Integrais Impróprios de 1ª especie

Integrais de funções integraveis e definidas em intervalos do tipo $[a,+\infty[$, $]-\infty, b]$ ou $]-\infty,+\infty[$

$$\int_a^{+\infty} f(x)\, dx$$

$\lim\limits_{t\to+\infty} \int_a^t f(x)\, dx$ = se o limite existe e é finito : $\int_a^{+\infty} f(x)\, dx$ é convengente

Se o lim não existe ou é $\pm\infty$: $\int_a^{+\infty} f(x)\, dx$ é divergente

Ex Slide 5

1

a) $\int_{\pi}^{+\infty} \cos(x)\, dx$

$$\lim_{t\to+\infty} \int_{\pi}^{t} \cos(x)\, dx = \lim_{t\to+\infty} \Big[\operatorname{sen}(x)\Big]_{\pi}^{T} = \lim_{t\to+\infty} \big(\operatorname{sen} t - \overset{0}{\operatorname{sen} \pi}\big) = \lim_{t\to+\infty} \operatorname{sen} t$$

não existe $\rightarrow \int_{\pi}^{+\infty} \cos(x)$ é divergente

b) $$\int_2^{+\infty} \frac{1}{(x+2)^2}\, dx = \lim_{t\to+\infty} \int_2^t \frac{1}{(x+2)^2}\, dx = \lim_{t\to+\infty} \int_2^t (x+2)^{-2} = \lim_{t\to+\infty} \left[\frac{(x+2)^{-1}}{-1}\right]_2^t = \lim_{t\to+\infty} \left[-\frac{1}{x+2}\right]_2^t = \lim_{t\to+\infty}\left(-\frac{1}{t+2} + \frac{1}{4}\right) = \frac{1}{4}$$

convengente

c) $$\int_1^{+\infty} \frac{(\ln x)^3}{x}\, d = \lim_{t\to+\infty} \int_1^t \frac{(\ln x)^3}{x}\, dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x}\cdot(\ln x)^3 = \lim_{t\to+\infty}\left[\frac{(\ln x)^4}{4}\right]_1^t = \lim_{t\to+\infty}\left(\frac{(\ln t)^4}{4} - 0\right) = +\infty$$

divengente

(2) Prove que $\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ e:

a) divergente se $\alpha \leq 1$ e convergente se $\alpha > 1$

$\alpha = 1$: $\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to+\infty} \left[\ln|x|\right]_1^t = \lim_{t\to+\infty} \left(\ln(t) - \ln(1)\right) = +\infty \leftarrow$ divergente

$$\alpha \neq 1: \int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x^\alpha}\,dx = \lim_{t\to+\infty} \int_1^t x^{-\alpha} = \lim_{t\to+\infty} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^t = \lim_{t\to+\infty} \left(\frac{t^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}\right) = \begin{cases} +\infty, & \alpha < 1 \\ \frac{1}{\alpha-1}, & \alpha > 1 \end{cases}$$

$$\boxed{\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \text{divergente se } \alpha \leq 1 \\ \frac{1}{\alpha-1} \text{ se } \alpha > 1 \end{cases}}$$

(3) $\int_0^{+\infty} e^{\beta x}\,dx$

$\underline{\beta = 0}$

$\int_0^{+\infty} e^0\,dx = \lim_{t\to+\infty} \left[x\right]_0^t = \lim_{t\to+\infty} t = +\infty$ divergente

$\underline{\beta \neq 0}$

$$\int_0^{+\infty} e^{\beta x}\,dx = \lim_{t\to+\infty} \frac{1}{\beta} \int_0^t \beta e^{\beta x}\,dx = \lim_{t\to+\infty} \frac{1}{\beta}\left[e^{\beta x}\right]_0^t = \lim_{t\to+\infty} \frac{1}{\beta}\left(e^{\beta t} - 1\right) = \begin{cases} +\infty, & \text{se } \beta > 0 \\ -\frac{1}{\beta}, & \text{se } \beta < 0 \end{cases}$$

$$\boxed{\int_0^{+\infty} e^{\beta x}\,dx \begin{cases} \text{divergente}, & \beta \geq 0 \\ -\frac{1}{\beta}, & \beta < 0 \end{cases}}$$

Slide 8

a) $\displaystyle\int_{-\infty}^{0} x e^{-x^2}\,dx = \lim_{t\to-\infty} -\frac{1}{2}\int_{-\infty}^{0} -2x e^{-x^2}\,dx = \lim_{t\to-\infty} -\frac{1}{2}\left[e^{-x^2}\right]_t^0 = \lim_{t\to-\infty} -\frac{1}{2}\left(1 - \overset{\to 0}{e^{-t^2}}\right) = -\frac{1}{2}(1-0) = -\frac{1}{2}$ convergente

c) $\displaystyle\int_{-\infty}^{0} \frac{4}{1+(x+1)^2}\,dx = \lim_{t\to-\infty} 4\int_t^0 \frac{1}{1+(x+1)^2} = \lim_{t\to-\infty} 4\left[\operatorname{arctg}(x+1)\right]_t^0 = \lim_{t\to-\infty} 4(\operatorname{arctg}(1) - \operatorname{arctg}(t+1)) = 4\left(\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right) = 3\pi$

$\displaystyle\int_{-\infty}^{0} a^x\,dx = -\int_{0}^{-\infty} a^x\,dx$

M.V

$x = -t$

$dx = -dt$

x	t
0	0
$-\infty$	$+\infty$

$\displaystyle = -\int_0^{+\infty} a^{-t}\cdot(-dt) = \int_0^{+\infty} a^{-t}\,dt = \int_0^{+\infty} a^{x}(-dx)$

$$\boxed{\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx}$$

Se um dos integrais divergir, então o integral diverge

Somando dois integrais convergentes resulta num integral convergente

$\displaystyle\int_a^{+\infty} f(x)\,dx$, $\displaystyle\int_b^{+\infty} f(x)\,dx$ com a mesma natureza (ambos convergentes ou divergentes), $a<b<+\infty$

$$\int_a^{+\infty} f(x)\,dx = \underbrace{\int_a^{b} f(x)\,dx}_{\text{constante}} + \int_b^{+\infty} f(x)\,dx$$

Ex Slide 13

②

c) $\int_{-\infty}^{+\infty} 2^x \, dx = \int_{-\infty}^{0} 2^x \, dx + \int_{0}^{+\infty} 2^x \, dx$

CA

- $\int_{-\infty}^{0} 2^x \, dx = \lim_{t\to-\infty} \int_{t}^{0} 2^x \, dx = \lim_{t\to-\infty} \left[\frac{2^x}{\ln(2)} \right]_t^0 = \lim_{t\to-\infty} \left(\frac{1}{\ln(2)} - \frac{\overset{\to 0}{2^t}}{\ln(2)} \right) = \frac{1}{\ln(2)}$ conv.

- $\int_{0}^{+\infty} 2^x \, dx = \lim_{t\to+\infty} \left[\frac{2^x}{\ln(2)} \right]_0^t = \lim_{t\to+\infty} \left(\frac{2^t}{\ln(2)} - \frac{1}{\ln(2)} \right) = +\infty$ div.

Então $\int_{-\infty}^{+\infty} 2^x \, dx$ é divergente

b) $\int_{-\infty}^{+\infty} \frac{1}{1+x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx + \int_{0}^{+\infty} \frac{1}{1+x^2} \, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

- $\int_{-\infty}^{0} \frac{1}{1+x^2} \, dx = \lim_{t\to-\infty} \left[\operatorname{arctg}(x) \right]_t^0 = \lim_{t\to-\infty} \left(\operatorname{arctg}(0) - \operatorname{arctg}(t) \right) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$ conv.

- $\int_{0}^{+\infty} \frac{1}{1+x^2} \, dx = \lim_{t\to+\infty} \left[\operatorname{arctg}(x) \right]_0^t = \lim_{t\to+\infty} \left(\operatorname{arctg}(t) - \operatorname{arctg}(0) \right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ conv.

Então $\int_{-\infty}^{+\infty} \frac{1}{1+x^2} \, dx$ converge e é π

Revisão teste

- Ex

se $\int_{-1}^{3} f(x) \, dx = 2$, $\int_{5}^{-1} f(x) \, dx = 1$. $\int_{3}^{5} f(x) \, dx = ?$

!!

$$\underbrace{\int_{-1}^{3} f(x)\,dx}_{2} + \int_{3}^{5} f(x)\,dx = \underbrace{\int_{-1}^{5} f(x)\,dx}_{-1} = 2 + \int_{3}^{5} f(x)\,dx = -1 = \int_{3}^{5} f(x)\,dx = -1-2 = \int_{3}^{5} f(x)\,dx = \underline{\underline{-3}}$$

$$\left(\int_{5}^{-1} f(x) = -\int_{-1}^{5} f(x) \right)$$

- Determinar o integral de $f(x) = \dfrac{\cos(2x)+x}{\operatorname{sen}(2x)+x^2}$?

quocientes → funções racionais ; → $\dfrac{u'}{u}$

$$\int \frac{\cos(2x)+x}{\operatorname{sen}(2x)+x^2}\,dx = \frac{1}{2}\int \frac{2\cos(2x)+2x}{\operatorname{sen}(2x)+x^2} = \frac{1}{2}\ln\left(\operatorname{sen}(2x)+x^2\right) + C,\ C \in \mathbb{R}$$

Critério de comparação

$$0 \leq f(x) \leq g(x)$$

- $\int_{0}^{+\infty} g(x)\,dx$ conv $\longrightarrow \int_{a}^{+\infty} f(x)\,dx$ conv. e
- $\int_{a}^{+\infty} f(x)\,dx$ div $\longrightarrow \int_{a}^{+\infty} g(x)$ div

Exs Ficha 3

(6) Natureza de :

a) $\displaystyle\int_1^{+\infty} \frac{\operatorname{sen}^2(x)}{x^{5/2}}\,dx$ $\qquad \dfrac{\operatorname{sen}^2 x}{x^{5/2}} \geq 0,\ \forall x \in [1, +\infty[$

$$0 \leq \frac{\operatorname{sen}^2 x}{x^{5/2}} \leq \frac{1}{x^{5/2}}$$

Relembrando:

$$\int_1^{+\infty} \frac{1}{x^\alpha} \begin{cases} \text{div} & \text{se } \alpha \leq 1 \\ \frac{1}{\alpha - 1} & \text{se } \alpha > 1 \end{cases}$$

$$\int_0^{+\infty} e^{\beta x} \begin{cases} \text{div} & \text{se } \beta \geq 0 \\ -\frac{1}{\beta} & \text{se } \beta < 0 \end{cases}$$

$\displaystyle\int_1^{+\infty} \frac{1}{x^{5/2}}$ é convergente porque é do tipo $\dfrac{1}{x^\alpha}$, com $\alpha = \frac{5}{2} > 1$

Pelo critério da comparação: $\displaystyle\int_1^{+\infty} \frac{\operatorname{sen}^2(x)}{x^{5/2}}\,dx$ é convergente

K) $\displaystyle\int_1^{+\infty} \frac{e^x}{x}\,dx$ $\qquad \dfrac{e^x}{x} > 0 \quad \forall x \in [1, +\infty[\qquad \dfrac{e^x}{x} > \dfrac{1}{x}$

$\displaystyle\int_1^{+\infty} \frac{1}{x}\,dx$ diverge porque é do tipo $\displaystyle\int_1^{+\infty} \frac{1}{x^\alpha}$, com $\alpha \leq 1$, logo pelo critério da comparação $\displaystyle\int_1^{+\infty} \frac{e^x}{x}$ também é divergente

f) $\displaystyle\int_1^{+\infty} \frac{x+2}{2x^{5/3}}\,dx$ $\qquad \dfrac{x+2}{2x^{5/3}} > \dfrac{x}{2x^{5/3}}, \quad \dfrac{x}{2x^{5/3}} = \dfrac{1}{2x^{2/3}}$

$\displaystyle\int_1^{+\infty} \frac{x}{2x^{5/3}} = \frac{1}{2}\int_1^{+\infty} \frac{1}{x^{2/3}}$ é divergente porque é do tipo $\dfrac{1}{x^\alpha}$ com $\alpha < 1$

pelo critério da comparação $\displaystyle\int_1^{+\infty} \frac{x+2}{2x^{5/3}}$ é divergente

Critério do Limite

$$L = \lim_{x \to +\infty} \frac{f(x)}{g(x)} \quad \text{com } f > 0 \text{ e } g > 0$$

- se $L \in \mathbb{R}^+$ então $\int_a^{+\infty} f(x)$ e $\int_a^{+\infty} g(x)$ têm a mesma natureza
- se $L = 0$ e $\int_a^{+\infty} g(x)$ é convergente então $\int_a^{+\infty} f(x)$ é convergente
- se $L = +\infty$ e $\int_a^{+\infty} g(x)$ é divergente então $\int_a^{+\infty} f(x)$ é divergente

Exs Ficha 3

6

f) $\int_1^{+\infty} \frac{x+2}{2x^{5/3}}\,dx$

Aplicando o critério do limite com $g(x) = \frac{1}{x^{5/3}}$

$\int_1^{+\infty} \frac{1}{x^{5/3}}$ converge $\left(\frac{1}{x^\alpha} \text{ com } \alpha > 1\right)$

$$L = \lim_{x \to +\infty} \frac{\frac{x+2}{2x^{5/3}}}{\frac{1}{x^{5/3}}} = \lim_{x \to +\infty} \frac{x+2}{2} = +\infty$$

$L = +\infty$ e $g(x)$ conv $\rightarrow$ Nada podemos concluir

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$\frac{x+2}{2x^{5/3}}$ se forem polinómios: do numerador e denominador usar maior grau

$$\frac{x+2}{2x^{5/3}} \longrightarrow \frac{x}{x^{5/3}} = \frac{1}{x^{2/3}} = g(x)$$

$$L = \lim_{x\to+\infty} \frac{\frac{x+2}{2x^{5/3}}}{\frac{1}{x^{2/3}}} = \lim_{x\to+\infty} \frac{x+2}{2x^{5/3}} \cdot x^{2/3} = \lim_{x\to+\infty} \frac{x+2}{2x} = \lim_{x\to+\infty} \frac{(x+2)'}{(2x)'} = \lim_{x\to+\infty} \frac{1}{2} = \frac{1}{2}$$

$L = \frac{1}{2}$ e $g(x)$ div $\left(\alpha = \frac{2}{3} < 1\right) \rightarrow f(x)$ e $g(x)$ têm a mesma natureza

$\therefore \int_F^1 \frac{x+2}{2x^{5/3}}\,dx$ é divergente

k) $\int_1^{+\infty} \frac{e^x}{x}\,dx$ $\rightarrow \frac{e^x}{x}$ vai para inf $\rightarrow$ n esperamos que o Limite seja convergente

$g(x) = \frac{1}{x}$ $\qquad L = \lim_{x\to+\infty} \frac{\frac{e^x}{x}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{xe^x}{x} = \lim_{x\to+\infty} e^x = +\infty$

$g(x)$ é div e $L = +\infty \longrightarrow \int_1^{+\infty} \frac{e^x}{x}$ é div. Pelo crit. limite
$\left(\frac{1}{x^\alpha}, \alpha = 1\right)$

Exs Slide 19 (também estão na ficha 3)

2) $\int_1^{+\infty} \frac{5x^2-3}{x^8+x-1}\,dx$ $\qquad g(x) = \frac{x^2}{x^8} = \frac{1}{x^6}$ (conv. pq $\alpha > 1$)

$$L = \lim_{x\to+\infty} \frac{\frac{5x^2-3}{x^8+x-1}}{\frac{1}{x^6}} = \lim_{x\to+\infty} \frac{5x^2-3}{x^8+x-1} \cdot x^6 = \lim_{x\to+\infty} \frac{5x^8-3x^6}{x^8+x-1} = \lim_{x\to+\infty} 5$$

$L = 5$ e g é conv

Logo, pelo C. Limite $\int_1^{+\infty} \frac{5x^2-3}{x^8+x-1}$ é convergente

3) $\int_0^{+\infty} e^{x^2}\, dx$

↳ tende para $+\infty$ (integral deve ser divergente)

$\int_0^{+\infty} e^{x}\, dx$ é div. (tipo $\int_0^{+\infty} e^{\beta x}$ com $\beta \geq 0$)

$$L = \lim_{x\to+\infty} \frac{e^{x^2}}{e^x} = \lim_{x\to+\infty} e^{x^2-x} = \lim_{x\to+\infty} e^{x(x-1)} = +\infty$$

$L = +\infty$ e $g(x)$ é div $\rightarrow$ Pelo C. Limite $\int_0^{+\infty} e^{x}\, dx$ é divergente

Relembrando:

$$\int_1^{+\infty} \frac{1}{x^\alpha} \begin{cases} \text{div} & \text{se } \alpha \leq 1 \\ \frac{1}{\alpha-1} & \text{se } \alpha > 1 \end{cases}$$

$$\int_0^{+\infty} e^{\beta x} \begin{cases} \text{div} & \text{se } \beta \geq 0 \\ -\frac{1}{\beta} & \text{se } \beta < 0 \end{cases}$$

4) $\int_1^{+\infty} \frac{\cos^2\left(\frac{1}{x}\right)}{x^7+2x+1}$

$0 \leq \frac{\cos^2\left(\frac{1}{x}\right)}{x^7+2x+1} \leq \frac{1}{x^7+2x+1} \leq \frac{1}{x^7}$, como $\frac{1}{x^7}$ converge ($\frac{1}{x^\alpha}$, com $\alpha > 1$)

então $\frac{\cos^2\left(\frac{1}{x}\right)}{x^7+2x+1}$ também converge, pelo Critério de Comparação

6) $\int_{-\infty}^{-1} \frac{x^3+3x}{2+x^2}\, dx = \int_{+\infty}^{1} \frac{-t^3-3t}{2+t^2}(-1)\, dt = \int_1^{+\infty} \frac{-t^3-3t}{2+t^2}\, dt$

$t = -x$

$dx = -dt$

$g(x) = \frac{t^3}{t^2} =$

$\Leftrightarrow g(x) = t$

$= \frac{1}{t^{-1}}$

$$\text{Lim} = \lim_{t\to+\infty} \frac{\frac{-t^3-3t}{2+t^2}}{t} =$$

$$= \lim_{x\to+\infty} \frac{-t^3-3t}{2t+t^3} = +1$$

Das fichas (1 e 2.1 + 2.2):

- um ex da materia mais facil que sai igualzinho no teste

- um ex q tem 2 alineas e na aula fizemos uma delas q sai mto parecido no teste

$L=1 \to$ mesma natureza e $g(x) = \frac{1}{t^{-1}}$ diverge logo pelo C. Limite : $\int_{-\infty}^{-1} \frac{x^3+3x}{2+x^2}$ é divergente

Um Integral converge absolutamente se $\int_a^{+\infty} |f(x)|$ converge

Ex: $\int_1^{+\infty} \frac{\text{sen } x}{x^2} \, dx$

N podemos usar C. Comparação pq $-\frac{1}{x^2} \leq \frac{\text{sen } x}{x^2} \leq \frac{1}{x^2}$

$\left|\frac{\text{sen } x}{x^2}\right| \leq \frac{1}{x^2}$ como $\int_1^{+\infty} \frac{1}{x^2}$ converge $\left(\frac{1}{x^\alpha}, \alpha > 0\right)$ então $\int_1^{+\infty} \left|\frac{\text{sen } x}{x^2}\right| dx$ é convergente (pelo Critério de Comparação)

então $\int_1^{+\infty} \frac{\text{sen } x}{x^2}$ é absolutamente convergente e se o integral é absolutamente convergente então $\int_1^{+\infty} \frac{\text{sen } x}{x^2}$ é convergente

Transformada de Laplace

$\mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t) \, dt$ nos pontos $s \in \mathbb{R}$ tais que o integral é convergente

$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{f(t)\} = F(s)$

Ex:

- $\mathcal{L}\{1\}(s) = \int_0^{+\infty} e^{-st} \cdot 1 \, dt = \lim_{n \to +\infty} \int_0^{n} e^{-st} \cdot 1 \, dt = \lim_{n \to +\infty} -\frac{1}{s} \int_0^{n} -s e^{-st}$

$= \lim_{n \to +\infty} \left[-\frac{1}{s} e^{-st} \right]_0^{n} = \lim_{n \to +\infty} \left(-\frac{1}{s} e^{-sn} + \frac{1}{s} e^{0} \right)$ se $s > 0$ o integral é convergente

$= 0 + \frac{1}{s} \cdot 1 = \frac{1}{s}$

- $\mathcal{L}\{e^{at}\}(s) = \int_0^{+\infty} e^{-st} \cdot e^{at} \, dt = \int_0^{+\infty} e^{t(-s+a)} \, dt = \lim_{n \to +\infty} \frac{1}{-s+a} \int_0^{n} (-s+a) e^{t(-s+a)} \, dt$

$= \lim_{n \to +\infty} \frac{1}{-s+a} \left[e^{t(-s+a)} \right]_0^{n} = \lim_{n \to +\infty} \frac{1}{-s+a} \left(e^{n(-s+a)} - e^{0} \right) =$

$-s + a < 0 \iff s > a$

$= \frac{1}{-s+a} (-1) = \frac{1}{s-a}, \quad s > a$

$$\boxed{\cosh(at) = \frac{e^{at} + e^{-at}}{2} \qquad \operatorname{senh}(at) = \frac{e^{at} - e^{-at}}{2}}$$

- $\mathcal{L}\{\cosh(at)\}(s) = \mathcal{L}\left\{ \frac{e^{at} + e^{-at}}{2} \right\} = \frac{1}{2} \left(\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\} \right) = \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right) =$

$s > |a|$

$= \frac{1}{2} \left(\frac{s+a+s-a}{(s-a)(s+a)} \right) = \frac{1}{2} \left(\frac{2s}{s^2 - a^2} \right) = \frac{s}{s^2 - a^2}$

• $\mathcal{L}\{e^{t^2}\}(s) = \int_0^{+\infty} e^{-st} \cdot e^{t^2}\, dt$. Será convergente? Usando o Crit. Limite com $g(t) = e^{t}$

$$L = \lim_{t \to +\infty} \frac{e^{-st} \cdot e^{t^2}}{e^{t}} = \lim_{t \to +\infty} e^{-st - t + t^2} = \lim_{t \to +\infty} e^{t(-s-1+t)} = +\infty$$ (s n funciona como variável)

Pelo Crit. Limite, $\int_0^{+\infty} e^{-st} \cdot e^{t^2}\, dt$ é divergente $\forall s \in \mathbb{R}$

Então não existe $\mathcal{L}\{e^{t^2}\}(s)$

Funções de ordem exponencial

- Funções polinomiais
- Funções limitadas (integráveis)
- $f(t) = t^n e^{at} \cos(bt)$, $n \in \mathbb{N}_0$, $a, b \in \mathbb{R}$
- 〃 〃 $\operatorname{sen}(bt)$, 〃 〃 〃

Se f é de ordem exponencial k, então $\lim_{t \to +\infty} e^{-st} f(t) = 0$ com $s > k$

Propriedades , $F(s) = \mathcal{L}\{f(t)\}(s)$, $s > s_F$

- $\mathcal{L}\{e^{\lambda t} f(t)\}(s) = F(s-\lambda)$ $\longrightarrow$ Deslocamento da Transformada

$$s > \lambda + s_F$$

Ex Ficha 3

(9) 1) $\mathcal{L}\{e^{2t}\cos(5t)\}(s) = \mathcal{L}\{\cos(5t)\}(s-2) = \frac{s-2}{(s-2)^2+5^2}$, $s > 0+2$ $= \frac{s-2}{(s-2)^2+25}$, $s > 2$

2) $\mathcal{L}\{t e^{3t}\}(s) = \mathcal{L}\{t\}(s-3) = \frac{1}{(s-3)^2}$, $s > 3$

- $\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$, $s > a s_F$ $\longrightarrow$ Transformada da Expansão / Contração

Ex:

$$\mathcal{L}\{\cos(5t)\}(s) = \frac{1}{5}\mathcal{L}\{\cos(t)\}\left(\frac{s}{5}\right) = \frac{1}{5}\frac{\frac{s}{5}}{\left(\frac{s}{5}\right)^2+1} = \frac{1}{25}\cdot\frac{s}{\frac{s^2+25}{25}} = \frac{s}{s^2+25}, \quad s > 0$$

derivada de ordem n!

- $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$, $s > s_F$ $\longrightarrow$ Derivada da Transformada

Ex :

(9) $\mathcal{L}\{t\cosh(t)\}(s) = (-1)^1 \left(\mathcal{L}\{\cosh(t)\}\right)'(s) = -1\left(\frac{s}{s^2-1}\right)' = -\frac{(s^2-1)-2s(s)}{(s^2-1)^2}$

$= \frac{s^2+1}{(s^2-1)^2}, \quad s > |1|$

- $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

Transformada da derivada

$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$

$\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - sf(0) - f'(0)$

$\mathcal{L}\{f'''(t)\}(s) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$

Ex:

$\mathcal{L}\{\text{sen}^2(t)\}(s) = ?$ *

$(\text{sen}^2 t)' = 2\,\text{sen}\,t \cdot \cos t = \text{sen}(2t) \rightarrow \mathcal{L}\{\text{sen}(2t)\} = \mathcal{L}\{(\text{sen}^2 t)'\}(s) = s\,\underbrace{\mathcal{L}\{\text{sen}^2 t\}(s)}_{F(s)} - \underbrace{\text{sen}^2(0)}_{f(0)}$

$= s\mathcal{L}\{\text{sen}^2 t\}(s)$

$\mathcal{L}\{\text{sen}(2t)\} = s\mathcal{L}\{\text{sen}^2 t\}(s) \Leftrightarrow$

$\Leftrightarrow \quad \frac{2}{s^2+4} = s\mathcal{L}\{\text{sen}^2 t\}(s) \Leftrightarrow$

$(=) \quad \mathcal{L}\{\text{sen}^2 t\} = \frac{s}{s(s^2+4)}, \quad s>0$

Função de grau unitário

$$H_a(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \qquad \text{se } a=0 \qquad H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t<0 \end{cases}$$ ← Função de Heaviside

- $\mathcal{L}\{H_a(t) \cdot f(t-a)\}(s) = e^{-as} F(s), \quad s > s_F$ ← Transformada do deslocamento

Ex:

(a)
- $\mathcal{L}\{(1-H_\pi(t)) \cdot (\text{sen } t)\}(s) = \mathcal{L}\{\text{sen } t - H_\pi(t) \cdot \text{sen } t\}(s) =$

$\text{sen}(t-\pi) = -\text{sen}(t)$

$= \mathcal{L}\{\text{sen } t\} - \mathcal{L}\{H_\pi(t) \cdot \text{sen } t\}(s) = \frac{1}{s^2+1} - \mathcal{L}\{-H_\pi(t) \cdot \text{sen}(t-\pi)\}(s)$

$= \frac{1}{s^2+1} + \mathcal{L}\{H_\pi(t) \cdot \text{sen}(t-\pi)\}(s) = \frac{1}{s^2+1} + e^{-\pi s} \mathcal{L}\{\text{sen } t\}$

$= \frac{1}{s^2+1} + e^{-\pi s} \cdot \frac{1}{s^2+1} = \frac{e^{-\pi s}+1}{s^2+1}, \quad s>0$

- $\mathcal{L}\{(t-2)^2 e^{2(t-2)} H_2(t)\}(s) = *$

$f(t) = t^2 e^{2t}$

$$* = e^{-2s} \mathcal{L}\{t^2 e^{2t}\}(s) = e^{-2s}(-1)^2 \mathcal{L}\{e^{2t}\}''(s) = e^{-2s}\left(\frac{1}{s-2}\right)'' =$$

$$= e^{-2s}\left(-\frac{1}{(s-2)^2}\right)' = e^{-2s}\left(\frac{2(s-2)}{(s-2)^4}\right) = e^{-2s} \cdot \frac{2}{(s-2)^3}, \quad s > 2$$

- $\mathcal{L}\{(f * g)(t) = F(s) \cdot G(s)$ ← Transformada da Convolução

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau$$

Ex: $(f * g)(t) = \int_0^t \tau \operatorname{sen}(t - \tau)\, dt$

$$\mathcal{L}\{(f * g)(t)\}(s) = \mathcal{L}\{t * \operatorname{sen} t\}(s) = \mathcal{L}\{t\}(s) \cdot \mathcal{L}\{\operatorname{sen}(t)\}(s) = \frac{1}{s^2} \cdot \frac{1}{s^2+1}, \quad s > 0$$

Ex

(11) Usando a transformada de Laplace, calcule

a) $\int_0^{+\infty} t^{10} e^{-2t}\, dt$

$$\mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t)\, dt$$

$$\mathcal{L}\{t^{10}\}(2) = \frac{10!}{2^{11}} \ , \ s > 0$$

b) $$\int_0^{+\infty} e^{-3t}\ t \cdot \operatorname{sen} t\ dt = \mathcal{L}\{t \cdot \operatorname{sen}(t)\}(3) = (-1)\left(\mathcal{L}\{\operatorname{sen}(t)\}\right)'(3) = -\left(\frac{1}{s^2+1}\right)'\Big|_{s=3}$$

$$= \frac{2s}{(s^2+1)^2}\Big|_{s=3} = \frac{6}{10^2} = \frac{3}{50}$$

(12) $f : \mathbb{R} \to \mathbb{R}$ diferenciavel . $f'(t) + 2f(t) = e^t$ e $f(0) = 2$. $f(t) = ?$

$$\mathcal{L}\{f'(t) + 2f(t)\}(s) = \mathcal{L}\{e^t\}(s) \Leftrightarrow \mathcal{L}\{f'(t)\} + 2\mathcal{L}\{f(t)\} = \mathcal{L}\{e^t\}$$

$$s\mathcal{L}\{f(t)\} - f(0) + 2\mathcal{L}\{f(t)\} = \frac{1}{s-1} \Leftrightarrow$$

$$\Leftrightarrow s\mathcal{L}\{f(t)\} + 2\mathcal{L}\{f(t)\} = \frac{1}{s-1} + f(0) \Leftrightarrow$$

$$\Leftrightarrow s\mathcal{L}\{f(t)\} + 2\mathcal{L}\{f(t)\} = \frac{1}{s-1} + 2 \Leftrightarrow (s+2)\mathcal{L}\{f(t)\} = \frac{1+2s-2}{s-1} \Leftrightarrow$$

$$\Leftrightarrow \mathcal{L}\{f(t)\} = \frac{2s-1}{(s-1)(s+2)}, \ s > -2 \quad \Leftrightarrow \quad f(t) = \mathcal{L}^{-1}\left\{\frac{2s-1}{(s-1)(s+2)}\right\}$$

Transformada de Laplace Inversa

$$\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$$

- grau do denominador tem de ser maior que grau do numerador
- $\mathcal{L}^{-1}\{F(s-\lambda)\} = e^{\lambda t}\,\mathcal{L}^{-1}\{F(s)\}$

Ex

10 $F(s) = \mathcal{L}\{f(t)\}(s) = \dfrac{1}{s^2+6s+9} = \dfrac{1}{(s+3)^2}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}(t) = e^{-3t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{-3t}\cdot t\,,\quad t>0$$

- $\mathcal{L}^{-1}\{FG\} = \mathcal{L}^{-1}\{F\} * \mathcal{L}^{-1}\{G\}$

Ex

10 $F(s) = \dfrac{1}{s^2+s-2}$

1) Com primitivas (mexer no denominador)

C.A $\quad s^2+s-2=0 \Leftrightarrow s=-2 \vee s=1$

$$F(s) = \frac{1}{(s+2)(s-1)} \qquad f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s-1)}\right\}(t) = \mathcal{L}^{-1}\left\{\overbrace{\frac{1}{s+2}}^{\mathcal{L}\{e^{-2t}\}}\cdot\overbrace{\frac{1}{s-1}}^{\mathcal{L}\{e^{t}\}}\right\} =$$

$$= e^{-2t} * e^{t} = \int_0^t e^{-2(t-\tau)}\cdot e^{\tau}\,d\tau$$

$$= \int_0^t e^{-2t+2\tau+\tau}\,d\tau = \int_0^t e^{-2t+3\tau}\,d\tau = e^{-2t}\int_0^t e^{3\tau}\,d\tau = e^{-2t}\,\frac{1}{3}\int_0^t 3e^{3\tau}\,dt = e^{-2t}\cdot\frac{1}{3}\left[e^{3\tau}\right]_0^t$$

$$= e^{-2t}\cdot\frac{1}{3}\left(e^{3t} - e^0\right) = e^{-2t}\cdot\frac{1}{3}\left(e^{3t} - 1\right) = e^{-2t}\left(\frac{1}{3}e^{3t} - \frac{1}{3}\right) = \frac{e^t - e^{-2t}}{3}$$

ou 2)

$$F(s) = \frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

$$\begin{cases} A+B = 0 \\ -A+2B = 1 \end{cases} \qquad \begin{cases} A = -B \\ 3B = 1 \end{cases} \qquad \begin{cases} A = -1/3 \\ B = 1/3 \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s-1)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{-1/3}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1/3}{s-1}\right\} = -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t = \frac{e^t - e^{-2t}}{3}$$

Ex 18

c) $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2-4s+6}\right\}$

C.A $s^2 - 4s + 6 = 0$ imp.

$s^2 - 4s + 6 = (s^2-4s) + 6 \Leftrightarrow (s^2-4s+4) + 6 - 4 \Leftrightarrow (s-2)^2 + 2$

Para $\mathcal{L}^{-1}$ mexemos no denominador ↗ convolução ↘ frações simples
ou
fazemos caso notavé

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{(s-2)^2+2}\right\}(t)$$

$$= 2\,\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+2}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+\sqrt{2}^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+\sqrt{2}^2}\right\} =$$

$$= 2e^{2t}\,\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+\sqrt{2}^2}\right\} - e^{2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+\sqrt{2}^2}\right\} = 2e^{2t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+\sqrt{2}^2}\right\} + 4e^{2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+\sqrt{2}^2}\right\} - e^{2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+\sqrt{2}^2}\right\}$$

$$= 2e^{2t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+\sqrt{2}^2}\right\} + 3e^{2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+\sqrt{2}^2}\right\} = 2e^{2t}\cdot\cos(\sqrt{2}\,t) + \frac{3e^{2t}}{\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+\sqrt{2}^2}\right\}$$

$$= 2e^{2t}\cdot\cos(\sqrt{2}\,t) + \frac{3e^{2t}}{\sqrt{2}}\cdot\operatorname{sen}(\sqrt{2}\,t)\,,\quad t \geq 0$$

d) $\mathcal{L}^{-1}\left\{\frac{2s}{(s-1)(s^2+2s+5)}\right\}$

C.A. $s^2+2s+5=0$ imp

$$\frac{2s}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5} = \frac{A(s^2+2s+5)+B(s^2-s)+C(s-1)}{(s-1)(s^2+2s+5)}$$

$$\begin{cases} A+B=0 \\ 2A-B+C=2 \\ 5A-C=0 \end{cases} \quad \begin{cases} A=-B \\ 2A+A+5A=2 \\ C=5A \end{cases} \quad \Big\} \; 8A=2 \quad \begin{cases} B=-1/4 \\ A=\frac{1}{4} \\ C=5/4 \end{cases} \rightarrow \mathcal{L}^{-1}\left\{\frac{1/4}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/4\,s+5/4}{s^2+2s+5}\right\}$$

$$= \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 1/4\,\mathcal{L}^{-1}\left\{\frac{s-5}{s^2+2s+5}\right\} = \frac{1}{4}e^{t} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s-5}{(s^2+2s+1)+4}\right\} = \frac{e^t}{4} - \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{s-5}{(s-1)^2+2^2}\right\}$$

$$= \frac{e^t}{4} - \frac{1}{4}e^{-t}\,\mathcal{L}^{-1}\left\{\frac{s-6}{s^2+2^2}\right\} = \frac{e^t}{4} - \frac{e^{-t}}{4}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{6}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}\right) =$$

$$= \frac{e^t}{4} - \frac{e^{-t}}{4}\Big(\cos(2t) - 3\operatorname{sen}(2t)\Big) = \frac{e^t}{4} - \frac{e^{-t}}{4}\cos(2t) + \frac{3}{4}e^{-t}\operatorname{sen}(2t) =$$

(20) $f''(t) - f'(t) = 2e^t \,, \quad f(0) = f'(0) = 0 \,, \quad f(t) = ?$

$$\mathcal{L}\left\{f''(t) - f'(t)\right\}(s) = \mathcal{L}\left\{2e^t\right\} \Leftrightarrow$$

$$\Leftrightarrow \mathcal{L}\left\{f''(t)\right\}(s) - \mathcal{L}\left\{f'(t)\right\}(s) = 2\mathcal{L}\left\{e^t\right\}(s) \Leftrightarrow$$

$$\Leftrightarrow s^2 F(s) - s\underbrace{f(0)}_{0} - \underbrace{f'(0)}_{0} - sF(s) + \underbrace{f(0)}_{0} = 2\,\frac{1}{s-1}$$

$$\Leftrightarrow (s^2 - s)F(s) = \frac{2}{s-1} \Leftrightarrow F(s) = \frac{2}{(s-1)(s^2-s)} \Leftrightarrow F(s) = \frac{2}{s(s-1)^2} \Leftrightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s(s-1)^2}\right\} \Leftrightarrow$$

$$\Leftrightarrow f(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s-1)^2}\right\} \Leftrightarrow f(t) = 2\left(1 * e^t . t\right)$$

$$\Leftrightarrow f(t) = 2\int_0^t 1 . e^{\tau}\tau \, d\tau \Leftrightarrow f(t) = 2\left(\tau e^{\tau}\Big|_0^{t} - \int_0^t e^{\tau}\, d\tau\right) \Leftrightarrow f(t) = 2\left(te^t - 0 - (e^t - e^0)\right) \Leftrightarrow$$

$u = \tau \quad dv = e^{\tau}$

$u' = 1 \quad v = e^{\tau}$

$$\Leftrightarrow f(t) = 2te^t - 2e^t + 2$$

Miniteste até aqui