Textbook Section 1.1:
8)

$$y'' + y = 3cos2x, y_1 = cos(x) - cos2x, y_2 = sin(x) - cos2x$$

a)

$$\frac{dy_1}{dx} = -sin(x) + 2sin(2x)$$

$$\frac{d^2y_1}{dx^2} = -cos(x) + 4cos2x$$

$$y_1'' + y_1 = (-cos(x) + 4cos2x) + (cos(x) - cos2x) = 3cos2x.$$

b)

$$\frac{dy_2}{dx} = cos(x) + 2sin(2x)$$

$$\frac{d^2y_2}{dx^2} = -sin(x) + 4cos2x$$

$$y_2'' + y_2 = (-sin(x) + 4cos2x) + (sin(x) - cos2x) = 3cos2x.$$

15)

$$y'' + y' - 2y = 0$$

$$Let\ y = e^{rx}$$

$$where\ r \in \mathbb{R}\ is\ a\ constant.$$

$$y' = re^{rx}$$

$$y'' = r^2e^{rx}$$

$$r^2e^{rx} + re^{rx} - 2re^{rx} = 0$$

$$(r^2 + r - 2)e^{rx} = 0$$

$$r^2 + r - 2 = 0$$

$$(r - 1) * (r + 2) = 0$$

$$r = 1\ or\ r = -2.$$

34) The accleration $\frac{dv}{dt}$ of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

$$\frac{dv}{dt} = \frac{250 - v}{t}$$

$$\frac{dv}{250-v} = \frac{dt}{t}$$
$$-ln(250-v) = ln(t) + C$$

We discard the absolute value sign on ln since we only care about non-negative time and non-negative velocity.

$$\frac{1}{250-v} = t + C$$

$$250 - v = \frac{1}{t+C}$$

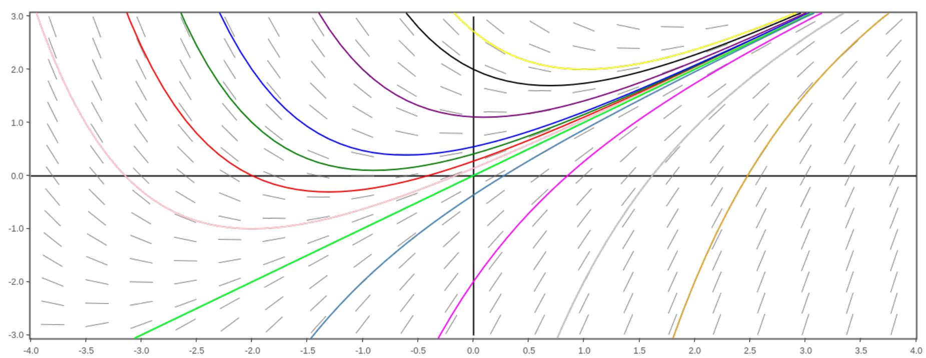
$$v = 250 - \frac{1}{t+C}$$

Assuming t is non-negative and v is non-negative, then C would have to be $\geq \frac{1}{250}$.

Textbook Section 1.2:

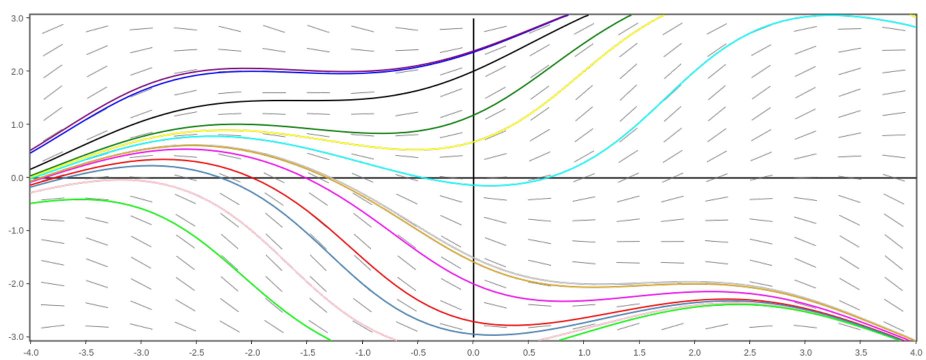
6)

$$\frac{dy}{dx} = x - y + 1$$



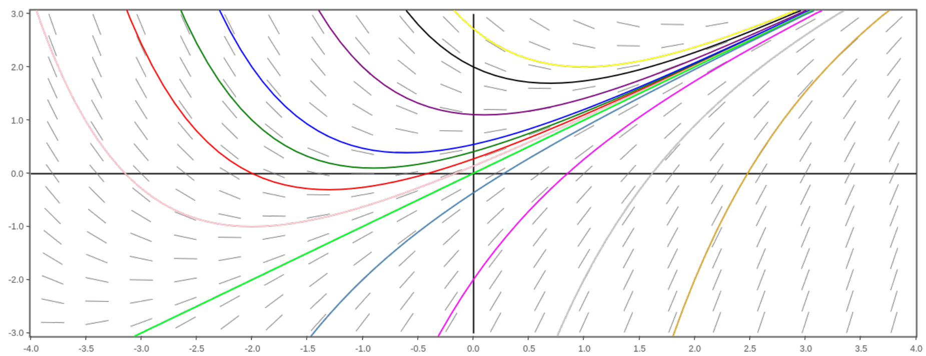
7)

$$\frac{dy}{dx} = sin(x) + sin(y)$$



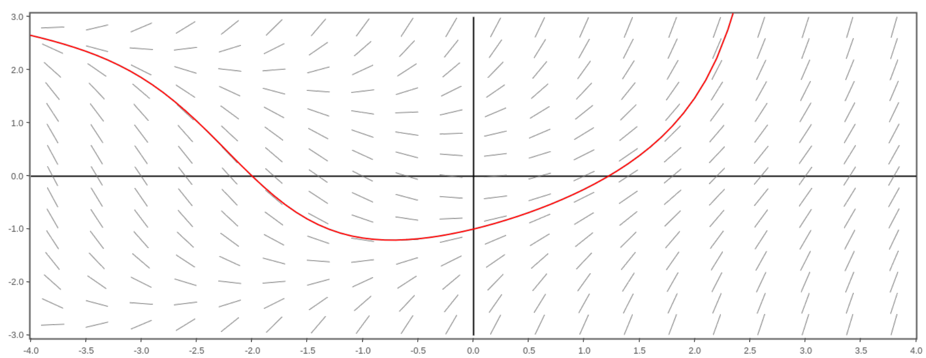
9)

$$\frac{dy}{dx} = x^2 - y - 2$$



24)

$$y' = x + \frac{1}{2}y^2, y(-2) = 0, y(2) = ?$$



$$y(2) \simeq 1.5$$

Textbook Section 1.4:

9)

$$(1 - x^2)\frac{dy}{dx} = 2y$$

$$(1 - x^2)\frac{dy}{dx} = 2y \iff \frac{dy}{2y} = \frac{dx}{1 - x^2}$$

$$\int \frac{1}{2y} dy = \int \frac{1}{1 - x^2} dx$$

$$\int \frac{1}{1 - x^2} dx$$

$$1 - x^2 = (1-x) * (1+x)$$

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} = \frac{A*(1+x)}{(1-x)*(1+x)} + \frac{B*(1-x)}{(1-x)*(1+x)}$$
$$= \frac{A + Ax + B - Bx}{1-x^2} = \frac{(A-B)x + (A+B)}{1-x^2} = \frac{0*x+1}{1-x^2}$$
$$\iff (A-B)x + (A+B) = 0*x+1 \iff A-B=0, A+B=1$$
$$A = B, 2A = 1 \iff A = 1/2$$

$$\int \frac{1}{1-x^2}dx = \int \frac{1/2}{1-x}dx + \int \frac{1/2}{1+x}$$
$$= -\frac{1}{2}\int \frac{-1}{1-x} + \frac{1}{2}\int \frac{1}{1+x}$$
$$= -\frac{1}{2}ln|1-x| + \frac{1}{2}ln|1+x|$$

$$\int \frac{1}{2y}dy = \int \frac{1}{1-x^2}dx$$
$$D = 2C$$
$$A = e^D$$
$$\iff \frac{1}{2}ln|y| = -\frac{1}{2}ln|1-x| + \frac{1}{2}ln|1+x| + C$$
$$\iff ln|y| = -ln|1-x| + ln|1+x| + D$$
$$\iff |y| = e^{ln(\frac{1}{|1-x|})+ln|1+x|+D}$$
$$\iff |y| = e^{ln(\frac{1}{|1-x|})} * e^{ln|1+x|} * e^D$$
$$\iff |y| = \frac{1}{|1-x|} * |1+x| * e^D$$
$$\iff |y| = A\frac{|1+x|}{|1-x|}$$
$$\iff y = \pm A\frac{|1+x|}{|1-x|}$$

Now if we plug in for positive x, we will get:

$$y = \pm A * \frac{1+x}{1-(-x)} \iff y = \pm A * \frac{1+x}{1+x} \iff y = \pm A * 1 \iff y = \pm A$$

And for negative x, we will get:

$$y = \pm A * \frac{1+(-x)}{1-x} \iff y = \pm A * \frac{1-x}{1-x} \iff y = \pm A * 1 \iff y = \pm A$$

So,

$$y = \pm A$$

Which works out if you plug it into the original ODE that we were given (note that D = 2C, which is why the constant of 2 appears on the rhs).

13)

$$y^3 \frac{dy}{dx} = (y^4 + 1)cos(x)$$

$$y^3 \frac{dy}{dx} = (y^4 + 1)cos(x)$$

$$\iff \frac{y^3}{y^4+1} dy = cos(x)dx$$

$$\iff \int \frac{y^3}{y^4+1} dy = \int cos(x)dx$$

$$\iff \frac{1}{4} \int \frac{4y^3}{y^4+1} dy = \int cos(x)dx$$

$$\iff \frac{1}{4} ln|y^4 + 1| = sin(x) + C$$

Notice that $y^4 + 1$ is always positive, so we can replace $|y^4 + 1|$ with $(y^4 + 1)$.

$$D = 4C$$

$$A = e^D$$

$$\frac{1}{4} ln(y^4 + 1) = sin(x) + C$$

$$\iff ln(y^4 + 1) = 4sin(x) + D$$

$$\iff y^4 + 1 = e^{4sin(x)+D}$$

$$\iff y^4 + 1 = A * e^{4sin(x)}$$

$$\iff y^4 = A * e^{4sin(x)} - 1$$

$$\iff y = \sqrt[4]{A * e^{4sin(x)} - 1}$$

22)

$$\frac{dy}{dx} = 4x^3 y - y, y(1) = -3$$

$$A = e^C$$

$$\frac{dy}{dx} = 4x^3y - y, y(1) = -3$$

$$\iff \frac{dy}{dx} = (4x^3 - 1) * y$$

$$\iff \frac{1}{y}dy = (4x^3 - 1)dx$$

$$\iff \int \frac{1}{y}dy = \int (4x^3 - 1)dx$$

$$\iff ln|y| = x^4 - x + C$$

$$\iff |y| = e^{x^4 - x + C}$$

$$\iff |y| = A * \frac{e^{x^4}}{e^x}$$

$$\iff y = \pm A * \frac{e^{x^4}}{e^x}$$

Notice that $\frac{e^{x^4}}{e^x}$ is always positive, so we can can rewrite this as:

$$y = A * \frac{e^{x^4}}{e^x}$$

Now, we have to find A.

$$y(1) = A * \frac{e^{1^4}}{e^1} = -3$$

$$A * \frac{e^{1^4}}{e^1} = A * \frac{e^1}{e^1} = A$$

$$\iff y(1) = A = -3$$

So,

$$A = -3$$

and

$$y = -3 * \frac{e^{x^4}}{e^x}$$

54) A tank is shaped like a vertical cyclinder; it initially contains water to a depth of 9ft, and a bottom plug is removed at time t = 0 (hours). After 1hr, the depth of the water has dropped to 4ft. How long does it take for all the water to drain from the tank?

$$y(0) = 9$$

$$y(1) = 4$$

$$D = \frac{C}{2}$$

$$A = \pi r_1^2$$

$$g = 32\frac{ft}{s^2}$$

$$a = \pi r_2^2$$

$$k = a\sqrt[2]{2g}$$

$$A * \frac{dy}{dt} = -k\sqrt[2]{y}$$

$$\iff \frac{1}{\sqrt[2]{y}}dy = -kAdt$$

$$\iff \int y^{-\frac{1}{2}}dy = \int -kAdt$$

$$\iff 2y^{1/2} = -kAt + C$$

$$\iff \sqrt[2]{y} = \frac{-kA}{2}t + D$$

$$\iff y = (\frac{-kA}{2}t + D)^2$$

$$\iff y = \frac{k^2A^2}{4} * t^2 - kAD * t + D^2$$

$$y(0) = D^2 = 9$$

$$\iff D = 3$$

$$y = \frac{k^2A^2}{4} * t^2 - kAD * t + D^2$$

$$\iff y = \frac{k^2A^2}{4} * t^2 - kA * 3 * t + 9$$

$$y(1) = \frac{k^2A^2}{4} - kA * 3 + 9 = 4$$

$$\iff k^2A^2 - kA * 12 + 20 = 0$$

$$M = kA$$

$$k^2A^2 - kA * 12 + 20 = 0$$

$$\iff M^2 - 12M + 20 = 0$$

$$\iff M = 10, M = 2 \iff M = 6 \pm 4$$

$$y = \frac{k^2 A^2}{4} * t^2 - kA * 3 * t + 9$$

$$\iff y = \frac{(6 \pm 4)^2}{4} * t^2 - (6 \pm 4) * 3 * t + 9$$

Now that we have y, we need to find when the value of t when $y = 0$.

$$y = \frac{(6 \pm 4)^2}{4} * t^2 - (6 \pm 4) * 3 * t + 9 = 0$$

Let's split this into two clauses ($M = 2$ and $M = 10$).

For $M = 10$:

$$y = \frac{(6 + 4)^2}{4} * t^2 - (6 + 4) * 3 * t + 9 = 0$$

$$\iff y = \frac{100}{4} * t^2 - 30 * t + 9 = 0$$

$$\iff y = 100 * t^2 - 120 * t + 36 = 0$$

$$t = \frac{3}{5}$$

For $M = 2$:

$$y = \frac{(6 - 4)^2}{4} * t^2 - (6 - 4) * 3 * t + 9 = 0$$

$$\iff t^2 - 6 * t + 9 = 0$$

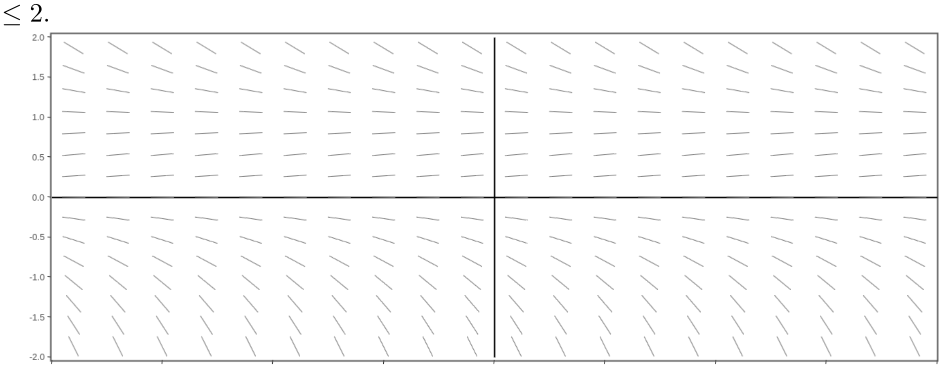
$$t = 3$$

So, the tank will become empty at either $t = \frac{3}{5}$ or $t = 3$ as given by the equation $y = \frac{(6 \pm 4)^2}{4} * t^2 - (6 \pm 4) * 3 * t + 9$.

Custom Problems:

Problem 1.

a) Construct a slope field for $y' = y(1 - y)$ on the interval $-2 \leq x$, $y \leq 2$.



b)