This is due Saturday 9/23 by 11:59 pm on Gradescope. Please either neatly write up your solutions or type them up. You can find a .tex template on Canvas. Your proofs should be written in complete sentences and paragraphs, using a combination of words and symbols. They should be **correct, clear, and concise**. You will be graded on all three, especially the first two!

1. **Directly from the definition of the the limit of a sequence**, show that $\lim_{n\to\infty} \sqrt{4+n} - \sqrt{n} = 0$. (So for this problem you can't use any of the theorems we proved about limits converging).

Solution:

$\lim_{n\to\infty} \sqrt{4+n} - \sqrt{n} = 0 \iff \forall \epsilon > 0, \exists M \in \mathbb{N}, M > 0$, such that if $n > M$, then $|a_n - 0| < \epsilon$

$$|\sqrt{4+n} - \sqrt{n}| < \epsilon$$

Since $4 + n > n \forall n \in \mathbb{R}$, then $\sqrt{4+n} > \sqrt{n} \forall n \in \mathbb{R}$. This means that $\sqrt{4+n} - \sqrt{n} > 0 \forall n \in \mathbb{R}$.

$$\begin{aligned}
&\implies \sqrt{4+n} - \sqrt{n} < \epsilon \\
&\implies \sqrt{4+n} < \epsilon + \sqrt{n} \\
&\implies 4 + n < \epsilon^2 + 2\epsilon\sqrt{n} + n \\
&\implies 4 < \epsilon^2 + 2\epsilon\sqrt{n} \\
&\implies 0 < \epsilon^2 + 2\epsilon\sqrt{n} - 4 \\
&\implies -2\epsilon\sqrt{n} < \epsilon^2 - 4 \\
&\implies \epsilon\sqrt{n} < -2\epsilon^2 + 2 \\
&\implies \sqrt{n} < -2\epsilon + \frac{2}{\epsilon} \\
&\implies n < 4\epsilon^2 + 2\frac{2}{\epsilon} \cdot (-2\epsilon) + \frac{4}{\epsilon^2} \\
&\implies n < 4\epsilon^2 - 8 + \frac{4}{\epsilon^2}
\end{aligned}$$

Let M equal $4\epsilon^2 - 8 + \frac{4}{\epsilon^2}$.

$\square$.

2. **Directly from the definition of the limit of a sequence**, show that $\left\{\sin\left(n\frac{\pi}{4}\right)\right\}_{n=0}^{\infty}$ does not converge.

Solution:

Claim: $\left\{\sin\left(n\frac{\pi}{4}\right)\right\}_{n=0}^{\infty}$ does not converge.

We will show this by contradition.

Assume it does converge:

$$\left|\sin\left(n\frac{\pi}{4}\right)\right| < \epsilon$$

$$\sin\left(n\frac{\pi}{4}\right) < \pm\epsilon$$

$$n\frac{\pi}{4} < \sin^{-1}(\pm\epsilon)$$

$$n < \frac{4}{\pi}\sin^{-1}(\pm\epsilon)$$

Well, the range of arcsin is contained within the interval $[-1,1]$ (if we assume a 1-1 mapping with *sin* as is standard/typical), so, the maximum of the right side becomes $\frac{4}{\pi}\cdot 1 = \frac{4}{\pi}$. We can just choose a natural number n greater than $\frac{4}{\pi}$ and our inequality is contradicted. Therefore the sequence does not converge.

3. Suppose that $\{a_n\}$ is a sequence that converges to some $a \in \mathbb{R}$, and that $b, c \in \mathbb{R}$. Suppose moreover that for all n we have that $b \leq a_n \leq c$. Show that $b \leq a \leq c$.

Solution:

$$\forall n \in \mathbb{N}, b \leq a_n \leq c$$
$$\implies b - a \leq a_n - a \leq c - a$$

Since $c > a$, $c - a \geq 0$.

Since $a > b$, $b - a \leq 0$.

$$\implies b - a \leq 0 \leq c - a$$
$$\implies b \leq a \leq c$$

$\square$.

4. Suppose that $\{a_n\}$ converges to a, where each $a_n \geq 0$, and that $k \in \mathbb{N}$. Show that $\{a_n^{1/k}\}$ converges to $a^{1/k}$. **Hint:** Use the geometric series, i.e that $(x-y)(x^{k-1} + x^{k-2}y + \ldots y^{k-1}) = x^k - y^k$.

Solution:

a_n converges to a

Claim: $a_n^{\frac{1}{k}}$ converges to $a^{\frac{1}{k}}$.

$$a_n - a < \epsilon$$
$$a_n < a + \epsilon$$
$$a_n^{\frac{1}{k}} < (a+\epsilon)^{\frac{1}{k}}$$

Let G be all of the terms involving ϵ (i.e. all of the terms that involve multiplying by a power of ϵ).

$$a_n^{\frac{1}{k}} < a^{\frac{1}{k}} + G$$
$$a_n^{\frac{1}{k}} - a^{\frac{1}{k}} < G$$

Since G is all terms involving ϵ, and ϵ is arbitrarily small, that must mean that $a_n^{\frac{1}{k}}$ converges to $a^{\frac{1}{k}}$.

5. Define a sequence $\{a_n\}$ by $a_1 = 1$ and for $n \geq 1$ by $a_{n+1} = \sqrt{a_n + 1}$. Show that $\{a_n\}$ converges and compute the limit of this sequence. **Hint:** Show that the sequence is monotone and bounded.

Solution:

The sequence is trivially monotone since it is adding a positive amount to itself at each step. So it must always be increasing.

Showing that is is bounded is trickier.

Let us take 2. This must necessarily be an upper bound for the sequence because of the fact that $1 + \sqrt{(g)}$ where $g < 3$ will always be less than $1 + 3 = 4$.

Since it is bounded and monotone, it must converge.

Now, what does it converge to?

The limit of this sequence is $\sqrt{e}$.

We can compute this by looking at the definition for e: $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$