1) Beck Excercise 3.1. Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, L \in \mathbb{R}^{p\times n}$, and $\lambda \in \mathbb{R}_{++}$. Consider the regularized least squres (RLS) problem

$$min_{x\in\mathbb{R}^n}||Ax-b||^2+\lambda||Lx||^2.$$

Show that the RLS problem has a unique solution if and only if $Null(A) \cap Null(L) = \{0\}$. Here, for a matrix B, Null(B) denotes the null space of B, $\{x : Bx = 0\}$.

We'll first prove the forward case using contradiction. Assume the RLS problem has a unique solution x^*. Suppose a non-zero vector v, where $v \in Null(A)$ and $v \in Null(L)$. This implies that $Av = 0$ and $Lv = 0$.

Evaluate $min_{x\in\mathbb{R}^n}||Ax-b||^2+\lambda||Lx||^2$ at x^*+v.

We get $min_{x\in\mathbb{R}^n}||A(x^*+v)-b||^2+\lambda||L(x^*+v)||^2$

$= min_{x\in\mathbb{R}^n}||Ax^*+Av-b||^2+\lambda||Lx^*+Lv||^2$

Recall that $Ax = 0$ and $Lx = 0$.

So we get $min_{x\in\mathbb{R}^n}||Ax^*-b||^2+\lambda||Lx^*||^2$.

Since both x^* and x^*+v map to the same thing, x^* is not unique if $Null(A)\cap Null(L) \neq \{0\}$.

Now for the backwards case. Assume $Null(A) \cap Null(L) = \{0\} \implies min_{x\in\mathbb{R}^n}||Ax-b||^2+\lambda||Lx||^2$

Take the spectral norm of $||Ax-b||^2+\lambda||Lx||^2$:

$(Ax-b)^T(Ax-b)+\lambda((Lx)^T(Lx))$

$\implies A^TAx^Tx-2b^TAx+b^Tb+\lambda L^TLx^Tx$

Set $\nabla f(x) = 0$.

$\implies 2xA^TA-2Ab^T+\lambda L^TL2x = 0$

$\implies 2xA^AA+\lambda L^TL2x = 2A^Tb$

$\implies x(A^TA+\lambda L^TL) = A^Tb$

Obviously $B^TB \succeq 0$ for any matrix B, therefore $A^TA \succeq 0$ and $\lambda L^TL \succeq 0$ (since $\lambda > 0$).

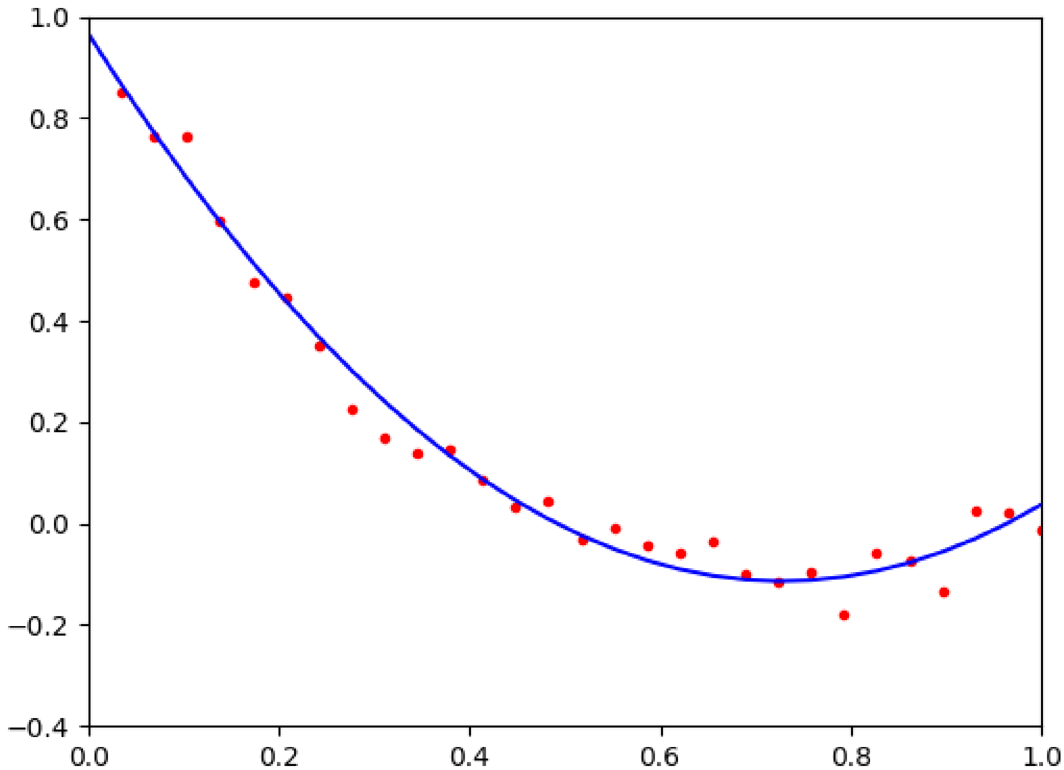
As a result $x = A^Tb(A^TA+\lambda L^TL)^{-1}$.

Since $(A^TA+\lambda L^TL)$ is invertible, the solution is unique.

2) Beck Excercise 3.2. Generate thirty points (x_i, y_i), $i = 1, 2, ..., 30$.

Find the quadratic function $y = ax^2+bx+c$ that best fits the points in the least squares sense. Indicate what are the parameters a, b, c found by the least squares solution and plot the points along with the derived quadratic function. The resulting plot should look like the one in Figure 3.5.

Note: For this problem, please also submit a copy of the code you used to solve the problem.



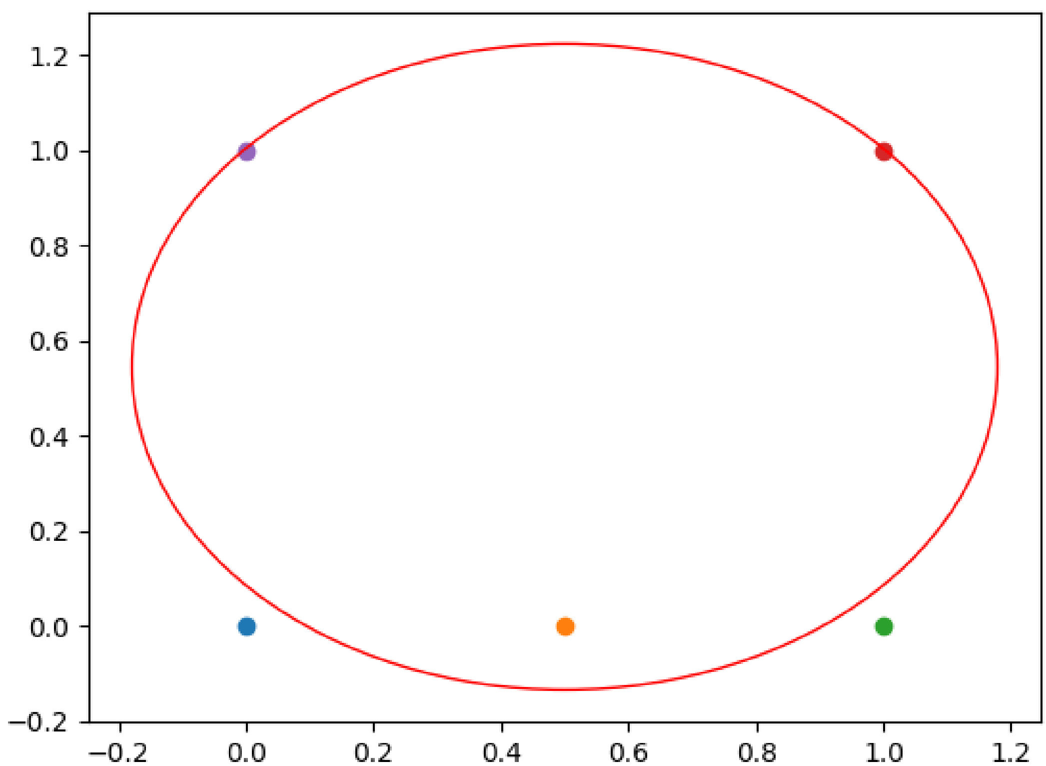

```
import numpy as np
from matplotlib import pyplot as plt

x = np.linspace(0, 1, 30)
y = 2*x**2 - 3*x + 1 + 0.05*np.random.randn(*x.shape)

A = np.zeros((x.size, 3))
for i in range(A.shape[1]):
    A[:,i] = x**i

coefficients = np.linalg.lstsq(A, y, rcond=None)[0]

print("{0:1.4f} x^2 + {1:1.4f} x + {2:1.4f}".format(coefficients[2], coefficient

plt.plot(x, y, 'r.', x, A @ coefficients, 'b')
```

```
plt.xlim([0, 1])
plt.ylim([-0.4, 1])
plt.show()
```

3) Beck Excercise 3.3. Write a function circle_fit whose input is an $n \times m$ matrix A, the columns of A are the m vectors in $\mathbb{R}^n$ to which the circle should be fitted. The call to the function will be of the form

(x, r) = circle_fit(A)

Note: For this problem, report the output (x, r) for this set of points and a plot of the circle together with the 5 points. Also submit a copy of the code you used to solve the problem.




```
from circle_fit import taubinSVD
import matplotlib.pyplot as plt

def circle_fit(points):
    xc, yc, r, sigma = taubinSVD(points)
```

```
    return ((xc, yc), r)

points = [[0, 0], [0.5, 0], [1, 0], [1, 1], [0, 1]]

(x, r) = circle_fit(points)


circle = plt.Circle(x, r, color='r', fill=False)

for point in points:
    plt.scatter(point[0], point[1])

plt.gca().add_patch(circle)

plt.show()
```

4) Beck Excercise 4.8. Let $f : \mathbb{R}^n -> \mathbb{R}$ be given by $f(x) = \sqrt{1 + ||x||^2}$. Show that $f \in C_1^{1,1}$.

Hint: Show that $0 \leq u^T \nabla^2 f(x) u \leq ||u||^2 \ \forall u \in \mathbb{R}^n$ and apply Theorem 4.20.

Theorem 4.20: Let $f \in C^2(\mathbb{R}^n)$, the following are equivalent:

a) $f \in C_L^{1,1}(\mathbb{R}^n)$

b) $||\nabla^2 f(x)|| \leq L \ \forall x \in \mathbb{R}^n$

$$\frac{d}{dx}(1 + ||x||^2)^{\frac{1}{2}}$$
$$= \frac{1}{2}(1 + ||x||^2)^{-\frac{1}{2}} \cdot 2x$$
$$= x \cdot (1 + ||x||^2)^{-\frac{1}{2}}$$

$$\frac{d}{dx}(x \cdot (1 + ||x||^2)^{-\frac{1}{2}})$$
$$= x \cdot \frac{d}{dx}((1 + ||x||^2)^{-\frac{1}{2}}) + \frac{d}{dx}(x) \cdot ((1 + ||x||^2)^{-\frac{1}{2}})$$
$$= x \cdot (-\frac{1}{2}(1 + ||x||^2)^{-\frac{3}{2}} \cdot 2x^T) + xx^T \cdot (1 + ||x||^2)^{-\frac{1}{2}}$$
$$= -(1 + ||x||^2)^{-\frac{3}{2}} \cdot xx^T + I_n \cdot (1 + ||x||^2)^{-\frac{1}{2}}$$

Let $a = (1 + ||x||^2)^{-\frac{1}{2}}$

$$= -a^3 xx^T + aI_n = aI_n - a^3 xx^T$$

Show that: $0 \leq u^T(aI_n - a^3 xx^T)u \leq ||u||^2 \ \forall u \in \mathbb{R}^n$

$$\implies au^T I_n u - a^3 u^T xx^T u$$
$$\implies a||u||^2 - a^3 u^T xx^T u$$

$$\implies a||u||^2 - a^3||u||^2||x||^2$$
$$\implies ||u||^2(a - a^3||x||^2)$$

$$||u||^2 \geq ||u||^2(a - a^3||x||^2) \geq 0$$
$$\implies 1 \geq a - a^3||x||^2 \geq 0$$
$$\implies 1 \geq (1 + ||x||^2)^{-\frac{1}{2}} - (1 + ||x||^2)^{-\frac{3}{2}}||x||^2 \geq 0$$

Note that if $||x||$ is getting smaller, the hessian gets bigger. Also, if $||x||$ is getting bigger, the hessian gets smaller. Also, the norm is by definition ≥ 0.

$$lim_{||x|| \to \infty} \frac{1}{\sqrt{1 + ||x||^2}} - \frac{||x||^2}{(1 + ||x||^2)^{\frac{3}{2}}}$$
$$\approx \frac{1}{\sqrt{||x||^2}} - \frac{||x||^2}{||x||^3}$$
$$\approx \frac{1}{||x||} - \frac{1}{||x||} = 0$$

The constants and whatnot don't matter since no matter what, you are ending up with a higher power of $||x||$ in the denominator, which means that term will always go to 0 as $||x||$ gets large.

The smallest possible value of $||x||$ is $(0, 0)$.

Plugging it in gives:

$$\frac{1}{\sqrt{1 + 0}} - \frac{0}{0 + 1} = 1$$

So the min is 0 and the max is 1, so the inequality holds and the function is in $C_1^{1,1}$.

5) Beck Excercise 5.2. Consider the Freudenstein and Roth test function

$$f(x) = f_1(x)^2 + f_2(x)^2, x \in \mathbb{R}^2$$

where

$$f_1(x) = -13 + x_1 + ((5 - x_2)x_2 - 2)x_2,$$
$$f_2(x) = -29 + x_1 + ((x_2 + 1)x_2 - 14)x_2.$$

i) Show that the function f has three stationary points. Find them and prove that one is a global minimizer, one is a strict local minimum, and the third is a saddle point.

$$\nabla f(x) = 2 f_1(x) \cdot \left(f_{1_{x_1}}(x)\hat{i} + f_{1_{x_2}}(x)\hat{j} \right) + 2 f_2 \cdot \left(f_{2_{x_1}}(x)\hat{i} + f_{2_{x_2}}(x)\hat{j} \right)$$

$$\nabla f(x) = (0, 0)$$

$$f_1(x) = -13 + x_1 + ((5 - x_2)x_2 - 2)x_2$$

$$f_2(x) = (-29 + x_1 + ((x_2 + 1)x_2 - 14)x_2)$$

$$f_1(x) = -13 + x_1 + 5x_2^2 - x_2^3 - 2x_2$$

$$f_{1_{x_1}}(x) = 1 \; ; \; f_{1_{x_2}}(x) = 10x_2 - 3x_2^2 - 2$$

$$f_2(x) = -29 + x_1 - 14x_2 + x_2^2 + x_2^3$$

$$f_{2_{x_1}}(x) = 1 \; ; \; f_{2_{x_2}}(x) = -14 + 2x_2 + 3x_2^2$$

$$2\left(f_1(\vec{x})\cdot\begin{bmatrix}1\\ -3x_2^2+10x_2-2\end{bmatrix} + f_2(\vec{x})\cdot\begin{bmatrix}1\\ 3x_2^2+2x_2-14\end{bmatrix}\right)=\vec{0}$$

$$\Rightarrow f_1(\vec{x})\cdot\begin{bmatrix}1\\ -3x_2^2+10x_2-2\end{bmatrix} + f_2(\vec{x})\cdot\begin{bmatrix}1\\ 3x_2^2+2x_2-14\end{bmatrix}=\vec{0}$$

$$f_1(\vec{x})+f_2(\vec{x}) =$$

$$(-13+x_1+5x_2^2-x_2^3-2x_2)$$

$$+(-29+x_1-14x_2+x_2^2+x_2^3)$$

$$=-42+2x_1-16x_2+6x_2^2=0$$

$$\Rightarrow 2x_1=-6x_2^2+16x_2+42$$

$$\Rightarrow x_1=-3x_2^2+8x_2+21$$

$$\Rightarrow f_1(\vec{x})=-f_2(\vec{x})$$

$$f_1(\vec{x}) \cdot \begin{bmatrix} 1 \\ -3x_2^2 + 10x_2 - 2 \end{bmatrix}$$

$$= f_1(\vec{x}) \cdot \begin{bmatrix} 1 \\ 3x_2^2 + 2x_2 - 14 \end{bmatrix} = 0$$

$$\Rightarrow f_1(\vec{x}) \cdot \left(\begin{bmatrix} 1 \\ -3x^2 + 10x_2 - 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3x_2^2 + 2x_2 - 14 \end{bmatrix} \right) = 0$$

$$\Rightarrow f_1(\vec{x}) \cdot \begin{bmatrix} 0 \\ -6x_2^2 + 8x_2 + 12 \end{bmatrix} = 0$$

$$\Rightarrow 2f_1(\vec{x}) \cdot \begin{bmatrix} 0 \\ -3x_2^2 + 4x_2 + 6 \end{bmatrix} = 0$$

$$\Rightarrow f_1(x) \cdot \begin{bmatrix} 0 \\ -3x_2^2 + 4x_2 + 6 \end{bmatrix} = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

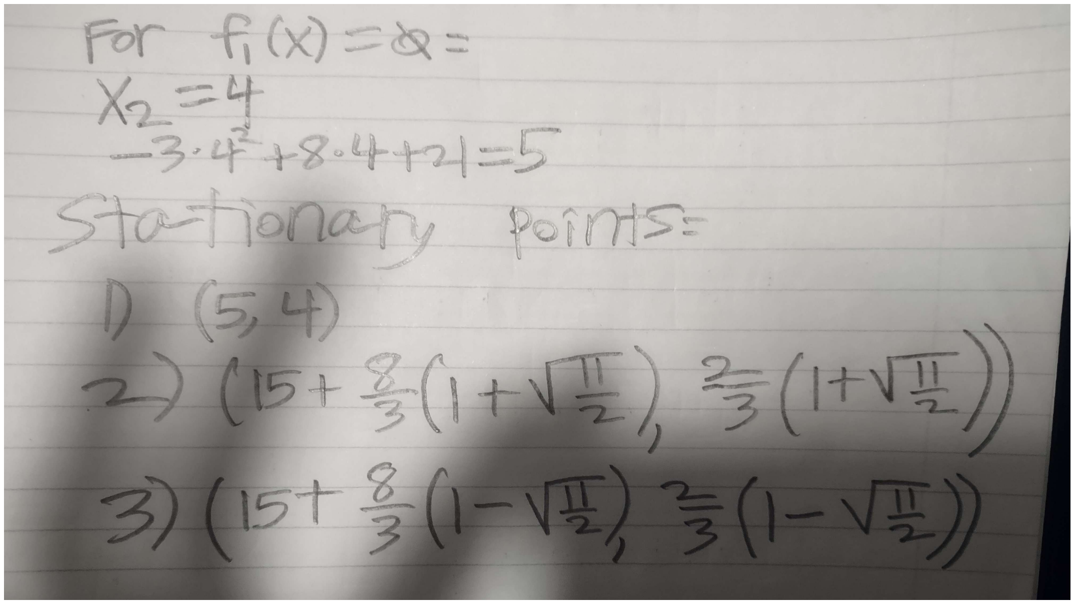
$a=-3,\ b=4,\ c=6$

$$\frac{-4 \pm \sqrt{16+18\cdot 4}}{-6} = \frac{4 \pm \sqrt{88}}{6}$$

$$= \frac{4 \pm 2\sqrt{22}}{6} = \frac{2 \pm \sqrt{22}}{3}$$

$$= \frac{2}{3}\left(1 \pm \sqrt{\frac{11}{2}}\right) = x_2$$

$$x_1 = -3x_2^2 + 8x_2 + 21$$



For $f_1(x) = 0$ =

$X_2 = 4$

$-3 \cdot 4^2 + 8 \cdot 4 + 21 = 5$

Stationary points:

1) $(5, 4)$

2) $\left(15 + \frac{8}{3}\left(1 + \sqrt{\frac{11}{2}}\right), \frac{2}{3}\left(1 + \sqrt{\frac{11}{2}}\right)\right)$

3) $\left(15 + \frac{8}{3}\left(1 - \sqrt{\frac{11}{2}}\right), \frac{2}{3}\left(1 - \sqrt{\frac{11}{2}}\right)\right)$

Now we have to classify these 3 stationary points.

$$\frac{\partial^2 f}{\partial x_2^2} = 2(5 - 8x_2 + 6x_2 - 28) = 2(-3x_2 - 23) = -6x_2 - 46$$

Evaluate $\frac{\partial^2 f}{\partial x_2^2}$ at the stationary points.

By looking at the sign, we get that:

(5, 4) is a saddle point

$\left(15 + \frac{8}{3}\left(1 + \sqrt{\frac{11}{2}}\right), \frac{2}{3}\left(1 + \sqrt{\frac{11}{2}}\right)\right)$ is a local minimum.

$\left(15 + \frac{8}{3}\left(1 - \sqrt{\frac{11}{2}}\right), \frac{2}{3}\left(1 - \sqrt{\frac{11}{2}}\right)\right)$ is a local minimum.

Plug into $f(x)$ to find out which is the global minimizer.

If we plug this into a calculator, we find that that he first local minimum gives ≈ 148.71 and the second local minimum gives ≈ 277.44.

That means that the global minimizer is:

$$\left(15 + \frac{8}{3}\left(1 + \sqrt{\frac{11}{2}}\right), \frac{2}{3}\left(1 + \sqrt{\frac{11}{2}}\right)\right)$$

since $148.71 < 277.44$.

ii) Use Python to employ the following three methods on the problem of minimizing f:

a) the gradient method with backtracking and parameters $(s, \alpha, \beta) = (1, 0.5, 0.5)$.

b) the hybrid Newton's method with parameters $(\alpha, \beta) = (0.5, 0.5)$

c) the damped Gauss-Newton method with a backtracking line search strategy with parameters $(s, \alpha, \beta) = (1, 0.5, 0.5)$.

All the algorithms should use the stopping criteria $||\nabla f(x)|| \leq 10^{-5}$. Each algorithm should be employed four times on the following four starting points: $(-50, 7)^T, (20, 7)^T, (20, -18)^T$, and $(5, -10)^T$. For each of the four starting points, compare the number of iterations and point to which each method converged. If a method did not coverge, explain why.

Note: For this problem, additionally submit a copy of the code you used to solve the problem.

6) Beck Excercise 5.3. Let f be a twice continuously differentiable function satisfying $LI_n \succeq \nabla^2 f(x) \succeq mI_n$ for some $L > m > 0$ and let x^* be the unique minimizer of f over $\mathbb{R}^n$.

i) Show that

$$f(x) - f(x^*) \geq \frac{m}{2}||x - x^*||^2$$

for any $x \in \mathbb{R}^n$.

First, the second order taylor expansion of f around x^* is:

$$f(x) \approx f(x^*) + \nabla f(x^*)^T(x - x^*) + \frac{1}{2}(x - x^*)^T \nabla^2 f(G)(x - x^*)$$

for some G in the line segment $[x, x^*]$.

Since x^* is the minimizer of f, $\nabla f(x^*) = 0$, so

$f(x) \approx f(x^*) + \frac{1}{2}(x - x^*)^T \nabla^2 f(L)(x - x^*)$

By the definition, $\nabla^2 f(G)$ is positive definite and the smallest eigenvalue $\geq m$.

So,

$$(x - x^*)^T \nabla f((G)(x - x^*)) \geq m||x - x^*||^2$$

Substituting this back into our taylor expansion gives:

$$f(x) \geq f(x^*) + \frac{m}{2}||x - x^*||^2$$

$$\implies f(x) - f(x^*) \geq \frac{m}{2}||x - x^*||^2$$

for any $x \in \mathbb{R}^n$.

ii) Let $\{x_k\}_{k \geq 0}$ be the sequence generated by the damped Newton's method with constant step-size $t_k = \frac{m}{L}$. Show that

$$f(x_k) - f(x_{k+1}) \geq \frac{m}{2L} \nabla f(x_k)^T (\nabla^2 f(x_k))^{-1} \nabla f(x_k).$$

for any $x \in \mathbb{R}^n$.

Damped Newton's method:

$$x_{k+1} = x_k - t_k(\nabla^2 f(x_k))^{-1} \nabla f(x_k)$$

With $t_k = \frac{m}{L}$, we have:

$$x_{k+1} = x_k - \frac{m}{L}(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$$

Let $\lambda_k^2 = \nabla f(x_k)^T(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$
Goal:

$$f(x_k) - f(x_{k+1}) \geq \frac{m}{2L}\lambda_k^2$$

Quadratic taylor approximation of f around x_k:

$$f(y) \approx f(x_k) + \nabla f(x_k)^T(y - x_k) + \frac{1}{2}(y - x_k)^T\nabla^2 f(x_k)(y - x_k)$$

Let $y = x_{k+1}$. We then get:

$$f(x_{k+1}) \approx f(x_k) + \nabla f(x_k)^T((x_k - \frac{m}{L}(\nabla^2 f(x_k))^{-1}\nabla f(x_k)) - x_k) + \frac{1}{2}((x_k - \frac{m}{L}(\nabla^2 f(x_k))^{-1}\nabla f(x_k)) - x_k)^T\nabla^2 f(x_k$$

Simplification:

$$f(x_{k+1}) \approx f(x_k) - \frac{m}{L}\nabla f(x_k)^T(\nabla^2 f(x_k))^{-1}\nabla f(x_k) + \frac{m^2}{2L^2}\nabla f(x_k)^T(\nabla^2 f(x_k))^{-1}\nabla^2 f(x_k)(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$$

Since $\nabla^2 f(x)$ is positive definite, we can simplify this to:

$$\frac{m^2}{2L^2}\nabla f(x_k)^T(\nabla^2 f(x_k))^{-1}\nabla f(x_k) \leq \frac{m}{2L}\nabla f(x_k)^T(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$$

We can deduce from this that:

$$f(x_k) - f(x_{k+1}) \geq \frac{m}{2L}\nabla f(x_k)^T(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$$

iii) Show that $x_k \to x^*$ as $k \to \infty$.
Combining the inequalities from i and ii, we get:

$$f(x_{k+1}) \leq f(x_k) - \frac{m}{2L}\nabla f(x_k)^T(\nabla^2 f(x_k))^{-1}\nabla f(x_k)$$

x^* is the global minimizer, so f is bounded below by $f(x^*)$. This means that $f(x_k)$ is decreasing and bounded below by $f(x^*)$. So, $f(x_k)$ converges to some value G.

This implies that $f(x_k) - f(x_{k+1})$ approaches 0 as $k \to \infty$.
Now, from our inequality,

$$\nabla f(x_k)^T(\nabla^2 f(x_k))^{-1}\nabla f(x_k) \to 0$$

as $k \to \infty$.
Because $\nabla^2 f(x)$ is positive definite and $\leq L$ and $\geq m$, that means that

$$\nabla f(x_k) \to 0$$

as $k \to \infty$.
But $\nabla f(x_k)$ is only 0 at x^*, so $x_k \to x^*$
as $k \to \infty$.