Math 3220-1: Homework 7, due 03/13/2024

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Problem 1. Let
$$C = \begin{bmatrix} 1 & -1 \\ 4 & -6 \\ -1 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$
Find CD and DC.

CD :

$2 \cdot \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} - 1 \cdot \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 14 \\ -4 \end{bmatrix}$

$0 \cdot \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix}$

$1 \cdot \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} + 3 \cdot \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -14 \\ 5 \end{bmatrix}$

So, $CD = \begin{bmatrix} 3 & -1 & -2 \\ 14 & -6 & -14 \\ -4 & 2 & 5 \end{bmatrix}$.

DC :

$1 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$-1 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} - 6 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So, $CD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem 2. Let
$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 5 \\ -2 & 2 \end{bmatrix}$$
Find $\det(A)$, $\det(B)$, A^{-1}, B^{-1}.

$\det(A) = 3 \cdot 1 - (-1 \cdot 2) = 3 + 2 = 5.$
$\det(B) = 2 \cdot 2 - (5 \cdot -2) = 4 + 10 = 14.$
$A^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}.$
$B^{-1} = \frac{1}{14} \begin{bmatrix} 2 & -5 \\ 2 & 2 \end{bmatrix}.$

Problem 3. Find the matrix of the linear transformation of $\mathbb{R}^2$ which reflects each point through the diagonal line $y = x$ (this transformation interchanges x and y coordinates of each point).

Let's look at what happens to the bases:
$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
Thus, the transformation matrix is: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Problem 4. Prove that if $K : \mathbb{R}^p \to \mathbb{R}^q$ and $L : \mathbb{R}^q \to \mathbb{R}^r$ are linear transformations, then

$$\|L \circ K\| \leq \|L\| \, \|K\|$$

Recall:

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|$$

Consider an arbitrary vector $x \in \mathbb{R}^p$ with $\|x\| = 1$.

First, observe that:

$$\|K(x)\| \leq \|K\|\|x\| = \|K\|$$

And:

$$\|L \circ K\| = \|L(K(x))\|$$

Applying the definition of the norm of L, we get that:

$$\|L(K(x))\| \leq \|L\|\|K(x)\|$$

But, we've established that $\|K(x)\| \leq \|K\|$, so:

$$\|L(K(x))\| \leq \|L\|\|K\|$$

Since this inequality holds for any x with $\|x\| = 1$, it also holds for the supremum of $\|(L \circ K)(x)\|$ over all such x, which is exactly $\|L \circ K\|$:

$$\|L \circ K\| \leq \|L\|\|K\|$$

$\square$