=1}^\infty$ is a Hamel Basis of X, define $X_n = Span((e_i)_{i=1}^n)$. Considering y_i by $y_i \in X_i$ and choose $dist(y_i, X_{i-1}) = 3^{-i}$.

Thus $(y_i)_{i=1}^n$ is Cauchy but can't has a limitation in any X_n as

$$d(y_{n+k+1}, X_n) \geq 3 - n - \sum_{i=1}^{k} 3^{-(n+i)} \geq 3^{-n} - \sum_{i=1}^{\infty} 3^{-(n+i)} = \frac{1}{2} 3^{-n} > 0$$

Thus we got a contradiction.