Math 3220-1: Homework 10 (corrected), due 04/22/2024

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Problem 1 *Find the degree* $n = 3$ *Taylor's Formula for the function* $f(x) = x^3 - x^2 - 4x + 4$ *with* $a = 1$ *.*

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

Now we have to find the derivatives of $f(x)$:

$$f'(x) = 3x^2 - 2x - 4$$
$$f''(x) = 6x - 2$$
$$f'''(x) = 6$$

Now, we evaluate the functions at $a = 1$:

$$f(1) = 0$$
$$f'(1) = -3$$
$$f''(1) = 4$$
$$f'''(1) = 6$$

Thus,

$$P_3(x) = -3x + 3 + 2(x-1)^2 + (x-1)^3$$

Problem 2 *Find the degree $n = 2$ Taylor Formula for $f(x,y) = x^2 + xy$ at the point $a = (1,2)$.*

$$\begin{aligned} P_2(x,y) = f(a) + \frac{\partial f}{\partial x}(a)(x - x_0) + \frac{\partial f}{\partial y}(a)(y - y_0) + \frac{1}{2} \cdot \bigg(\frac{\partial^2 f}{\partial x^2}(a)(x - x_0)^2 \\ + 2\frac{\partial^2 f}{\partial x \partial y}(a)(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(a)(y - y_0)^2 \bigg) \end{aligned} \tag{1}$$

Now, let's calculate the derivatives:

$$\frac{\partial f}{\partial x} = 2x + y$$

$$\frac{\partial f}{\partial y} = x$$

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

Evaluating at $a = (1,2)$:

$$f(1,2) = 3$$

$$\frac{\partial f}{\partial x}(1,2) = 4$$

$$\frac{\partial f}{\partial y} = 1$$

$$\frac{\partial^2 f}{\partial x^2}(1,2) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y}(1,2) = 1$$

$$\frac{\partial^2 f}{\partial y^2}(1,2) = 0$$

Plugging these in gives:

$$P_2(x,y) = 3 + 4(x-1) + (y-2) + (x-1)^2 + (x-1)(y-2) \tag{2}$$

Problem 3 *Find all points of relative maximum and relative minimum and all saddle points for* $f(x,y) = y^3 + y^2 + x^2 - 2xy - 3y$.

Since we need to find the zero points of the first derivatives, let's first compute the first partial derivatives:

$$\frac{\partial f}{\partial x}(x,y) = 2x - 2y$$

$$\frac{\partial f}{\partial y}(x,y) = 3y^2 + 2y - 2x - 3$$

We now need to solve for:

$$2x - 2y = 0$$

and:

$$3y^2 + 2y - 2x - 3$$

From $2x - 2y = 0$, we get that $x = y$.

Substituting $x = y$ into the second equation gives $3y^2 - 3 = 0$. This turns into $y^2 = 1$, which means $y = \pm 1$.

Substituting $y = 1$ into $x = y$ gives the critical point $(1,1)$.

Similarly, substituting $y = -1$ into $x = y$ gives the critical point $(-1,-1)$.

Computing the second derivatives gives:

$$f_{xx}(x,y) = 2$$

$$f_{xy}(x,y) = -2$$

$$f_{yy}(x,y) = 6y + 2$$

The Hessian is thus:

$$\begin{bmatrix} 2 & -2 \\ -2 & 6y+2 \end{bmatrix}$$

Evaluating the determinant $\nabla = f_{xx}f_{yy} - fxy^2$ at the critical points gives:

At $(1,1)$:

$$f_{yy}(1,1) = 8$$

$$\nabla = 12$$

At $(-1,-1)$:

$$f_{yy}(-1,-1) = -4$$

$$\nabla = -12$$

For $(1,1)$, since $\nabla > 0$ and $f_{xx} > 0$, this point is a relative minimum.

For $(-1,-1)$, since $\nabla < 0$, this point is a saddle point.

Problem 4 *Prove Corollary 9.5.6 (hint: you may want to use the fact that if $U \subset \mathbb{R}^p$ is an open and connected set, then every two points* **a** *and* **b** *of U can be joined by a piecewise linear path).*

First, let's recall what Corollary 9.5.6 is (taken from 9.5 pdf page 3): Suppose U is connected and f is a differentiable function on U. If $\nabla f(x) = 0 \; \forall x \in U$, then f is a constant function.

Let a and b be any two points in U. Since U is connected and open, there exists a piecewise linear path connecting a and b.

We can think of this path as a continuous function $p : [0, 1] \to U$ such that $p(0) = a$ and $p(1) = b$ and where each segment of p is a straight line.

If we compose this function with f and use the chain rule, we get:

$$(f \circ p)'(t) = \nabla f(p(t)) \cdot p'(t); \forall t \in [0, 1]$$

Given that $\nabla f = 0 \; \forall x \in U$, that means that $f(p(t)) = 0 \; \forall t \in [0, 1]$.

That means we can rewrite the above as:

$$(f \circ p)'(t) = 0 \cdot p'(t) = 0; \forall t \in [0, 1]$$

Since $(f \circ p)'(t) = 0$ over $[0, 1]$, by the Fundamental Theorem of Calculus, $f \circ p$ must be constant over $[0, 1]$. This means that $f(p(0)) = f(p(1))$, i.e. $f(a) = f(b)$.

Since a and b are arbitrary points in U and we've shown that $f(a) = f(b)$, it clearly means that f must be constant over U.