Math 3220-1: Homework 9, due 04/10/2024

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Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation

$$f(x, y) = (e^x \cos y, e^x \sin y).$$

(a) Show that f is one-to-one on the set $A = \{(x, y) \in \mathbb{R}^2 : 0 < y < 2\pi\}$.
(b) What is the set $B = f(A)$?
(c) If g is the inverse function, find $Dg(0, 1)$.

a) Suppose $f(x_1, y_1) = f(x_2, y_2)$. Then:

$$(e^{x_1} \cos(y_1), e^{x_1} \sin(y_1)) = (e^{x_2} \cos(y_2), e^{x_2} \sin(y_2))$$

This gives us two equations:

$$e^{x_1} \cos(y_1) = e^{x_2} \cos(y_2)$$
$$e^{x_1} \sin(y_1) = e^{x_2} \sin(y_2)$$

Dividing the second equation by the first (assuming none of the cos terms are 0, which is guarenteed by $0 < y < 2\pi$):

$$\frac{\sin(y_1)}{\cos(y_1)} = \frac{\sin(y_2)}{\cos(y_2)} \implies \tan(y_1) = \tan(y_2)$$

Since y_1 and y_2 are in $(0, 2\pi)$, tan is injective. This means that $y_1 = y_2$.
This means we can reduce the equation to:

$$e^{x_1} = e^{x_2} \implies x_1 = x_2$$

Thus, $(x_1, y_1) = (x_2, y_2)$ and f is one-to-one.
b) Since x can be any real number, x^x covers all positive real numbers.
Since $y \neq 0$ and $y \neq 2\pi$, the second term cannot be 0.
So, the set B is $\{(u, v) \in \mathbb{R}^2 : u > 0\} \cup \{(u, v) \in \mathbb{R}^2 : u = 0, v > 0\}$.
c) To find $Dg(0, 1)$, we have to find a point (x_0, y_0) in A such that $f(x_0, y_0) = (0, 1)$, compute $Df(x_0, y_0)$, and then invert it.

$$Df(x, y) = \begin{bmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{bmatrix}$$

$$Df\left(0, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The inverse is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Thus, $Dg(0, 1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Problem 2. Use the Inverse Function Theorem to determine the points of $\mathbb{R}$ near which the sin function has a smooth local inverse function. What is the derivative of the inverse function when it exists?

$\sin : \mathbb{R} \to \mathbb{R}$ is continuously differentiable everywhere.

The derivative of $\sin(x)$ is $\cos(x)$.

To find where the sin function has a smooth local inverse, we need to determine where $\cos(x) \neq 0$.

$\cos(x) = 0$ when $x = \frac{\pi}{2} + k\pi$ for any integer k. So, the sin function will have a smooth local inverse at points where $x \neq \frac{\pi}{2} + k\pi$.

Now, we have to find the derivative of the inverse function.

If $y = \sin^{-1}(x)$, then:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Given $y = \sin(x)$, and $\frac{dy}{dx} = \cos(x)$, we have:

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\cos(y)}$$

Since $y = \sin^{-1}(x)$ and using the identity that $\sin^2(x) + \cos^2(x) = 1$, we can write:

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$$

So, the derivative of the inverse sin function is:

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1 - x^2}}$$

The derivative exists and is valid for $-1 < x < 1$, correponding to where the sin function is defined and smooth, excluding the points where $\cos(x) = 0$.

Problem 3. Are there any points on the graph of the equation $x^3+3xy^2+2y^3=1$ where it may not be possible to solve for y as a smooth function of x in some neighborhood of the point?

We'll use the Implicit Function Theorem for this.

We need to find where the partial derivative of the function with respect to y, $\frac{\partial}{\partial y} = 6xy + 6y^2$, equals 0; because at these points the Implicit Function Theorem does not guarentee existence of a smooth function.

$6xy + 6y^2 = 0$ when $y = 0$ or $y = -x$.

When $y = 0$, substituting into the original equation $x^3 + 3xy^2 + 2y^3 = 1$ gives $x^3 = 1$, so $x = 1$. Thus, the point of interest is $(1, 0)$.

When $y = -x$, substituting into the original equation gives $-4x^3 = 1$, resulting in $x^3 = -\frac{1}{4}$.

Thus, it might not be possible to solve for a smooth function of x in some neighborhood of $(1, 0)$. At this point, the Implicit Function Theorem does not guarentee existence of a smooth function because the partial derivative with respect to y is 0.

The other place is the intersection of $y = -x$ and $x^3 = -\frac{1}{4}$. This intersection occurs at $x = \left(-\frac{1}{4}\right)^{1/3}$. Thus, a smooth function of x is also not guarenteed in the neighborhood of $(\left(-\frac{1}{4}\right)^{1/3}, -\left(-\frac{1}{4}\right)^{1/3})$.

Problem 4. Consider the set S described by the equation

$$xz + yz + \sin(x + y + z) = 0.$$

Can S be represented as a smooth parameterized 2-surface near the point $(0, 0, 0)$? (justify your answer). If so, find an equation of a tangent space to S at this point.

Let's first find the gradient of the function:

$$\nabla F = (z + \cos(x + y + z), z + \cos(x + y + z), x + y + \cos(x + y + z))$$

Evaluated at the point $(0, 0, 0)$ gives $(1, 1, 1)$.

Since the gradient is non-zero at $(0, 0, 0)$, the set S can be represented as a smooth parameterized 2-surface near this point.

Let's now find it. The tangent space is given by:

$$\nabla F(a, b, c) \cdot (x - a, y - b, z - c) = 0$$

Substituting in $\nabla F(0, 0, 0) = (1, 1, 1)$ and $(a, b, c) = (0, 0, 0)$ gives:

$$1 \cdot (x - 0) + 1 \cdot (y - 0) + 1 \cdot (z - 0) = 0$$

which simplifies to:

$$x + y + z = 0$$

This is the equation for the tangent space to S at $(0, 0, 0)$.

Problem 5. For the system of equations

$$x^2 + y^2 - z^2 = 0,$$
$$x + y + z = 0,$$

at which points of the solution set S is there a neighborhood in which S is a smooth curve? At each such point find an equation of the tangent line.

First let's compute the gradients:

$$(2x, 2y, -2z)$$
$$(1, 1, 1)$$

For these to be parallel, then $x = y = -z$ must be true. From the second equation we are given, $x + y + z = 0$, we have that $-z = x + y$. This means that $x = y = x + y$, which is only true if $x = y = -z = 0$. This is the zero case from before, which means that the two vectors are always linearly independent except at $(0, 0, 0)$.

So, at $(0, 0, 0)$, the condition for the existence of a smooth curve is not satisfied, but it is satisfied everywhere else.

Substituting $z = -x - y$ into the first equation gives $-2xy = 0 \implies xy = 0$. This means that $y = 0$ when $x \neq 0$. Using the second equation gives $z = -x$.

So, the points of the solution set S are of the form $(x, 0, -x)$ for $x \neq 0$.

This means that $\nabla F = (2x, 0, 2x)$.

Now, to compute the normal of the tangent space, we need to compute $\nabla F(x, 0, -x) \times \nabla G(x, 0, -x) = (2x, 0, 2x) \times (1, 1, 1)$.

This works out to $(-2x, 0, 2x)$.

Using the point-direction form of a line, we eventually get:

$$\frac{z + x}{2x} = t$$

This means that $x = x$, $y = 0$, and $z = 2xt - x$.

For each value of t, this set of equations describes the coordinates of a point on the tangent line at $(x, 0, -x)$ on the solution set S.