This is due Saturday 12/2 by 11:59 pm on Gradescope. Please either neatly write up your solutions or type them up. You can find a .tex template on Canvas. Your proofs should be written in complete sentences and paragraphs, using a combination of words and symbols. They should be **correct, clear, and concise**. You will be graded on all three, especially the first two!

1. (4) Show that if $\{f_n\}$ is a sequence of *integrable* functions $f_n : [a,b] \to \mathbb{R}$ converging *uniformly* to $f : [a,b] \to \mathbb{R}$ that f is integrable and $\lim_{n\to\infty} \int_a^b f_n \, dx = \int_a^b f \, dx$. **Hint:** Using uniform convergence you can make $|f(x) - f_n(x)|$ as small as you like on $[a,b]$, so for any ϵ for n big enough you can show that $f_n - \epsilon < f$ on $[a,b]$, similarly that $f < f_n + \epsilon$.

Solution:

We first need to show that f is integrable. Since f_n are integrable on $[a,b]$ and $f_n \to f$ uniformly, f must also be bounded. Since the uniform limit of continuous functions is continuous, and continuous functions are integrable, f is integrable if the f_n are continuous.

Now we have to bound the difference between f and f_n using uniform convergence. By the definition of uniform convergence, for every $\epsilon > 0$, $\exists$ an $N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in [a,b]$, we have $|f(x) - f_n(x)| < \frac{\epsilon}{b-a}$. This implies $f_n(x) - \frac{\epsilon}{b-a} < f(x) < f_n(x) + \frac{\epsilon}{b-a}$ $\forall x \in [a,b]$ and $n \geq N$.

Integrating the inequalities $f_n(x) - \frac{\epsilon}{b-a} \leq f(x) \leq f_n(x) + \frac{\epsilon}{b-a}$ over $[a,b]$, we get

$$\int_a^b (f_n(x) - \frac{\epsilon}{b-a})dx \leq \int_a^b f(x)dx \leq \int_a^b (f_n(x) + \frac{\epsilon}{b-a})dx.$$

Simplifying these, we have:

$$\int_a^b f_n(x)dx - \epsilon \leq \int_a^b f(x)dx \leq \int_a^b f_n(x)dx + \epsilon.$$

Since the above inequality holds $\forall n \geq N$, taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \int_a^b f_n(x)dx - \epsilon \leq \int_a^b f(x)dx \leq \lim_{n\to\infty} \int_a^b f_n(x)dx + \epsilon.$$

Since ϵ is arbitrary, it follows that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

This finishes the proof as we have shown that f is integrable and the limit of the integrals of the sequence $\{f_n\}$ is equal to the integral of the limit function f.

2. (4) Observe that the function defined for $x \neq 0$ by $f(x) = x/|x|$ and $f(0) = 0$ is differentiable except at 0, and $f'(x) = 0$, which is integrable on $[-1,1]$. But $\int_{-1}^1 f' \, dx = 0$, while $f(1) - f(-1) = 2$. Why does this not contradict the first fundamental theorem of calculus (5.3.1 in your text)?

Solution:

The situation described does not contradict the First Fundamental Theorem of Calculus due to the nature of the function f and its derivative f'. Let's break down the situation and theorem to see why:

The function f is defined as $f(x) = \frac{x}{|x|}$ for $x \neq 0$ and $f(0) = 0$.

The derivative of f is 0 $\forall x \neq 0$. This is because $f(x)$ is a piecewise constant function (1 for $x > 0$, 0 for $x = 0$, and -1 for $x < 0$).

Since $f'(x) = 0$ $\forall x \neq 0$, the integral $\int_{-1}^{1} f'(x)dx$ equals 0.

$f(1) = 1$ and $f(-1) = -1$, so $f(1) - f(-1) = 2$.

The First Fundamental Theorem of Calculus states that if a function f is continuous on $[a, b]$ and has an antiderivative F on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Why there isn't a contradiction:

f is not continuous at $x = 0$. The theorem requires the function to be continuous over the entire interval.

Since f is not continuous at 0, its derivative f' does not meet the continuity requirement of the theorem over the interval $[-1, 1]$.

Therefore, the theorem does not meet the requirements to be applied and so no contradiction occurs when the theorem's conclusion does not hold.

3. (4) Compute the derivative with respect to x of $\int_{1/x}^{x} e^{-t^2}\, \mathrm{d}t$.

Solution:

We will compute the derivative with respect to x of the integral $\int_{1/x}^{x} e^{-t^2}\, \mathrm{d}t$, we will use the Leibniz rule for differentiating an integral with variable limits.

The Leibniz rule states that if we have an integral of the form $\int_{a(x)}^{b(x)} f(t, x)dt$, then its derivative with respect to x is given by:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t, x)dt = f(b(x), x) \cdot b'(x) - f(a(x), x) \cdot a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(t, x)dt.$$

In our case, $f(t, x) = e^{-t^2}$, $a(x) = \dfrac{1}{x}$, and $b(x) = x$. Thus, the derivative of the integral is:

$$\frac{d}{dx}\int_{\frac{1}{x}}^{x} e^{-t^2} dt = e^{-x^2} \cdot \frac{d}{dx}(x) - e^{-(\frac{1}{x})^2} \cdot \frac{d}{dx}\left(\frac{1}{x}\right).$$

$$\frac{d}{dx}(x) = 1.$$

$$\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}.$$

Substituting these into the equation gives:

$$\frac{d}{dx}\int_{\frac{1}{x}}^{x} e^{-t^2} dt = e^{-x^2} \cdot 1 - e^{-(\frac{1}{x})^2} \cdot \left(-\frac{1}{x^2}\right) = e^{-x^2} + \frac{e^{-\frac{1}{x^2}}}{x^2}$$

Therefore, the derivative of $\int_{\frac{1}{x}}^{x} e^{-t^2} dt$ with respect to x is $e^{-x^2} + \dfrac{e^{-\frac{1}{x^2}}}{x^2}$.

4. (4) Recall that for $a \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}$ we defined a^x by $a^x = \exp(\ln(a)x)$. Compute the derivative with respect to x of a^x.

Solution:

To compute the derivative of a^x with respect to x, where a is a positive real number and x is any real number, and a^x is defined as $\exp(\ln(a) \cdot x)$, we can use the chain rule.

Let $f(x) = \ln(a) \cdot x$ and $g(f) = \exp(f)$. Then the derivative with respect to x of $f(x)$ and $g(f)$ is:

$$f'(x) = \ln(a)$$

since $\ln(a)$ is a constant.

$$g'(f) = \exp(f)$$

since $\dfrac{d}{dx}\exp(x) = \exp(x)$.

Thus, applying the chain rule, we get:

$$\frac{dy}{dx} = g'(f(x)) \cdot f'(x) = \exp(\ln(a) \cdot x) \cdot \ln(a) = a^x \ln(a).$$

5. (4) Note that your solution to the previous problem gives that a^x is strictly monotone, and hence has an inverse. Call this inverse $\log a(x)$. Compute the derivative with respect to x of $\log a(x)$, and show that for any $x \in \mathbb{R}_{>0}$ that $\log a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Solution:

To find the derivative of the inverse function of a^x, $log_a(x)$, and to show that $log_a = \dfrac{\ln(x)}{\ln(a)}$ for any $x \in \mathbb{R}_{>0}$, we have to follow a few steps.

First let's find the deriative of $log_a(x)$ with respect to x. The deriative of the function $log_a(x)$ can be found using the formula:

$$\frac{d}{dx} log_a(x) = \frac{1}{\frac{d}{dy} a^y}$$

where $y = log_a(x)$ (i.e. $x = a^y$).

We already know from the previous problem that the derivative of a^x with respect to x is $a^x \ln(a)$. Then, substituting $x = a^y$ gives:

$$\frac{d}{dx} log_a(x) = \frac{1}{a^{log_a(x)} \ln(a)} = \frac{1}{x \ln(a)}.$$

Now, let's prove that $log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

By definition, $alog_a(x) = x$. Thus, $log_a(x)$ is the power to which a must be raised to get x.

Also, $e^{\ln(x)} = x$.

By the property of logarithms, $\ln(a^b) = b \ln(a)$, we can write $\ln(x)$ as $\ln(a^{log_a(x)}) = log_a(x) \ln(a)$

Solving for $log_a(x)$, we get that $log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Therefore, we have shown both that the derivative of $log_a(x)$ with respect to x is $\dfrac{1}{x\ln(x)}$ and that $log_a(x) = \dfrac{\ln(x)}{\ln(a)}$ for any $x \in \mathbb{R}_{>0}$. This formula is consistent with the change of base formula for logarithms.