Textbook Section 2.1:

22) Suppose that at time $t = 0$, half of a "logistic" population of 100,000 persons have hard a certain rumor, and that the number of those who have heard it is then increasing at a rate of 1000 persons per day. How long will it take for this rumor to spread to 80% of the population? (Suggestion: Find the value of k by substituting $P(0)$ and $P'(0)$ in the logistic equation, Eq. (3).)

Equation 3:

$$\frac{dP}{dt} = kP(M - P)$$

where $k = b$ and $M = \frac{a}{b}$ are constants.
$M = 100,000,\ P(0) = 50,000,\ P'(0) = 1,000.$

$$1,000 = k \cdot 50,000(100,000 - 50,000) \implies 1 = k \cdot 50(50,000)$$

$$\implies k = \frac{1}{2,500,000}$$

$$\frac{dP}{dt} = kP(M - P)$$

$$\int \frac{dP}{P(100,000 - P)} = \int k \cdot dt$$

$$-\frac{ln(|\frac{100,000}{P} - 1|)}{100,000} + C = kt$$

$$ln(|\frac{100,000}{P} - 1|) + C = -100,000kt$$

$$ln(|\frac{100,000}{P} - 1|) = -100,000kt + C$$

$$e^{ln(|\frac{100,000}{P} - 1|)} = e^{-100,000kt+C}$$

$$|\frac{100,000}{P} - 1| = e^{-100,000kt+C}$$

$$|\frac{100,000}{P} - 1| = A \cdot e^{-100,000kt}$$

$$\frac{100,000}{P} = 1 \pm (A \cdot e^{-100,000kt})$$

$$P = \frac{100,000}{1 \pm (A \cdot e^{-100,000kt})}$$

$$P(0) = 50,000$$

$$\implies P = \frac{100,000}{1 + (1 \cdot e^{-100,000kt})}$$

$$\implies P = \frac{100,000}{1 + e^{-100,000kt}}$$

$$P(t) = \frac{100,000}{1 + e^{-100,000kt}}$$

80% of population $= 0.8 \cdot 100,000 = 80,000$ Solve for t:

$$P(t) = 80,000$$

$$\frac{100,000}{1 + e^{-100,000kt}} = 80,000$$

$$\frac{1}{1 + e^{-100,000kt}} = \frac{4}{5}$$

$$\frac{5}{4} = 1 + e^{-100,000kt}$$

$$\frac{5}{4} - 1 = e^{-100,000kt}$$

$$\frac{1}{4} = e^{-100,000kt}$$

$$ln(\frac{1}{4}) = -100,000kt$$

$$-\frac{ln(\frac{1}{4})}{100,000} = kt$$

$$t = -\frac{ln(\frac{1}{4})}{100,000k}$$

$$t \approx 34.657$$

80% of people will have heard a certain rumor after approximately 34.657 days.

Textbook Section 2.2:

5) $\frac{dx}{dt} = x^2 - 4$

$$\frac{dx}{dt} = 0$$

$$\implies x^2 - 4 = 0$$

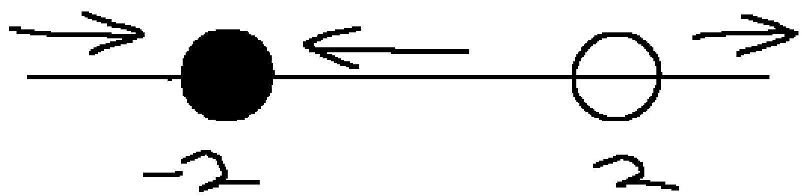
$$\implies (x+2)(x-2) = 0$$

$$x = \pm 2$$

$$x'(-3) = 5$$

$$x'(0) = -4$$

$$x'(3) = 5$$

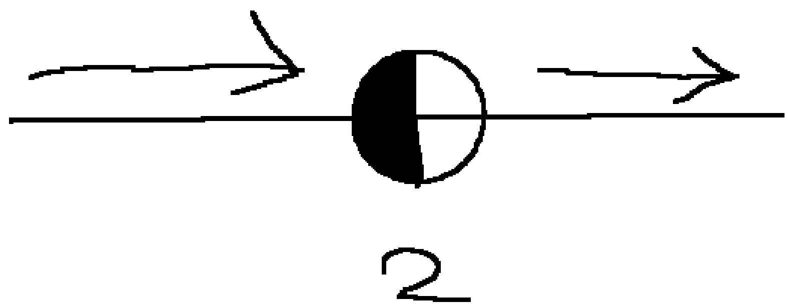


$$\int \frac{dx}{x^2-4} = \int dt$$
$$\frac{ln(|x-2|) - ln(|x+2|)}{4} + C = t$$
$$ln(|x-2|) - ln(|x+2|) + C = 4t$$
$$ln(|\frac{x-2}{x+2}|) = 4t + C$$
$$|\frac{x-2}{x+2}| = e^{4t+C}$$
$$\frac{x-2}{x+2} = A \cdot e^{4t}$$
$$x - 2 = (A \cdot e^{4t})(x+2)$$
$$(-A \cdot e^{4t})x + x - 2 = 2A \cdot e^{4t}$$
$$(-A \cdot e^{4t})x + x = 2A \cdot e^{4t} + 2$$
$$(1 - A \cdot e^{4t})x = 2A \cdot e^{4t} + 2$$
$$x = \frac{2A \cdot e^{4t} + 2}{1 - A \cdot e^{4t}}$$

7) $\frac{dx}{dt} = (x-2)^2$

$$\frac{dx}{dt} = 0$$
$$\implies (x-2)^2 = 0$$
$$x = 2$$

$$x'(1) = 1$$
$$x'(3) = 1$$



$$\int \frac{dx}{(x-2)^2} = \int dt$$
$$-\frac{1}{x-2} + C = t$$
$$-t + C = \frac{1}{x-2}$$
$$x - 2 = -\frac{1}{t+C}$$
$$x = 2 - \frac{1}{t+C}$$

9) $\frac{dx}{dt} = x^2 - 5x + 4$

$$\frac{dx}{dt} = 0$$

$$\implies x^2 - 5x + 4 = 0$$

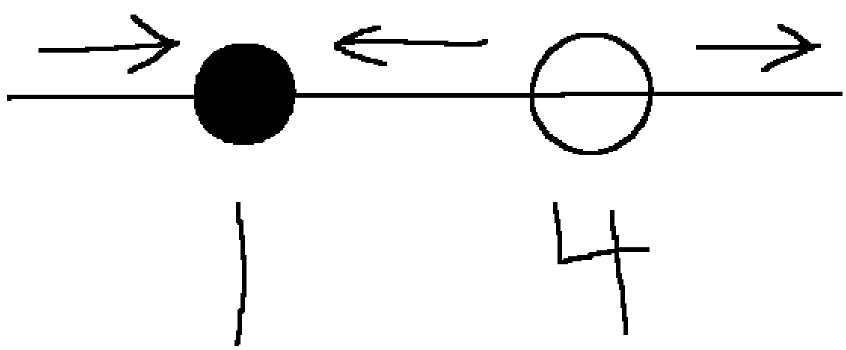
$$\implies (x-4)(x-1) = 0$$

$$x = 1, x = 4$$

$$x(0) = 4$$

$$x(2) = -2$$

$$x(5) = 4$$



$$\int \frac{dx}{(x-4)(x-1)} = \int dt$$

$$\frac{ln(|\frac{3}{x-1} - 1|)}{3} + C = t$$

$$ln(|\frac{3}{x-1} - 1|) + C = 3t$$

$$ln(|\frac{3}{x-1} - 1|) = 3t + C$$

$$|\frac{3}{x-1} - 1| = e^{3t+C}$$

$$|\frac{3}{x-1} - 1| = A \cdot e^{3t}$$

$$\frac{3}{x-1} - 1 = A \cdot e^{3t}$$

$$\frac{3}{x-1} = 1 + A \cdot e^{3t}$$

$$\frac{3}{1 + A \cdot e^{3t}} = x - 1$$

$$x = 1 + \frac{3}{1 + A \cdot e^{3t}}$$

Textbook Section 2.3:

2) Suppose that a body moves through a resisting medium with resistance proportional to its velocity v, so that $\frac{dv}{dt} = -kv$. a) Show that its velocity and position at time t are given by

$$v(t) = v_0 e^{-kt}$$

and

$$x(t) = x_0 + (\frac{v_0}{k})(1 - e^{-kt})$$

$$\int \frac{dv}{v} = \int -k \cdot dt$$

$$\implies \int \frac{dv}{v} = -k \cdot \int dt$$

$$\implies ln(|v|) + C = -kt$$

$$\implies ln(|v|) = -kt + C$$

$$\implies |v| = e^{-kt+C}$$

$$\implies |v| = v_0 \cdot e^{-kt}$$

Assuming velocity is always positive:

$$v = v_0 \cdot e^{-kt}$$

$$\frac{dx}{dt} = v_0 \cdot e^{-kt}$$

$$\int dx = \int v_0 \cdot e^{-kt} dt$$

$$\int dx = v_0 \cdot \int e^{-kt} dt$$

$$x = C - \frac{v_0}{k} e^{-kt}$$

$$C = x_0 + \frac{v_0}{k}$$

$$x = x_0 + \frac{v_0}{k} e^{-kt}(1 - e^{-kt})$$

3) Suppose that a motorboat is moving at $40\frac{ft}{s}$ when its motor suddenly quits, and that $10s$ later the boat has slowed to $20\frac{ft}{s}$. Assume, as in Problem 2, that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

Assume $x(t) = x_0 + (\frac{v_0}{k})(1 - e^{-kt})$ and $v(t) = v_0 e^{-kt}$.

$v_0 = 40$.

Solve for k:

$$v(10) = 20$$

$$\implies 40e^{-k\cdot 10} = 20$$

$$\implies e^{-k\cdot 10} = \frac{1}{2}$$

$$\implies -k \cdot 10 = ln(\frac{1}{2})$$

$$\implies k = -\frac{ln(\frac{1}{2})}{10}$$

$$v(t) = 40e^{\frac{ln(\frac{1}{2})}{10}\cdot t}$$

Solve for the t of when the boat stops:

$$v(t) = 0$$

$$\implies 40e^{\frac{ln(\frac{1}{2})}{10}\cdot t} = 0$$

$$\implies e^{\frac{ln(\frac{1}{2})}{10}\cdot t} = 0$$

$$\implies e^{\frac{ln(\frac{1}{2})}{10}\cdot t} = ln(0)$$

It is undefined. So we must use an improper integral from 0 to ∞

$$x(t) = \int_0^\infty 40e^{\frac{ln(\frac{1}{2})}{10}\cdot t} dt$$

$$x(t) = 40 \cdot \int_0^\infty e^{\frac{ln(\frac{1}{2})}{10}\cdot t} dt$$

$$x(t) = \frac{400}{ln(\frac{1}{2})} e^{\frac{ln(\frac{1}{2})}{10}\cdot t}|_0^\infty$$

$\frac{ln(\frac{1}{2})}{10} < 0$, so $\lim_{t\to\infty} e^{\frac{ln(\frac{1}{2})}{10}\cdot t} = 0$.

$$\implies \frac{400}{ln(\frac{1}{2})}e^{\frac{ln(\frac{1}{2})}{10}\cdot t}|_0^\infty = 0 - \frac{400}{ln(\frac{1}{2})}e^{\frac{ln(\frac{1}{2})}{10}\cdot 0}$$

$$= -\frac{400}{ln(\frac{1}{2})} \approx 577.078$$

The boat coasts approximately 577.078 feet.

9) A motorboat weighs $32,000lb$ and its motor provides a thrust of $5000lb$. Assume that the water resistance is 100 pounds for each foot per second of the speed v of the boat. Then

$$1000\frac{dv}{dt} = 5000 - 100v.$$

If the boat starts from rest, what is the maximum velocity that it can attain?

$$1000\frac{dv}{dt} = 5000 - 100v$$

$$\implies \frac{dv}{dt} = 5 - \frac{1}{10}v$$

Find stationary points:

$$\frac{dv}{dt} = 0$$

$$5 - \frac{1}{10}v = 0$$

$$v = 50$$

When it hits the stationary point, its velocity will be 50. This isn't a minimum since plugging in a non-zero $\frac{dv}{dt}$ (e.g. $\frac{dv}{dt} = 1$) yields at least one value of v that is less than 50.

10) A woman bails out of an airplane at an altitude of $10,000ft$, falls freely for $20s$, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance $pv\frac{ft}{s^2}$, taking $p = 0.15$ without the parachute and and $p = 1.5$ with the parachute. (Suggestion: First determine her height above the ground and velocity when the parachute opens.)

$$g = 32.174\frac{ft}{s^2}$$

$$v(t) = (v_0 + \frac{g}{p})e^{-pt} - \frac{g}{p}$$

$$x(t) = -\frac{g}{p}t - \frac{1}{p}(v_0 + \frac{g}{p})e^{-pt} + C$$

$$C = x_0 + \frac{1}{p}(v_0 + \frac{g}{p})$$

$$x(t) = x_0 - \frac{g}{p}t + \frac{1}{p}(v_0 + \frac{g}{p})(1 - e^{-pt})$$

$$x_0 = 10,000; v_0 = 0$$

$$\frac{g}{p} = \frac{32.174}{0.15}; p = 0.15$$

$$x(20) = 10,000 - \frac{32.174}{0.15}(20) + \frac{1}{0.15}(\frac{32.174}{0.15})(1 - e^{-0.15 \cdot 20}) \approx 7068.9ft$$

$$v(20) = \frac{32.174}{0.15}e^{-0.15 \cdot 20} - \frac{32.174}{0.15} \approx -203.8\frac{ft}{s}$$

Solve for t:

$$7068.9 - \frac{32.174}{1.5}t + \frac{1}{1.5}(-203.8 + \frac{32.174}{1.5})(1 - e^{-1.5t}) = 0$$

$$t \approx 323.9 seconds$$

The total time it takes is $323.9 + 20 = 353.9$ seconds.

12) It is proposed to dispose of nuclear wastes – in drums with weight $W = 640lb$ and volume $8ft^3$ – by dropping them into the ocean ($v_0 = 0$). The force equation for a drum falling through water is

$$m\frac{dv}{dt} = -W + B + F_R,$$

where the buoyant force B is equal to the weight (at $62.5\frac{lb}{ft^3}$) of the volume of water displaced by the drum (Archimedes' principle) and F_R is the force of water resistance, found empirically to be $1lb$ for each foot per second of the velocity of a drum. If the drums are likely to burst upon an impact of more than $75\frac{ft}{s}$, what is the maximum depth to which they can be dropped in the ocean without likelihood of bursting?

$$W = 640lb, B = 500lb, F_R = -v, m = \frac{640}{32} - 20slugs$$

$$20\frac{dv}{dt} = -6400 + 500 - v$$

$$20\frac{dv}{dt} = -140 - v$$

$$\int \frac{dv}{140 + v} = -\frac{1}{20}\int dt$$

$$ln(140 + v) = -\frac{t}{20} + C$$

$$140 + v = C \cdot e^{-0.05t}$$

The drums start at rest, so $v(0) = 0$. Therefore $C = 140$.

$$v(t) = 140(e^{-0.05t} - 1)$$

Find time t when $v = -75\frac{ft}{s}$

$$v(t) = -75$$
$$\implies 140(e^{-0.05t} - 1) = -75$$
$$\implies t \approx 15.35$$

$$x(t) = \int_0^{15.35} 140(e^{-0.05t} - 1)$$
$$\implies x(t) \approx -649$$

The maximum safe depth for the drums is approximately 649ft.