

1) Beck Exercise 6.9. Let  $a, b \in \mathbb{R}^n, a \neq b$ . For what values of  $\mu$  is the set:

$$S_\mu = \{x \in \mathbb{R}^n : \|x - a\|_2 \leq \mu \|x - b\|_2\}$$

convex?

The set  $S_\mu$  is convex if for any two points  $x, y$  in the set and any  $\lambda \in [0, 1]$ , the point  $z = \lambda x + (1 - \lambda)y$  is also in the set. In other words, our goal is to show:

$$\|\lambda x + (1 - \lambda)y - a\|_2 \leq \mu \|\lambda x + (1 - \lambda)y - b\|_2$$

$\forall x, y \in S_\mu$  and  $\lambda \in [0, 1]$ .

Given  $x, y \in S_\mu$ , we have:

$$\|x - a\|_2 \leq \mu \|x - b\|_2$$

and

$$\|y - a\|_2 \leq \mu \|y - b\|_2$$

We now need to show that for  $z = \lambda x + (1 - \lambda)y$ ,

$$\|z - a\|_2 \leq \mu \|z - b\|_2$$

By the triangular inequality, we get:

$$\|\lambda x + (1 - \lambda)y - a\|_2 \leq \lambda \|x - a\|_2 + (1 - \lambda) \|y - a\|_2$$

and

$$\mu \|\lambda x + (1 - \lambda)y - b\|_2 \geq \mu(\lambda \|x - b\|_2 + (1 - \lambda) \|y - b\|_2)$$

Given that  $\|x - a\|_2 \leq \mu \|x - b\|_2$  and  $\|y - a\|_2 \leq \mu \|y - b\|_2$ , we have  $\lambda \|x - a\|_2 + (1 - \lambda) \|y - a\|_2 \leq \mu(\lambda \|x - b\|_2 + (1 - \lambda) \|y - b\|_2)$ . This implies:  $\|\lambda x + (1 - \lambda)y - a\|_2 \leq \mu \|\lambda x + (1 - \lambda)y - b\|_2$ .

This means that  $S_\mu$  is convex for all values of  $\mu$  in  $\mathbb{R}$ .

2) Beck Exercise 6.10. Let  $C \subset \mathbb{R}^n$  be a nonempty convex set. For each  $x \in C$ , define the normal cone of  $C$  at  $x$  by

$$N_C(x) = \{w \in \mathbb{R}^n : \langle w, y - x \rangle \leq 0 \forall y \in C\}$$

and define  $N_C(x) = \emptyset$  when  $x \notin C$ . Show that  $N_C(x)$  is a closed convex cone.

Let's split this into two parts:

Showing that  $N_C(x)$  is convex and showing that  $N_C(x)$  is a cone.

Let's first do convexity:

We have to show that for any two points  $w_1, w_2 \in N_C(x)$  and any  $\lambda \in [0, 1]$ , the point  $\lambda w_1 + (1 - \lambda)w_2$  is also in  $N_C(x)$ .

$$\langle w_1, y - x \rangle \leq 0 \forall y \in C$$

and

$$\langle w_2, y - x \rangle \leq 0 \forall y \in C.$$

Therefore, for any  $\lambda \in [0, 1]$  and any  $y \in C$ ,

$$\langle \lambda w_1 + (1 - \lambda)w_2, y - x \rangle = \lambda \langle w_1, y - x \rangle + (1 - \lambda) \langle w_2, y - x \rangle \leq 0$$

which implies that  $\lambda w_1 + (1 - \lambda)w_2 \in N_C(x)$ .

Now, for the cone:

We have to show that for any two points  $w_1, w_2 \in N_C(x)$  and any  $\mu \leq 0$ , the point  $\mu w$  is also in  $N_C(x)$ .

$$\langle w, y - x \rangle \leq 0 \forall y \in C$$

Therefore,

$$\langle \mu w, y - x \rangle = \mu \langle w, y - x \rangle \leq 0$$

which implies that  $\mu w \in N_C(x)$ .

Now, we have to show that  $N_C(x)$  is closed.

Let  $\{w_k\}$  be a sequence in  $N_C(x)$  that converges to some  $w$ . We have to show that  $w \in N_C(x)$ .

We have:

$$\langle w_k, y - x \rangle \leq 0 \forall y \in C$$

Taking the limit as  $k \rightarrow \infty$ , we have:

$$\langle w, y - x \rangle \leq 0 \forall y \in C$$

which implies that  $w \in N_C(x)$ . Therefore,  $N_C(x)$  is a closed convex cone.

3) Let  $a \in \mathbb{R}^n - \{0\}$ , what is the distance between the two parallel hyperplanes

$$\{x \in \mathbb{R}^n : a^T x = b_1\} \text{ and } \{x \in \mathbb{R}^n : a^T x = b_2\}?$$

The distance between any two hyperplanes can be found by taking any one point from one of the hyperplanes and taking the distance between that point and the other hyperplane.

Let's take some point  $x_1$  on the first hyperplane such that  $a^T x_1 = b_1$ . We have to find the distance between  $x_1$  and the second hyperplane. The distance from a point  $x_0$  to the hyperplane  $\{x \in \mathbb{R}^n : a^T x = b\}$  is given by:

$$\frac{|a^T x_0 - b|}{\|a\|}.$$

Plugging in  $x_1$  as the point, we get:

$$\frac{|a^T x_1 - b_2|}{\|a\|} = \frac{|b_1 - b_2|}{\|a\|}.$$

This is the equation for the distance between the two hyperplanes.

4) Beck Exercise 7.4. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable convex function. Show that for any  $\epsilon > 0$ , the function

$$g_\epsilon(x) = f(x) + \epsilon \|x\|^2$$

is coercive.

A function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is called coercive if  $\lim_{\|x\| \rightarrow \infty} g(x) = \infty$ .

Since  $f$  is convex and continuously differentiable, it must grow at least as fast as any of its linear approximations. So, for any  $x_0 \in \mathbb{R}^n$ , we get:

$$f(x) \geq f(x_0) + \nabla f(x_0)^T (x - x_0).$$

If we set  $x_0 = 0$ , we get:

$$f(x) \geq f(0) + \nabla f(0)^T x.$$

Now, consider  $g_\epsilon(x)$ :

$$g_\epsilon(x) = f(x) + \epsilon \|x\|^2 \geq f(0) + \nabla f(0)^T x + \epsilon \|x\|^2.$$

Let's plug in the limit  $\|x\| \rightarrow \infty$ .  $\epsilon \|x\|^2$  dominates everything else, so we get:

$$\lim_{\|x\| \rightarrow \infty} g_\epsilon(x) \geq \lim_{\|x\| \rightarrow \infty} (f(0) + \nabla f(0)^T x + \epsilon \|x\|^2) = \infty.$$

Therefore, the function  $g_\epsilon$  is coercive.

5) Beck Exercise 7.5. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Prove that  $f$  is convex if and only if for any  $x \in \mathbb{R}^n$  and  $d \neq 0$ , the one-dimensional function  $g_{x,d}(t) = f(x + td)$  is convex.

We have to prove two separate implications:

1. If  $f$  is convex, then for any  $x \in \mathbb{R}^n$  and  $d \neq 0$ , the function  $g_{x,d}(t) = f(x + td)$  is convex.

2. If for any  $x \in \mathbb{R}^n$  and  $d \neq 0$ , the function  $g_{x,d}(t) = f(x + td)$  is convex, then  $f$  is convex.

Ok, let's start with 1. Assume that  $f$  is convex. Then, for any  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ , we have:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Let  $g_{x,d}(t) = f(x + td)$ . We want to show that for any  $t_1, t_2 \in \mathbb{R}$  and  $\lambda \in [0, 1]$ , we have:

$$g_{x,d}(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda g_{x,d}(t_1) + (1 - \lambda)g_{x,d}(t_2).$$

If we substitute in the definition of  $g_{x,d}$ , we have:

$$f(x + (\lambda t_1 + (1 - \lambda)t_2)d) \leq \lambda f(x + t_1 d) + (1 - \lambda)f(x + t_2 d).$$

Thus this inequality follows from the convexity of  $f$ .

Now, let us handle the second implication:

Assume that for any  $x \in \mathbb{R}^n$  and  $d \neq 0$ , the function  $g_{x,d}(t) = f(x + td)$  is convex. Then, for any  $t_1, t_2 \in \mathbb{R}$  and  $\lambda \in [0, 1]$ , we get:

$$f(x + (\lambda t_1 + (1 - \lambda)t_2)d) \leq \lambda f(x + t_1 d) + (1 - \lambda)f(x + t_2 d).$$

We want to show that  $f$  is convex. Let  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ . Let  $d = y - x$  and  $t_1 = 0, t_2 = 1$ . Then:

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &= f(x + \lambda(y - x)) \\ &= f(x + \lambda d) \leq \lambda f(x) + (1 - \lambda)f(x + d) = \lambda f(x) + (1 - \lambda)f(y). \end{aligned}$$

Therefore,  $f$  is convex.