Proofs for Two Properties in Online Resource Allocation

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Property 1. *The K types of resources are allocated sequentially, i.e., $r_{k'}(t) = B_{k'}, \forall k' < k$ is a necessary condition for $r_k(t) > 0$.*

Proof. The KKT conditions are given as:

$$\begin{aligned} Ma_k(t)h'\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right) + Q_k(t) + \tau_k - v_k =& 0, \quad (1)\\ v_k r_k(t) =& 0, \quad (2)\\ \tau_k[B_k - r_k(t)] =& 0, \quad (3)\\ v_k \geq& 0, \quad (4)\\ \tau_k \geq& 0. \quad (5) \end{aligned}$$

Assume that the optimal resource allocation policy $\boldsymbol{r}(t)$ satisfies

$$-\frac{1}{p_{k+1}(t)} < M \cdot h'\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right) \leq -\frac{1}{p_k(t)} \quad (6)$$

then for $k' = 1, 2, \ldots, k-1$, $\tau_{k'}$ has to be positive to satisfy Eqs. [(1),(4),(5),(6)], then $r_{k'}$ must equal to $B_{k'}$ given Eq. (3). Meanwhile, for $k' = k+1, k+2, \ldots, K$, $v_{k'}$ has to be positive to satisfy Eqs. [(1),(4),(5),(6)], then $r_{k'}$ must equal to zero given Eq. (2). □

Property 2. *If $M \cdot h'\left(C(t) + \sum_{k'=1}^{k-1} a_{k'}(t) r_{k'}(t)\right) < -\frac{1}{p_k(t)}$ and $B_k > 0$, the k-th resource will be allocated, i.e, $r_k(t) > 0$. Moreover, if $0 < Q_k(t) \leq \frac{Ma_k(t)}{4}$, then $r_k(t)$ satisfies*

$$r_k(t) = min\left\{\frac{\gamma(t) - C(t) - \sum_{k'=0}^{k-1} a_{k'}(t)B_{k'}}{a_k(t)}, B_k\right\}, \quad (7)$$

where $\gamma(t) = \ln(-1 + \frac{Ma_k(t) + \sqrt{M^2 a_k(t)^2 - 4Ma_k(t)Q_k(t)}}{2Q_k(t)})$; if $Q_k(t) = 0$, we have $r_k(t) = B_k$.

Proof. If $M \cdot h'\left(C(t) + \sum_{k'=1}^{k-1} a_{k'}(t) r_{k'}(t)\right) < -\frac{1}{p_k(t)}$ and $B_k > 0$, then $r_k(t)$ has to be positive given Eqs. [(1),(2),(3),(4),(5)], otherwise τ_k will be zero and Eq. (1) is not satisfied.

Moreover, according to Eqs. [(2),(3),(4),(5)], when $0 < r_k(t) < B_k$, i.e., $\tau_k = v_k = 0$, r_k satisfies

$$Q_k(t) + Ma_k(t)h'\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right) = 0. \quad (8)$$

By calculating, we have

$$h'\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right) = \frac{-\exp(C(t) + \sum_{k'=0}^{k-1} a_{k'}(t)B_{k'} + a_k(t)r_k(t))}{\left(1 + \exp\left(C(t) + \sum_{k'=0}^{k-1} a_{k'}(t)B_{k'} + a_k(t)r_k(t)\right)\right)^2}.$$

Let S denote $\exp\left(C(t) + \sum_{k'=0}^{k-1} a_{k'}(t)B_{k'} + a_k(t)r_k(t)\right)$. Then Eq. (8) becomes:

$$Q_k(t)s^2 + (2Q_k(t) - Ma_k(t))s + Q_k(t) = 0. \quad (9)$$

As a further step, we define the left side of the above equation as function $\beta(s)$. If $Q_k(t) > \frac{Ma_k(t)}{4}$, i.e, there is no solution to Eq. (9) and $Q_k(t) + Ma_k(t)h'\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right) \geq 0$. Then the minimum of $Q_k(t)r_k(t) + M \cdot h\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right)$ appears at $r_k(t) = 0$. If $Q_k(t) \leq \frac{Ma_k(t)}{4}$, i.e., Eq. (9) has two solutions $s1, s2 = -1 + \frac{Ma_k(t)}{2Q_k(t)} \pm \frac{\sqrt{M^2 a_k(t)^2 - 4Ma_k(t)Q_k(t)}}{2Q_k(t)}$(assuming $s1 < s2$). From $s1, s2$, it can be easily get two $r_k(t)$-s, which are denoted by ${r_k}^1(t)$ and ${r_k}^2(t)$ respectively. By calculating the gradient of $\beta(s)$, It can be observed that ${r_k}^1(t)$ is local maxima and ${r_k}^2(t)$ is local minima. Therefore, ${r_k}^2(t)$ is the optimal solution which ensures that the objective function $Q_k(t)r_k(t) + M \cdot h\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right)$ is the smallest when $Q_k(t) \leq \frac{Ma_k(t)}{4}$. When $Q_k(t) = 0$ and $r_k(t) > 0$, we have $v_k = 0$ given Eq. (2), and Eq. (1) can be simplified to $Ma_k(t)h'\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right)) + \tau_k = 0$, then $\tau_k = -Ma_k(t)h'\left(C(t) + \boldsymbol{a}^\mathsf{T}(t)\boldsymbol{r}(t)\right) \neq 0$. Therefore, according to Eq. (3), we have $r_k(t) = B_k$ when $Q_k(t) = 0$. □

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