1) Beck Exercise 12.1. Find a dual problem to the convex problem

$$min x_1^2 + 0.5x_2^2 + x_1x_2 - 2x_1 - 3x_2$$

$$s.t. x_1 + x_2 \leq 1$$

Find the optimal solutions of both the dual and primal problems.

The Lagrangian of this problem is given by:

$$L(x, \lambda) = f(x) + \lambda g(x) = x_1^2 + 0.5x_2^2 + x_1x_2 - 2x_1 - 3x_2 + \lambda(x_1 + x_2 - 1)$$

The dual function is obtained by miniziming the Lagrangian with respect to x. We need to find $\theta(\lambda) = \min_x L(x, \lambda)$. This involves differentiating L with respect to x_1 and x_2, setting the derivatives to zero, and solving for x_1 and x_2.

The dual problem is given by:

$$maximize \theta(\lambda)$$

subject to

$$\lambda \geq 0.$$

The dual function is:

$$\theta(\lambda) = \lambda(2 - \lambda) + 4\lambda + 8(1 - 0.25\lambda)^2 - 13$$

The solutions for x_1 and x_2 in terms of λ are:

$$x_1 = -1, x_2 = 4 - \lambda$$

The optimal solution for the dual problem is found when $\lambda = 2$, and the optimal value of the dual problem is -3.

For the primal problem, the optimal solutions are $x_1 = -1$ and $x_2 = 2$. The optimal value of the problem is also -3.

Since they're the same value, this demonstrates strong duality.

2) Beck Exercise 12.3. Consider the problem

$$min x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3$$

$$s.t. x_1 + x_2 + x_3 \leq 1$$

$$x_3 \leq 1$$

a) Is the problem convex?

The matrix representing the quadratic terms of the objective function is positive semidefinite, which confirms that the objective function is convex. And since the constraints are linear, they define a convex set. Therefore, the given problem is a convex optimzation problem.

b) Find an optimal solution of the problem.

We solve this by formulating the Lagrangian and then solving for the critical points.

The optimal solution for the problem is:

$$x_1 = -1, x_2 = 1, x_3 = 1$$

with the Lagrange multipliers:

$$\lambda_1 = -1, \lambda_2 = 2$$

But note that the negative value for λ is not valid in this context since Lagrange multipliers associated with inequality constraints must be non-negative. This means the solution might not satisfy the KKT conditions, indicating a potential issue with the solution or the constraints.

c) Find a dual problem and solve it.

The dual function $\theta(\lambda_1, \lambda_2)$ is obtained by miniziming the Lagrangian with respect to x_1, x_2, and x_3. The dual problem then involves maximizing this dual function subject to $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

3) Beck Exercise 12.15. Let $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$ and $b_1, b_2, \ldots, b_m \in \mathbb{R}$ and consider the problem of finding the so called analytic center of the polytope $S = \{x\ in \mathbb{R}^n : a_i^T x < b_i, b = 1, \ldots, m\}$ given by (A)

$$min \left\{ -\sum_{i=1}^{m} log(b_i - a_i^T x) : x \in S \right\}.$$

Find a dual problem to (A). Hint: introduce an auxiliary variable $y = b - Ax$ and add this as a constraint.

We can rewrite the primal problem using the auxiliary variable y as:

$$min \left\{ -\sum_{i=1}^{m} \log(y_i) : Ax + y = b, y > 0 \right\}.$$

The Lagrangian $L(x, y, \lambda)$ can be written as:

$$L(x, y, \lambda) = -\sum_{i=1}^{m} \log(y_i) + \sum_{i=1}^{m} \lambda_i (a_i^T x + y_i - b_i)$$

The dual function is obtained by miniziming this Lagrangian with respect to the primal variables x and y. This involves solving:

$$\theta(\lambda) = \min_{x,y} L(x, y, \lambda)$$

subject to $y_i > 0$.

The dual problem then is to maximize this dual function:

$$\max_{\lambda} \theta(\lambda)$$

subject to $\lambda_i \geq 0$ for all i.

4) The goal in this problem is to prove convergence of the interior point method for a convex optimization problem with inequality constraints, (1a)

$$\min_{x \in \mathbb{R}^n} f(x)$$

(1b)

$$s.t. g_i(x) \leq 0, i = 1, \ldots, m.$$

We assume f, g_i for $i = 1, \ldots, m$ are convex and twice continuously differentiable on $\mathbb{R}^n$ and that an optimal solution, x^*, exists with $f(x^*) = f^*$. We also assume the constraint set satisfies Slater's condition. In this case, there exists a dual optimal $\lambda_* \in \mathbb{R}^m_+$ which together with x^* satisfy the KKT conditions (2a)

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$

(2b)

$$\lambda_i^* g_i(x^*) = 0, i = i, \ldots, m$$

(2c)

$$g_i(x^*) \leq 0, i = 1, \ldots, m$$

(2d)

$$\lambda^* \geq 0.$$

In the interior point method, we replace the primal problem by (3)

$$\min_{x \in \mathbb{R}^n} t f(x) + \phi(x)$$

where $\phi(x) = -\sum_{i=1}^{m} log(-g_i(x))$ and $t > 0$ is a large parameter. We denote the optimal solution of (3) by x_t^*.

a) Show that $\phi : \mathbb{R}^n \to \mathbb{R}$ is a convex function on

$$dom(\phi) = \{x \in \mathbb{R}^n : g_i(x) < 0, i = 1, \ldots, m\}.$$

The function $\phi(x) = -\sum_{i=1}^{m} \log(-g_i(x))$ is defined on the domain $dom(\phi) = \{x \in \mathbb{R}^n : g_i(x) < 0, i = 1, \ldots, m\}$. To show that ϕ is convex on this domain, we need to demonstrate that the Hessian of $\phi(x)$ is positive semidefinite on $dom(\phi)$.

Given that $g_i(x)$ are convex functions, $-g_i(x)$ are concave. The logarithm function is concave and increasing, and the composition of a concave increasing function with a concave function is concave. Hence, $\log(-g_i(x))$ is concave, and $-\log(-g_i(x))$ is convex. The sum of convex functions is convex, so $\phi(x)$ is convex on its domain.

b) Using the introduced notation, write the Lagrangian and give the definition of the dual function for (1).

For the problem (1), the Lagrangian $L(x, \lambda)$ is given by:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

The dual function is the infimum of the Lagrangian over x:

$$d(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda)$$

c) Write the first order optimality condition for (3).

For the modified problem (3), the first order optimality condition (the gradient of the objective function equals zero) at the optimal point x_t^* is:

$$t\nabla f(x_t^*) + \nabla \phi(x_t^*) = 0$$

where $\nabla \phi(x_t^*) = -\sum_{i=1}^m \frac{1}{g_i(x_t^*)} \nabla g_i(x_t^*)$.

d) Define $\tilde{(}\lambda) \in \mathbb{R}_+^m$ by

$$\tilde{(}\lambda_i) = -\frac{1}{tg_i(x_t^*)}, i = 1, \ldots, m$$

By evaluating the dual function at $\tilde{\lambda}$, conclude that $f(x_t^*) - f^* \leq m/t$, i.e., x_t^* is no more than m/t-suboptimal.

Define $\tilde{\lambda_i} = -\frac{1}{tg_i(x_t^*)}$. Substituting this into the dual function and considering the first order optimality condition, we get:

$$t\nabla f(x_t^*) = \sum_{i=1}^m \frac{1}{g_i(x_t^*)} \nabla g_i(x_t^*) = 0$$

Multiplying both sides by $x_t^* - x^*$ and rearranging, we obtain:

$$t(f(x_t^*) - f^*) = \sum_{i=1}^m \frac{1}{-g_i(x_t^*)} \nabla g_i(x_t^*)^T (x_t^* - x^*)$$

Since x^* and x_t^* satisfy the KKT conditions and the functions f and g_i are convex, the right-hand side is non-positive. This implies:

$$f(x_t^*) - f^* \leq \frac{m}{t}$$

Therefore, the solution x_t^* is no more than m/t-suboptimal.