Lincoln Sand

H1.

128.19:

Consider an ellipse with the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a is the major axis and b is the semi-minor axis. The foci are at $(c, 0)$ and $(-c, 0)$, where $c = \sqrt{a^2 - b^2}$.

Let $P(x, y)$ be a point on the ellipse. We want to prove that the sum of squares of the distance from P to the foci is constant and always equal to $2a^2$.

Distance from P to the first focus: The distance PF_1 is given by $PF_1 = \sqrt{(x-c)^2 + y^2}$.

Distance from P to the second focus: The distance PF_2 is given by $PF_2 = \sqrt{(x+c)^2 + y^2}$.

We need to prove that $PF_1^2 + PF_2^2$ is constant for any point on the ellipse.

$$PF_1^2 = (x-c)^2 + y^2 = x^2 - 2cx + c^2 + y^2$$
$$PF_2^2 = (x+c)^2 + y^2 = x^2 + 2cx + c^2 + y^2$$

$$PF_1^2 + PF_2^2 = 2x^2 + 2c^2 + 2y^2$$

Since P lies on the ellipse, substituting $y^2 = b^2 - \frac{b^2}{a^2}x^2$ gives:

$$PF_1^2 + PF_2^2 = 2x^2 + 2c^2 + 2b^2 - \frac{2b^2}{a^2}x^2$$
$$PF_1^2 + PF_2^2 = 2a^2 + 2c^2 - 2c^2$$
$$PF_1^2 + PF_2^2 = 2a^2$$

128.26:

We can rewrite the given equation as:

$$y^2 = px - \frac{p}{2a}x^2$$

This can be written in standard form where $b^2 = \frac{p}{2a}$.

We will prove in the following order:

1. NP is perpendicular to the axis of the ellipse.
2. PG is the minimum distance from G to the curve.
3. PG is perpendicular to the tangent at P.

1. By construction, NP is drawn perpendicular to the major axis AA'. Any line perpendicular to the major axis and passing through a point must also pass through the corresponding point on the ellipse. This is true by the reflective property of ellipses.

2. For PG to be the minimum distance from G to the ellipse, it must be perpendicular to the curve at P. In an ellipse, the shortest distance from a point outside the ellipse to the ellipse itself is along the line that passes through the nearest focus. Since NP is perpendicular to the major axis and intersects the ellipse at P, it implies that NP is the shortest distance from N to the ellipse, and hence PG must be the shortest distance from G to the ellipse.

3. Given $y^2 = x(p - \frac{p}{2a}x)$, let's differentiate this with respect to x to find the slope of the tangent line.

We get:

$$2y\frac{dy}{dx} = p - \frac{p}{a}x$$

$$\implies \frac{dy}{dx} = \frac{p - \frac{p}{a}x}{2y}$$

At point P, the line NP is vertical, so x is a constant and y varies. Since NP is perpendicular to the major axis AA', it is also the normal to the ellipse at point P. Therefore, the slope of the tangent at P must be horizontal, which means $\frac{dy}{dx}$ at P must be 0.

168.1:

We will use the formula:

$$crd\left(\frac{\theta}{2}\right)^2 = R^2 - \left(\frac{crd(\theta)}{2}\right)^2$$

$$crd(30^\circ)^2 = R^2 - \left(\frac{crd(60^\circ)}{2}\right)^2$$

$$\implies crd(30^\circ) = \sqrt{R^2 - \left(\frac{R}{2}\right)^2}$$

$$\implies crd(30^\circ) = \frac{R\sqrt{3}}{2} = \frac{60\sqrt{3}}{2}$$

Using the same forumula, We get:

$$crd(15^\circ) = \sqrt{60^2 - \left(\frac{crd(30^\circ)}{2}\right)^2}$$

$$crd(7.5^\circ) = \sqrt{60^2 - \left(\frac{crd(15^\circ)}{2}\right)^2}$$

168.4:

Let's consider a cyclic quadrilateral $ABCD$ inscribed in a circle, where:

$$\angle ADB = 180^\circ - \alpha$$

$$\angle BDC = 180^\circ - \beta$$

$$\angle ADC = 180^\circ - (\alpha + \beta)$$

The sides AD and BC are not adjacent and thus will form the diagonals of the quadrilateral when connected. The other sides are AB, CD, BD, and AC.

Now, according to Ptolemy's theorem:

$$AD \cdot BC + AB \cdot CD = AC \cdot BD$$

In terms of chord lengths in a circle of radius R, where $R = 60$ to fit the ancient Greek chord system, we can express the sides as follows:

$$AD = crd(180^\circ - \alpha)$$

$$BC = crd(180^\circ - \beta)$$

$$AB = crd(\beta)$$

$$CD = crd(\alpha)$$

$$AC = crd(180^\circ - (\alpha + \beta))$$

$$BC = crd(\alpha + \beta)$$

Substituting this into Ptolemy's theorem gives us:

$$crd(180^\circ - \alpha) \cdot crd(180^\circ - \beta) + crd(\alpha) \cdot crd(\beta)$$

$$= crd(180^\circ - (\alpha + \beta)) \cdot crd(\alpha + \beta)$$

We need to prove that:

$$120 \cdot crd(180^\circ - (\alpha + \beta))$$

$$= crd(180^\circ - \alpha) \cdot crd(180^\circ - \beta) - crd(\alpha) \cdot crd(\beta)$$

To align this with the equation derived from Ptolemy's theorem, we need to consider the property of chord lenghts where $crd(\theta) = crd(360^\circ - \theta)$. This implies that $crd(\alpha + \beta) = crd(180^\circ - (\alpha + \beta))$ in a semicircle.

Thus, the equation becomes:

$$crd(180^\circ - \alpha) \cdot crd(180^\circ - \beta) + crd(\alpha) \cdot crd(\beta)$$

$$= crd(180^\circ - (\alpha + \beta)) \cdot crd(180^\circ - (\alpha + \beta))$$

Rearranging and multiplying both sides by 120 gives:

$$120 \cdot [crd(180^\circ - \alpha) \cdot crd(180^\circ - \beta) - crd(\alpha) \cdot crd(\beta)]$$
$$= 120 \cdot [crd(180^\circ - (\alpha + \beta)^2)]$$

This equation demonstrates the sum formula.

168.22:

First, let's confirm this is a valid triangle. The sum of any two sides should be larger than the third side.

$$4 + 7 > 10$$
$$4 + 10 > 7$$
$$7 + 10 > 4$$

Method 1: Heron's formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$.
For our triangle, $s = \frac{21}{2} = 10.5$

$$A = \sqrt{10.5(10.5-4)(10.5-7)(10.5-10)}$$

Method 2: Heron's alternative formula

$$A = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

$$A = \frac{1}{4}\sqrt{(4+7+10)(-4+7+10)(4-7+10)(4+7-10)}$$

Both methods give approximately 10.93 square units.