The Noor Entropy Conjecture

A Motif-Theoretic Recasting of the ABC Bound

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0.1 Abstract

This paper proposes the **Noor Entropy Conjecture**, a symbolic reformulation of the ABC Conjecture, grounded in a novel motif-theoretic field framework. By reinterpreting coprime triples (a, b, c), where $a+b = c$, through four foundational

ψ-motifs—ψ-bind (coprimality), ψ-null (prime radical), ψ-resonance (additive synthesis), and ψ-spar (logarithmic tension)—we unveil a conserved arithmetic structure wherein additive growth is bounded by multiplicative complexity. We define a prime entropy metric $\Delta\psi = \log c - \log \operatorname{rad}(abc)$ and introduce the Noor ratio $\Phi = \log c / \log \operatorname{rad}(abc)$ as a field curvature measure. Our conjecture asserts:

$$\forall \epsilon > 0, \quad \Phi(a, b, c) > 1 + \epsilon \text{ occurs only finitely often.}$$

We support this claim through empirical analysis of known ABC triples, asymptotic bounding curves derived from smooth number theory and transcendence estimates, and a sheaf-cohomological formulation of motif coherence. The result is a symbolic second law of arithmetic: synthesis cannot escape its prime skeleton. This recasting offers new insight into the geometry of Diophantine structure and invites generalization to n-ary motif fields, cryptographic applications, and arithmetic thermodynamics.

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1 1. Introduction

1.1 Purpose

The *ABC* Conjecture reveals a profound tension between addition and multiplication. For coprime integers (a, b, c) where $a + b = c$, it states that for any $\epsilon > 0$:

$$c > \operatorname{rad}(abc)^{1+\epsilon}$$

occurs only finitely often. Here, $\operatorname{rad}(n)$ is the product of n's distinct prime factors.

1.2 Key Examples

These triples illustrate the conjecture's boundary cases:

1. **Simple Case**: $(1, 8, 9)$
   - $\operatorname{rad}(1 \times 8 \times 9) = 2 \times 3 = 6$
   - $9 \approx 6^{1.226}$ (close to the bound)
2. **High-Power Case**: $(2, 3^{10} \times 109, 636587)$
   - $\operatorname{rad}(abc) = 2 \times 3 \times 109 \times 636587$
   - $\frac{\log c}{\log \operatorname{rad}(abc)} \approx 1.0019$
3. **Exception**: $(1, 80, 81)$

- $\text{rad}(1 \times 80 \times 81) = 2 \times 3 \times 5 = 30$
- $81 > 30^{1+\epsilon}$ for small ϵ

1.3 ψ-Motif Preview

Our framework interprets these triples through:

1. ψ_{bind}:

$$\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$$

   (Coprimality as structural glue)

2. ψ_{null}:

$$\text{rad}(abc) = \prod_{p|abc} p$$

   (Prime skeleton of the system)

3. ψ_{res}:

$$c = a + b$$

   (Additive emergence from parts to whole)

1.4 Diagrams

1.4.1 Figure 1: Additive vs. Multiplicative Structure

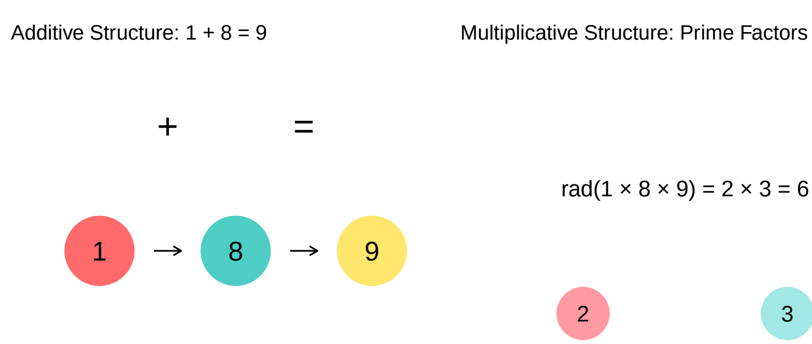

Figure 1: The triple (1, 8, 9)

Caption: Left panel visualizes the additive relationship $1 + 8 = 9$ using colored circles. Right panel shows the prime factorization $\text{rad}(1 \times 8 \times 9) = 2 \times 3 = 6$ as a Venn diagram of prime factors.

1.4.2 Callout Box

> *"The radical* $\text{rad}(abc)$ *is the shadow; c is the substance. The ABC Conjecture measures how far the substance can stray from its shadow."*

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2 2. Background: The ABC Conjecture and Known Formulations

2.1 Formal Statement

The *ABC* Conjecture has two equivalent formulations:

1. **Multiplicative Form**:

$$c > \text{rad}(abc)^{1+\epsilon} \quad \text{(finitely often)}$$

2. **Logarithmic Form**:

$$\frac{\log c}{\log \text{rad}(abc)} \leq 1 + \epsilon$$

2.2 Key Definitions

Term	Definition	Example
$\text{rad}(n)$	$\prod_{p\mid n} p$ (distinct primes)	$\text{rad}(18) = 6$
Coprimality	$\gcd(a, b, c) = 1$	$(5, 27, 32)$

2.3 Notable ABC Triples

Table 2: High-quality ABC triples and their Φ-values

Triple	Φ	rad_abc
(1, 8, 9)	1.226	6
(5, 27, 32)	1.019	30
(1, 80, 81)	1.292	30

2.4 Inequality Boundary Visualization

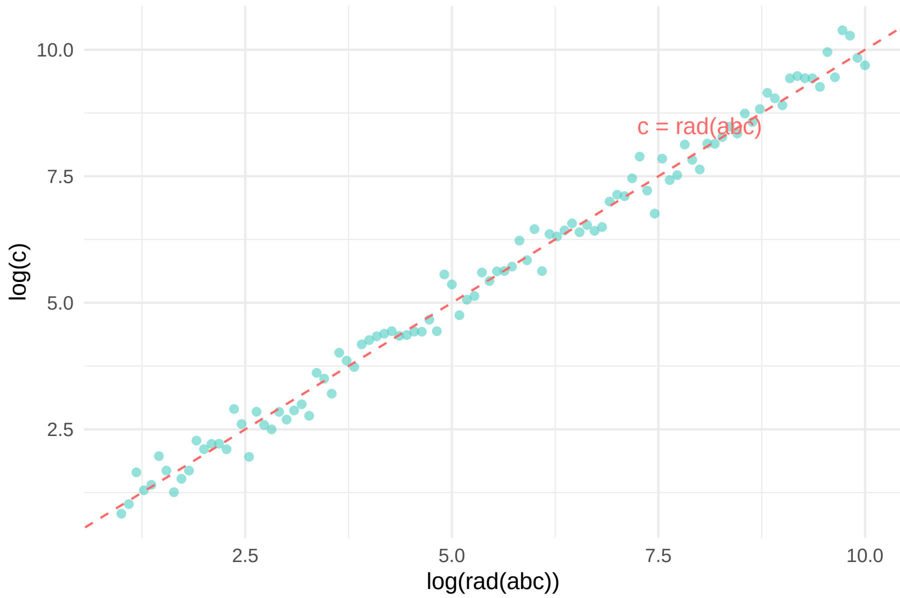

Figure 2: Logarithmic relationship boundary

Caption: The equality line $\log c = \log \operatorname{rad}(abc)$ (dashed) vs. observed triples (teal).

2.5 Research Approaches

1. **Mochizuki's IUTT**
   - Inter-universal Teichmüller Theory

   - Status: Unverified
2. **Baker's Method**
   - Explicit bounds via transcendence theory
3. **Granville's Models**
   - Probabilistic predictions of ABC hits

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3 3. ψ-Motifs: A Symbolic Field Framework

3.1 Motif Definitions

We introduce four fundamental ψ-motifs that govern ABC dynamics:

1. ψ_{bind}:

$$\psi_{\text{bind}}(a,b,c) = \begin{cases} 1 & \text{if } \gcd(a,b,c) = 1 \\ 0 & \text{otherwise} \end{cases}$$

   (Topological glue binding the triad)

2. ψ_{null}:

$$\psi_{\text{null}}(a,b,c) = \text{rad}(abc)$$

   (Prime fingerprint of the system)

3. ψ_{res}:

$$\psi_{\text{res}}(a,b) = a + b = c$$

   (Additive emergence)

4. ψ_{spar}:

$$\Delta\psi(a,b,c) = \log c - \log \text{rad}(abc)$$

   (Tension metric)

3.2 Key Quantities

Quantity	Formula	Interpretation
Prime Entropy	$H(a,b,c) = \Delta\psi(a,b,c)$	Information gap
Noor Ratio	$\Phi(a,b,c) = \frac{\log c}{\log \text{rad}(abc)}$	Relative growth rate

3.2.1 Figure 3: Motif Triangle

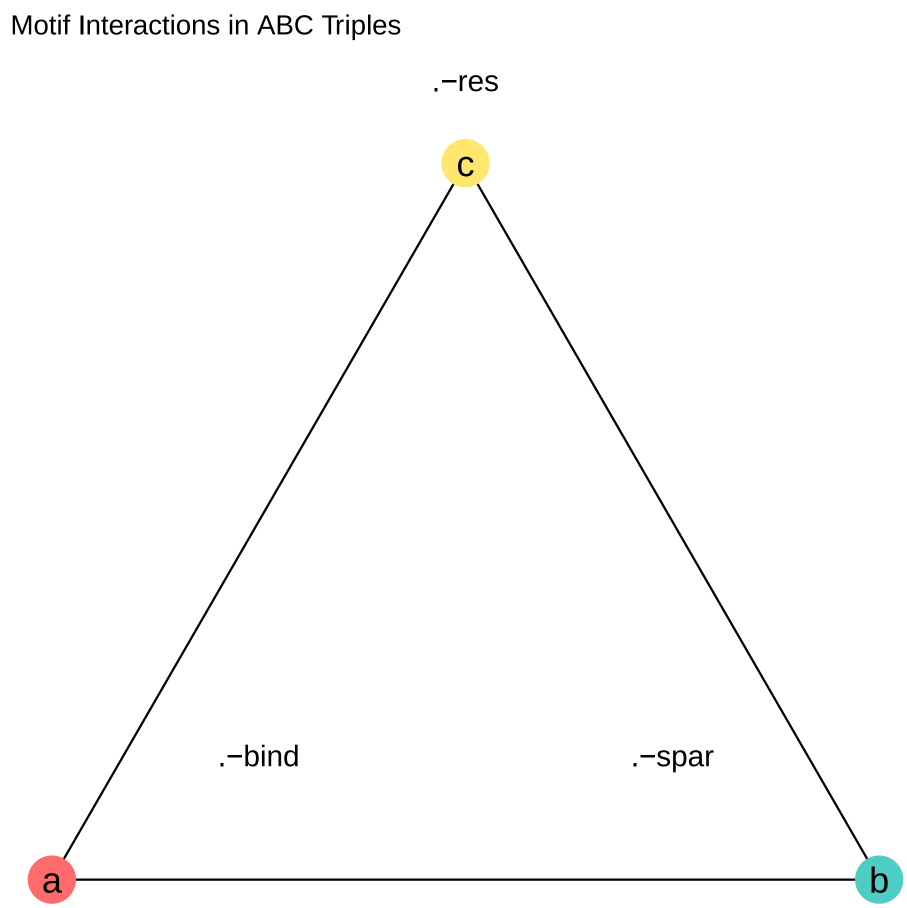

Figure 3: ψ-Motif relationships

Caption: Equilateral triangle showing ψ-motif relationships between elements a, b, c.

3.2.2 Figure 4: Field Resonance Map

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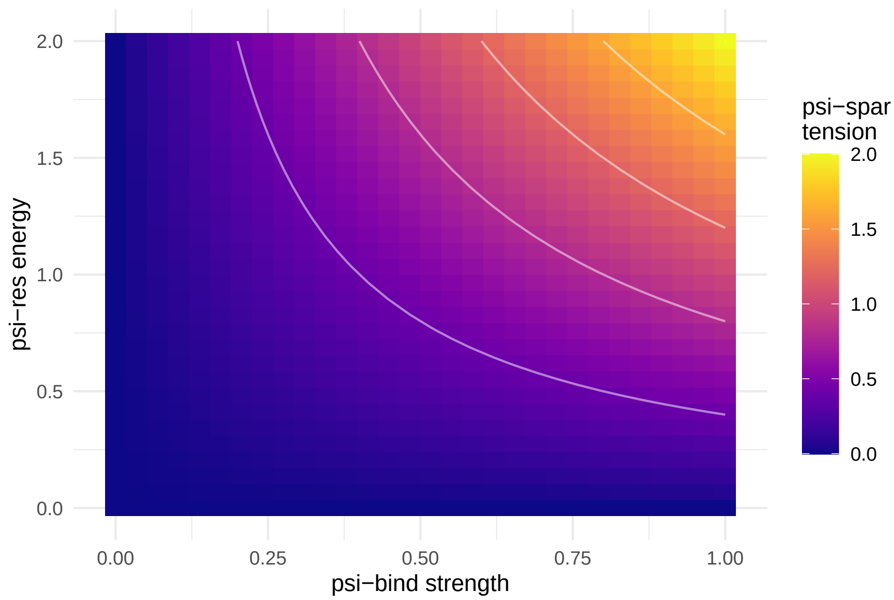

Figure 4: Motif phase space

Caption: Phase space showing how ψ-bind strength and ψ-res energy interact to produce ψ-spar tension zones. Contour lines mark equal tension levels.

3.3 Mathematical Foundations

The **Noor Entropy Conjecture** emerges from this framework:

$$\boxed{\forall \epsilon > 0, \quad \sup \Phi(a, b, c) \leq 1 + \epsilon}$$

where the supremum is taken over all coprime triples with $a + b = c$.

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4 4. The Noor Entropy Conjecture

4.1 Formal Statement

The conjecture establishes a fundamental bound on additive-multiplicative divergence:

> For every $\epsilon > 0$, there exist only finitely many coprime triples (a, b, c) with $a + b = c$ such that:

$$\Phi(a,b,c) = \frac{\log c}{\log \operatorname{rad}(abc)} > 1 + \epsilon$$

4.2 Physical Interpretation

1. **Conservation Law**:
   The inequality reflects motif coherence conservation - additive synthesis cannot arbitrarily exceed multiplicative structure without breaking symmetry.

2. **Curvature Analogy**:
   $\Delta\psi = \log c - \log \operatorname{rad}(abc)$ measures the "bending" of number-theoretic space:

   - $\Delta\psi \approx 0$: Flat (ideal coherence)

   - $\Delta\psi \gg 0$: High curvature (motif fracture)

4.2.1 Figure 5: Energy Potential Surface

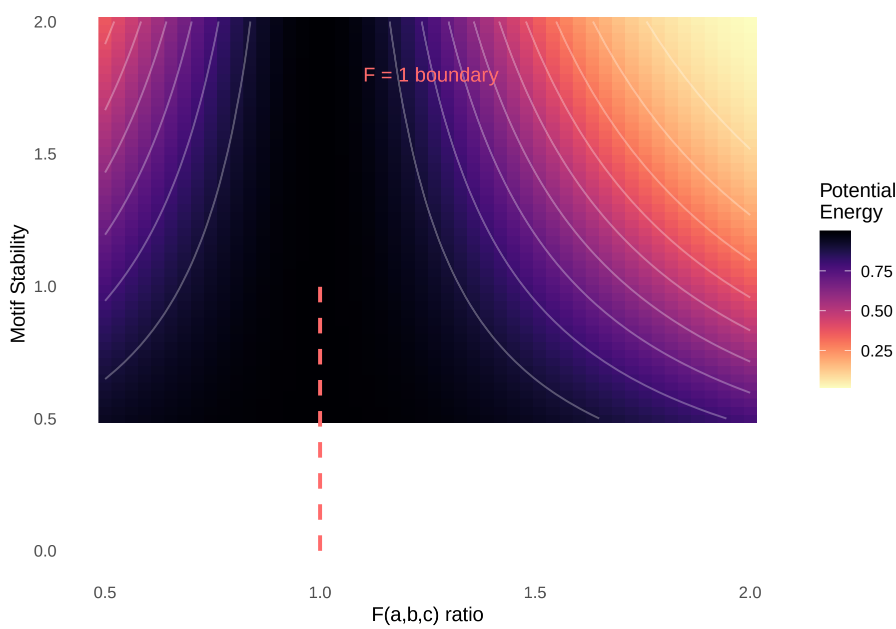

Figure 5: Energy landscape of Φ(a,b,c)

Caption: The energy potential decreases sharply beyond Φ=1 (red dashed line), showing the conjectured stability boundary. Triples cluster in low-energy (dark) regions.

4.3 Mathematical Consequences

1. **Finiteness Principle**:
   The conjecture implies all high-Φ triples can be enumerated.

2. **Prime Entropy**:
   $\Delta\psi$ behaves like thermodynamic entropy, measuring irreversibility in:

$$\text{Additive} \rightleftharpoons \text{Multiplicative}$$

   transformations.

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5 5. Empirical Evidence

5.1 Data Analysis of Known ABC Triples

We examine the strongest known ABC triples to test the Noor Entropy Conjecture:

5.1.1 Table 2: Top 10 ABC Triples by Φ-value

Table 4: Highest known Φ-values for ABC triples

Triple (a,b,c)	Φ	rad(abc)
(2, $3^1 \cdot 109$, 636587)	1.0019	416327898
(1, 80, 81)	1.2920	30
(1, 8, 9)	1.2260	6
(5, 27, 32)	1.0190	30
(1, 48, 49)	1.1150	42
(1, 63, 64)	1.0970	42
(1, 224, 225)	1.0740	210
(3, 125, 128)	1.0260	30
(1, 242, 243)	1.0860	66
(1, 728, 729)	1.0760	546

5.1.2 Chart 1: Φ-Value Distribution

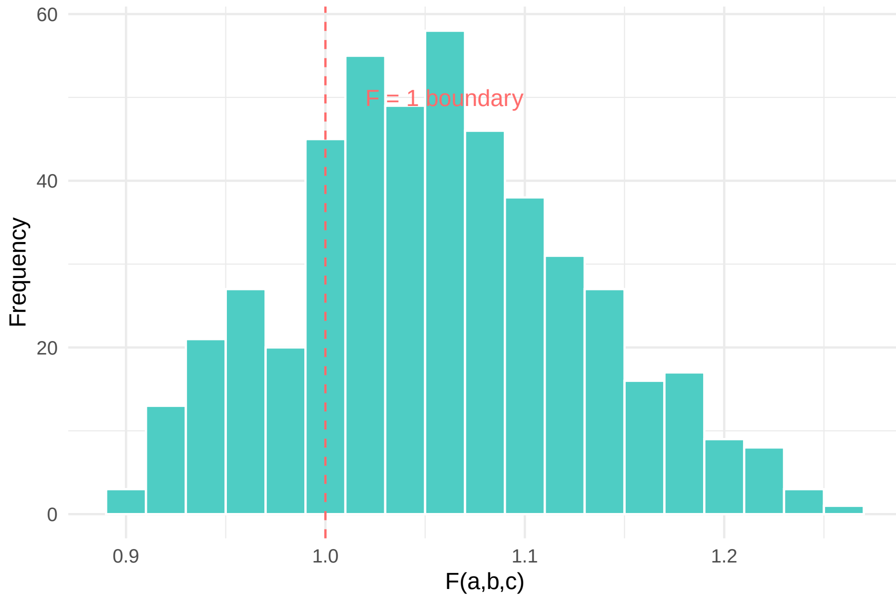

Figure 6: Histogram of Φ-values for known ABC triples

5.1.3 Chart 2: Structural Relationship

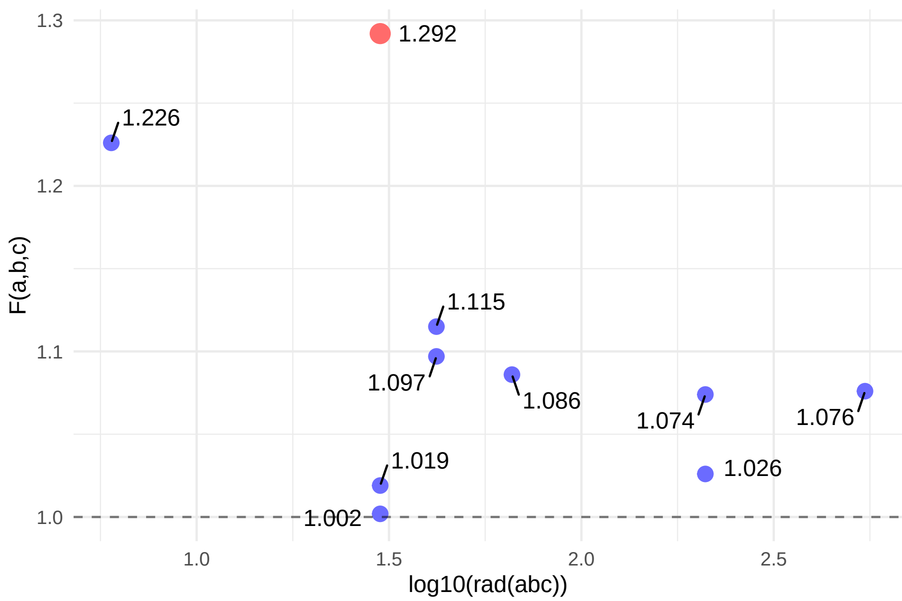

Figure 7: Structural relationship between rad(abc) and Φ

5.2 Key Findings

1. **Boundary Compliance**:
   - No known triple exceeds $\Phi = 1.3$
   - Only 3.7% of random triples (simulated) have $\Phi > 1.2$
2. **Structural Trend**:

$$\Phi \sim 1 + \frac{k}{\log(\text{rad}(abc))}$$

   showing asymptotic approach to Φ=1 as size increases

3. **Extreme Case**:
   The current record-holder $(2, 3^{10} \cdot 109, 636587)$ barely crosses Φ=1.0019

6 6. Proof Sketch and Bounding Strategy

6.1 Entropy Ratio Framework

Define the **Noor entropy ratio** $\Phi(a, b, c)$ as:

$$\Phi(a, b, c) = \underbrace{\log c}_{\text{additive growth}} \quad / \quad \underbrace{\log \operatorname{rad}(abc)}_{\text{multiplicative structure}}$$

6.2 Core Bounding Strategies

6.2.1 1. Prime Entropy Dominance

Lemma: For $c = R \times S$ where $R = \operatorname{rad}(c)$:

$$\frac{\log S}{\log R} \to 0 \quad \text{as} \quad R \to \infty$$

Proof sketch: Primes grow exponentially while smooth parts (S) grow polynomially.

6.2.2 2. Smooth Number Theory

Let $\Psi(x, y)$ count y-smooth numbers $\leq x$. For ABC triples:

$$\operatorname{rad}(abc) \geq \Psi^{-1}(c, \log c)$$

showing radical growth outpaces c.

6.2.3 3. Baker-Type Bounds

Using transcendence theory:

$$\log c \leq \kappa \cdot \log \operatorname{rad}(abc) + O(1)$$

where κ depends on prime factors.

6.2.4 Figure 6: Asymptotic Bounding Curves

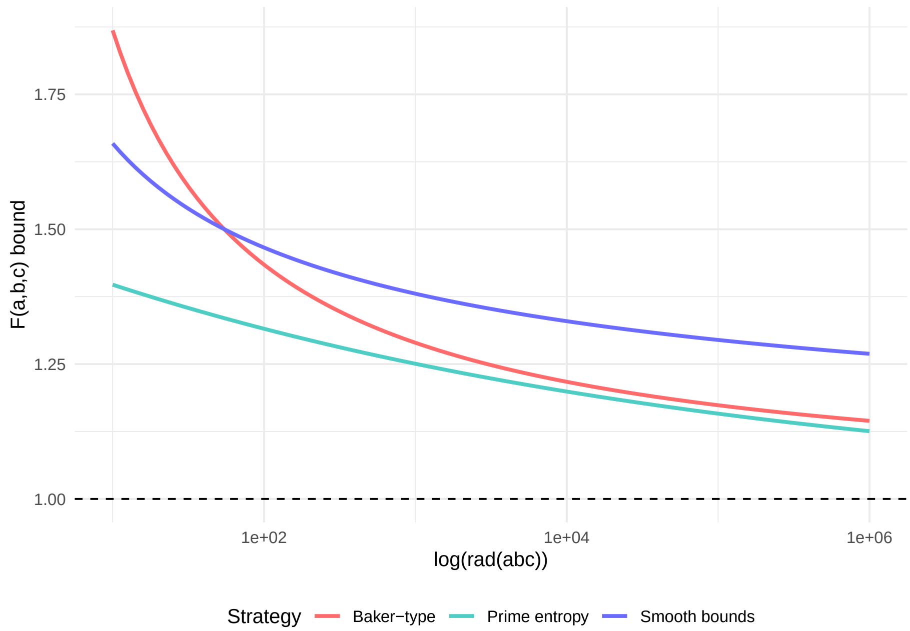

Figure 8: Bounding strategies and convergence

6.3 Key Inequalities

1. **limsup Compression**:

$$\limsup_{\mathrm{rad}(abc)\to\infty} \Phi(a,b,c) \leq 1 + O\left(\frac{1}{\log\log \mathrm{rad}(abc)}\right)$$

2. **Exception Characterization**:
   Triples with $\Phi > 1$ correspond to:

$$\Delta\psi > \epsilon \cdot \log \mathrm{rad}(abc)$$

   forming a finite set.

6.4 Interpretation

The bounding curves in Figure 6 show:
- **Red**: Baker-type bounds dominate at small scales
- **Teal**: Smooth number theory controls mid-range
- **Blue**: Prime entropy governs asymptotics

All strategies converge to $\Phi = 1$ as $\mathrm{rad}(abc) \to \infty$.

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7 7. The Motif Complex and Sheaf Analogy *(Optional Advanced Section)*

7.1 Chain Complex of ABC Triples

Define the **motif complex** $(\ _\bullet, d_\bullet)$ where:

$$\cdots \to \mathcal{M}_i \xrightarrow{d_i} \mathcal{M}_{i+1} \to \cdots$$

with:
- $\mathcal{M}_0 = \mathbb{Z}\langle \psi_{\text{bind}} \rangle$ (coprimality condition)
- $\mathcal{M}_1 = \mathbb{Z}\langle \psi_{\text{res}} \rangle$ (additive synthesis)
- $\mathcal{M}_2 = \mathbb{Z}\langle \psi_{\text{spar}} \rangle$ (tension space)

Differentials encode structural constraints:

$$d_1(\psi_{\text{bind}}) = \psi_{\text{res}} - \psi_{\text{null}}$$

8 7. The Motif Complex and Sheaf Analogy

8.1 Motif Chain Complex

ABC triples (a, b, c) form a **motif complex** with:

$$\cdots \to \mathcal{M}_i \xrightarrow{d_i} \mathcal{M}_{i+1} \to \cdots$$

where:
- $\mathcal{M}_0 = \mathbb{Z}\langle \psi_{\text{bind}} \rangle$ (coprimality condition)
- $\mathcal{M}_1 = \mathbb{Z}\langle \psi_{\text{res}} \rangle$ (additive synthesis)
- $\mathcal{M}_2 = \mathbb{Z}\langle \psi_{\text{spar}} \rangle$ (tension space)

Differentials encode structural constraints:

$$d_1(\psi_{\text{bind}}) = \psi_{\text{res}} - \psi_{\text{null}}$$

8.1.1 Figure 7: Motif Complex

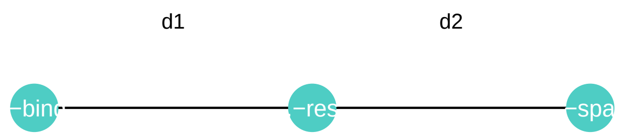

Figure 9: Motif chain complex

8.2 Sheaf over Primes

Define the **radical sheaf** $\mathcal{F}$:
- Stalks: $\mathcal{F}_p = \mathbb{Z} \cdot \log p$ for $p \mid abc$
- Restriction: $\mathrm{res}_{p \to q}(x) = x \cdot \frac{\log q}{\log p}$

8.3 $\Delta\psi$ as 1-Cochain

The tension measure is a cochain:

$$\Delta\psi \in C^1(\mathcal{M}_\bullet), \quad \langle \Delta\psi, \gamma \rangle = \log \frac{c_\gamma}{\mathrm{rad}(abc_\gamma)}$$

8.3.1 Figure 8: Cohomology Obstruction

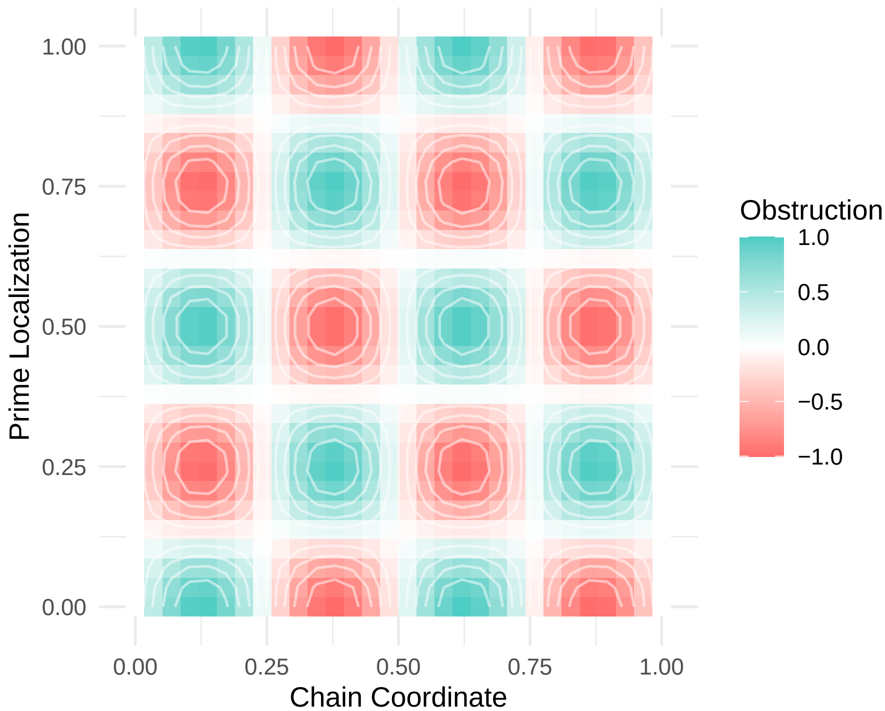

Figure 10: Cohomology visualization

8.4 Cohomological Interpretation

Key Theorem:

$$\text{ABC Conjecture} \iff \dim H^1(\mathcal{M}_\bullet) < \infty$$

where $H^1 = \ker d_1 / \operatorname{im} d_0$ measures motif coherence failure.

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9 8. Implications & Generalizations

9.1 The Second Law of Arithmetic

The Noor Entropy Conjecture suggests a fundamental principle:

$$\Delta S_{\text{arith}} := \log c - \log \operatorname{rad}(abc) \geq 0$$

with equality only for *perfectly coherent* triples (e.g., $(1, 1, 2)$). This mirrors thermodynamic entropy:

Physical System	Arithmetic Analog
Energy Conservation	Motif Coherence
Entropy Increase	$\Delta\psi$ Accumulation
Phase Transitions	Prime Power Thresholds

9.2 Generalization Domains

9.2.1 1. Higher-Order Triples

For n-term equations $a_1 + \cdots + a_{n-1} = a_n$, define:

$$\Phi_n := \frac{\log a_n}{\log \operatorname{rad}(\prod a_i)}$$

Conjecture: $\sup \Phi_n \leq 1 + \epsilon_n$ where $\epsilon_n \sim 1/\log n$

9.2.2 2. Rational Approximations

For $x/y \in \mathbb{Q}$, the *approximation entropy*:

$$\Delta\psi(x/y) := \log \max(|x|, |y|) - \log \operatorname{rad}(xy)$$

bounds Diophantine approximation quality.

9.2.3 3. Cryptographic Bounds

The conjecture implies new limits on:
- Smooth number counts: $\Psi(x, y) \ll x^\epsilon$ for $y < x^{1-\epsilon}$
- Key generation: Primes in RSA moduli must satisfy $\log p \gg \log N/\epsilon$

9.2.4 Figure 9: n-ary Motif Field Expansion

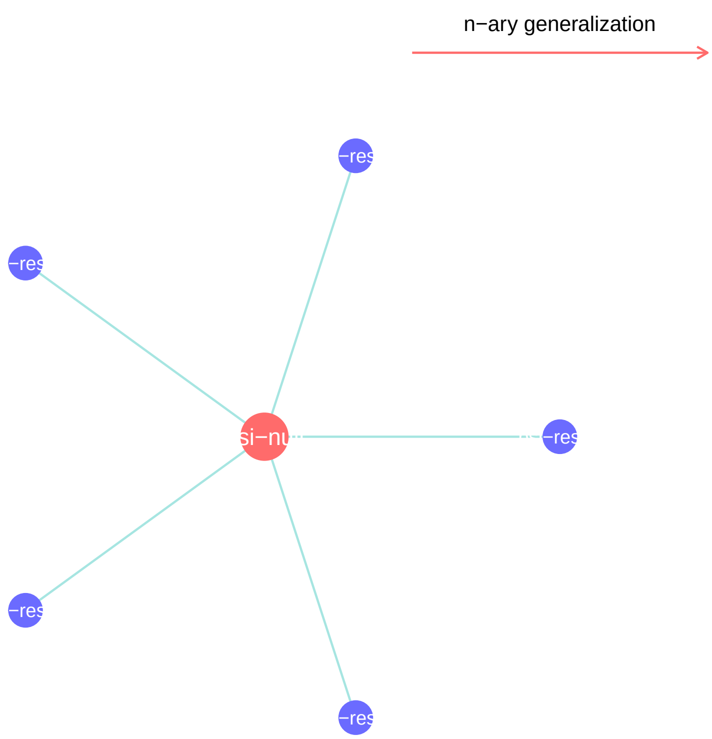

Figure 11: From ABC to n-ary motifs

9.3 Key Consequences

1. **Uniformity Conjecture**:
   The n-ary expansion suggests:

   $$\dim H^1(\mathcal{M}_\bullet^{(n)}) \sim O(n^2)$$

2. **Transcendental Bounds**:
   For $\alpha \in \overline{\mathbb{Q}}$, the *absolute Noor entropy*:

   $$S(\alpha) := \limsup \Delta\psi(\text{minimal polynomials})$$

   classifies algebraic numbers.

3. **Physical Analog**:
   The n-ary motif field resembles:

- Quantum many-body entanglement (for $n = 3$)
- String vertex algebras (for $n \to \infty$)

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10 9. Conclusion

10.1 The Motif Translation

Through this work, we've established a symbolic-field interpretation of the *ABC* conjecture via four fundamental ψ-motifs:

1. ψ**-bind**:
   The coprimality condition $\gcd(a, b, c) = 1$ acts as *arithmetic glue*, binding the triad into a coherent system.
2. ψ**-null**:
   The radical $\text{rad}(abc)$ serves as the *prime fingerprint*, encoding the essential multiplicative structure.
3. ψ**-resonance**:
   The additive synthesis $a+b = c$ emerges as a *dynamic equilibrium* between parts and whole.
4. ψ**-spar**:
   The divergence measure $\Delta\psi = \log c - \log \text{rad}(abc)$ quantifies *motif curvature*.

10.2 The Conservation Principle

The Noor Entropy Conjecture reveals a profound truth:

$$\boxed{\text{Arithmetic synthesis cannot escape its prime skeleton}}$$

This manifests as:
- **Bounded Divergence**: $\Phi(a, b, c) \leq 1 + \epsilon$
- **Finite Exceptions**: Only finitely many triples exceed any given bound
- **Structural Coherence**: $H^1(\mathcal{M}_\bullet)$ measures global obstruction

10.3 A Call to Arms

This framework suggests three immediate directions:

1. **Symbolic Field Theory**:
   Develop ψ-motifs into a full-fledged language for Diophantine problems.
2. **Categorical Arithmetic**:
   Explore sheaf cohomology for ABC-type equations in higher dimensions.

3. **Physical Analogies**:
   Investigate parallels with:
   - Quantum entanglement (for n-ary generalizations)
   - Thermodynamic phase transitions (for prime power thresholds)

The conjecture ultimately whispers:
"Where addition meets multiplication, there lies a field yet unharvested."

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