Lincoln Sand

L1.

318.9:

Solve the following in three variables due to Abu Kamil:

$$x < y < z$$

$$x^2 + y^2 = z^2$$

$$xz = y^2$$

$$xy = 10$$

(Begin by setting $y = \frac{10}{x}$, $z = \frac{100}{x^3}$, and substituting in the first equation.)

If we substitute the two hinted equalities into $x^2+y^2 = z^2$, we get:

$$x^2 + \left(\frac{10}{x}\right)^2 = \left(\frac{100}{x^3}\right)^2$$

Solving for x yields:

$$x = 2^{1/4}\sqrt{5}(-1+\sqrt{5})^{1/4}$$

Rewriting the hinted equations and substituting into $xz = y^2$ for y instead of x yields a value of

$$y = \frac{2^{3/4}\sqrt{5}}{(-1+\sqrt{5})^{1/4}}$$

Using back substitution, we finally get:

$$z = \frac{2 \cdot 2^{1/4}\sqrt{5}}{(-1+\sqrt{5})^{3/4}}$$

If we plug these values into the four initial equations, we find that it satisfies all of them.

318.16:

Show that one can solve $x^3+d = cx$ by intersecting the hyperbola $y^2 - x^2 + \frac{d}{c}x = 0$ with the parabola $x^2 = \sqrt{c}y$. Sketch the two conics. Find sets of values for c and d for which these conics do not intersect, intersect once, and intersect twice.

From $y^2 - x^2 + \frac{d}{c}x = 0$, we get $y^2 = x^2 - \frac{d}{c}x$.

From $x^2 = \sqrt{c}y$, we get $y = \frac{x^2}{\sqrt{c}}$.

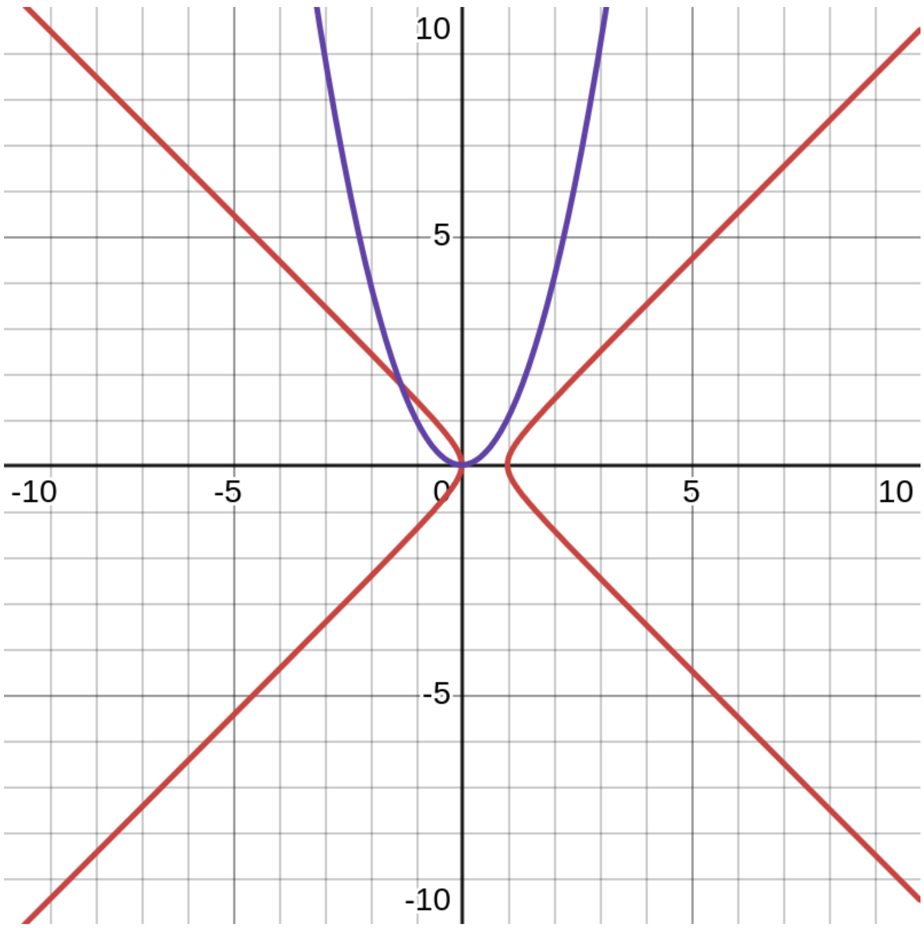
We now set $x^2 - \frac{d}{c}x = \frac{x^4}{c}$ and solve for x. This gives us the solutions of $x^3 + d = cx$ through simple algebraic manipulation.

Simple algebraic manipulation:

$$x^2-\frac{d}{c}x = \frac{x^4}{c} \implies cx^2-dx = x^4 \implies cx-d = x^3 \implies x^3+d = cx$$

We equate $x^2 - \frac{d}{c}x$ from the hyperbola to $\frac{x^4}{c}$ from the parabola.

We end up with $x^4 - cx^2 + dx = 0$. The number of intersections depends on the discriminant when calculating the roots. If the discriminant is non-real, there is no intersection. If it's 0, some of the roots are repeated and thus we get one intersection. If the discriminant is non-zero and real, there are two intersections.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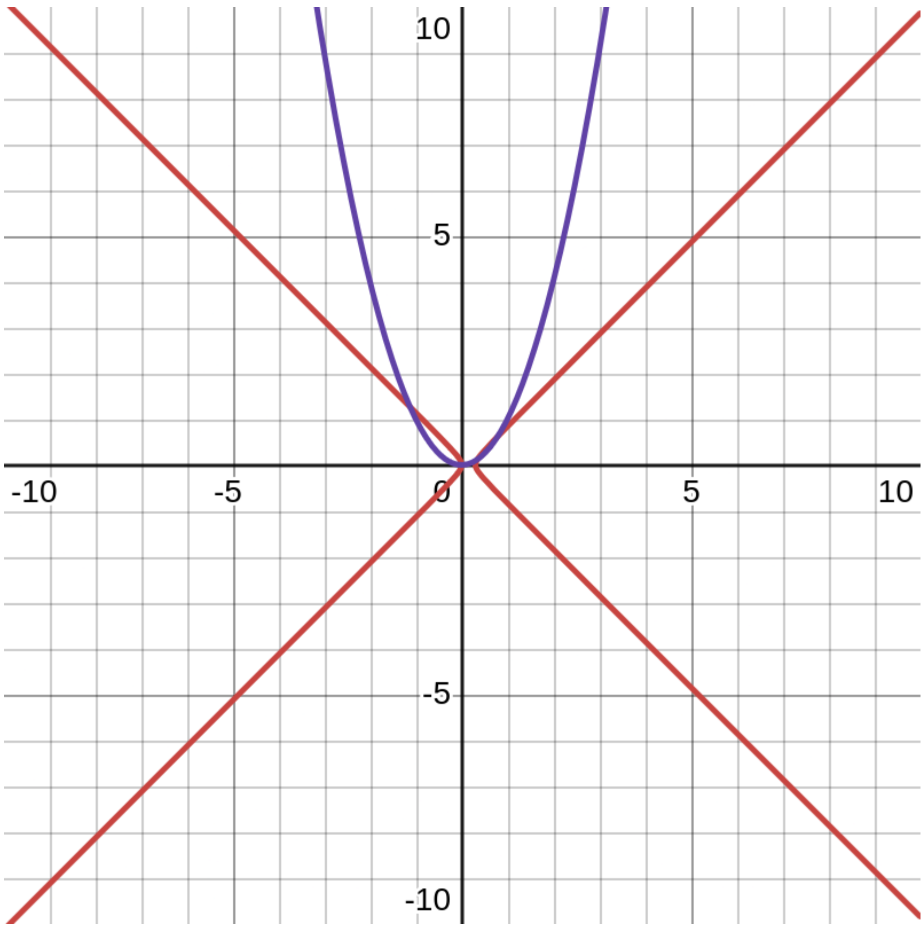
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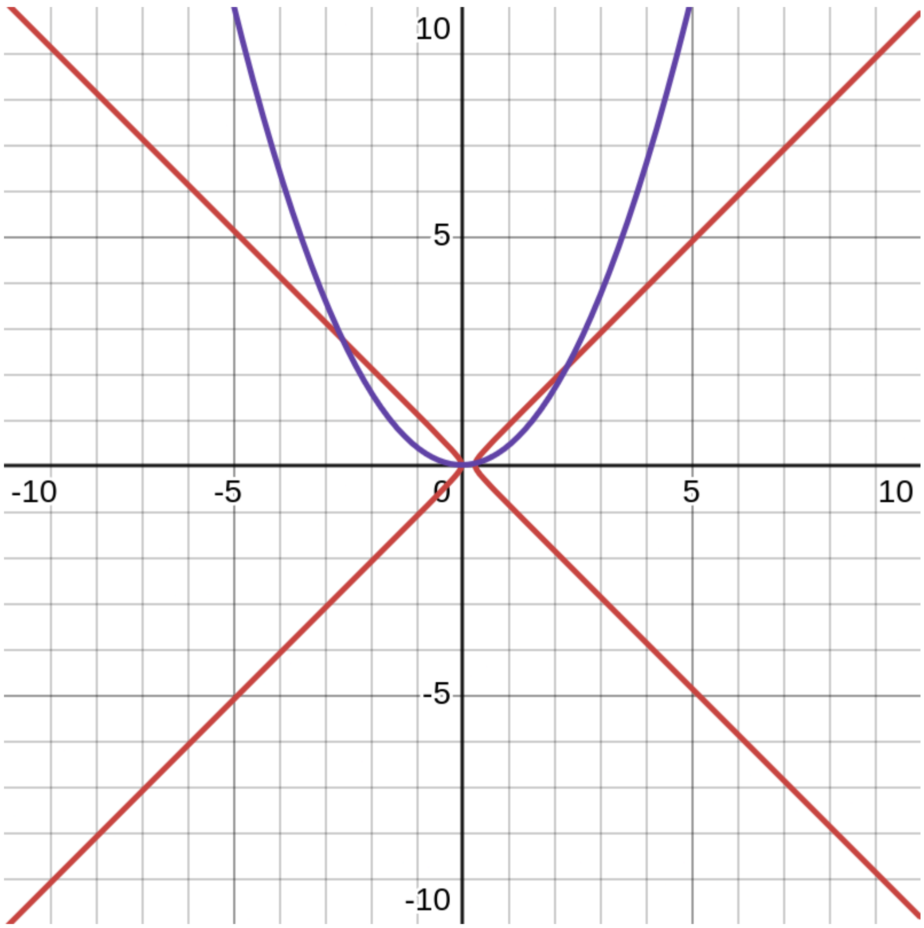

359.2:

Let's denote m as the man, w as the wolf, g as the goat, and c for the cabbage.

We start with $m = w = g = c = 0$ and we want to end up with $m = w = g = c = 1$.

$g \neq c$ when $m \neq g$. $w \neq g$ when $m \neq g$.

Step 1:

Move the goat to the right bank.

$m = g = 1$ and $w = c = 0$.

Step 2:

Return the man alone.

$g = 1$ and $m = w = c = 0$.

Step 3:

Take the wolf to the right bank.

$m = g = w = 1$ and $c = 0$.

Step 4:

Bring the goat back to the left bank.

$m = g = c = 0$ and $w = 1$.

Step 5:

Move the cabbage to the right bank.

$g = 0$ and $m = w = c = 1$.

Step 6:

Return the man alone.

$m = g = 0$ and $w = c = 1$.

Step 7:

Bring the goat to the right bank.

$m = w = g = c = 1$.

359.31:

The Fibonacci sequence (the sequence of rabbit pairs) is determined by the recursive rule $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Show that

$$F_{n+1} \cdot F_{n-1} = F_n^2 - (-1)^n$$

and that

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$$

We will show the first identity $F_{n+1} \cdot F_{n-1} = F_n^2 - (-1)^n$ using mathematic induction.

Base case:

For $n = 1$:

$$F_2 \cdot F_0 = F_1^2 - (-1)^1$$

$$2 \cdot 1 = 1^2 + 1$$
$$2 = 2$$

The base case holds.
The inductive case:
Assume $F_{k+1} \cdot F_{k-1} = F_k^2 - (-1)^k$ for some $k \in \mathbb{Z}$.
Using the reursive definition:

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{k+1} \cdot F_k = (F_{k+1} + F_k) \cdot F_k - (-1)^{k+1}$$
$$F_{k+1} \cdot F_k = F_{k+1}^2 + F_k \cdot F_{k+1} - (-1)^{k+1}$$
$$F_{k+1} \cdot F_k = F_{k+1}^2 + F_{k+1} \cdot F_k - (-1)^{k+1}$$

Simplifying gives:

$$F_{k+1}^2 - (-1)^{k+1} = F_{k+1} \cdot F_k$$

Which completes the proof.
For the second identity $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$, consider the ratio $L = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$.
We know that $F_n = F_{n-1} + F_{n-2}$. So,

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{F_{n-1} + F_{n-1}}{F_{n-1}}$$

This can be simplified to $L = 1 + \frac{1}{L}$. Solving for L gives:

$$L^2 = L + 1$$
$$L^2 - L - 1 = 0$$

Applying the quadratic formula gives:

$$L = \frac{1 \pm \sqrt{5}}{2}$$

Since the ratio L must be positive since all Fibonacci numbers are positive, this becomes $\frac{1+\sqrt{5}}{2}$.

359.40:

From Oresme's Tractatus de configurationibus qualitatum et motuum: Show geometrically that the sum of the series

$$48 \cdot 1 + 48 \cdot \frac{1}{4} \cdot 2 + 48 \left(\frac{1}{4}\right)^2 \cdot 4 + \cdots + 48 \left(\frac{1}{4}\right)^n \cdot 2^n + \ldots$$

is equal to 96.

Each term of the series can be viewed as a rectangle with height $48\left(\frac{1}{4}\right)^n$ and width 2^n.

The rectangle's height quarters every step while its width doubles. This means each successive rectangle's area halves.

Arranging these rectangles side by side makes them resemble a right triangle.

we can rearrange the rectangles so that it is a shape with height 48 (that of the first rectangle) and whose width is the limit of $1 + \frac{1}{2} + \frac{1}{4} + \ldots$, which is 2.

So, the area would be $48 \cdot 2 = 96$.

418.33:

Solve $x^3 + 21x = 9x^2 + 5$ completely by first using the substitution $x = y + 3$ to eliminate the term in x^2 and then solve the resulting equation in y.

Substituting in $x = y + 3$ and doing simplication to eliminate the y^2 term gives:

$$y^3 - 6y + 4 = 0$$

Using the cubic equation gives solutions for y as 2, $-1+\sqrt{3}$, and $-1-\sqrt{3}$. Plugging this back into $x = y+3$

gives:

$$x = 5$$
$$x = 2 + \sqrt{3}$$
$$x = 2 - \sqrt{3}$$

Plugging these back into the original equation verifies that they are valid solutions.