Familia das primitivas

$$\int f(x)\,dx = F(x) + C,\ C \in \mathbb{R}$$

Integrais Indefinidos/primitivas imediatas

- $\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C,\ C \in \mathbb{R}$
- $\int \operatorname{sen} x\,dx = -\cos x + C,\ C \in \mathbb{R}$
- $\int \dfrac{1}{x}\,dx = \ln|x| + C,\ C \in \mathbb{R}$
- $\int \cos x\,dx = \operatorname{sen} x + C,\ C \in \mathbb{R}$
- $\int \sec^2 x\,dx = \operatorname{tg} x + C,\ C \in \mathbb{R}$
- $\int \operatorname{cosec}^2 x\,dx = -\operatorname{cotg} x + C,\ C \in \mathbb{R}$
- $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \operatorname{arcsen} x + C,\ C \in \mathbb{R}$
- $\int \dfrac{1}{1+x^2}\,dx = \operatorname{arctg} x,\ C \in \mathbb{R}$
- $\int e^x\,dx = e^x + C$
- $\int \sec x \cdot \operatorname{tg} x\,dx = \sec x + C,\ C \in \mathbb{R}$
- $\int \operatorname{cosec} x \cdot \operatorname{cotg} x\,dx = -\operatorname{cosec} x$
- $\int a^x\,dx = \dfrac{a^x}{\ln a} + C,\ C \in \mathbb{R}$
- $\int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$

Ficha 2

1)

a) $\int 3x^2 + 5x + 7\,dx = 3\int x^2\,dx + 5\int x\,dx + \int 7\,dx = 3\dfrac{x^3}{3} + 5\dfrac{x^2}{2} + 7x$

$= x^3 + \dfrac{5x^2}{2} + 7x + C,\ C \in \mathbb{R}$

b) $\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \dfrac{x^{4/3}}{4/3} + C,\ C \in \mathbb{R}$

f) $\int \dfrac{1}{x^7}\,dx = \int x^{-7}\,dx = \dfrac{x^{-6}}{-6} + C,\ C \in \mathbb{R}$

6) A primitiva de $f(x) = \dfrac{1}{x^2} + 1$ que se anula em $x = 2$?

$\int \dfrac{1}{x^2} + 1\,dx = \int x^{-2}\,dx + \int 1\,dx = \dfrac{x^{-1}}{-1} + x + C = \underbrace{-\dfrac{1}{x} + x + C}_{F(x)},\ C \in \mathbb{R}$

$F(2) = 0 \Leftrightarrow -\dfrac{1}{2} + 2 + C = 0 \Leftrightarrow \dfrac{3}{2} + C = 0 \Leftrightarrow C = -\dfrac{3}{2}$

$$f(x) = -\frac{1}{x} + x - \frac{3}{2}$$

Integrais indefinidos

$$\int f(g'(x))\, g'(x)\, dx = F(g(x)) + c, \; c \in \mathbb{R}$$

Ex:

- $\int x \cdot \operatorname{sen}(x^2)\, dx = \frac{1}{2}\int 2x \cdot \operatorname{sen}(x^2) = \frac{1}{2}(-\cos(x^2)) + c$
- $\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\int \frac{2x}{x^2+1}\, dx = \frac{\ln|x^2+1|}{2} + c, \; c \in \mathbb{R}$

Ficha 2

2)

a) $$\int \frac{e^{\operatorname{arcsen} x}}{\sqrt{1-x^2}}\, dx = \int \underbrace{\frac{1}{\sqrt{1-x^2}}\, e^{\operatorname{arcsen} x}}_{f(x)'\, e^{f(x)}}\, dx = e^{\operatorname{arcsen} x} + c, \; c \in \mathbb{R}$$

c) $$\int \frac{1}{x}\cos(\ln x)\, dx = \int f'(x)\cos(f(x))\, dx = \operatorname{sen}(f(x)) = \operatorname{sen}(\ln x) + c, c \in \mathbb{R}$$

e) $$\int \frac{e^{3x}}{(e^{3x}-2)^6}\, dx = \int e^{3x}(e^{3x}-2)^{-6}\, dx = \frac{1}{3}\int \underbrace{3e^{3x}}_{u'}\underbrace{(e^{3x}-2)^{-6}}_{u^p} = \frac{1}{3}\,\frac{u^{p+1}}{p+1}$$

$$= \frac{1}{3}\,\frac{(e^{3x}-2)^{-5}}{-5} = \frac{-(e^{3x}-2)^{-5}}{15} + c, \; c \in \mathbb{R}$$

g) $$\int \frac{1}{x\sqrt{1-\ln^2 x}}\, dx = \int \frac{1}{x}\cdot\frac{1}{\sqrt{1-\ln^2 x}}\, dx = \int \frac{\frac{1}{x}}{\sqrt{1-\ln^2 x}}\, dx = \int \frac{u'}{\sqrt{1-u^2}}\, dx$$

$$= \operatorname{arcsen} u = \operatorname{arcsen}(\ln x) + c, \; c \in \mathbb{R}$$

h) $$\int e^x\sqrt{1+e^x}\, dx = \int e^x(1+e^x)^{1/2}\, dx = \int u' \cdot u^p\, dx = \frac{u^{p+1}}{p+1} = \frac{(1+e^x)^{3/2}}{3/2} + c, \; c \in \mathbb{R}$$

k) $$\int \frac{1+\cos x}{x+\operatorname{sen} x}\, dx = \int \frac{u'}{u} = \ln|u| = \ln(x+\operatorname{sen} x) + c, \; c \in \mathbb{R}$$

o) $$\int \frac{\cos(\ln(x^2))}{x}\, dx = \int \frac{1}{x}\cos(\ln x^2)\, dx = \frac{1}{2}\int \overset{u'}{\frac{2}{x}}\, \overset{\cos(u)}{\cos(\ln x^2)}\, dx = \frac{\operatorname{sen}(\ln x^2)}{2} + c$$

$$(\ln x^2)' = \frac{2x}{x^2} = \frac{2}{x}$$

p) $\displaystyle\int \frac{1}{e^x+9e^{-x}}\,dx = \int \frac{1}{e^x(1+9e^{-2x})}\,dx = \int \frac{e^{-x}}{1+(3e^{-x})^2}\,dx = -\frac{1}{3}\int \frac{-3e^{-x}}{1+(3e^{-x})^2}\,dx$

↑ Fizemos diferente de como tá nas soluções mas é equivalente

$\displaystyle = -\frac{1}{3}\int \frac{u'}{1+u^2} = \frac{1}{3}\operatorname{arctg} u = -\frac{1}{3}\operatorname{arctg}(3e^{-x}) + C$

q) $\displaystyle\int \frac{\operatorname{sen}(\operatorname{arctg} x)}{1+x^2}\,dx = \int \underbrace{\frac{1}{1+x^2}}_{\operatorname{arctg}' x} \cdot \operatorname{sen}(\operatorname{arctg} x)\,dx = -\cos(\operatorname{arctg} x) + C,\ C \in \mathbb{R}$

r) $\displaystyle\int \frac{\arccos(x) - x}{\sqrt{1-x^2}}\,dx = \int \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx =$

$\displaystyle = -\int -\frac{1}{\sqrt{1-x^2}}\cdot\arccos(x)\,dx - \int x(1-x^2)^{-1/2}\,dx = -\frac{(\arccos)^2}{2} - \left(-\frac{1}{2}\right)\int -2x(1-x^2)^{-1/2}\,dx$

$\displaystyle = -\frac{(\arccos)^2}{2} + \frac{1}{2}\cdot\frac{(1-x^2)^{1/2}}{1/2} = -\frac{\arccos^2 x}{2} + \sqrt{1-x^2} + C,\ C \in \mathbb{R}$

Primitivação por partes

Sejam f e g duas funções diferenciáveis:

$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + g'(x)\cdot f(x) \Leftrightarrow$

$\Leftrightarrow f'(x)\,g(x) = (f(x)\,g(x))' - f(x)\,g'(x)$

Integrando ambos os membros:

$\int f'(x)\,g(x)\,dx = \int (f(x)\,g(x))'\,dx - \int f(x)\,g'(x)\,dx \Leftrightarrow$

$\Leftrightarrow \int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx$

Ex: $\displaystyle\int \underbrace{x}_{f}\cdot\underbrace{\cos(x)}_{g'}\,dx = \underbrace{x}_{f}\,\underbrace{\operatorname{sen} x}_{g} - \int \underbrace{x'}_{f'}\,\underbrace{\operatorname{sen} x}_{g}\,dx = x\operatorname{sen} x - \int \operatorname{sen} x\,dx$

$= x\operatorname{sen} x + \cos x + C,\ C \in \mathbb{R}$

*

Dicas Miniteste

Domínio Contradomínio

Lagrange + Rolle + Bolzano + Weierstrass – compacto → tem max e mínimo

$\displaystyle f'(c) = \frac{f(b)-f(a)}{b-a}$

f(x) tem 2 zeros – f'(x) tem 1 zero lá

$f(x) = 5x^9 + 2x - \operatorname{sen} x$ nº Max de zeros?

$f'(x) = \underbrace{45x^8}_{\geq 0} + 2 - \underbrace{\cos x}_{\in[-1,1]} > 0 \rightarrow$ estritamente crescente $\rightarrow f(x)$ tem 1 zero

$f(0)=0$

---

*

Ficha 2

Ex 9.

d) $\int e^{-3x}(2x+3)\,dx = \int v' \cdot u = v \cdot u - \int v \cdot u' = -\frac{1}{3}e^{-3x}(2x+3) - \int -\frac{1}{3}e^{-3x} \cdot 2\,dx$

$v' = e^{-3x}\,dx \qquad u = 2x+3$

$v = -\frac{1}{3}e^{-3x} \qquad u' = 2$

$= -\frac{1}{3}e^{-3x}(2x+3) - \frac{2}{3}\int -e^{-3x}\,dx = \frac{1}{3}e^{-3x}(2x+3) - \frac{2}{9}\int -3e^{-3x}\,dx$

$= -\frac{1}{3}e^{-3x}(2x+3) - \frac{2}{9}e^{-3x} + C, \quad C \in \mathbb{R}$

m) $\int x \arcsen(x^2)\,dx = \int u' \cdot v = u \cdot v - \int u \cdot v' = \frac{x^2}{2} \cdot \arcsen(x^2) - \int \frac{x^2}{2} \cdot \frac{2x}{\sqrt{1-x^4}}$

$v = \arcsen(x^2) \qquad u' = x$

$v' = \frac{2x}{\sqrt{1-x^4}} \qquad u = \frac{x^2}{2}$

$= \frac{x^2 \cdot \arcsen(x^2)}{2} - \int x^3 \cdot (1-x^4)^{-1/2}\,dx$

$= \frac{x^2 \cdot \arcsen(x^2)}{2} + \frac{1}{4}\int \underbrace{-4x^3}_{u'} \underbrace{(1-x^4)^{-1/2}}_{u^p,\ p=-1/2}\,dx$

$= \frac{x^2 \cdot \arcsen(x^2)}{2} + \frac{1}{4} \cdot \frac{(1-x^4)^{1/2}}{1/2} = \frac{x^2 \cdot \arcsen(x^2)}{2} + \frac{(1-x^4)^{1/2}}{2}$

$= \frac{x^2 \cdot \arcsen(x^2) + \sqrt{1-x^4}}{2} + C, \quad C \in \mathbb{R}$

e) $\int \ln^2 x\,dx = \int 1 \cdot \ln^2 x = x \cdot \ln^2 x - \int x \cdot 2\ln x \cdot \frac{1}{x} = x \cdot \ln^2 x - 2\int \ln x\,dx$

$u = \ln^2 x \qquad v' = 1$

$u' = 2\ln x \cdot \frac{1}{x} \qquad v = x\,dx$

$u = \ln x \qquad v' = 1$

$u' = \frac{1}{x} \qquad v = x$

$= x \cdot \ln^2 x - \ln x \cdot x - 2\int \frac{1}{x} \cdot \cancel{x} = x\ln^2 x - \ln x \cdot x + 2x$

i) $\int \cos(\ln(x))\,dx = \int \underbrace{1}_{f'} \cdot \underbrace{\cos(\ln(x))}_{g}\,dx = x \cos(\ln(x)) + \int x \cdot \frac{1}{x} \sen(\ln(x))\,dx$

$f' = 1 \qquad g = \cos(\ln(x))$

$f = x \qquad g' = \frac{1}{x}(-\sen(\ln(x)))$

$= x \cdot \cos(\ln(x)) + \int 1 \cdot \sen(\ln(x))\,dx =$

$= x \cdot \cos(\ln(x)) + x \cdot \sen(\ln(x)) - \int x \cdot \frac{1}{x} \cdot \cos((\ln x))\,dx$

$$= x \cdot \cos(\ln(x)) + x\,\text{sen}(\ln(x)) - \int \cos(\ln(x))\,dx$$

$$= \int \cos(\ln(x))\,dx = x\cos(\ln(x)) + x\,\text{sen}(\ln(x)) - \int \cos(\ln(x))\,dx \;(=)$$

$$(=)\; 2\int \cos(\ln(x))\,dx = x\cos(\ln(x)) + x\,\text{sen}(\ln(x))$$

$$(=)\; \int \cos(\ln(x))\,dx = \frac{1}{2}\left(x\cos(\ln(x)) + x\,\text{sen}(\ln(x))\right)$$

o) $$\int \text{arctg}\, x\,dx = \int \underbrace{1}_{f'} \cdot \underbrace{\text{arctg}\, x}_{g}\,dx = \underbrace{x}_{f} \cdot \underbrace{\text{arctg}\, x}_{g} - \int \underbrace{x}_{f} \cdot \underbrace{\frac{1}{1+x^2}}_{g'}\,dx$$

$$= x\,\text{arctg}\, x - \int \frac{x}{1+x^2}\,dx = x \cdot \text{arctg}\, x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = x\,\text{arctg}\, x - \frac{1}{2}\ln|1+x^2| +$$

Slide 12

4) $$\int \sec x\,dx = \int \frac{\sec x \cdot (\sec x + \text{tg}\, x)}{\text{tg}\, x + \sec x}\,dx = \int \frac{(\sec x + \text{tg}\, x)'}{\sec x + \text{tg}\, x} =$$

$$= \ln|\sec x + \text{tg}\, x| + C,\; C \in \mathbb{R}$$

Nota: $$(\sec x + \text{tg}\, x)' = \sec' x + \text{tg}' x = \sec x \cdot \text{tg}\, x + \sec^2 x$$
$$= \sec x\,(\text{tg}\, x + \sec x)$$

5) $$\int \text{cosec}\, x\,dx = -\int \frac{-\text{cosec}\, x\,(\text{cotg}\, x + \text{cosec}\, x)}{\text{cotg}\, x + \text{cosec}\, x}\,dx = \int \frac{(\text{cosec}\, x + \text{cotg}\, x)'}{\text{cosec}\, x + \text{cotg}\, x}\,dx$$

$$= \ln|\text{cosec}\, x + \text{cotg}\, x| + C,\; C \in \mathbb{R}$$

Nota: $$(\text{cosec}\, x + \text{cotg}\, x)' = -\text{cosec}\, x \cdot \text{cotg}\, x - \text{cosec}^2 x = -\text{cosec}\, x\,(\text{cotg}\, x + \text{cosec}\, x)$$

Primitivação de Funções Trigonométricas

- $$\int \text{sen}\, x \cdot \cos^n x\,dx = -\int \underbrace{-\text{sen}\, x}_{\cos' x} \cos^n x\,dx = -\frac{\cos^{n+1} x}{n+1} + C,\; C \in \mathbb{R}$$
- $$\int \underbrace{\cos x}_{\text{sen}' x} \cdot \text{sen}^n x\,dx = \frac{\text{sen}^{n+1}}{n+1} + C,\; C \in \mathbb{R}$$
- $$\int \sec^2 \text{tg}^n x\,dx = \frac{\text{tg}^{n+1} x}{n+1} + C,\; C \in \mathbb{R}$$

> Potências ímpares de seno e cosseno

- $$\int \text{sen}^3 x\,dx = \int \text{sen}\, x \cdot \text{sen}^2 x\,dx = \int \text{sen}\, x \cdot (1 - \cos^2 x)\,dx =$$

$$= \int \text{sen}\, x\,dx - \int \underbrace{\text{sen}\, x}_{-\cos'} \cdot \cos^2 x\,dx = -\cos x + \frac{\cos^3 x}{3} + C,\; C \in \mathbb{R}$$

coloca-se em evidencia uma unidade da potência ímpar e usa-se a formula fundamental da trigonometria $sen^2 x = 1 - cos^2 x \Leftrightarrow cos^2 x = 1 - sen^2 x$

> Potências pares de seno e cosseno

- $\int sen^4 x \, dx = \int (sen^2 x)^2 dx = \int \left(\frac{1 - cos\, x}{2}\right)^2 dx = \frac{1}{4} \int 1 - 2cos(2x) + cos^2(2x) dx$

$= \frac{1}{4} \int 1 - 2 cos(2x) + \frac{1 + cos(4x)}{2} dx =$

$= \frac{1}{4} \left(\int 1 \, dx - \int 2 cos(2x) dx + \frac{1}{2} \int (1 + cos(4x)) dx \right)$

Usam-se as fórmulas $cos^2(x) = \frac{1 + cos(2x)}{2}$ e $sen^2(x) = \frac{1 + sen(2x)}{2}$

> Fatores do tipo $sen(nx)$ ou $cos(nx)$

- $\int sen(3x) cos(4x) \, dx = \int \frac{1}{2} (sen(3x + 4x) + sen(3x - 4x)) \, dx =$

$= \frac{1}{2} \int sen(7x) + sen(-x) \, dx = \frac{1}{2} \int sen(7x) - \frac{1}{2} \int sen\, x \, dx =$

$= \frac{1}{2} \cdot \frac{1}{7} \int \underbrace{7}_{(7x)'} sen(7x) \, dx + \frac{1}{2} cos\, x = \frac{1}{14}(-cos(7x)) + \frac{1}{2} cos\, x + C, \; C \in \mathbb{R}$

$= -\frac{1}{14} cos(7x) + \frac{1}{2} cos\, x + C, \; C \in \mathbb{R}$

Usam-se as fórmulas
- $sen\, x \, sen\, y = \frac{1}{2}\left(cos(x-y) - cos(x+y)\right)$
- $cos\, x \, cos\, y = \frac{1}{2}\left(cos(x+y) + cos(x-y)\right)$
- $sen\, x \, cos\, y = \frac{1}{2}\left(sen(x+y) + sen(x-y)\right)$

> Potências (pares ou ímpares) de $tg\, x$ ou $cotg\, x$

- $\int tg^4 x \, dx = \int tg^2 x \cdot tg^2 x \, dx = \int (sec^2 x - 1) tg^2 x \, dx =$

$= \int sec^2 x \cdot tg^2 x - tg^2 x \, dx = \int (tg\, x)' \cdot tg^2 x - \int (sec^2 x - 1) \, dx$

$= \frac{tg^3 x}{3} - \int sec^2 x \, dx + \int 1 \, dx = \frac{tg^3 x}{3} - tg\, x + x + C, \; C \in \mathbb{R}$

coloca-se $tg^2 x$ ou $cotg^2 x$ em evidencia e usam-se as fórmulas $tg^2 x = sec^2 x - 1$ e $cotg^2 x = cosec^2 x - 1$

Ex Ficha 2

12)

e) $\int sen^5 x \cdot cos^2 x \, dx = \int sen\, x \cdot sen^4 x \cdot cos^2 x \, dx =$

$= \int sen\, x \, (sen^2 x)^2 \cdot cos^2 x \, dx = \int sen\, x \, (1 - cos^2 x)^2 \cdot cos^2 x \, dx =$

$= \int sen\, x \, (1 - 2cos^2 x + cos^4 x) \, cos^2 x \, dx = \int sen\, x \, (cos^2 x - 2cos^4 x + cos^6 x) dx$

$= -\int -sen\, x \cdot cos^2 x \, dx + 2\int -sen\, x \cdot cos^4 x \, dx - \int -sen\, x \cdot cos^6 x \, dx$

$= -\frac{cos^3 x}{3} + 2\frac{cos^5 x}{5} - \frac{cos^7 x}{7} + C, \; C \in \mathbb{R}$

u) $\int cos\, x \cdot cos(5x) \, dx = \int \frac{1}{2} (cos(x + 5x) + cos(x - 5x)) dx$

$= \frac{1}{2} \int cos(6x) + cos(-4x) \, dx = \frac{1}{2} \cdot \frac{1}{6} \int 6\, cos(6x) + \frac{1}{2} \cdot \frac{1}{4} \int 4\, cos(4x) \, dx$

$= \frac{1}{2} sen(6x) + \frac{1}{8} sen(4x) + C, \; C \in \mathbb{R}$

Primitivação por substituição

Seja $f: I \to \mathbb{R}$ primitivável com primitiva F

$\varphi: J \to \mathbb{R}$ diferenciável $\varphi(J) \subset I$

$(F \circ \varphi)$ é diferenciável em J e $\forall_t \in J, (F \circ \varphi)'(t) = F'(\varphi(t)) \, \varphi'(t)$

$= f(\varphi(t)) \varphi'(t) = (f \circ \varphi)(t) \, \varphi'(t)$

ou seja, $F \circ \varphi$ é uma primitiva de $(f \circ \varphi) \varphi'$

Portanto $\int f(\varphi(t)) \, \varphi'(t) \, dt = F(\varphi(t)) + C, \; C \in \mathbb{R}$

Seja $H = F \circ \varphi$ primitiva de $(f \circ \varphi)\varphi'$. Admitimos que φ é invertível podemos concluir que

$H = F \circ \varphi \quad , \quad H \circ \varphi^{-1} = F$

Portanto $H \circ \varphi^{-1}$ é uma primitiva de f

Ex

- $\int \frac{sen(\sqrt{x})}{\sqrt{x}} dx = \int \frac{1}{\sqrt{x}} sen(\sqrt{x}) \, dx = 2 \int \frac{1}{2\sqrt{x}} sen(\sqrt{x}) \, dx$

$= 2 \int (\sqrt{x})' sen(\sqrt{x}) \, dx = -2\, cos(\sqrt{x}) + C, \; C \in \mathbb{R}$

Relembrar primitivas quase imediatas

$\int g'(x) \cdot f(g(x)) \, dx = F(g(x)) + C$

• $\int \operatorname{sen}(\sqrt{x})\,dx \xrightarrow{\text{primitivação por substituição}} \int \operatorname{sen}(t) \cdot 2t\,dt =$

$t = \sqrt{x}$, $t \geq 0 \Leftrightarrow$
$\Leftrightarrow t^2 = x \rightarrow \varphi(t) = t^2$
$\varphi(t)$ é invertível, $t \geq$

$x = \varphi(t)$
$\frac{dx}{dt} = \varphi'(t) \Leftrightarrow$
$\Leftrightarrow dx = \varphi'(t)\,dt$
$\Leftrightarrow dx = 2t\,dt$

$$= 2\int \underbrace{t}_{f} \cdot \underbrace{\operatorname{sen} t}_{g'}\,dt = 2\left[t(-\cos t) - \int \underbrace{1}_{f'} \cdot \underbrace{(-\cos t)}_{g}\,dt\right] =$$

$$= 2\left(-t\cos t + \int \cos t\,dt\right) = -2t\cos t + 2\operatorname{sen} t + c,\ c \in \mathbb{R}$$

$$= -2\sqrt{x}\cos(\sqrt{x}) + 2\operatorname{sen}(\sqrt{x}) + c,\ c \in \mathbb{R}$$

Passos da integração por substituição

1) Identificar a expressão a substituir

$$x = \varphi(t)$$

2) Identificar o Domínio de φ e garantir que $\varphi(t)$ é invertível

3) Substituir no integral

$$x \rightarrow \varphi(t), \quad dx \rightarrow \varphi'(t)\,dt$$

4) Calcular o integral obtido

5) Voltar à variável original

$$t \longrightarrow \varphi^{-1}(x)$$

Exs:

$$\int \frac{\operatorname{sen}(\sqrt{x})}{\sqrt{x}}\,dx \xrightarrow{\text{Substituição}} \int \frac{\operatorname{sen} t}{\cancel{t}}\,2\cancel{t}\,dt = \int 2\operatorname{sen} t\,dt$$

$\sqrt{x} = t \Leftrightarrow x = t^2$, $t > 0$
$\underbrace{\varphi(t) = t^2}_{\text{invertível porque } t > 0}$
$dx = \varphi' t\,dt$
$dx = 2t\,dt$

$$= 2\int \operatorname{sen} t\,dt = -2\cos t + c = -2\cos(\sqrt{x}) + c,\ c \in \mathbb{R}$$

Ficha 2

10)

f) $\int x^2 \sqrt{1-x}\, dx = \int (1-t)^2 . t . -2t\, dt = \int (1-2t^2+t^4).t(-2t)\,dt$

$= \int (-2t^2+4t^4-2t^6)\,dt = -2\int t^2 + 4\int t^4 - 2\int t^6$

$= -2\frac{t^3}{3} + 4\frac{t^5}{5} - 2\frac{t^7}{7} + C$

$= -2\frac{(\sqrt{1-x})^3}{3} + 4\frac{(\sqrt{1-x})^5}{5} - 2\frac{(\sqrt{1-x})^7}{7} + C,\ C \in \mathbb{R}$

Substituição

$t = \sqrt{1-x}$, $t \geq 0$

$t^2 = 1-x \Leftrightarrow x = \underbrace{1-t^2}_{\varphi(t)}$

$\varphi(t) = 1-t^2,\ t \geq 0$

$\varphi'(t) = -2t \rightarrow$ sempre negativo

$\varphi(t)$ é estrita/ decrescente logo invertível

$dx = -2t\,dt$

n) $\int x(2x+5)^{10}\,dx = x.\frac{(2x+5)^{11}}{22} - \int \frac{(2x+5)^{11}}{22} . 1\,dx = \frac{x(2x+5)^{11}}{22} - \frac{1}{22}\int (2x+5)^{11}$

$f = x$ $\quad f' = 1$

$g' = (2x+5)^{10}$ $\quad g = \int (2x+5)^{10} = \frac{1}{2}\int 2(2x+5)^{10} = \frac{1}{2} . \frac{(2x+5)^{11}}{11}$

$= \frac{x(2x+5)^{11}}{22} - \frac{1}{22}\frac{(2x+5)^{12}}{12} + C, \quad C \in \mathbb{R}$

Primitivação por partes

ou

por substituição

$\int x(2x+5)^{10}\,dx = *$

seja $t = 2x+5 \Leftrightarrow \underbrace{\frac{t-5}{2}}_{\varphi(t)} = x$, $t \in \mathbb{R}$

$\varphi'(t) = \frac{2}{2^2} = \frac{1}{2}$, $dx = \frac{1}{2}dt$

$* = \int \frac{t-5}{2} . t^{10} . \frac{1}{2}dt = \frac{1}{4}\int t-5 . t^{10}dt = \frac{1}{4}\int t^{11} - 5t^{10}\,dt =$

$= \frac{1}{4}\left(\frac{t^{12}}{12} - \frac{5t^{11}}{11}\right) + C, \quad C \in \mathbb{R}$

$= \frac{1}{4}\left(\frac{(2x+5)^{12}}{12} - \frac{5(2x+5)^{11}}{11}\right) + C = \frac{1}{4}\left(\frac{(2x+5)^{12}}{12} - \frac{(10x+25)^{11}}{11}\right) + C,\ C \in \mathbb{R}$

p) $\int \frac{\ln(x)}{x\sqrt{1+\ln x}}\,dx = \int \frac{t}{e^t\sqrt{1+t}}\,e^t\,dt = \int \frac{t}{\sqrt{1+t}}\,dt =$

$\left.\begin{array}{l} t = \ln x \Leftrightarrow e^t = x \;,\; t \in \mathbb{R} \\ dx = e^t dt \end{array}\right]$

$\left[\begin{array}{l} u = \sqrt{1+t} \Leftrightarrow u^2 = 1+t \\ \quad \Leftrightarrow u^2 - 1 = t \end{array}\right.$ → como $u>0$, u^2-1 é invertível para $u>0$

$t' = 2u\,du$

$= \int \frac{u^2-1}{u}\,2u\,du = 2\int u^2 - 1\,du = 2\frac{u^3}{3} - u + C = 2\frac{\sqrt{1+t}^3}{3} - \sqrt{1+t} + C$

$= 2\frac{\sqrt{1+\ln x}^3}{3} - \sqrt{1+\ln x} + C, C \in \mathbb{R}$

q) $\int \frac{1+tg^2 x}{\sqrt{tg(x)-1}}\,dx = \int \frac{1+t^2}{\sqrt{t-1}} \cdot \frac{1}{1+t^2}\,dt = \int \frac{1}{\sqrt{t-1}}\,dt = \int (t-1)^{-1/2}\,dt =$

$= \frac{(t-1)^{1/2}}{\frac{1}{2}} + C$

$t = tg\, x \Leftrightarrow arctg\, t = x \;,\; x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$x' = \frac{1}{1+t^2}\,dt$

$= \frac{(tg\,x - 1)^{1/2}}{1/2} + C = 2\sqrt{tg\,x - 1} + C, C \in \mathbb{R}$

Substituições trigonométricas

- $\sqrt{a^2 + x^2}$, $a \in \mathbb{R}^+$

Relembrando:

$\times a^2 \left(\begin{array}{l} 1 + tg^2 x = sec^2 x \\ a^2 + (a + tg\, x)^2 = (a sec\, x)^2 \end{array}\right.$

Mudança de variável:

$x = a.\,tg(t) \Leftrightarrow arctg\left(\frac{x}{a}\right) = t \quad , t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\varphi(t) = a\, tg(t) \Leftrightarrow \varphi'(t) = a\, sec^2 t\, dt$

Ex 10

d) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \left| \begin{array}{l} a^2 = 4 \Leftrightarrow a = 2 \;(a>0) \\ \text{Mudança de variável} \\ x = a.tg\,t \Leftrightarrow x = 2tg\,t \\ x' = 2sec^2 t\; dt \end{array} \right| = \int \frac{1}{4tg^2 t\sqrt{4(tg^2 t + 1)}} \cdot 2sec^2 t\, dt$

$= \int \frac{1}{4tg^2 t \cdot \sqrt{4sec^2 t}} \cdot 2\,sec^2 t\; dt = \int \frac{2\,sec^2 t}{4tg^2 t\,.\,2sec\, t}\,dt = \int \frac{sec\, t}{4\,tg^2\, t}\,dt$

$= \frac{1}{4}\int \frac{1}{cos\,t} \cdot \frac{cos^2 t}{sen^2 t}\,dt = \frac{1}{4}\int \frac{cos\, t}{sen^2 t}\,dt = \frac{1}{4}\int \frac{cos\, t}{sen\, t} \cdot \frac{1}{sen\, t}\,dt =$

$$= \frac{1}{4}\int \operatorname{cotg} t \cdot \operatorname{cosec} t \, dt = -\frac{1}{4}\operatorname{cosec} t + c$$

$$= -\frac{1}{4}\sqrt{1+\operatorname{cotg}^2 t} = -\frac{1}{4}\sqrt{1+\frac{1}{tg^2 t}} = -\frac{1}{4}\sqrt{1+\frac{1}{\left(\frac{x}{2}\right)^2}}$$

$$= -\frac{1}{4}\sqrt{1+\frac{4}{x^2}} = -\frac{1}{4}\sqrt{\frac{x^2+4}{x^2}} + c, \quad c \in \mathbb{R}$$

$1+\operatorname{cotg}^2 t = \operatorname{cosec}^2 t \iff$

$\iff \operatorname{cosec} t = \sqrt{1+\operatorname{cotg}^2 t}$

$1-\cos^2\alpha = sen^2\alpha \longrightarrow a^2-(sen^2 x) = a^2\cos^2 x$

$1+tg^2\alpha = sec^2\alpha \longrightarrow a^2+(a\,tg\,x)^2 = a^2 sec^2 x$

$sec^2\alpha - 1 = tg^2\alpha \longrightarrow a^2 sec^2 x - a^2 = a^2 tg^2 x$

$\sqrt{a^2-x^2} \longrightarrow x = a\,sen\,t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sqrt{a^2+x^2} \longrightarrow x = a\,tg\,t, \quad t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$\sqrt{x^2-a^2} \longrightarrow x = a\,sec\,t, \quad t \in \left]0, \frac{\pi}{2}\right[$

$\sqrt{a^2-b^2x} = \sqrt{b^2\left(\frac{a^2}{b^2}-x^2\right)} = b\sqrt{\frac{a^2}{b^2}-x^2} = b\sqrt{\left(\frac{a}{b}\right)^2-x^2} \longrightarrow x = \frac{a}{b}\,sen\,t$

10)

j) $$\int \frac{1}{x^2\sqrt{9-x^2}}\,dx \xrightarrow[\substack{\varphi(t) = 3\,sen\,t \text{, inventível} \\ dx = 3.\cos t\,dt}]{\substack{x=\sqrt{9}\,sen\,t,\ t\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\\ \iff x = 3\,sen\,t}} \int \frac{1}{9\,sen^2 t\sqrt{9-9\,sen^2 t}}\, 3.\cos t\,dt$$

$$= \int \frac{1}{9\,sen^2 t\sqrt{9(1-sen^2 t)}}\,3\cos t\,dt = \int \frac{1}{9\,sen^2 t \cdot 3\sqrt{\cos^2 t}}\,3\cos t\,dt =$$

$$= \int \frac{1}{9\,sen^2 t \cdot 3\cos t} \cdot 3\cos t\,dt = \int \frac{1}{9}\operatorname{cosec}^2 t\,dt = -\frac{1}{9}\operatorname{cotg} t + c$$

$$= -\frac{1}{9}\sqrt{\operatorname{cosec}^2 t - 1} + c = -\frac{1}{9}\sqrt{\left(\frac{3}{2}\right)^2 - 1} + c$$

$$= -\frac{1}{9}\sqrt{\frac{9-x^2}{x^2}} + c, \quad c \in \mathbb{R}$$

$x = 3\,sen\,t \iff sen\,t = \frac{x}{3}$

$\iff \operatorname{cosec} t = \frac{x}{3}$

$\operatorname{cotg}^2 t + 1 = \operatorname{cosec}^2 t \iff$

$\iff \operatorname{cotg}^2 t = \operatorname{cosec}^2 t - 1 \iff$

$\iff \operatorname{cotg} t = \sqrt{\operatorname{cosec}^2 t - 1}$

b) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx \xrightarrow[\underbrace{x=1\sec t}_{\varphi(t)},\ t\in]0,\frac{\pi}{2}[]{} \int \frac{1}{\sec t\sqrt{\sec^2 t-1}}\sec t\cdot tg\,t\,dt$

$\varphi'(t)=\sec t\, tg\, t$

$dx=\sec t\, tg\, t\, dt$

$$= \int \frac{1}{\sec t\sqrt{tg^2 t}}\sec t\, tg\, t\, dt = \int \frac{1}{\sec t\cdot tg\, t}\cdot \sec t\, tg\, t\, dt = \int 1\, dt = t+c$$

$$= \arccos\left(\frac{1}{x}\right)+c,\ c\in\mathbb{R}$$

$x=\sec t$

$\Leftrightarrow x=\frac{1}{\cos t}$

$\Leftrightarrow t=\arccos\left(\frac{1}{x}\right)$

Integração de funções racionais

$f(x)=\frac{N(x)}{D(x)}$, N e D polinómios

grau N > grau D → fração imprópria

grau D > grau N → fração própria

Só conseguimos integrar frações próprias!

$N(x)=D(x)\,Q(x)+R(x) \Leftrightarrow \frac{N(x)}{D(x)}=Q(x)+R(x)$

Ex:

$$\int \frac{x^3+2x^2-3}{x^2-1}\,dx = \int \Big(\underbrace{x+2}_{\text{polinomio}}+\overbrace{\frac{x-1}{x^2-1}}^{\text{fração própria}}\Big)\,dx$$

$$\begin{array}{rrr|l} x^3+2x^2 & & -3 & x^2-1 \\ \cline{4-4} -x^3 & +x & & x+2 \\ \hline & 2x^2+x & -3 & \\ & -2x^2 & +2 & \\ \cline{2-3} & x & -1 & \end{array}$$

Fração Simples: $\frac{A}{(x-a)^p}$ ou $\frac{Bx+C}{(x^2+\beta x+\gamma)^q}$, $\frac{x-1}{x^2+1}$ pq x^2+1 não tem raizes reais

Fração não simples: $\frac{x-1}{x^2-1}$ pq x^2-1 admite raizes reais

Para primitivar, as frações próprias devem ser convertidas na soma de frações simples

Ex: $\dfrac{2x^2+2x^4+1}{x^2+x^4}$ grau(N) = grau(D) → n é propria → $\dfrac{2(x^2+x^4)+1}{x^2+x^4} = 2 + \dfrac{1}{x^2+x^4}$

polinomio de grau 0
+ frasão própria

$\dfrac{1}{x^2+x^4}$, Como converter a frasão própria na soma de frações simples:

1) fatorizar o denominador:

$$x^2+x^4 = x^2(1+x^2) \quad *$$

• Cada fator do tipo $(x-a)^\alpha$ vai dar origem a somas de frações simples:

$$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \cdots + \frac{D}{(x-a)^\alpha}$$

• Cada fator do tipo $(x^2+bx+c)^2$ origina a soma de frações simples:

$$\frac{Ax+B}{x^2+bx+c} + \frac{Cx+D}{(x^2+bx+c)^2} + \cdots + \frac{Fx+E}{(x^2+bx+c)^\alpha}$$

$$* \quad \frac{1}{x^2(x^2+1)} = \underset{(x(x^2+1))}{\frac{A}{x}} + \underset{(x^2+1)}{\frac{B}{x^2}} + \underset{(x^2)}{\frac{Cx+D}{x^2+1}}$$

$$= \frac{Ax(x^2+1)+B(x^2+1)+Cx^3+Dx^2}{x^2(x^2+1)} = \frac{A(x^3+x)+B(x^2+1)+Cx^3+Dx^2}{x^2(x^2+1)}$$

$$= \frac{(A+C)x^3+(B+D)(x^2)+Ax+B}{x^2(x^2+1)} = \frac{1}{x^2(x^2+1)}$$

$$\therefore \begin{cases} A+C=0 \\ B+D=0 \\ A=0 \\ B=1 \end{cases} \Leftrightarrow \begin{cases} C=0 \\ D=-1 \\ A=0 \\ B=1 \end{cases} \Rightarrow \frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}$$

11)

d) $\displaystyle\int \frac{1}{x^3+8}\,dx$

frasão propria
=
denominador
n tá fatorizado

$-2^3+8=0 \rightarrow$

$$\begin{array}{c|cccc} & 1 & 0 & 0 & 8 \\ -2 & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

$\rightarrow x^3+8=(x+2)(x^2+2x+4)$

$x^2+2x+4=0 \Leftrightarrow x = \dfrac{2 \pm \sqrt{2^4-4.4}}{2}$ impossivel

$$\frac{1}{x^3+8} = \frac{1}{(x+2)(x^2-2x+4)}$$

$$\frac{1}{(x+2)(x^2-2x+4)} = \underset{(x^2+2x+4)}{\frac{A}{x+2}} + \underset{(x+2)}{\frac{Bx+C}{x^2-2x+4}} = \frac{A(x^2-2x+4)+Bx(x+2)+C(x+2)}{(x+2)(x^2-2x+4)}$$

$$= \frac{A(x^2-2x+4)+B(x^2+2x)+C(x+2)}{(x+2)(x^2-2x+4)} = \frac{(A+B)(x^2)+(-2A+2B+C)(x)+(4A+2C)}{(x+2)(x^2-2x+4)}$$

$$\begin{cases} A+B=0 \\ -2A+2B+C=0 \\ 4A+2C=1 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ 2B+2B=-C \\ \end{cases} \Leftrightarrow \begin{cases} 4B=-C \\ -4B-8B=1 \end{cases} \begin{cases} \\ -12B=1 \end{cases}$$

$$\begin{cases} A=1/12 \\ C=1/3 \\ B=-1/12 \end{cases} \rightarrow \int \frac{1}{x^3+8}\,dx = \int \frac{1/12}{x+2} + \frac{1/12\,x+1/3}{x^2-2x+4}\,dx =$$

$$= \frac{1}{12}\int \frac{1}{x+2}\,dx - \frac{1}{12}\int \frac{x-4}{x^2-2x+4}\,dx = \frac{1}{12}\ln|x+2| - \frac{1}{12}\left(\int \frac{x}{x^2-2x+4}\,dx + \int \frac{-4}{x^2-2x+4}\,dx\right)$$

$$= \frac{1}{12}\ln|x+2| - 1/12\left(\int \frac{x-1}{x^2-2x+4}\,dx + \int \frac{-3}{x^2-2x+4}\,dx\right)$$

$$= \frac{1}{12}\ln|x+2| - 1/12\left(\frac{1}{2}\int \frac{2x-2}{x^2-2x+4}\,dx + \int \frac{-3}{x^2-2x+4}\,dx\right)$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln|x^2-2x+4| - \frac{1}{12}\int \frac{-3}{x^2-2x+4}\,dx$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln(x^2-2x+4) + \frac{1}{4}\int \frac{1}{(x-1)^2+3}\,dx$$

C.A

$$x^2-2x+4 = (x^2-2x)+4 = (x^2-2x+1)-1+4 = (x-1)^2+3$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln(x^2-2x+4) + \frac{1}{4}\int \frac{1}{3\left(\left(\frac{x-1}{\sqrt{3}}\right)^2+1\right)}\,dx =$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln(x^2-2x+4) + \frac{1}{12}\times\sqrt{3}\int \frac{1/\sqrt{3}}{3\left(\left(\frac{x-1}{\sqrt{3}}\right)^2+1\right)}\,dx =$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln(x^2-2x+4) + \frac{\sqrt{3}}{12}\operatorname{arctg}\left(\frac{x-1}{\sqrt{3}}\right) + c, \quad c \in \mathbb{R}$$

11)

n) $\int \frac{1}{x(x^2+1)^2}\,dx$

· Fração própria

· denominador tá fatorizado

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} =$$

$$\frac{A(x^2+1)^2 + Bx(x(x^2+1)) + C(x(x^2+1)) + D(x^2) + Ex}{x(x^2+1)^2} =$$

$$= \frac{A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + D(x^2) + Ex}{x(x^2+1)^2} =$$

$$\begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ E=0 \\ A=1 \end{cases} \Leftrightarrow \begin{cases} B=-1 \\ C=0 \\ 2-1+D=0 \\ E=0 \\ A=1 \end{cases} \quad \begin{cases} B=-1 \\ C=0 \\ D=-1 \\ E=0 \\ A=1 \end{cases} \rightarrow \int \frac{1}{x(x^2+1)^2}\,dx = \int \frac{1}{x} + \frac{-1x}{x^2+1} + \frac{-1x}{(x^2+1)^2}$$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx = \ln|x| - \frac{1}{2}\int \frac{2x}{(x^2+1)}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \int x(x^2+1)^{-2}\,dx = \ln|x| - \frac{\ln|x^2+1|}{2} - \frac{1}{2}\int 2x(x^2+1)^{-2}\,dx$$

$$= \ln|x| - \frac{\ln|x^2+1|}{2} - \frac{1}{2}\,\frac{(x^2+1)^{-1}}{-1} = \ln|x| - \frac{\ln|x^2+1|}{2} + \frac{1}{2(x^2+1)} + c, \quad c \in \mathbb{R}$$

Revisões do Capítulo

Exs Revisão da Ficha 2

13) Primitiva de $f(x) = tg\, x$ que passa em $(\pi, 3)$?

$$\int tg\, x\, dx = \int \frac{sen\, x}{\cos x}\,dx = \underbrace{-\ln|\cos x|}_{F(x)} + c, \quad c \in \mathbb{R}$$

$F(\pi) = 3$

$$-\ln|\cos\pi| + c = 3 \Leftrightarrow -\underbrace{\ln|-1|}_{0} + c = 3 \Leftrightarrow c = 3 \rightarrow F(x) = -\ln|\cos x| + 3$$

16)

d) $\int x^2 . \operatorname{arctg} x \, dx = \int v' . u = u.v - \int u'.v = \operatorname{arctg} x . \frac{x^3}{3} - \int \frac{1}{1+x^2} . \frac{x^3}{3} \, dx$

$u = \operatorname{arctg} x \qquad v' = x^2$

$u' = \frac{1}{1+x^2} \qquad v = \frac{x^3}{3}$

fração imprópria

$= \frac{x^3}{3} \operatorname{arctg} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx$

fração própria e simples

C.A

$$\begin{array}{r|l} x^3 & x^2+1 \\ \cline{2-2} -x^3 - x & x \\ \hline -x & \end{array}$$

$= \frac{x^3 \operatorname{arctg}}{3} - \frac{1}{3} \int \left(x + \frac{-x}{x^2+1} \right) dx$

$= \frac{x^3 \operatorname{arctg}}{3} + \frac{1}{3} \left(\int x \, dx + \int \frac{x}{x^2+1} \, dx \right) =$

$= \frac{x^3 \operatorname{arctg}}{3} - \frac{1}{3} \frac{x^2}{2} + \frac{1}{6} \int \frac{2x}{x^2+1} \, dx =$

$= \frac{x^3}{3} \operatorname{arctg} x - \frac{x^2}{6} + \frac{1}{6} \ln |x^2+1| + C, \ C \in \mathbb{R}$

f) $\int \frac{x^2}{\sqrt{1+x^3}} \, dx = \int x^2 . (1+x^3)^{-1/2} \, dx = \frac{1}{3} \int 3x^2 . (1+x^3)^{-1/2} = \frac{1}{3} . \frac{(1+x^3)^{1/2}}{1/2}$

$= \frac{1}{3} . \frac{\sqrt{1+x^3}}{\frac{1}{2}} = \frac{2}{3} \sqrt{1+x^3} + C, \ C \in \mathbb{R}$

o) $\int \cos(\sqrt{x}) \, dx = \int \cos(t) . 2t \, dt = \int v' . u = u.v - \int u'.v$

$t = \sqrt{x} \Leftrightarrow x = t^2$

$\varphi(t) = t^2 \quad \varphi'(t) = 2t \, dt$

$u = 2t \qquad v' = \cos(t)$

$u' = 2 \qquad v = \operatorname{sen}(t)$

$= 2t . \operatorname{sen}(t) - \int 2 \operatorname{sen}(t) = 2t \operatorname{sen}(t) + 2\cos t = 2\sqrt{x} . \operatorname{sen}(\sqrt{x}) + 2\cos(\sqrt{x}) + C$

p) $\int \operatorname{tg}^3 x \, dx = \int \operatorname{tg} x . \operatorname{tg}^2 x \, dx = \int \operatorname{tg} x . \sec^2 x - 1 \, dx$

$= \int \operatorname{tg} x . \sec^2 x - \operatorname{tg} x \, dx = \int \operatorname{tg} x . \sec^2 x \, dx - \int \operatorname{tg} x \, dx$

$= \int \operatorname{tg} x . (\operatorname{tg} x)' \, dx - \int \operatorname{tg} x = \frac{\operatorname{tg}^2 x}{2} - \int \operatorname{tg} x = \frac{\operatorname{tg}^2}{2} - \int \frac{\operatorname{sen} x}{\cos x} \, dx$

$= \frac{\operatorname{tg}^2}{2} + \ln |\cos x| + C, \ C \in \mathbb{R}$

c) $\int \frac{1}{x^2 \sqrt{x^2 - 9}} \, dx \xrightarrow[dx = 3\sec(t)\operatorname{tg}(t) \, dt]{\substack{\text{Substituição} \\ x = 3\sec t, \ t \in]0, \frac{\pi}{2}[}} \int \frac{1}{9\sec^2 t . \sqrt{9(\sec^2 t - 1)}} 3\sec(t) \operatorname{tg}(t) \, dt$

$$= \int \frac{3\sec(t)\,tg(t)}{9\sec^2 t \cdot 3tg(t)}\,dt = \int \frac{1}{9\sec t}\,dt = \frac{1}{9}\int \cos t\ dt = \frac{1}{9}\,sen\,t + c$$

CA

$$\frac{x}{3} = \frac{1}{\cos t} \Leftrightarrow \cos t = \frac{3}{x} \Leftrightarrow \left(\frac{3}{x}\right)^2 + sen^2 t = 1 \Leftrightarrow sen^2 t = 1 - \frac{9}{x^2} \Leftrightarrow$$

$$= sen\ t = \sqrt{\frac{x^2 - 9}{x^2}}$$

$$= \frac{1}{9}\sqrt{\frac{x^2-9}{x^2}} + c,\ c \in \mathbb{R}$$

Fração imprópria

$$b) \int \frac{1}{\sqrt{x} - \sqrt[4]{x}}\,dx = \int \frac{4t^3}{t^2 - t}\,dt = 4\int \frac{t^2}{t-1}\,dt = 4\int \left(t + 1 + \frac{1}{t-1}\right)dt =$$

divisão

$t = \sqrt[4]{x}$, $t^4 = x$

$dx = 4t^3\ dt$

$$= 4\left(\frac{t^2}{2} + t + \ln|t-1|\right) + c$$

$$= 4\left(\frac{\sqrt{x}}{2} + \sqrt[4]{x} + \ln|\sqrt[4]{x} - 1|\right) + c,\ c \in \mathbb{R}$$

Aula q faltei

f contínua em $[a,b] \rightarrow$ integrável em $[a,b]$

f contínua em $[a,b]$ exceto alguns pontos definidos $\rightarrow$ integrável em $[a,b]$

Teorema fundamental do cálculo

$$\left(\int_2^x (x^2+3)\,dx\right)' = x^2 + 3$$

Teorema do valor médio

F é contínua em $[a,b]$, $\exists_c \in [a,b]$:

$$\int_a^b F(x)\,dx = F(c)(b-a)$$

$$H(x) = \int_{g_1(x)}^{g_2(x)} dt(t)\,dt \qquad H'(x) = ?$$

$$H(x) = \int_{g_1(x)}^{a} f(t)\,dt + \int_{a}^{g_2(x)} f(t)\,dt \quad , \quad a \in \mathbb{R}$$

$$H(x) = -\int_{a}^{g_1(x)} f(t)\,dt + \int_{a}^{g_2(x)} f(t)\,dt \quad , \quad a \in \mathbb{R}$$

$$H'(x) = \underbrace{\left(\int_{g_2(x)}^{a} f(t)\,dt \right)'}_{(A)} - \underbrace{\left(\int_{a}^{g_1(x)} f(t)\,dt \right)'}_{(B)}$$

(A) Seja $F(x) = \int_a^x f(t)\,dt \qquad G(x) = \int_a^{g_2 x} f(t)\,dt$

$G = (F \circ g_2)x = F(g_2(x))$

$G'(x) = \left(\int_a^{g_2(x)} f(t)\,dt \right)' = F'(g_2(x)) \cdot g'(x) = \left(\int_a^x f(g_2(t))\,dt \right)' \cdot g_2'(x)$

$= f(g_2(x))\, g_2'(x)$

$$\boxed{\left(\int_{g_1(x)}^{g_2(x)} f(t)\,dt \right)' = f(g_2(x))'\, g_2'(x) - f(g_1(x))\, g_1'(x)}$$

Ficha 3 (ou 2 parte 2)

2)

f) $F(x) = \int_{x^2}^{1+e^{3x}} \operatorname{sen}(t^2)\,dt$

$F'(x) = \operatorname{sen}(1+e^{3x})^2 (1+e^{3x})' - \operatorname{sen}(x^4)(x^2)'$

$= 3e^{3x} \operatorname{sen}(1+e^{3x})^2 - 2x \operatorname{sen}(x^4)$

h) $F(x) = \int_{\cos x}^{x^3} \ln(t^2+1)\,dt$

$F'(x) = \ln(x^6+1)(x^3)' - \ln(\cos^2 x + 1)(\cos x)'$

$$= 3x^2 \ln(x^6+1) + \operatorname{sen} x \cdot \ln(\cos^2 x + 1)$$

4)

$$F(x) = \int_0^{x^2} \operatorname{sen}(t^2)\, dt \quad , \quad F'\left(\sqrt[4]{\tfrac{\pi}{4}}\right) = ?$$

$$F'(x) = \operatorname{sen}(x^4)\,(x^2)' = 2x \cdot \operatorname{sen}(x^4)$$

$$F'\left(\sqrt[4]{\tfrac{\pi}{4}}\right) = 2 \cdot \sqrt[4]{\tfrac{\pi}{4}} \cdot \operatorname{sen}\left(\left(\sqrt[4]{\tfrac{\pi}{4}}\right)^4\right) = 2\sqrt[4]{\tfrac{\pi}{4}} \cdot \operatorname{sen}\left(\tfrac{\pi}{4}\right) = 2 \cdot \sqrt[4]{\tfrac{\pi}{4}} \cdot \frac{\sqrt{2}}{2}$$

$$= \underbrace{\sqrt{2}}_{\sqrt[4]{2^2}} \cdot \sqrt[4]{\tfrac{\pi}{4}} = \sqrt[4]{\tfrac{4\pi}{4}} = \sqrt[4]{\pi}$$

Seja $G(x) = \int_a^x f(t)\, dt$, $x \in [a,b]$, Pelo TFCI G é primitiva de f então tem-se: $G(x) = F(x) + C$, $C \in \mathbb{R}$, onde F é primitiva de f.

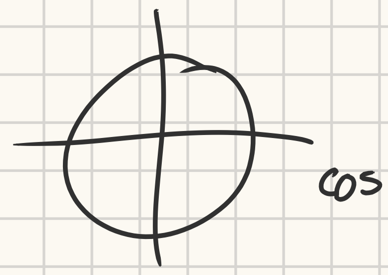
Se $x = a$, $G(a) = F(a) + C$ e como $G(a) = \int_a^a f(t)\, dt = 0$ então:
$F(a) = -C \Leftrightarrow C = -F(a)$

Se $x = b$, $G(b) = F(b) + C$ e como $G(b) = \int_a^b f(t)\, dt$ tem-se que:

$$\int_a^b f(t)\, dt = G(b) = F(b) + C = F(b) - F(a)$$

$$\boxed{\therefore \int_a^b f(t)\, dt = F(b) - F(a)}$$

Formula de Barrow



Ex 12)

e) $$\int_0^1 \frac{1}{1+t^2}\, dt = \Big[\operatorname{arctg}(t)\Big]_0^1 = \operatorname{arctg}(1) - \operatorname{arctg}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

i) $$\int_{-\pi}^{0} \operatorname{sen}(3x)\, dx = \frac{1}{3}\int_{-\pi}^{0} 3\operatorname{sen}(3x)\, dx = \frac{1}{3}\Big[-\cos(3x)\Big]_{-\pi}^{0} = \frac{1}{3}\big(-\cos(0) + \cos(-3\pi)\big) =$$

$$= \frac{1}{3}(-1-1) = -\frac{2}{3}$$

n) $$\int_e^{e^2} \frac{1}{x(\ln x)^2}\, dx = \int_e^{e^2} \frac{1}{x} \cdot \ln^{-2} x = \left[\frac{\ln(x)^{-1}}{-1}\right]_e^{e^2} = \left[-\frac{1}{\ln(x)}\right]_e^{e^2}$$

$$= -\frac{1}{\ln(e^2)} + \frac{1}{\ln(e)} = -\frac{1}{2} + 1 = \frac{1}{2}$$

p) $$\int_1^2 \frac{1}{x^2+2x+5}\, dx = *$$

CA) $x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4 = 4\left(\frac{(x+1)^2}{4}+1\right) = 4\left(\left(\frac{x+1}{2}\right)^2+1\right)$

$$* \int_1^2 \frac{1}{4\left(\left(\frac{x+1}{2}\right)^2+1\right)} dx = \frac{1}{4}\int_1^2 \frac{1}{\left(\frac{x+1}{2}\right)^2+1} dx = \frac{1}{2}\int_1^2 \frac{1/2}{\left(\frac{x+1}{2}\right)^2+1} dx = \frac{1}{2}\left[\operatorname{arctg}\left(\frac{x+1}{2}\right)\right]_1^2$$

$$= \frac{1}{2}\left(\operatorname{arctg}\left(\frac{3}{2}\right) - \operatorname{arctg}(1)\right) = \frac{1}{2}.\operatorname{arctg}\left(\frac{3}{2}\right) - \frac{\pi}{4}$$

13)

d) $\int_1^e x \ln x \, dx = \int_1^e v'.u = u.v - \int_1^e u'.v = \ln x . \frac{x^2}{2}\Big]_1^e - \int_1^e \frac{1}{x}.\frac{x^2}{2} dx$

$u = \ln x \quad u' = \frac{1}{x}$

$v' = x \quad v = \frac{x^2}{2}$

$$= \left(\ln(e).\frac{e^2}{2}\right) - \left(\ln(1).\frac{1^2}{2}\right) - \int_1^e \frac{x}{2} dx$$

$$= \frac{e^2}{2} - 0 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1^2}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$$

e) $\int_1^e \ln^2 x \, dx = \int_1^e 1 . \ln^2 x \, dx = \ln^2 x . x\Big|_1^e - \int_1^e x . 2\ln x . \frac{1}{x}$

$u = \ln^2 x \quad v' = 1$

$u' = 2\ln x . \frac{1}{x} \quad v = x$

$$= e\ln(e) - \ln(1) - \int_1^e x . 2\ln x . \frac{1}{x}$$

$$= e\ln(e) - 2\int_1^e \ln x =$$

$$= e - 2\left[x \ln x\right]_1^e - \int_1^e x \frac{1}{x} dx$$

$$= e - 2\left(\left(e.1 - \ln(1)\right) - \int_1^e 1 \, dx\right)$$

$$= e - 2\left((e) - [x]_1^e\right) = e - 2(e - e + 1) = e - 2$$

a) $\int_{-\ln 2}^{\ln 2} \frac{1}{e^x+4} dx \xrightarrow[dx = \frac{1}{t}dt]{t = e^x \Leftrightarrow \ln(t) = x} \int_{\frac{1}{2}}^2 \frac{1}{t+4}\frac{1}{t} dt = *$

x	t
$-\ln 2$	$e^{-\ln 2} = e^{\ln 2^{-1}} = \frac{1}{2}$
$\ln 2$	$e^{\ln 2} = 2$

CA)

$\frac{1}{(t+4).t} = \frac{A}{t+4} + \frac{B}{t} = \frac{tA + B(t+4)}{(t+4)t}$ $\begin{cases} A+B=0 \\ 4B=1 \end{cases}$ $\begin{cases} A = -\frac{1}{4} \\ B = \frac{1}{4} \end{cases}$

$$* \int_{1/2}^2 \frac{-1/4}{t+4} + \frac{\frac{1}{4}}{t} dt = \frac{1}{4}\int_{1/4}^2 \frac{1}{t+4} - \frac{1}{t} dt = -\frac{1}{4}\left[\ln|t+4| - \ln|t|\right]_{1/2}^2 =$$

$= -\frac{1}{4}\left(\ln(6) - \ln(2) - \ln(9/2) + \ln(1/2)\right) = -\frac{1}{4}\ln\left(\frac{3}{9}\right) = -\frac{1}{4}\ln\left(\frac{1}{3}\right) = \frac{1}{4}\ln(3)$

e) $\int_0^1 \sqrt{4-x^2}\,dx = \int_0^{\pi/6} \sqrt{4-(2\operatorname{sen} t)^2}\cdot 2\cos t\,dt = \int_0^{\pi/6} 2\sqrt{\cos^2 t}\cdot 2\cos t\,dt$

$x = 2\operatorname{sen} t$, $dx = 2\cos t\,dt$ $\qquad = \int_0^{\pi/6} 2\cos t \cdot 2\cos t\,dt = 4\int_0^{\pi/6} \cos^2 t\,dt$

x	$t = \operatorname{arcsen}\left(\frac{x}{2}\right)$
0	$\operatorname{arcsen}(0) = 0$
1	$\operatorname{arcsen}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$= 4\int_0^{\pi/6} \frac{1+\cos(2t)}{2}\,dt = 2\int_0^{\pi/6} 1 + \cos(2t)\,dt$

$= 2\int_0^{\pi/6} 1\,dt + 2\int_0^{\pi/6} \cos(2t)\,dt = \left[2t\right]_0^{\pi/6} + \int_0^{\pi/6} 2\cos(2t)\,dt$

$= \left[2t\right]_0^{\pi/6} + \left[\operatorname{sen}(2t)\right]_0^{\pi/6} = 2\left(\frac{\pi}{6} - 0\right) + \operatorname{sen}\left(\frac{2\pi}{6}\right) - \operatorname{sen}(0)$

$= \frac{2\pi}{6} + \operatorname{sen}\left(\frac{\pi}{3}\right) - 0 = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

Slide 25

$a > 0$, $f : [a, -a]$

1 se f é par, então $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$? ← Pode ser Importante

$f(x) = f(-x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx \;*$$

Mudança de variavel em

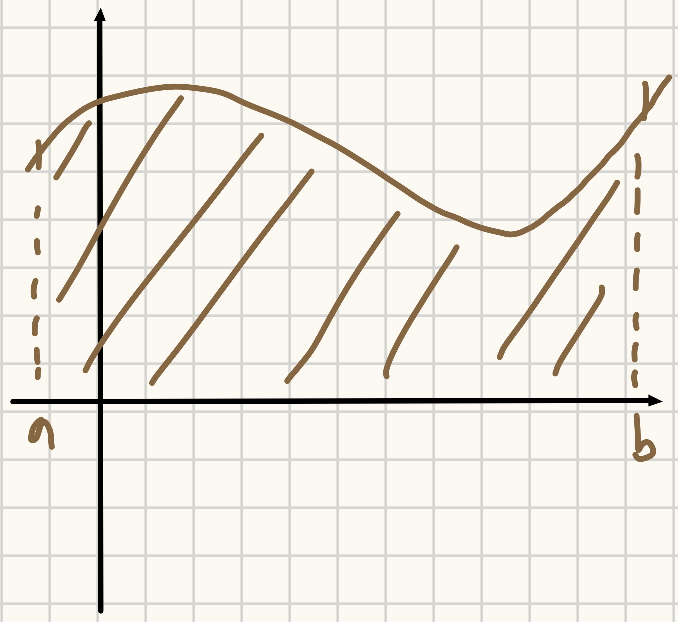
$\int_0^{-a} f(x)\,dx$ $\qquad x = -t$, $dx = -dt$

x	t
0	0
$-a$	a

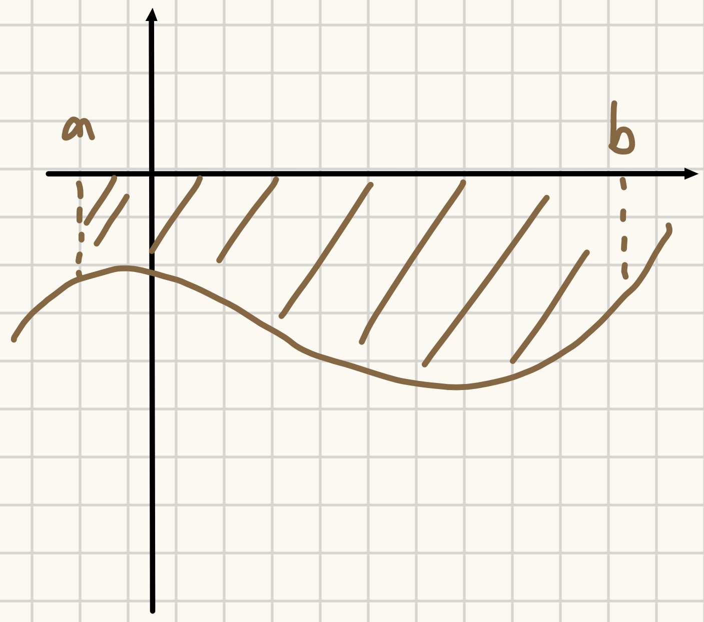
$$= -\int_0^a f(-t)(-dt) = -\underbrace{-\int_0^a f(t)\,dt}_{\text{porque } f \text{ é par}}$$

$$* = \int_0^a f(t)\,dt + \int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

Calculo de Areas



$$\int_a^b f(x)\,dx$$



$$\int_a^b (-f(x))\,dx$$

(17) Area da região limitada entre $x=0$, $x=2$, eixo de abcissas e grafico de $g(x) = \dfrac{e^{2x}+1}{e^x+1}$

$g(x) > 0$

$$\int_0^2 \frac{e^{2x}+1}{e^x+1}\,dx \xrightarrow[\substack{t = e^x \Leftrightarrow x = \ln(t),\ t>0 \\ dx = \frac{1}{t}\,dt}]{\text{Mudança de Variavel}} \int_1^{e^2} \frac{t^2+1}{t+1}\cdot\frac{1}{t}\,dt$$

x	t
0	1
2	e^2

$$= \int_1^{e^2} \frac{t^2+1}{t^2+t}\,dt = *$$ ← Fração imprópria (acho?)

e.A

$$\begin{array}{l|l} t^2+1 & t^2+t \\ \hline -t^2-t & 1 \\ \hline -t+1 & \end{array}$$

$$* = \int_1^{e^2} \left(1 + \frac{-t+1}{t^2+t}\right) dt$$

$$\frac{-t+1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{A(t+1) + B(t)}{t(t+1)}$$

$$\begin{cases} A+B = -1 \\ A = 1 \end{cases} \Leftrightarrow \begin{cases} B = -2 \\ A = 1 \end{cases}$$

$$\int_1^{e^2} 1 + \frac{1}{t} - \frac{2}{t+1}\,dt = \int_1^{e^2} 1\,dt + \int_1^{e^2} \frac{1}{t} - 2\int_1^{e^2} \frac{1}{t+1}\,dt$$

$$= [t]_1^{e^2} + [\ln(t)]_1^{e^2} - 2[\ln(t+1)]_1^{e^2}$$

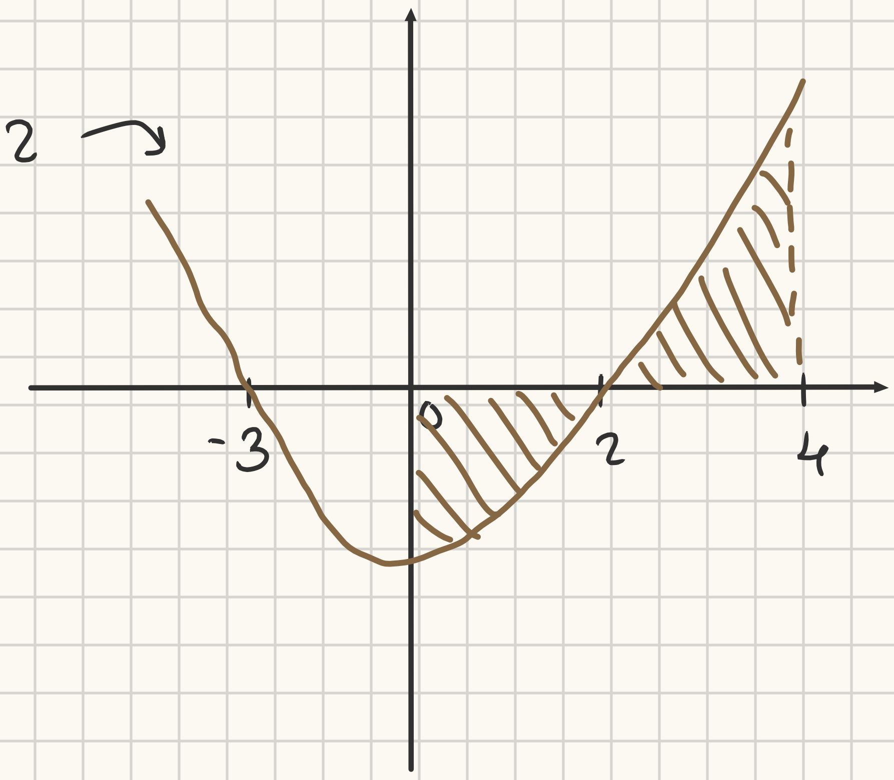
$$= e^2 - 1 + \ln(e^2) - \ln(1) - 2\ln(e^2+1) + 2\ln(2)$$

$$= e^2 - 1 + 2 - 0 - 2\ln(e^2+1) + 2\ln(2) = e^2 + 1 + 2\ln\left(\frac{2}{e^2+1}\right)$$

Ex: Área limitada entre

$f(x) = (x+3)(x-2)$, eixo das abcissas, $x = 0$ e $x = 4$

$f(x) = 0 \Leftrightarrow (x+3) = 0 \vee (x-2) = 0 \Leftrightarrow x = -3 \vee x = 2 \rightarrow$



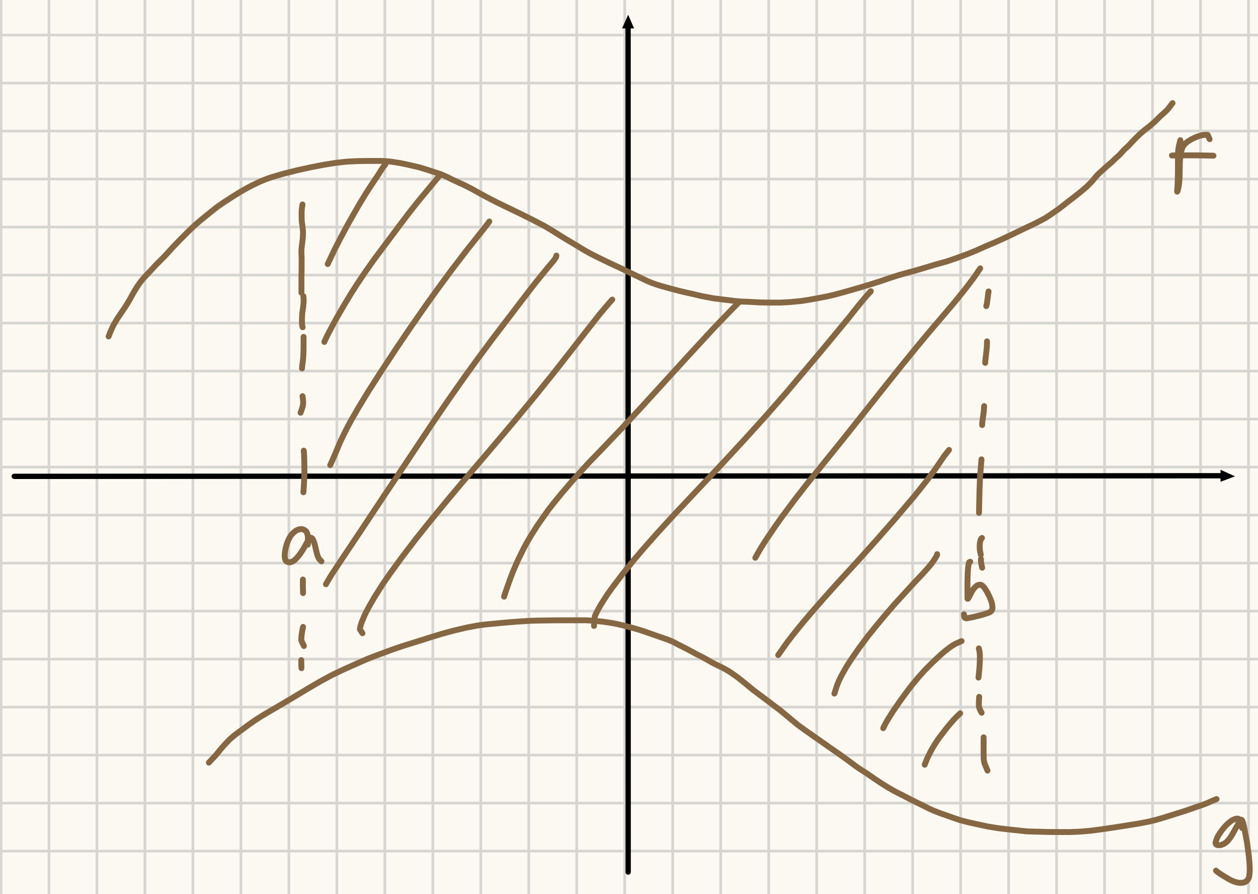
$f(x) = x^2 + x - 6$

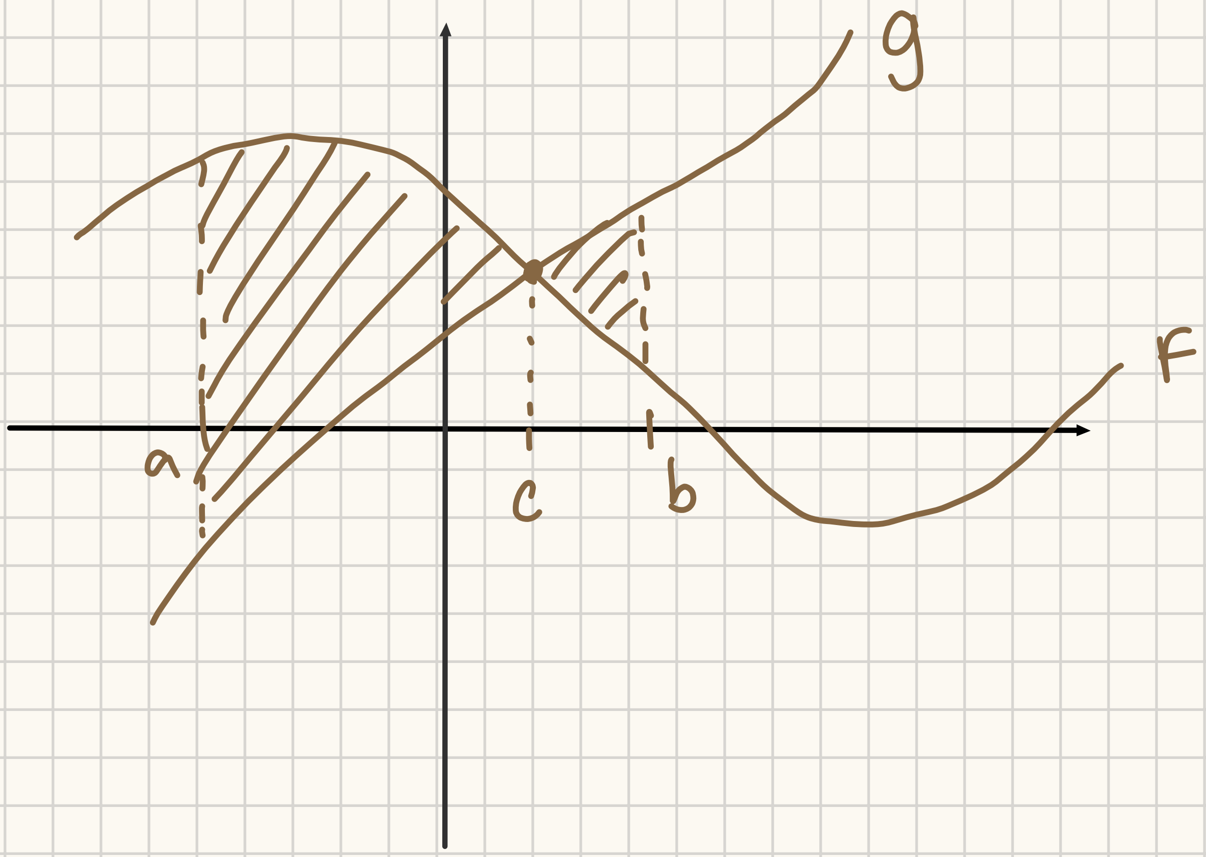
$$A = -\int_0^2 (x^2 + x - 6)\, dx + \int_2^4 (x^2 + x - 6)\, dx$$

$$= -\left[\frac{x^3}{3} + \frac{x^2}{2} - 6x\right]_0^2 + \left[\frac{x^3}{3} + \frac{x^2}{2} - 6x\right]_2^4$$

$$= \left(-\frac{8}{3} + \frac{4}{2} - 12\right) + \left(\frac{64}{3} + \frac{16}{2} - 24 - \frac{8}{3} - \frac{4}{2} + 12\right)$$

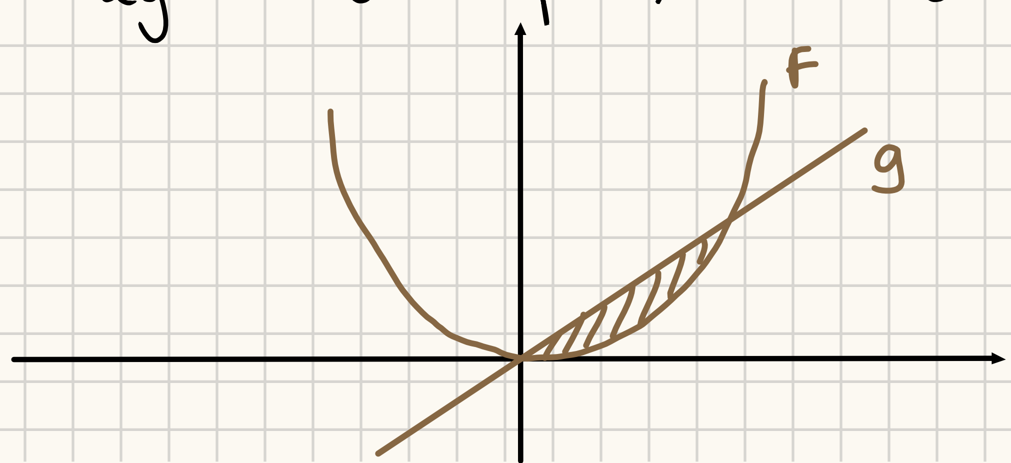
$$A = \int_a^b f(x) - g(x)\, dx$$





$$A = \int_a^c f(x) - g(x)\, dx + \int_c^b g(x) - f(x)\, dx$$

(20) Área da região entre $f(x) = x^2$ e $g(x) = x$?

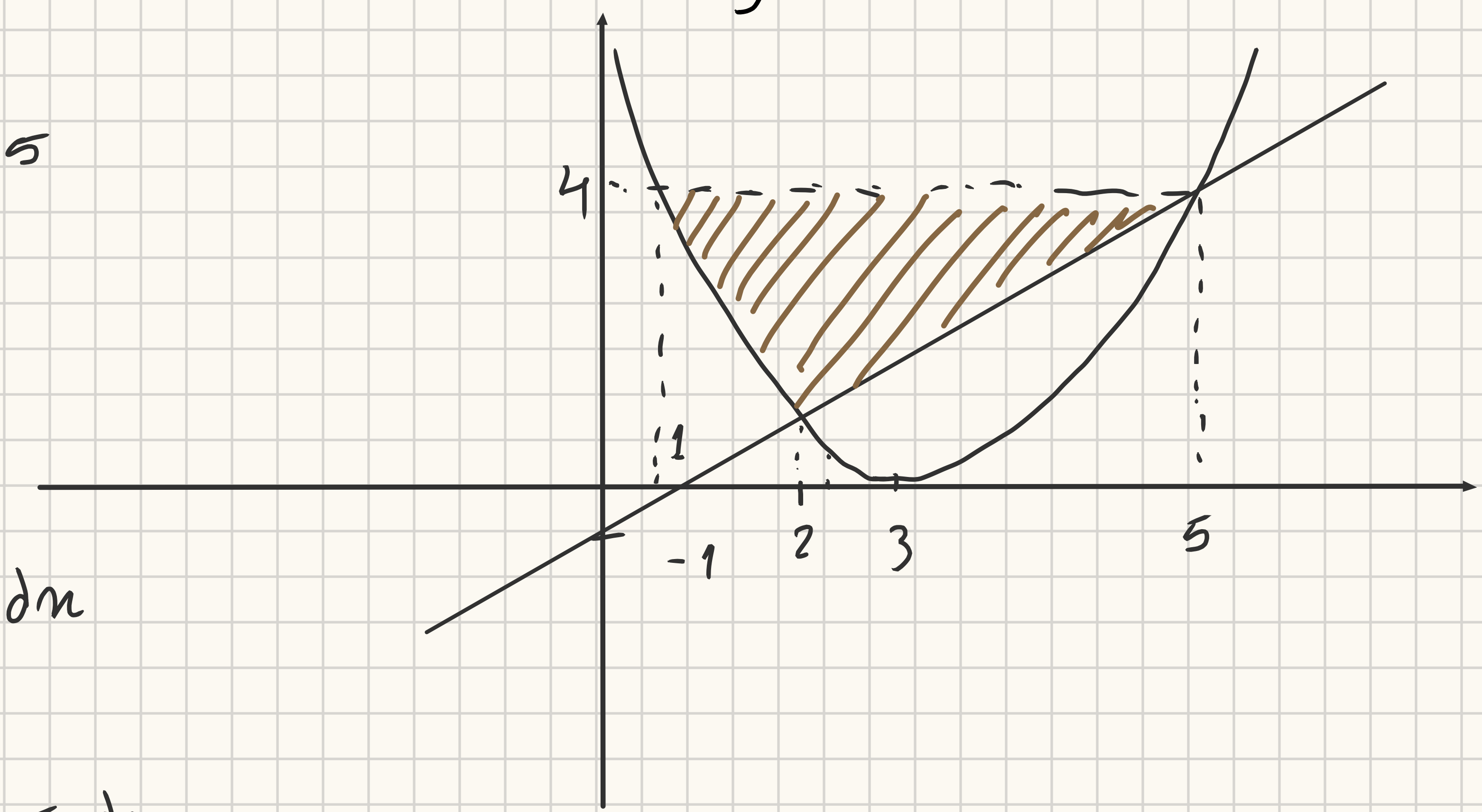


ver onde se intersetam:

$$x^2 = x \Leftrightarrow x^2 - x = 0 \Leftrightarrow x(x-1) = 0 \Leftrightarrow x = 0 \vee x = 1$$

$$A = \int_0^1 g(x) - f(x)\, dx = \int_0^1 x - x^2\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

25 $A = \left\{ (x,y) \in \mathbb{R}^2 : y \geq (x-3)^2, y \geq x - 1, y \leq 4 \right\}$



- $(x-3)^2 = x - 1 \Leftrightarrow x = 2 \vee x = 5$
- $x - 1 = 4 \Leftrightarrow x = 5$
- $(x-3)^2 = 4 \Leftrightarrow x = 1 \vee x = 5$

$$A = \int_1^2 4 - (x-3)^2\, dx + \int_2^5 4 - x + 1\, dx$$

$$= \int_1^2 4\, dx - \int_1^2 (x-3)^2\, dx + \int_2^5 -x+5\, dx$$

$$= \left[4x\right]_1^2 - \left[\frac{(x-3)^3}{3} \right]_1^2 + \left[-\frac{x^2}{2} + 5x \right]_2^5 =$$

$$= 4 + \frac{1}{3} - \frac{8}{3} - \frac{25}{2} + 25 + \frac{4}{2} - 10 = \frac{37}{6}$$

21 Area delimitada por $f(x) = e^{2x+1}$, $g(x) = x e^{2x+1}$, $x = -1$ e $x = -\frac{1}{2}$

$e^{2x+1} = xe^{2x+1} \Leftrightarrow x = 1 \rightarrow$ a partir de $x = 1 \rightarrow g(x) > f(x)$

$$\int_{-1}^{-\frac{1}{2}} e^{2x+1} - xe^{2x+1}\, dx = \int_{-1}^{-1/2} e^{2x+1}\, dx - \int_{-1}^{-1/2} xe^{2x+1}\, dx =$$

$$= \frac{1}{2} \int_{-1}^{-1/2} 2e^{2x+1}\, dx - \int_{-1}^{-1/2} xe^{2x+1}\, dx = \frac{1}{2} \left[e^{2x+1} \right]_{-1}^{-1/2} - \left(\left. x \frac{e^{2x+1}}{2} \right]_{-1/2}^{-1} - \int_{-1/2}^{-1} \frac{e^{2x+1}}{2}\, dx \right)$$

$u = x \quad v' = e^{2x+1}$

$$= \frac{1}{2}\left(e^0 - e^{-1}\right) - \left(-\frac{1}{2} \cdot \frac{1}{2} + \frac{e^{-1}}{2} \right) + \left. \frac{1}{4} e^{2x+1} \right]_{-1}^{-1/2}$$

$$= \frac{1}{2} - \frac{1}{2e} + \frac{1}{4} - \frac{1}{2e} + \frac{1}{4} - \frac{1}{4e} = 1 - \frac{5}{4e}$$

Ex:

$$f(x) = \frac{x}{(x+1)(x^2+1)}$$

fração própria ✓
denominador n pode ser mais fatorizado

a) $\int f(x)\, dy = ?$

$$\frac{x}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)} = \frac{A(x^2+1) + B(x^2+x) + C(x+1)}{(x+1)(x^2+1)}$$

$$\begin{cases} A+B=0 \\ B+C=1 \\ A+C=0 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ -A-A=1 \\ C=-A \end{cases} \begin{cases} B=\frac{1}{2} \\ A=-\frac{1}{2} \\ C=\frac{1}{2} \end{cases}$$

$$\int -\frac{1/2}{(x+1)} + \frac{\frac{1}{2}x+\frac{1}{2}}{(x^2+1)}\, dx = -\frac{1}{2}\int \frac{1}{x+1}\, dx + \frac{1}{2}\int \frac{x+1}{x^2+1}\, dx$$

$$= -\frac{1}{2}\ln|x+1| + \frac{1}{4}\int \frac{2x}{x^2+1}\, dx + \frac{1}{2}\int \frac{1}{x^2+1}\, dx = -\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\operatorname{arctg}(x) + c,\ c\in\mathbb{R}$$

b) Valor da area da região delimitada por f, eixo abcisses, $x=0$ e $x=1$

$f(x)=0 \Leftrightarrow x=0 \wedge x\neq -1$

$$A = \int_0^1 \frac{x}{(x+1)(x^2+1)} = \left[-\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\operatorname{arctg}\right]_0^1$$

$$= -\frac{1}{2}\ln(2) + \frac{1}{4}\ln(2) + \frac{1}{2}\operatorname{arctg}(1) - \left(-\frac{1}{2}\ln(1) + \frac{1}{4}\ln(1) + \frac{1}{2}\operatorname{arctg}(0)\right)$$

$$= -\frac{1}{2}\ln(2) + \frac{1}{2}\ln(\sqrt{2}) + \frac{\pi}{8} + 0 - 0 - 0 = \frac{1}{2}\ln\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi}{8}$$