Participation-Adaptive Pricing*

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Abstract

A large number of buyers with single unit demand have a common value for the good being sold. We describe a new class of mechanisms, which we refer to as *participation-adaptive pricing:* Each prospective buyer indicates whether or not they want to buy a good, available goods are rationed among those who wish to purchase, and the market price is an increasing function of the number of buyers who wish to purchase. We characterize a class of pricing rules for which, as the number of buyers grows large, the expected market price converges to the expected value, regardless of the buyers' information and equilibrium strategies.

Keywords: Mechanism design, common value auction, full surplus extraction, large market, robustness, Bayes correlated equilibrium, coarse correlated equilibrium.

JEL Classification: C72, D44, D82, D83.

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1 Introduction

Economists have long extolled the virtues of prices to aggregate private information and efficiently coordinate trade. But an element that is often missing from this story is a detailed mechanism by which the private information of large numbers of individual traders is incorporated into prices. In particular, the standard models of competitive markets take as a premise that traders are price takers, meaning that each individual trader has no effect on the price, even though the price incorporates the traders' information in the aggregate. A seemingly more natural approach would be to model an economy with a large but finite number of traders, where each trader makes strategic choices based on their private information and has the ability, however small, to influence the market price. But are there reasonable models of the game in the finite market such that equilibrium behavior approximates the competitive model, when the market is large?

In a classic article, Wilson (1977) argued that a simple mechanism, the first-price auction, achieves that approximate aggregation of private information through prices in large finite markets. The setting is one in which a single unit of a good is available, the buyers all have the same value for that good, but each buyer only observes the value with conditionally independent noise, i.e., the so-called "mineral rights" model. If the distribution of the value and the noise term are held fixed as the number of buyers grows large, the market price converges to the true value.

Wilson's finding suggested that first-price auctions can microfound the aggregation of private information through prices, and provide a model for price discovery in competitive markets. Subsequent work by Milgrom (1979), Pesendorfer and Swinkels (1997), Kremer (2002) and Bali and Jackson (2002) delivered similar conclusions for second-price auctions, English auctions, and other "standard" mechanisms. However, these results rely on the assumption that the individual buyers are "informationally small," in the sense that as the market grows large, each individual buyer's information is negligible compared to the rest of the market. Moreover, there are settings for which the expected price does not converge to the expected value, even when the number of buyers grows large (Engelbrecht-Wiggans, Milgrom, and Weber, 1983; Bergemann, Brooks, and Morris, 2017).

More recently, Du (2018) and Brooks and Du (2021) constructed different classes of mechanisms, distinct from the first-price auction, for which the expected price *does* converge to the expected value, regardless of the form of the buys' information, and regardless of which equilibrium is being played. In the case of Brooks and Du (2021), the mechanisms have the form of "proportional auctions," in which each buyer's probability of receiving the good is proportional to their bid, and the market price depends on the sum of the bids.

In this paper, we demonstrate that an even simpler class of mechanisms suffices to microfound information aggregation through prices in competitive markets, regardless of the information structure and equilibrium. We enrich the setting to allow for uncertainty about the value, the number of buyers, as well as the number of units that are available For example, our model accommodates scenarios in which there are enough units to serve a constant fraction of the potential buyers, even as the number of potential buyers grows large. All we assume is that the fraction of the market that can be served is bounded away from one. In the mechanism, each buyer simply indicates whether or not they wish to buy the good at a market price.[1] If more buyers wish to buy than there are units, then ties are broken randomly, and otherwise all orders are filled. The market price, in turn, is determined as a function of the number of buyers who wish to buy. We characterize pricing rules for which, in the limit as the number of buyers goes to infinity (in probability), all of the goods are sold, and the expected price is equal to the expected value. This limit is attained regardless of the sequences of buyers' information structures and the equilibria being played.

In fact, the optimal pricing rules have an astonishingly simple form, that does not depend at all on the fundamentals of the economy, except through an upper bound on the support of the value and an upper bound on the fraction of the market that can be served. Let us suppose that there is enough of the good to satisfy the demand of at most a fraction $\kappa \in (0, 1)$ of the market. The pricing rule depends on two parameters, $\alpha \in (\kappa, 1)$ and $\epsilon > 0$. We define the *participation rate* to be the share of potential buyers who wish to buy. The price is zero until the participation rate exceeds $\alpha - \epsilon$. Moreover, if the participation rate is greater than $\alpha + \epsilon$, then the price is equal to the given upper bound on the value. For intermediate participation rates, the price rises linearly from zero to the upper bound. We refer to the mechanism consisting of the aforementioned allocation and pricing rules as *participation-adaptive pricing*.

Our main result is that if we take a sequence of markets where the number of potential buyers N is going to infinity in probability, then as long as ϵ goes to zero slower than $1/N$, then the expected price will converge to the expected value. Moreover, the optimal rate, which is achieved when ϵ is on the order of $1/\sqrt{N}$, is unimprovable, and is the same rate as achieved by the proportional auctions of Brooks and Du (2021).

Participation-adaptive pricing induces a competitive outcome in large markets, by which we mean that that the expected price is equal to the expected value, and trade occurs with probability arbitrarily close to one. However, it is not necessarily the case that the equilibrium price aggregates private information. For one thing, in contrast with Wilson (1977), it need not even be the case that the join of the players information identifies the value, even in the limit when the number of buyers grows large. More than that, it could be that some buyers have information about the value that is not incorporated into the price, though these buyers have a negligible effect on the equilibrium outcome. The distinction between competitive pricing and information aggregation was previously made by Kremer (2002) in the context of the mineral rights model.

There are two features of participation-adaptive pricing that are essential to achieving the competitive limit. One is that the price rises continuously, as a function of the fraction of the buyers who participate. As a result, the price that a buyer expects to pay if they opt in is close to the price if they opt out; the buyers are essentially price takers. This ensures that when the market is large, the interim expected values of buyers who are opting out cannot be above the equilibrium market price with high probability. The second property is that intermediate prices occur for only a narrow window of participation rates. This ensures that there is little uncertainty about the participation rate when the market is large. In particular, if the participation rate were too low, the price would be extremely low, and more buyers would want to opt in, and if the participation rate were too high, then the price

[1] In other words, a buyer submits a *market order* to buy one unit of the good. A market order is distinguished from a *limit order* which buys or sells at a specific price.

would be extremely high, and more buyers would want to opt out. This feature eliminates the possibility of a *winner's curse*: All else equal, a buyer is more likely to be allocated the good if the participation rate is low, so if buyers are more likely to opt in when their interim values are high, being allocated would be a negative signal about the value. But because there is little uncertainty about the participation rate when the market is large, any such inference is negligible. Somewhat surprisingly, participation-adaptive pricing also rules out any winner's curse that might arise through correlation between the number of potential buyers and the value, such as that described in Lauermann and Wolinsky (2017, 2022). Our main result shows that the competitive outcome is still obtained even when the number of potential buyers and the value are both uncertain and correlated.

The rest of this paper is organized as follows. Section 2 describes our model. Section 3 presents two examples illustrating potential problems that are avoided by participation-adaptive pricing. Section 4 presents our main result. Section 5 is a discussion and conclusion.

2 Model

Let the state space be $\Theta = [0,1] \times \mathbb{Z}_+ \times \mathbb{Z}_+$. A state $\theta = (v, N, K) \in \Theta$ means that there are K units of a good, and there are N buyers with a unit-demand and a pure common value v for the good. Let $\mu \in \Delta(\Theta)$ be a distribution over the states. We suppose $\mu(\{N \leqslant \overline{N}\}) = 1$ for some $\overline{N} \in \mathbb{Z}_+$.[2]

The buyers' private information about the state is described by an information structure $I = (S, \sigma)$, where S_i is a finite set of signals (or types) for buyer i, $S = \prod_{i=1}^{\overline{N}} S_i$, and $\sigma \in \Delta(\Theta \times S)$ is the joint distribution of the states and signals such that $\mathrm{marg}_\Theta\, \sigma = \mu$. Moreover, we require each S_i contains a null type $\varnothing$; if $s_i = \varnothing$, then buyer i is not present. Thus, for consistency we also require that for every (v, N, K, s) in the support of σ, we have $N = |\{i : s_i \neq \varnothing\}|$.

A market mechanism is a tuple $M = (A, q, t)$, where A_i is a set of actions for buyer i, $A = \prod_{i=1}^{\overline{N}} A_i$ are the action profiles, and $q_i : A \to [0,1]$ is the allocation rule, and $t_i : A \to \mathbb{R}$ is the transfer rule. The allocation rule must satisfy $\sum_i q_i(a) \leqslant K$ for all $a \in A$. Buyer i's utility is $u_i = vq_i(a) - t_i(a)$. We assume for every buyer i there exists an opt-out (or *participation secure*) action $0 \in A_i$ such that $q_i(0, a_{-i}) = t_i(0, a_{-i}) = 0$ for all $a_{-i} \in A_{-i}$.[3] To model the absence of some buyers, for every i we also include a null action $\varnothing$ in A_i; if $a_i = \varnothing$ then buyer i is not present in the mechanism, and hence $q_i(\varnothing, a_{-i}) = t_i(\varnothing, a_{-i}) = 0$ for all $a_{-i} \in A_{-i}$.

The opt-out action is a strategic choice in the mechanism, while the null action is a modeling device to represent the absence of a buyer. Thus we restrict the (behavioral) strategy for buyer i to be a mapping $b_i : S_i \to \Delta(A_i)$ such that $b_i(\varnothing \mid \varnothing) = 1$ and $b_i(\varnothing \mid s_i) = 0$ for $s_i \neq \varnothing$, where $b_i(a_i|s_i)$ is the probability of action a_i given signal/type s_i.

[2] In Theorem 1 below, we will consider a sequence of priors μ_l where the number of buyers goes to infinity in probability, and the corresponding upper bounds $\overline{N}_l$ go to infinity as well.

[3] The notion of participation security is adapted from Brooks and Du (2021) and Brooks and Du (2023), as are other features of the model.

Buyer i's expected utility given a strategy profile b is

$$U_i(M, I, b) = \int_{v,N,K,s,a} (vq_i(a) - t_i(a)) \prod_i b_i(a_i|s_i)\sigma(dv, dN, dK, ds).$$

A (Bayes-Nash) equilibrium for the Bayesian game (M, I) is a strategy profile b such that $U_i(M, I, b) \geqslant U_i(M, I, (b'_i, b_{-i}))$ for all strategy b'_i and all buyer i. Let $\mathcal{E}(M, I)$ be the set of equilibria for (M, I). Because M and I are finite the set of equilibria is always non-empty.

Let $R(M, I, b)$ be the expected revenue at an equilibrium b:

$$R(M, I, b) = \int_{v,N,K,s,a} \sum_i t_i(a) \prod_i b_i(a_i|s_i)\sigma(dv, dN, dK, ds).$$

Define the *revenue guarantee* of a mechanism M as the minimum expected revenue over all information structures I and all equilibria b of (M, I):

$$G(M) = \inf_I \inf_{b \in \mathcal{E}(M,I)} R(M, I, b).$$

The efficient surplus is

$$\int_{v,N,K} Kv\, \mu(dv, dN, dK).$$

In this paper we focus on a particular class of mechanisms, $M(p) = (A, q, t)$, where $A_i = \{\varnothing, 0, 1\}$, (1 means to opt in, and 0 means to opt out)

$$\begin{aligned} q_i(a) &= \begin{cases} \min(K/n(a), 1) & a_i = 1, \\ 0 & a_i \in \{0, \varnothing\}, \end{cases} \\ t_i(a) &= q_i(a)p(x), \quad x = n(a)/N(a), \end{aligned} \tag{1}$$

where $n(a) = |\{i : a_i = 1\}|$ is the number of buyers who opt in, $N(a) = |\{i : a_i \neq \varnothing\}|$ is the potential number of buyers, x is the participation rate, and $p : [0, 1] \to \mathbb{R}$ is a pricing function. For mechanisms of this form, one can interpret opting in as a market order to buy one unit of the good, at whatever is the prevailing market price. In contrast, the bid in a first-price or second-price auction should be interpreted as a limit order to buy a unit at a fixed price (cf. Jovanovic and Menkveld, 2022).

3 Examples

Before describing the pricing functions that attain the competitive limit, let us first consider two natural candidates, and explain why they would fail to induce competitive outcomes even when N is large. We assume $K = 1$ and a commonly known N in this section for simplicity. This will serve to motivate the pricing rules that we propose.

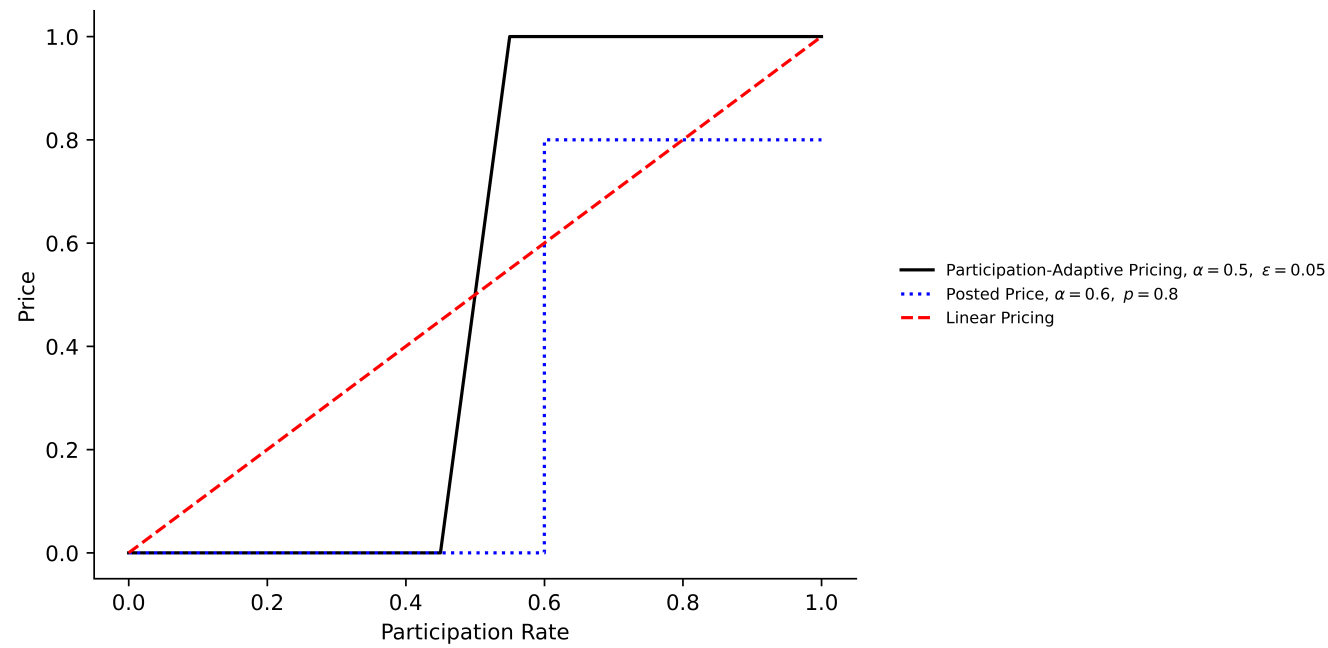

Figure 1: Pricing rules.

3.1 Posted Price

The first candidate is a class of "posted price" rules:

$$p_\alpha(x) = \begin{cases} 0 & x \leqslant \alpha, \\ p & x > \alpha, \end{cases}$$

where α is a participation cutoff and $p \in [0, 1]$ is a posted price. This rule is depicted in Figure 1. It can be viewed as a generalization of the conventional posted price mechanism, which is obtained when $\alpha = 0$.

Taking this pricing rule as given, we now exhibit an information structure for which the expected price is bounded away from the expected value, no matter how large is the market. First, consider the case where $p < 1$. Suppose $v \in \{0, 1\}$, both values equally likely. Furthermore, assume that the buyers have full information, meaning that for all i, $s_i = 1$ when $v = 1$ and $s_i = 0$ when $v = 0$. An equilibrium is that all buyers opt in if and only if $v = 1$. Therefore, regardless of N, revenue is $p/2$, which is strictly below the expected value of $1/2$, no matter how large is the market.

Now consider the case where $p = 1$. Suppose the information structure is such that only $\lfloor \alpha N \rfloor$ buyers have a full information about the value, and the rest of the buyers have no information beyond the prior distribution. Then it is an equilibrium for the $\lfloor \alpha N \rfloor$ buyers with full information to opt in if $v = 1$ and opt out otherwise, and the buyers with no information always opt out. Under such an equilibrium, the participation rate is always just below α, and hence the market price is 0.

The reason for the low revenue in these cases is that the pricing function has a sudden jump from 0 to p, and the limited number of prices precludes price discovery. To remedy these flaws, the pricing rules that we propose vary continuously in the participation rate.

3.2 Linear Pricing

Another natural pricing function is the linear rule $p(x) = x$. For comparison, this rule is also depicted in Figure 1.

Let us construct an information structure where the linear pricing fails to obtain the efficient surplus even when N is large. Again, suppose that $v \in \{0, 1\}$ and both are equally likely. Suppose N is even and let $S_i = \{0, 1, u\}$. If $v = 1$, then exactly $1/2$ fraction of the buyers (uniformly drawn from the set of all buyers) observe the uninformative signal $s_i = u$, and the other $1/2$ of the buyers observe the perfectly informative signal $s_i = 1$. Likewise, if $v = 0$, then exactly $1/2$ fraction of the buyers (uniformly drawn from the set of all buyers) observe the uninformative signal $s_i = u$, and the other $1/2$ observe the perfectly informative signal $s_i = 0$.

We claim that for this information structure, it is an equilibrium for the buyers to opt in if $s_i = 1$ and to opt out otherwise. The equilibrium constraints for $s_i = 1$ and $s_i = 0$ are trivial, because under the proposed strategies, the price for the $s_i = 1$ types is just $1/2$, so they strictly prefer to opt in, whereas for the $s_i = 0$ types the price is 0, but the value is zero too, so they are happy to opt out. For the uninformed $s_i = u$ types, the payoff from opting in is

$$\frac{1}{2}\left(1-\left(\frac{1}{2}+\frac{1}{N}\right)\right)\frac{1}{\frac{N}{2}+1}+\frac{1}{2}\left(0-\frac{1}{N}\right) < \frac{1}{4}\frac{2}{N}-\frac{1}{2}\frac{1}{N}=0.$$

As this payoff is non-positive, opt out is optimal for the $s_i = u$ types in equilibrium.

In this equilibrium, the price is positive only if $v = 1$, but the price is $1/2 < 1$, so the expected price is bounded away from the expected value regardless of N. In effect, there is a winner's curse that keeps the $s_i = u$ types from opting in. Were an uninformed buyer to opt in, they would win with probability 1 when $v = 0$ and obtain a net payoff of $-1/N$, but they would only win with probability $2/(N + 2)$ when $v = 1$ and obtain a net payoff of $1/2 - 1/N$ conditional on winning. The net payoff is negative. In contrast, the rules we propose in the next section force the equilibrium participation rate to be in a narrow window with high probability. This effectively shuts down any updating about the value from the fact that one is allocated the good, and thereby precludes a winner's curse.

4 Main result

We now present pricing rules that guarantee competitive outcomes in large markets. Let

$$p_{\alpha,\epsilon}(x) = \begin{cases} 0 & x \leqslant \alpha - \epsilon, \\ \frac{x-(\alpha-\epsilon)}{2\epsilon} & \alpha - \epsilon \leqslant x \leqslant \alpha + \epsilon, \\ 1 & x \geqslant \alpha + \epsilon, \end{cases} \tag{2}$$

where $0 \leqslant \alpha - \epsilon$ and $\alpha + \epsilon \leqslant 1$. We call the mechanism $M(p_{\alpha,\epsilon})$ (cf. equation (1)) *participation-adaptive pricing*. An example of such a rule is depicted in Figure 1 for comparison. Note that linear pricing is a special case of participation-adaptive pricing, and so is posted price when the price is 1.

The following is our main result:

Theorem 1. *Let $\{\mu_l : l \in \mathbb{Z}_+\}$ be a sequence of state distributions and let (v_l, N_l, K_l) be the corresponding sequence of random variables. Consider a sequence of participation-adaptive pricing $\{M(p_{\alpha,\epsilon_l}) : l \in \mathbb{Z}_+\}$. Suppose ϵ_l converges to 0 and $\epsilon_l N_l$ converges in probability to ∞ as $l \to \infty$, and $K_l/N_l < \alpha$ with probability one for every l. Then there exists a sequence of random variables $\{\delta_l : l \in \mathbb{Z}_+\}$ such that $G(M(p_{\alpha,\epsilon_l})) \geqslant \mathbb{E}[K_l(v_l - \delta_l)]$ for every l and δ_l converges in probability to zero as $l \to \infty$.*

The proof of Theorem 1 relies on methodology that was previously developed in Du (2018), Brooks and Du (2021), and related work. In particular, Brooks and Du (2023) present a general theory for informationally robust mechanism design. For any mechanism, a lower bound on performance can be computed using an object known as the *strategic virtual objective*. In the present context, the strategic virtual objective depends on an arbitrary parameter $\beta > 0$, and is defined as

$$\lambda((v,N,K),a) = \sum_i t_i(a) + \beta \sum_i \mathbb{I}_{a_i=0} \left[v(q_i(a_i+1, a_{-i}) - q_i(a)) - (t_i(a_i+1,a_{-i}) - t_i(a)) \right]. \quad (3)$$

The strategic virtual objective is essentially the objective in a Lagrangian for minimizing expected revenue subject to obedience constraints. Importantly, we drop all obedience constraints except those for buyers who are "recommended" in equilibrium to opt out, and β is the Lagrange multiplier on those obedience constraints. The following result is an immediate consequence of Theorem 1 of Brooks and Du (2023):

Lemma 1. *The revenue guarantee of participation-adaptive pricing is at least*

$$\int_{v,N,K} \min_{a:N(a)=N} \lambda((v,N,K),a)\mu(dv,dN,dK). \quad (4)$$

For the sake of completeness, we will sketch the logic behind the lower bound (4). In any information structure and equilibrium, whenever buyer i chooses to opt out, they must prefer opting out to opting in. Thus, the difference in expected utility utility

$$\int_{v,N,K,s,a} \mathbb{I}_{a_i=0}(v(q_i(a_i+1,a_{-i}) - q_i(a)) - (t_i(a_i+1,a_{-i}) - t_i(a))) \prod_j b_j(a_j|s_j)\sigma(dv,dN,dK,ds)$$

must be non-positive. As a result, for any $\beta \geqslant 0$, we can obtain a lower bound on revenue by adding these terms multiplied by β to expected revenue. This is equivalent to the assertion that in any information structure and equilibrium, expected revenue is at least

$$\int_{v,N,K,s,a} \lambda((v,N,K),a) \prod_j b_j(a_j|s_j)\sigma(dv,dN,dK,ds).$$

This expression must be weakly greater than what we obtain by, for each (v,N,K), replacing the integrand with the minimum of $\lambda((v,N,K),a)$ across all a with $N(a) = N$, which is precisely (4).

We can now complete the proof of our main result.

Proof of Theorem 1. Consider a state distribution μ and a participation-adaptive pricing rule $p_{\alpha,\epsilon}$. The strategic virtual objective of the mechanism is:

$$\begin{aligned}
&\lambda((v,N,K),a)\\
&= n(a)\min\left(\frac{K}{n(a)},1\right)p_{\alpha,\epsilon}\left(\frac{n(a)}{N}\right)+\beta v(N-n(a))\min\left(\frac{K}{n(a)+1},1\right)\\
&\quad-\beta(N-n(a))\min\left(\frac{K}{n(a)+1},1\right)p_{\alpha,\epsilon}\left(\frac{n(a)+1}{N}\right)\\
&= \beta v(N-n(a))\min\left(\frac{K}{n(a)+1},1\right)\\
&\quad+\left(n(a)\min\left(\frac{K}{n(a)},1\right)-\beta(N-n(a))\min\left(\frac{K}{n(a)+1},1\right)\right)p_{\alpha,\epsilon}\left(\frac{n(a)}{N}\right)\\
&\quad-\beta(N-n(a))\min\left(\frac{K}{n(a)+1},1\right)\left(p_{\alpha,\epsilon}\left(\frac{n(a)+1}{N}\right)-p_{\alpha,\epsilon}\left(\frac{n(a)}{N}\right)\right).
\end{aligned}$$

Let us write $n \equiv n(a)$. Clearly, $\lambda((v,N,K),a)$ depends only on n, so with a slight abuse of notation, we will write it as $\lambda((v,N,K),n)$, which we minimize over $n \in \{0,1,\ldots,N\}$.

When $(n+1)/N \leqslant \alpha-\epsilon$, $p_{\alpha,\epsilon}$ is zero, and $\lambda((v,N,K),n)$ is decreasing in n. Thus,

$$\begin{aligned}
\min_{n\leqslant N(\alpha-\epsilon)-1}\lambda((v,N,K),n) &= v\beta K\frac{N-n}{n+1}\bigg|_{n=\lfloor N(\alpha-\epsilon)\rfloor-1}\\
&\geqslant Kv-K\left|1-\beta\frac{N-n}{n+1}\right|\Bigg|_{n=\lfloor N(\alpha-\epsilon)\rfloor-1},
\end{aligned}$$

where we used the fact that $\lfloor N(\alpha-\epsilon)\rfloor \geqslant K$ with probability 1 in μ, and hence the min constraint in the allocation is not binding.

Likewise, when $n/N \geqslant \alpha+\epsilon$, $p_{\alpha,\epsilon}$ is one, and $\lambda((v,N,K),n)$ is increasing in n, and hence

$$\begin{aligned}
\min_{n\geqslant N(\alpha+\epsilon)}\lambda((v,N,K),n) &= K\left(1-(1-v)\beta\frac{N-n}{n+1}\right)\Bigg|_{n=\lceil N(\alpha+\epsilon)\rceil}\\
&\geqslant Kv-2K\left|1-\beta\frac{N-n}{n+1}\right|\Bigg|_{n=\lceil N(\alpha+\epsilon)\rceil}.
\end{aligned}$$

When $(n+1)/N > \alpha-\epsilon$ and $n/N < \alpha+\epsilon$, we have

$$\begin{aligned}
\lambda((v,N,K),n) &\geqslant \beta vK\frac{N-n}{n+1}-\left|K-\beta K\frac{N-n}{n+1}\right|-\beta K\frac{N-n}{n+1}\frac{1}{N\epsilon}\\
&\geqslant Kv-2K\left|1-\beta\frac{N-n}{n+1}\right|-\beta K\frac{N-n}{n+1}\frac{1}{N\epsilon}.
\end{aligned}$$

We set[4]

$$\beta=\frac{\alpha}{1-\alpha}.$$

[4]Thus, the value of the multiplier β is tightly connected to the parameters of the pricing rule and the participation rates that minimize the lower bound on revenue. At first glance, this seems at odds with the

Then for $\lfloor N(\alpha-\epsilon)\rfloor - 1 \leqslant n \leqslant \lceil N(\alpha+\epsilon)\rceil$, we have

$$\begin{aligned}\left|1-\beta\frac{N-n}{n+1}\right| &= \frac{|(1-\alpha)(n+1)-\alpha(N-n)|}{(1-\alpha)(n+1)} \\ &\leqslant \frac{|n-\alpha N|+(1-\alpha)}{(1-\alpha)(n+1)} \\ &\leqslant \frac{\epsilon N+2}{(1-\alpha)(N(\alpha-\epsilon)-1)} = \frac{\epsilon+2/N}{(1-\alpha)(\alpha-\epsilon-1/N)}.\end{aligned}$$

We now set $v = v_l$, $N = N_l$, $K = K_l$ and $\epsilon = \epsilon_l$, according to the hypotheses of the theorem. Define

$$\delta_l = 2\cdot\frac{\epsilon_l+2/N_l}{(1-\alpha)(\alpha-\epsilon_l-1/N_l)} + \left(1+\frac{\epsilon_l+2/N_l}{(1-\alpha)(\alpha-\epsilon_l-1/N_l)}\right)\frac{1}{N_l\epsilon_l}. \tag{5}$$

Clearly δ_l converges in probability to zero as $l \to \infty$. Moreover, the previous arguments imply that

$$\lambda((v_l, N_l, K_l), n) \geqslant K_l(v_l - \delta_l).$$

The theorem then follows from Lemma 1. □

Consistent with our previous discussion, the error bound (5) demonstrates that there are tradeoffs in choosing ϵ for a fixed N. In particular, making ϵ smaller reduces uncertainty about the participation rate, but it also increases the size of the price jump when another buyer opts in. In fact, examining (5), it is clear that the optimal balance between these two forces is achieved when ϵ is on the order of $1/\sqrt{N}$. This observation immediately yields the following corollary:

Corollary 1. *Consider a sequence of state distributions $\{\mu_N : N \in \mathbb{Z}_+\}$ where the number N of buyers and units K_N of good are commonly known in μ_N, and $K_N/N \leqslant \kappa$ for some $\kappa \in (0,1)$. Let $\epsilon_N = 1/\sqrt{N}$ and $\alpha \in (\kappa, 1)$. Then there exists a constant $C > 0$ such that*

$$\left|\mathbb{E}[v_N] - \frac{G(M(p_{\alpha,\epsilon_N}))}{K_N}\right| \leqslant \frac{C}{\sqrt{N}} \tag{6}$$

for every N.

In Corollary 1, the distribution of value in μ_N may vary with N. Nonetheless, the difference between the market price and the value is still guaranteed to vanish in expectation as N gets large, and the constant C in (6) is independent of the value distribution. Moreover, the convergence rate of $1/\sqrt{N}$ is unimprovable, since it is the rate given by the guarantee-maximizing mechanisms of Brooks and Du (2021) (i.e., proportional auctions) for a fixed distribution of the common value, a commonly known number of buyers, and a single unit

analysis of Brooks and Du (2023), who emphasize that the Lagrange multiplier on obedience constraints corresponds to a choice of units for actions, and can be normalized to any value. But Brooks and Du (2023) analyze a limit of mechanisms in which the number of actions can be arbitrarily large, whereas participation-adaptive pricing has only two actions. When we constrain the number of actions in the mechanism, the nominal value of the Lagrange multiplier matters.

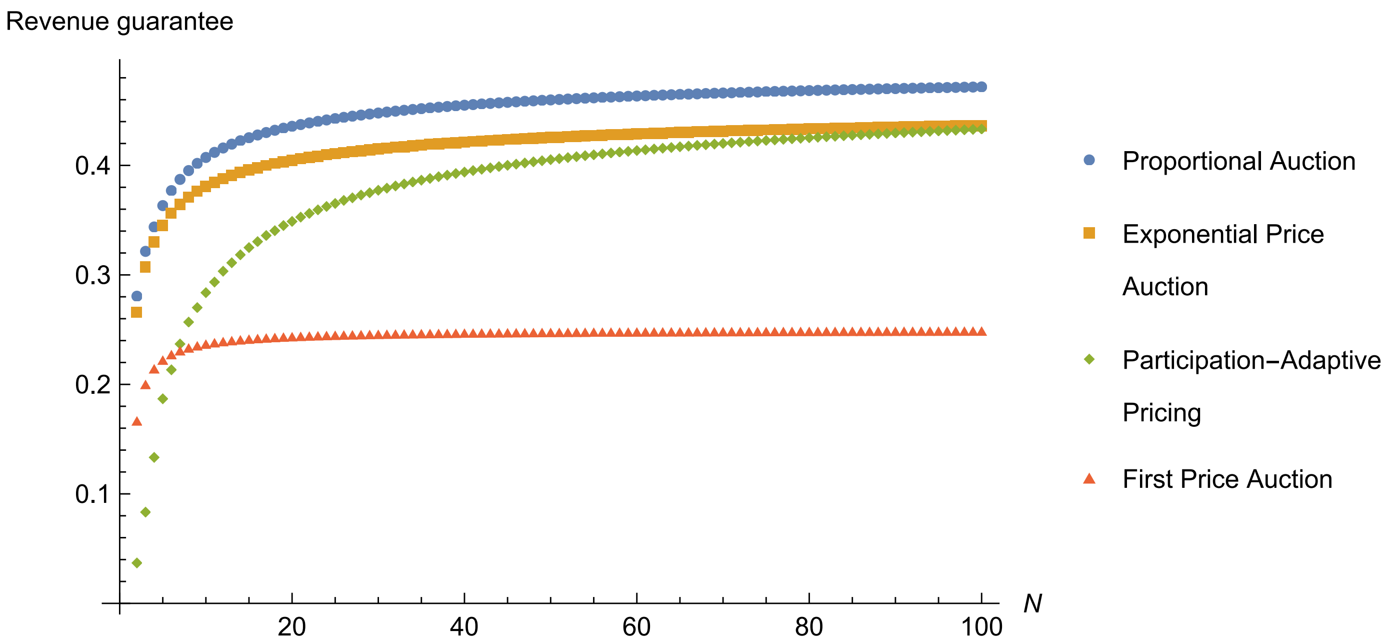

Figure 2: Comparison of revenue guarantees.

of the good.[5] The $1/\sqrt{N}$ convergence rate to full surplus is significantly better than the $1/\log(N)$ rate for the exponential price auction in Du (2018).[6] Needless to say, participation-adaptive pricing also achieves the optimal rate even when multiple units are for available, a case which is not covered in the prior literature.

As an illustration, in Figure 2, we have plotted the revenue guarantees of participation-adaptive pricing for a setting in which $v \sim U[0,1]$, $K = 1$, N is deterministic, $\alpha = 1/2$, and $\epsilon = 1/\sqrt{N}$. For comparison, we have also plotted the revenue guarantees of the optimal proportional auction of Brooks and Du (2021), the exponential price auction of Du (2018), and the first-price auction. As we can see, even for moderate values of N, participation-adaptive pricing outperforms the first-price auction, although it is still outperformed by the proportional auction. Around $N = 100$, participation-adaptive pricing overtakes the exponential price auction. Although it is still dominated by proportional auctions (as it must be), the gap is reduced to about 20% of the efficient surplus. As Theorem 1 and Corollary 1 show, this gap must go to zero as N goes to infinity, at a rate of $1/\sqrt{N}$, as the revenue guarantees of both proportional auctions and participation-adaptive pricing converge to full surplus.

[5]It is interesting to note that participation-adaptive pricing can be viewed as a kind of "restricted" proportional auction, where bids are only allowed in $\{0, 1\}$, and the pricing rule has the simple piecewise-linear form.

[6]Du (2018) proves that $1/\log(N)$ is a lower bound on the rate of convergence for the exponential price auction. The true rate could be higher.

5 Discussion

Our contention is that participation-adaptive pricing, of the form described in Theorem 1, provides a robust link between finite-agent models where agents have price impact and large market models where agents are price takers. Moreover, this linkage is obtained regardless of the structure of private information and the equilibrium being played. While the same can be said of the exponential price and proportional auctions, in the special case of a single good, it is quite striking that competitive outcomes can also be obtained with mechanisms in which the buyers have just two actions, opt in or opt out, and the pricing rule has the simple form that is a piecewise linear approximation of a step function. Moreover, participation-adaptive pricing also achieves competitive outcomes in large markets even when many units are available and when there is potentially correlation between the value, the number of participants, and the number of units. One can either interpret these mechanisms literally, as a formal institution that could be implemented by a market designer, or as a metaphor for the process by which the market price is aggregated in a large, decentralized market. To our knowledge, the prior literature has not identified mechanisms that guarantee approximate competitive outcomes in large markets for such a diverse array of environments.

In practice, the simple structure of participation-adaptive pricing might make it easier for buyers to optimize and for the economy to converge to an equilibrium. In particular, the only equilibrium constraints used in the proof of Theorem 1 are that buyers who opt out should not want to deviate to opting in. This deviation is obviously equivalent to abandoning one's equilibrium strategy altogether and just opting in, regardless of one's private information. More broadly, the same argument would apply for any *coarse Bayes correlated equilibrium* under participation adaptive pricing, by which we mean any joint distribution over fundamentals and actions such that each player prefers their equilibrium strategy to any alternative strategy that always plays a fixed action. As Hartline, Syrgkanis, and Tardos (2015) show, no-regret learning dynamics are guaranteed to converge to a coarse Bayes correlated equilibrium in the long run. Thus, no-regret learning by buyers who face participation-adaptive pricing in a large market setting will necessarily lead to a competitive outcome.[7]

One apparent limitation of participation-adaptive pricing is its dependence on the number of potential buyers N, through the participation rate. In contrast, the first-price auction's rules do not depend on the number of potential buyers. However, there is a way of weakening this dependence, while still obtaining competitive outcomes in the large market limit. In particular, for the conclusion of Theorem 1 to hold, it would suffice to know some integer M that is a lower bound on N, as long as M becomes arbitrarily large, and K/M is bounded away from one. The market price can be set as a function of $n(a)/M$, and if that quantity is greater than one, the market price is $\overline{v}$. Under these assumptions, it is straightforward to extend our proof to show that the competitive outcome obtains if M goes to infinity.

[7] It is natural to ask, what would actually happen in equilibrium? This obviously depends on the particular form of information. However, the proof of Theorem 1 shows that if the participation rate is not in the band $[\alpha - \epsilon, \alpha + \epsilon]$, then the lower bound on revenue would be higher than the value. As a result, in equilibrium with a large market, the probability that the participation rate is outside $[\alpha - \epsilon, \alpha + \epsilon]$ must be close to zero, and the economy spends most of its time with intermediate prices.

A more substantive limitation of the current analysis is that the supply of the good is treated as exogenous. A natural direction for future work would be to consider two-sided markets, consisting of buyers and sellers, and where both sides must choose to participate in order for trade to take place. It is our hope that participation-adaptive pricing can be used to construct market mechanisms will facilitate efficient trade in such settings.

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