This is due Saturday 9/23 by 11:59 pm on Gradescope. Please either neatly write up your solutions or type them up. You can find a .tex template on Canvas. Your proofs should be written in complete sentences and paragraphs, using a combination of words and symbols. They should be **correct, clear, and concise**. You will be graded on all three, especially the first two!

1. Show that if D and D' are dedekind cuts, then $D + D'$ is also a Dedekind cut. Recall that $D + D' = \{x \in \mathbb{Q} : x = d + d', \text{ for some } d \in D, d' \in D'\}$. Do these directly from the definition of dedekind cuts. **Remark:** This is one of the things that needs to be checked in the course of showing that the set of Dedekind cuts forms an ordered field.

Solution:

A dedekind cut is a set with a least upper bound. So D has least upper bound d and D' has least upper bound d'. Since we sum up all of the entries of D and D' to get the entries of $D+D'$, then we get a least upper bound of $d+d'$. And since we have a least upper bound for $D+D'$, then $D+D'$ is a dedekind cut.

2. Show that for S a subset of $\mathbb{R}$ the following are equivalent:

   1. For all $x, y \in \mathbb{R}$ with $x < y$, there is an $s \in S$ so that $x < s < y$.
   2. For all $x \in \mathbb{R}$ and for all $\epsilon \in \mathbb{R}_{>0}$ (this means the nonnegative reals), there is an $s \in S$ so that $|x - s| < \epsilon$.

   (Subsets of the reals that satisfies these equivalent properties are called *dense*)

Solution:

Claim 1) $\iff$ 2).

I should note that for the rest of this proof, I am assuming S is a set such that either 1) or 2) holds. In the case that this isn't true, neither would hold because of the fact that the Archimedes' principle would fail. I am concerning myself with the non-trivial case: showing that they must both be true if one of them is.

$$x < y \implies 0 < y - x$$

By Archimedes' principle

$$\implies \frac{1}{n} < y - x$$

for some $n \in \mathbb{N}$, where $n > 0$.

$$\implies x + \frac{1}{n} < y$$

and

$$x < x + \frac{1}{n}$$

So,

$$x < x + \frac{1}{n} < y$$

Let $s = x + \frac{1}{n}$ and we get 1).

Also,

$$x < x + \frac{1}{n} < y < y + \frac{1}{n}$$
$$\implies x < y + \frac{1}{n}$$
$$x - y < y + \frac{1}{n}$$

and

$$y - x < y + \frac{1}{n}$$

That means:

$$|y - x| < |y| + \frac{1}{n}$$

Note that $|y| + \frac{1}{n}$ is the definition of all positive real numbers.

If we let $|y| + \frac{1}{n}$ be ϵ and have $s = x$, we get 2) with minimal rewriting.

3. Compute the supremum and infimum of the following sets. Prove that your answers are correct (remember for us that $\mathbb{N}$ doesn't contain 0).

   (a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$

   (b) $\{n/(5n + 1) : n \in \mathbb{N}\}$

Solution:

(a) The infimum is 0 and the supremum is 1.

$$0 < \frac{m}{n} < 1$$

Let us set m to the $inf(\mathbb{N}) = 1$. Then we increase n to be arbitrarily high. As n approaches inf $(sup(\mathbb{N}))$, then $\frac{1}{n} approaches 0$. This must be the infimum since we are maximizing the difference between n and m where $n > m$. It can't be larger or else it would fall within the set.

For the opposite case, we want to decrease the difference between n and m where $n > m$. Since we are dealing with $\mathbb{N}$, the smallest difference between two natural numbers where they aren't equal is 1. If we set m to be arbitrarily large and then set $n = m + 1$, then as m gets larger, the closer that $\frac{m}{m+1}$ gets to 1. So the supremum must be 1. It can't be smaller than 1 or else it would fall within the set.

(b) The infimum is $\frac{1}{6}$ and the supremum is $\frac{1}{5}$.

Similar reasoning to above holds.

If we minimize the difference between n and $5n + 1$, then $n = 1$. That leaves us with $\frac{1}{6}$. Since $\frac{1}{6}$ is a lower bound and in the set, it must be the infimum.

If we maximize the difference, then we have n be arbitrarily big, which means $\frac{n}{5n+1}$ will approach $\frac{1}{5}$. The supremum of the set can't be lower or else it would be in the set. Therefore the supremum of the set must be $\frac{1}{5}$.

4. Let A, B be non-empty subsets of $\mathbb{R}$. Prove that if $A \subseteq B$, then

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Solution:

Let's consider A. Since A is a subset of B, then the infimum of A must be a lower bound of B. But let us consider the case of $B = \{1, 2\}; A = \{2\}$. In this case, the infimum of A is greater than the infimum of B. Therefore, since the infimum of B is a lower bound of A, but the infimum of B can be greater than the infimum of A, then the infimum of A must be greater than or equal to the infimum of B.

Now, for the case of the supremum. Since B is a superset of A, that means that the supremum of B must be an upper bound of A. But let us consider the case of $B = \{1, 2\}; A = \{1\}$. In this case, the supremum of A is less than the supremum of B. Therefore, since the supremum of B is an upper bound of A, but the supremum of A can be less than the supremum of B, then the supremum of A must be less than or equal to the supremum of B.

5. Let A and B be non-empty subsets of $\mathbb{R}$. Prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Solution:

Let $C = A \cup B$

Since $A \subseteq C$, $sup(A) \leq sup(C)$. Since $B \subseteq C$, $sup(B) \leq sup(C)$.

But the only way that this can be true is if $sup(C) = \max\{\sup A, \sup B\}$ because otherwise $sup(C) < sup(A)$ or $sup(C) < sup(B)$.

For instance, if $sup(A) < sup(B)$, then $sup(C) = sup(B)$. This satisfies that $sup(B) \leq sup(C)$ since $sup(C) = sup(B)$ and $sup(A) \leq sup(C)$ since $sup(A) < sup(B)$.

Now, if $sup(B) < sup(A)$, then $sup(C) = sup(A)$. This satisfies that $sup(A) \leq sup(C)$ since $sup(C) = sup(A)$ and $sup(B) \leq sup(C)$ since $sup(B) < sup(A)$.

And in the csae that $sup(A) = sup(B)$, trivially both $sup(A) \leq sup(C)$ and $sup(B) \leq sup(C)$ are satisfied because $sup(A) = sup(B) = sup(C)$.