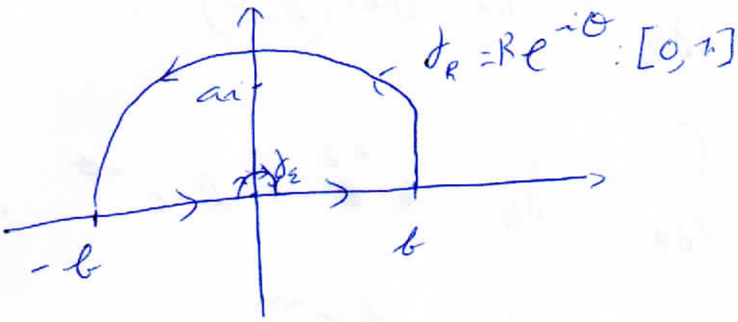
1) $\int_{-\infty}^{\infty} \frac{sen(mx)}{x(x^2+a^2)^2}\, dx \qquad m \in \mathbb{N}$

$= Im\left(\int_{-\infty}^{\infty} \frac{e^{imx}}{x(x^2+a^2)^2}\, dx\right)$

è analitica se $x(x^2+a^2)^2 \neq 0$ — $x \neq 0$ polo singolo, $x \neq ia$ polo doppio, $x \neq -ia$ polo doppio

$f(z) = \frac{e^{imx}}{x(x^2+a^2)^2}\, dx$

$\int_\gamma f(z_0)\, dz = 2\pi i \left(Res(f, ia)\right)$



~~$Res(f, ia) =$~~

~~$f(z)\Big|_{z=ia} =$~~

$Res(f, ia) = \lim_{z \to ia} \frac{d}{dz} \frac{e^{imz}}{z(z+ia)^2 \cancel{(z-ia)^2}} \cancel{(z-ia)^2}$

$= \lim_{z \to ia} \frac{im\, e^{imz} \cdot z(z+ia)^2 - e^{imz}(3z+ia)(z+ia)}{z^2(z+ia)^4}$

$= \frac{e^{imz}\cancel{(z+ia)}\left[imz(z+ia) - 3z+ia\right]}{z^2(z+ia)^{\cancel{4}3}}$

$= \frac{e^{-am}\left[+am \cdot 2\cancel{i}a + \cancel{3}2\cancel{i}a\right]}{-a^2(\cancel{i}a^3)} = \frac{2e^{am}(am+1)\ \cancel{e^{-am}(-2ia)}}{-a^4}$

$$\int_\gamma f(z)\,dz = \underbrace{\int_{-\infty b}^{-\varepsilon} + \int_{\varepsilon}^{+\infty b}} + \int_{-\gamma_\varepsilon} + \int_{\gamma_R}$$

$$\cancel{\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty}}$$

$$\lim_{\substack{\varepsilon\to 0\\ b\to+\infty}} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} = I$$

$$\int_{\gamma_\varepsilon} = \pi i \cdot \operatorname{Res}(f,0) = \pi i\, \cancel{x} \cdot \frac{e^{imx}}{\cancel{x}(x^2+a^2)}\Bigg|_{x=0} = \frac{\pi i}{a^2}$$

$$\int_{\gamma_R} : \gamma_R : z \to b\cdot e^{i\theta} \quad \text{con } \theta\in[0,\pi].$$

$$\lim_{b\to+\infty} \int \frac{e^{im(b\cdot e^{i\theta})}}{be^{i\theta}(b^2e^{2i\theta}+a^2)}\, i b e^{i\theta}\, d\theta$$

$$\cancel{= \lim_{b\to+\infty} \int e^{imb(\cos\theta + i\,\text{sen}\,\theta)}}$$

$$\lim_{b\to+\infty} \left| \int \frac{e^{imb(\cos\theta + i\,\text{sen}\,\theta)}}{b^3 e^{3i\theta}\left(1+\frac{a^2}{b^2e^{2i\theta}}\right)}\, i b e^{i\theta}\, d\theta \right|$$

$$< \lim_{b\to+\infty} \left| \int \frac{e^{-mb\,\text{sen}\,\theta}\, i b}{b^3 e^{3i\theta}}\, d\theta \right| \to 0$$

$$\int_\gamma f(z)\,dz = \frac{2e^{-am}(am+1)}{-a^4} = I - \frac{\pi i}{a^2}$$

$$I = \frac{2e^{-am}(am+1)}{-a^4} + \frac{\pi i}{a^2} \Rightarrow \operatorname{Im}(I) = \boxed{\frac{\pi}{a^2}}$$

a) $f(z) = \dfrac{\cancel{\operatorname{sen} z}}{} \dfrac{1}{\operatorname{sen} z}$

è analitica ovunque tranne i punti in cui $\operatorname{sen} z = 0$

$\Downarrow$

$z = k\pi \quad k \in \mathbb{Z}$

b) è sviluppabile in Laurent, perché è analitica in un intorno $B_r(0) - \{0\}$.

$$\operatorname{sen} z = \sum_{n=0} (-1)^{n} \frac{z^{2n+1}}{(2n+1)!}$$

$$\frac{1}{\operatorname{sen} z} = \frac{1}{x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots} \sim \frac{1}{x} \qquad \operatorname{Res}(f, 0) = 1$$

c)

$w = \frac{1}{z}$

$\lim_{w \to 0} \dfrac{1}{\operatorname{sen}\left(\frac{1}{w}\right)}$ non è definito tale limite $\Rightarrow$ non è sviluppabile in Laurent.

$w = 0$ è una singolarità essenziale

$$f(z) = \frac{1}{\cos\left(\frac{1}{z}\right)}$$

è analitica se $\cos\left(\frac{1}{z}\right) \neq 0$

$$\frac{1}{z} \neq \frac{\pi}{2} + k\pi$$

$$z \neq \frac{2}{\pi + 2k\pi} \qquad k \in \mathbb{Z}$$

b) non è sviluppabile in $z=0$

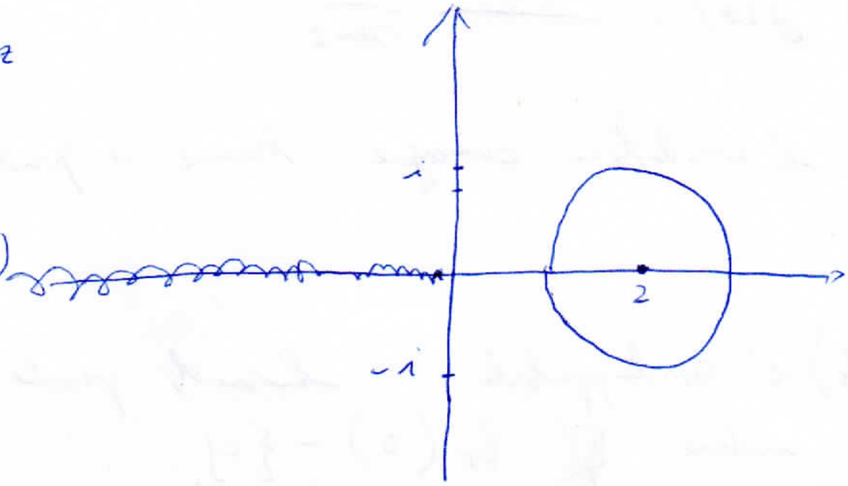
c) $w = \frac{1}{z}$

$$f(w) = \frac{1}{\cos(w)} \qquad \cos(w) = \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n}}{(2n)!}$$

$$\operatorname{Res}(f(w)) = \frac{1}{1 - \frac{w^2}{2}} \cdot w = 0$$

3) $\displaystyle\int_\gamma \frac{\ln(z)}{(z-2)^3(z^2+1)}\,dz$

$z \neq -t$ con $t \in [0,+\infty)$



$$\frac{\ln(z)}{(z-2)^3(z^2+1)} = g(z)$$

$(z-2)^3(z^2+1) = 0$

- $z = 2$ ordine 3
- $z = i$ ordine 1 ma fuori da γ_R
- $z = -i$ ordine 1. fuori da γ_R

$$\int_\gamma \frac{\ln(z)}{(z-2)^3(z^2+1)} = \frac{\cancel{2}\pi i}{\cancel{2}} \cdot \frac{d^2}{dz^2}\left(\frac{\ln(z)}{(z^2+1)\cancel{(z-2)^3}} \cdot \cancel{(z-2)^3}\right)$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}} \frac{d}{dz}\left(\frac{\frac{z^2+1}{z} - 2z\ln(z)}{(z^2+1)^2}\right)$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}} \frac{d}{dz}\left(\frac{1}{z(z^2+1)} - \frac{2z\ln(z)}{(z^2+1)^2}\right) \quad \cancel{4z(z^2+1)}$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}}\left(\frac{(3z^2+1)\cdot(-1)}{(z^3+z)^2} - \frac{(z^2+1)^{\cancel{2}}\cdot 2(\ln(z)+1) - 8z\ln(z)}{(z^2+1)^{\cancel{4}3}}\right)$$

$$= 2\pi i\left(-\frac{3z^2+1}{z^2(z^2+1)^2} - \frac{2z^2-6z^2\ln(z)+2\ln(z)+2}{(z^2+1)^3}\right)\Bigg|_{z=2}$$

$$= 2\pi i\left(-\frac{13}{100} - \frac{8-24\ln(2)+2\ln(2)+2}{125}\right)$$

$$= \cancel{2}\pi i\left(\frac{-65-40-88\ln(2)}{500}\right) = -\cancel{2}\pi i\,\frac{125+88\ln(2)}{500} = -\frac{1}{\cancel{10}\,20}i - \frac{44}{\cancel{125}\,250}\ln(2)$$

3) $\displaystyle\int_\gamma \frac{\ln(z)}{(z-2)^3(z^2+1)}\,dz = \cancel{-\frac{44}{125}\ln(2) - \frac{1}{10}i}$

$\displaystyle -i\left(\frac{22}{125}\ln(2) + \frac{1}{20}\right)$