Proofs in Online Resource Allocation

Tongtong Lin, Jiahang Qiu, and Liqun Fu

School of Informatics, Xiamen University, Xiamen 361005, China
Email: {tongtong, jiahang}@stu.xmu.edu.cn, liqun@xmu.edu.cn

Lemma 1. *For any control parameter $M \geq 0$, the drift-plus-penalty expression of problem $\mathscr{P}_1$ is upper bounded by*

$$\begin{aligned}\Delta \boldsymbol{V}(t) + M \cdot \mathbb{E}\{h\left(C(t) + \boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right)|\boldsymbol{Q}(t)\} \\ \leq D + \sum_{k=1}^{K} Q_k(t)\mathbb{E}\left\{r_k(t)|\boldsymbol{Q}(t)\right\} \\ + M \cdot \mathbb{E}\{h\left(C(t) + \boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right)|\boldsymbol{Q}(t)\},\end{aligned} \tag{1}$$

where D is a positive constant that satisfies

$$D \geq \frac{1}{2}\sum_{k=1}^{K} Q_k(t)\mathbb{E}\left\{(r_k(t) - \bar{r}_k)^2|\boldsymbol{Q}(t)\right\}, \forall t = 1, 2, \cdots, T.$$

Proof. According to the definitions of $\boldsymbol{Q}_t$ and $\Delta\boldsymbol{V}(t)$, we have

$$\begin{aligned}&\Delta \boldsymbol{V}(t) = \mathbb{E}\left\{\boldsymbol{V}(t+1) - \boldsymbol{V}(t)|\boldsymbol{Q}(t)\right\} \\ =&\frac{1}{2}\sum_{k=1}^{K}\mathbb{E}\left\{{Q_k}^2(t+1) - {Q_k}^2(t)|\boldsymbol{Q}(t)\right\} \\ =&\frac{1}{2}\sum_{k=1}^{K}\mathbb{E}\left\{\left(\max\left\{Q_k(t) + r(t) - \bar{r}_k, 0\right\}\right)^2 - {Q_k}^2(t)\middle|\boldsymbol{Q}(t)\right\} \\ \overset{(a)}{\leq}&\frac{1}{2}\sum_{k=1}^{K}\mathbb{E}\left\{\left(Q_k(t) + r_k(t) - \bar{r}_k\right)^2 - {Q_k}^2(t)\middle|\boldsymbol{Q}(t)\right\} \\ =&\sum_{k=1}^{K}Q_k(t)\mathbb{E}\left\{r_k(t)|\boldsymbol{Q}(t)\right\} - \sum_{k=1}^{K}Q_k(t)\mathbb{E}\left\{\bar{r}_k(t)|\boldsymbol{Q}(t)\right\} \\ +&\frac{1}{2}\sum_{k=1}^{K}\mathbb{E}\left\{\left(r_k(t) - \bar{r}_k\right)^2\middle|\boldsymbol{Q}(t)\right\} \\ \overset{(b)}{\leq}&\sum_{k=1}^{K}Q_k(t)\mathbb{E}\left\{r_k(t)|\boldsymbol{Q}(t)\right\} + D,\end{aligned}$$

where in inequality (a) , the fact $(\max\{X, 0\})^2 \leq X^2$ is exploited. In inequality (b), the term $\sum_{k=1}^{K} Q_k(t)\mathbb{E}\left\{\bar{r}_k(t)|\boldsymbol{Q}(t)\right\}$ is removed because it is positive, and due to the first constraint in $\mathscr{P}_1$, the term $\frac{1}{2}\sum_{k=1}^{K}\mathbb{E}\left\{\left(r_k(t) - \bar{r}_k\right)^2|\boldsymbol{Q}(t)\right\}$ has an upper bound which is denoted as D. Adding the term $M \cdot \mathbb{E}\{h\left(C(t) + \boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right)|\boldsymbol{Q}(t)\}$ to both sides, the above lemma can be obtained. □

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Property 1. *The K types of resources are allocated sequentially, i.e., $r_{k'}(t) = B_{k'}, \forall k' < k$ is a necessary condition for $r_k(t) > 0$.*

Proof. The KKT conditions are given as:

$$Ma_k(t)h'\left(C(t) + \boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right) + Q_k(t) + \tau_k - v_k = 0, \tag{2}$$

$$v_k r_k(t) = 0, \tag{3}$$

$$\tau_k[B_k - r_k(t)] = 0, \tag{4}$$

$$v_k \geq 0, \tag{5}$$

$$\tau_k \geq 0, \tag{6}$$

$$\forall k = 1, \ldots, K, \tag{7}$$

Assume that the optimal resource allocation policy $\boldsymbol{r}(t)$ satisfies

$$-\frac{1}{p_{k+1}(t)} < M \cdot h'\left(C(t) + \boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right) \leq -\frac{1}{p_k(t)} \tag{8}$$

then for $k' = 1, 2, \ldots, k-1$, $\tau_{k'}$ has to be positive to satisfy Eqs. [(2),(5),(6),(8)], then $r_{k'}$ must equal to $B_{k'}$ given Eq. (4). Meanwhile, for $k' = k+1, k+2, \ldots, K$, $v_{k'}$ has to be positive to satisfy Eqs. [(2),(5),(6),(8)], then $r_{k'}$ must equal to zero given Eq. (3). □

Property 2. *If $M \cdot h'\left(C(t) + \sum_{k'=1}^{k-1} a_{k'}(t)r_{k'}(t)\right) < -\frac{1}{p_k(t)}$ and $B_k > 0$, the k-th resource will be allocated, i.e, $r_k(t) > 0$. Moreover, if $0 < Q_k(t) \leq \frac{Ma_k(t)}{4}$, then $r_k(t)$ satisfies*

$$r_k(t) = min\left\{\frac{\gamma(t) - C(t) - \sum_{k'=0}^{k-1} a_{k'}(t)B_{k'}}{a_k(t)}, B_k\right\}, \tag{9}$$

where $\gamma(t) = \ln(-1 + \frac{Ma_k(t) + \sqrt{M^2a_k(t)^2 - 4Ma_k(t)Q_k(t)}}{2Q_k(t)})$; if $Q_k(t) = 0$, we have $r_k(t) = B_k$.

Proof. If $M \cdot h'\left(C(t) + \sum_{k'=1}^{k-1} a_{k'}(t)r_{k'}(t)\right) < -\frac{1}{p_k(t)}$ and $B_k > 0$, then $r_k(t)$ has to be positive given Eqs. [(2),(3),(4),(5),(6)], otherwise τ_k will be zero and Eq. (2) is not satisfied.

Moreover, according to Eqs. [(3),(4),(5),(6)], when $0 < r_k(t) < B_k$, i.e., $\tau_k = v_k = 0$, r_k satisfies

$$Q_k(t) + Ma_k(t)h'\left(C(t) + \boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right) = 0. \tag{10}$$

By calculating, we have

$$
\begin{aligned}
&h'\left(C(t)+\boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right) \\
&= \frac{-\exp(C(t)+\sum\limits_{k'=0}^{k-1} a_{k'}(t)B_{k'}+a_k(t)r_k(t))}{\left(1+\exp\left(C(t)+\sum\limits_{k'=0}^{k-1} a_{k'}(t)B_{k'}+a_k(t)r_k(t)\right)\right)^2}.
\end{aligned}
$$

Let S denote $\exp\left(C(t)+\sum\limits_{k'=0}^{k-1} a_{k'}(t)B_{k'}+a_k(t)r_k(t)\right)$. Then Eq. (10) becomes:

$$
Q_k(t)s^2+(2Q_k(t)-Ma_k(t))s+Q_k(t)=0. \tag{11}
$$

As a further step, we define the left side of the above equation as function $\beta(s)$. If $Q_k(t)>\frac{Ma_k(t)}{4}$, i.e, there is no solution to Eq. (11) and $Q_k(t)+Ma_k(t)h'\left(C(t)+\boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right)\geq 0$. Then the minimum of $Q_k(t)r_k(t)+M\cdot h\left(C(t)+\boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right)$ appears at $r_k(t)=0$. If $Q_k(t)\leq\frac{Ma_k(t)}{4}$, i.e., Eq. (11) has two solutions $s1,s2=-1+\frac{Ma_k(t)}{2Q_k(t)}\pm\frac{\sqrt{M^2a_k(t)^2-4Ma_k(t)Q_k(t)}}{2Q_k(t)}$(assuming $s1<s2$). From $s1,s2$, it can be easily get two $r_k(t)$-s, which are denoted by ${r_k}^1(t)$ and ${r_k}^2(t)$ respectively. By calculating the gradient of $\beta(s)$, It can be observed that ${r_k}^1(t)$ is local maxima and ${r_k}^2(t)$ is local minima. Therefore, ${r_k}^2(t)$ is the optimal solution which ensures that the objective function $Q_k(t)r_k(t)+M\cdot h\left(C(t)+\boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right)$ is the smallest when $Q_k(t)\leq\frac{Ma_k(t)}{4}$. When $Q_k(t)=0$ and $r_k(t)>0$, we have $v_k=0$ given Eq. (3), and Eq. (2) can be simplified to $Ma_k(t)h'\left(C(t)+\boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right))+\tau_k=0$, then $\tau_k=-Ma_k(t)h'\left(C(t)+\boldsymbol{a}(t)^{\mathsf{T}}\boldsymbol{r}(t)\right)\neq 0$. Therefore, according to Eq. (4), we have $r_k(t)=B_k$ when $Q_k(t)=0$. $\square$