Lincoln Sand

M1.

579.9: Find the relationship of the fluxions using Newton's rules for the equation $y^2 - a^2 - x\sqrt{a^2 - x^2} = 0$. Put $z = x\sqrt{a^2 - x^2}$.

First, let's differentiate the equation implicitly with respect to time:

Given the equation $y^2 - a^2 - x\sqrt{a^2 - x^2} = 0$, we differentiate implicitly $2y\dot{y} - \frac{d}{dt}(x\sqrt{a^2 - x^2}) = 0$.

Now we need to differentiate $z = x\sqrt{a^2 - x^2}$:

we apply the product rule $\dot{z} = \dot{x}\sqrt{a^2 - x^2} + x\frac{d}{dt}(\sqrt{a^2 - x^2})$.

Using the chain rule on $\sqrt{a^2 - x^2}$, we get $-\frac{x\dot{x}}{\sqrt{a^2 - x^2}}$.

Substituting back in, we get $\dot{z} = \dot{x}\sqrt{a^2 - x^2}\left(1 - \frac{x^2}{a^2 - x^2}\right)$.

Simplifying this gives $\dot{z} = \dot{x}\sqrt{a^2 - x^2}\frac{a^2 - 2x^2}{a^2 - x^2}$.

Substituting back again gives us $2y\dot{y} - \dot{x}\sqrt{a^2 - x^2}\frac{a^2 - 2x^2}{a^2 - x^2} = 0$.

Solving for $\dot{y}$ finally yields $\dot{y} = \frac{\dot{x}\sqrt{a^2 - x^2}(a^2 - 2x^2)}{2y(a^2 - x^2)}$.

579.24: Given the curve $y^q = x^p$ $(q > p > 0)$, show using the transmutation theorem that

$$\int_0^{x_0} y dx = \frac{q x_0 y_0}{p + q}$$

Note that from $y^q = x^p$, it follows that $q dy/y = p dx/x$ and therefore that $z = y - x dy/dx = [(q - p)/q]y$.

We know from above that $y = x^{p/q}$ and $q\frac{dy}{y} = p\frac{dx}{x}$. This implies that $\frac{dy}{dx} = \frac{p}{q}\frac{y}{x}$.

z is defined as $z = y - x\frac{dy}{dx}$. Substituting from above, we get that $z = \frac{q-p}{q}y$.

We can rewrite this as $z = \frac{q-p}{q} x^{p/q}$.

For the integral, using the transmutation theorem, we get that $\int_0^{x_0} y dx = \frac{q}{q-p} \int_0^{x_0} z dx = \frac{q}{q-p} \int_0^{x_0} x^{p/q} dx$.

This becomes $\frac{q-p}{p} \left(\frac{x^{p/q+1}}{(p/q+1)} \right)_0^{x_0} = \frac{q-p}{q} \frac{x_0^{p/q+1}}{p/q+1}$.

Substituting back gives us $\int_0^{x_0} y dx = \frac{q}{q-p} \left(\frac{q-p}{q} \frac{x_0^{p/q+1}}{p/q+1} \right)$. Simplifying gives us $\frac{q x_0 y_0}{p+q}$ as expected.

579.25: Prove the quotient rule $d(\frac{x}{y}) = \frac{ydx - xdy}{y^2}$ by an argument using differentials.

Given $z = \frac{x}{y}$, we can write this as $z = x \cdot y^{-1}$.

Now, we apply the product rule to get $dz = d(x \cdot y^{-1}) = dx \cdot y^{-1} + x \cdot d(y^{-1})$.

We can calculate $d(y^{-1})$ using the chain rule to get $-y^{-2} \cdot dy$.

Substituting back gives us $dz = dx \cdot y^{-1} + x \cdot -y^{-2} \cdot dy = \frac{dx}{dy} - \frac{x \cdot dy}{y^2}$.

Using basic algebra to turn this into a common denominator, we get $dz = \frac{y \cdot dx - x \cdot dy}{y^2}$. Which is the quotient rule we wanted to prove.

M2. Recall Napier's logarithm $Nlog(x) = m$ if $10^7(1 - 10^{-7})^m = x$. Show that

$$Nlog(x) + Nlog(y) = Nlog(xy) + Nlog(1)$$

$$10^7(1 - 10^{-7})^m = x$$
$$10^7(1 - 10^{-7})^n = y$$

Multiplying these together:

$$10^7(1 - 10^{-7})^m \cdot 10^7(1 - 10^{-7})^n = 10^7 \cdot 10^7(1 - 10^{-7})^{m+n}$$

We can write $10^7 \cdot 10^7$ as 10^14, but this means we have to divide by 10^7 to match Napier's logarithm. So, we have $10^7(1-10^{-7})^{m+n}$.

Now, if we take Napier's logarithm of both sides, we get $Nlog(xy) = m + n$.

Now, we have to handle $Nlog(1)$.

For $x = 1$, we have $10^7(1-10^{-7})^m = 1$. The only power of any number that will equal 1 is 0. So, $m = 0$. Thus, $Nlog(1) = 0$.

Substituting back in gives us:

$$Nlog(x) + Nlog(y) = m + n$$

$$Nlog(xy) + Nlog(1) = m + n + 0$$

Thus, the relation:

$$Nlog(x) + Nlog(y) = Nlog(xy) + Nlog(1)$$

holds for Napier's logarithm.

M3. Show that the binomial series gives

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \dots$$

Then use

$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

to derive Newton's series for $\sin^{-1}(x)$.

Part 1: Binomal series for $\frac{1}{\sqrt{1-t^2}}$

First, let's expand it using the binomal series expansion for a power of $(1-x)$. The binomal series expansion for $(1-x)^n$ is given by:

$$(1-x)^n = \sum_{i=0}^{\infty} \binom{n}{i} (-x)^i$$

where $\binom{n}{i} = \frac{n(n-1)...(n-i+1)}{i!}$.

For $n = -\frac{1}{2}$ and $x = t^2$, we get:

$$\sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i} (-1)^i t^{2i}$$

The binomial coefficient $\binom{-\frac{1}{2}}{i}$ simplifies to:

$$\frac{(-1)^i (1 \cdot 3 \cdot 5 \ldots (2i-1))}{2^i \cdot i!}$$

Notice that $\frac{(-1)^i (1 \cdot 3 \cdot 5 \ldots (2i-1))}{2^i \cdot i!}$ simplifies to:

$$\frac{(-1)^i (1 \cdot 3 \cdot 5 \ldots (2i-1))}{2 \cdot 4 \cdot 6 \ldots (2n)}$$

This means we have:

$$(1-t^2)^{-1/2} = \sum_{i=0}^{\infty} \frac{(-1)^i (1 \cdot 3 \cdot 5 \ldots (2i-1))}{2 \cdot 4 \cdot 6 \ldots (2n)} t^{2n}$$

We can write this as:

$$\frac{1}{\sqrt{1-t^2}} = \sum_{i=0}^{\infty} \frac{(-1)^i (1 \cdot 3 \cdot 5 \ldots (2i-1))}{2 \cdot 4 \cdot 6 \ldots (2n)} t^{2n}$$

Part 2: Newton's Series for $\sin^{-1}(x)$

Given $\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}$, substituting the expansion from part 1 gives:

$$\sin^{-1}(x) = \int_0^x (1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \ldots)dt$$

Integrating by terms yields:

$$\sin^{-1}(x) = \left(t + \frac{1}{2}\frac{t^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{t^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{t^7}{7} + \ldots \right)_0^x$$

This yields:

$$x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \dots$$

This is Newton's series for the arcsine function.

M4. Use Fermat's method of ad-equation to find the slope of the curve $f(x) = x^2 - \sqrt{x}$ at $x > 0$.

Let's increment x by a very small value e:

$$f(x+e) = (x+e)^2 - \sqrt{x+e}$$

$$(x+e)^2 = x^2 + 2xe + e^2$$

For $\sqrt{x+e}$, using the first terms of the Taylor series, we get:

$$\sqrt{x+e} \approx \sqrt{x} + \frac{1}{2\sqrt{x}}e$$

Substituting back gives:

$$f(x+e) \approx x^2 + 2xe + e^2 - \sqrt{x} - \frac{1}{2\sqrt{x}}e$$

Now we subtract $f(x)$ and factor out e:

$$f(x+e) - f(x) = 2xe + e^2 - \frac{1}{2\sqrt{x}}e = e\left(2x + e - \frac{1}{\sqrt{x}}\right)$$

Now we apply adequality. Using adequality, we simplify it to:

$$e(2x - \frac{1}{\sqrt{x}})$$

Then we divide out the e to finally get:

$$f'(x) = 2x - \frac{1}{2\sqrt{x}}$$

M5. Use Newton's version of Newton's method to approximate the root of $x^2 - 2 = 0$ to an accuracy of eight decimal places.

Let's first list the steps involved in Newton's version of Newton's method (according to the class notes):

1. Take the current approximation x_i.
2. Consider a small change p such that $x_{i+1} = x_i + p$.
3. Substitute x_{i+i} into the equation $x^2 - 2 = 0$ and ignore the higher-order terms of p.
4. Solve for p and update the approximation for x_{i+1}.
5. Repeat until $|p| < 10^{-8}$.

For $f(x) = x^2 - 2$, the linearized equation around x_i is:

$$(x_i - p)^2 - 2 = x_i^2 - 2x_i p + p^2 - 2$$

Since p is small, we ignore p^2 and simplify it to:

$$x_i^2 - 2x_i p - 2 = 0$$

Now, let's solve for p.

$$2x_i p = 2x_i^2 \implies p = \frac{2 - x_i^2}{2x_i}$$

Now, we are trying to approximate $\sqrt{2}$. So let's pick $x_0 = 1.4$ and start iterating.

Iteration 1:

$$x_0 = 1.4$$
$$p_1 = \frac{2 - 1.4^2}{2 \cdot 1.4} \approx 0.014285714285714379$$
$$x_1 = x_0 + p_1 \approx 1.414285714285714379$$

Iteration 2:

$$p_2 = \frac{2 - 1.414285714285714379^2}{2 \cdot 1.414285714285714379} \approx -0.000072150072150011227$$

$$x_2 = x_1 + p_2 \approx 1.4142135642135643$$

Iteration 3:

$$p_3 = \frac{2 - 1.4142135642135643^2}{2 \cdot 1.4142135642135643} \approx -1.8404691290714918 \cdot 10^{-9}$$

$$x_3 = x_2 + p_3 \approx 1.4142135623730951$$

Since $p_3 < 10^{-8}$, x_3 is out final approximation of the root ($\sqrt{2}$).

Note: I used a calculator and had to manually type the numbers above, so if there are any typos/mistakes, I apologize.

M6. Essay on Modern Mathematics Proposal

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Working title: Infinity and Sets: Georg Cantor's Controversial Set theory

Essay topic description: I want to write about the birth of set theory by Georg Cantor and its controversy.

Interesting fact: Cantor came up with the idea of different "sizes" of infinity with his famous diagonal argument (to prove there were more reals than rationals).

Style manual I will use:

MLA

Two internet references:

1. Ferreirós, J. (2020, June 18). The early development of set theory. Stanford Encyclopedia of Philosophy.
https://plato.stanford.edu/entries/settheory-early/

2. Set theory from Cantor to Cohen. (n.d.).
https://booksite.elsevier.com/samplechapters/9780444516213/sample.pdf

Two journal/book references:

1. Zenkin, Alexander (2004), "Logic Of Actual Infinity And G. Cantor's Diagonal Proof Of The Uncountability Of The Continuum", The Review of Modern Logic, vol. 9, no. 30, pp. 27-80

2.
https://www.math.uwaterloo.ca/ xzliu/cantor-set.pdf

(will figure out how to MLA citate this second source properly later)