Section 3.4:

2) Determine the period and frequency of the simple harmonic motion of a body of mass 0.75 kg on the end of a spring with spring constant 48 N/m.

The period T of a simple harmonic oscillator, is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Plugging in the values, we get:

$$T = 2\pi\sqrt{\frac{0.75}{48}} \approx 2\pi \cdot 0.125 \approx 0.785s$$

Frequency f is given by:

$$f = \frac{1}{T}$$

So,

$$f = \frac{1}{0.785} \approx 1.27Hz$$

Thus, the period is approximately 0.785 seconds and the frequency is approximately 1.27 hertz.

5) Two pendulums are of lengths L_1 and L_2 and - when located at their respective distances R_1 and R_2 from the center of the earth - have periods p_1 and p_2. Show that

$$\frac{p_1}{p_2} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}$$

.

The period of a simple pendulum is given by:

$$p = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum and g is the acceleration due to gravity.

$$g = \frac{GM}{R^2}$$

where G is the gravitational constant, M is the mass of the earth, and R is the distance from the center of the earth.

So,

$$p = 2\pi\sqrt{\frac{L}{\frac{GM}{R^2}}}$$

So,

$$p_1 = 2\pi\sqrt{\frac{L_1}{\frac{GM}{R_1^2}}}$$

$$p_2 = 2\pi\sqrt{\frac{L_2}{\frac{GM}{R_2^2}}}$$

So,

$$\frac{p_1}{p_2} = \frac{2\pi\sqrt{\frac{L_1}{\frac{GM}{R_1^2}}}}{2\pi\sqrt{\frac{L_2}{\frac{GM}{R_2^2}}}}$$

Simplifying this gives us,

$$\frac{p_1}{p_2} = \sqrt{\frac{L_1 R_2^2}{L_2 R_1^2}}$$

Then, multiply both the numerator and denominator by $R_1\sqrt{L_1}$:

$$\frac{p_1}{p_2} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}$$

15) A mass m is attached to both a spring (with given spring constant k) and a dashpot (with given damping constant c). The mass is set in motion with initial position x_0 and initial velocity v_0. Find the position function x(t) and determine whether the motion is overdamped, critically damped, or underdamped. If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt} cos(\omega_1 t - \alpha_1)$. Also, find the undamped position function $u(t) = C_0 cos(\omega_0 t - \alpha_0)$ that would result if the mass were set in motion with the same initial position and velocity, but with the dashpot disconnected (so c = 0). Finally, construct a figure that illustrates the effect of damping by comparing the graphs of x(t) and u(t).

$m = \frac{1}{2}$, c = 3, k = 4, $x_0 = 2$, $v_0 = 0$

The equation of motion for a damped harmonic oscillator is:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Plugging in the given constants, we have:

$$\frac{1}{2}\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 4x = 0$$

First find the characteristic equation of the differential equation:

$$\frac{1}{2}r^2 + 3r + 4 = 0$$

Solving for r, we get:

$$r = -3 \pm \sqrt{5}$$

Since the roots are both real and distinct, the motion is overdamped. The general solution for overdamped systems is:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Now, solve for C_1 and C_2:

$$x(0) = 2 = C_1 + C_2$$

$$v(0) = 0 = C_1 r_1 + C_2 r_2$$

Substituting the values of r_1 and r_2, we get:

$$0 = C_1(-3 + \sqrt{5}) + C_2(-3 - \sqrt{5})$$

Since we have two equations and two unknowns, we can solve for C_1 and C_2.

We get $C_1 = \frac{2(\sqrt{5}-1)}{4}$ and $C_2 = \frac{2(3-\sqrt{5})}{4}$. Therefore, the position function is:

$$x(t) = \frac{2(\sqrt{5}-1)}{4} e^{(-3+\sqrt{5})t} + \frac{2(3-\sqrt{5})}{4} e^{(-3-\sqrt{5})t}$$

Now, for the undamped case, the equation of motion is:

$$\frac{1}{2}\frac{d^2u}{dt^2} + 4u = 0$$

Yielding the characteristic equation:

$$\frac{1}{2}r^2 + 4 = 0$$

So, $r = \pm 2i$, which gives a frequency of $\omega_0 = 2$. The general solution for the undamped case is:

$$u(t) = C_0 cos(\omega_0 t - \alpha_0)$$

Solving for C_0 and α_0:

$$u(0) = 2 = C_0 cos(\alpha_0)$$

$$v(0) = 0 = -C_0 \omega_0 sin(\alpha_0)$$

We get $C_0 = 2$ and $\alpha_0 = 0$ from the above equations. So, the undamped equation is:

$$u(t) = 2cos(2t)$$

Graph: https://www.desmos.com/calculator/4bv9ucyrbo

21) A mass m is attached to both a spring (with given spring constant k) and a dashpot (with given damping constant c). The mass is set in motion with initial position x_0 and initial velocity v_0. Find the position function x(t) and determine whether the motion is overdamped, critically damped, or underdamped. If it is underdamped, write the position function in the

form $x(t) = C_1e^{-pt}cos(\omega_1 t - \alpha_1)$. Also, find the undamped position function $u(t) = C_0cos(\omega_0 t - \alpha_0)$ that would result if the mass were set in motion with the same initial position and velocity, but with the dashpot disconnected (so c = 0). Finally, construct a figure that illustrates the effect of damping by comparing the graphs of x(t) and u(t).

m = 1, c = 10, k = 125, $x_0 = 6$, $v_0 = 50$

We have the equation:

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 125x = 0$$

Assume a solution of the form:

$$x(t) = e^{rt}$$

Substituting in the equation, we get the characteristic equation:

$$r^2 + 10r + 125 = 0$$

The roots are $r = -5 \pm 10i$, so the motion is underdamped.

So, the solution will be of the form:

$$x(t) = e^{-5t}(C_1cos(10t) + C_2sin(10t))$$

Using the initial conditions of $x(0) = 6$ and $v(0) = 50$:

$$x(0) = 6 = C_1$$

$$v(0) = 50 = -5C_1 + 10C_2$$

Using the above equations, we get:

$$C_2 = 8$$

Thus, the solution is:

$$x(t) = e^{-5t}(6cos(10t) + 8sin(10t))$$

This can be written in our desired form as:

$$x(t) = C_1e^{-pt}cos(\omega_1 t - \alpha_1)$$

where:

$$C_1 = \sqrt{6^2 + 8^2} = 10$$

$$p = 5$$

$$\omega_1 = 10$$

$$\alpha_1 = \arctan\left(\frac{8}{6}\right) = \arctan\left(\frac{4}{3}\right)$$

Now, for undamped motion:

$$\frac{d^2u}{dt^2} + 125u = 0$$

The solution will be of the form:

$$u(t) = C_0 cos(\omega_0 t - \alpha_0)$$

where $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{125} = 5\sqrt{5}$.

Using the initial conditions from before,

$$u(0) = 6 = C_0 cos(\alpha_0)$$

$$v(0) = 50 = -C_0 \omega_0 sin(\alpha_0)$$

Giving us:

$$\alpha_0 = 0; C_0 = 6$$

Thus, the undamped position function is:

$$u(t) = 6cos(5\sqrt{5}t)$$

Graph: https://www.desmos.com/calculator/cc0xenguxa

Section 3.6:

2) Equation 8: $x(t) = Ccos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} cos\omega t$.

Express the solution of the given initial value problem as a sum of two oscillation as in Equation 8. Throughout, primes denote derivatives with respect to time t. Then graph the solution function x(t) in such a way that you can identify and label its period.

x" + 4x = 5 sin 3t; x(0) = x'(0) = 0

The homogenous equation $x'' + 4x = 0$ has the characteristic equation:

$$r^2 + 4 = 0$$

So, $r = \pm 2i$, so the homogenous solution is:

$$x_h(t) = C_1 cos(2t) + C_2 sin(2t)$$

For the particular solution, we assume it is of the form:

$$x_p(t) = Acos(3t) + Bsin(3t)$$

Since the inhomogenous term is $5sin(3t)$, then $x_p(t) = Bsin(3t)$.

$$B'' sin(3t) + 4Bsin(3t) = 5sin(3t)$$

$$\implies x_p(t) = -sin(3t)$$

So, the general solution is:

$$x(t) = C_1 cos(2t) + C_2 sin(2t) - sin(3t)$$

We can find C_1 and C_2 using the initial conditions:

$$x(0) = C_1 - 0 = 0; C_1 = 0$$

$$v(0) = 2C_2 - 3 = 0; C_2 = \frac{3}{2}$$

This gives a final solution to the initial value problem of:

$$x(t) = \frac{3}{2} sin(2t) - sin(3t)$$

Graph: https://www.desmos.com/calculator/irukqqk6ky
The period is 2π.

11) Find and plot both the steady periodic function $x_{sp}(t) = Ccos(\omega t - \alpha)$ of the given differential equation and the actual solution $x(t) = x_{sp} + x_u(t)$ that satisfies the given initial conditions.

x" + 4x' + 5x = 10 cos 3t; x(0) = x'(0) = 0

First we need to find the steady periodic function x_{sp} of:

$$x'' + 4x' + 5x = 10cos(3t)$$

We assume a solution of the form:

$$x_{sp}(t) = Ccos(3t - \alpha)$$

Plugging this back into the differential equation, we get:

$$-C9cos(3t - \alpha) - 4C3sin(3t - \alpha) + 5Ccos(3t - \alpha) = 10cos(3t)$$

Solving for the constants, we get:

$$C = \frac{5}{2}; \alpha = 0$$

So the steady periodic function is:

$$x_{sp}(t) = \frac{5}{2} cos(3t)$$

Now, we have to find the full solution x(t). The general solution of $x'' + 4x' + 5x = 0$ is:

$$x_h(t) = e^{-2t}(C_1 cos(t) + C_2 sin(t))$$

So, the complete solution is:

$$x(t) = \frac{5}{2} cos(3t) + e^{-2t}(C_1 cos(t) + C_2 sin(t))$$

Using the initial conditions $x(0) = v(0) = 0$, we get:

$$C_1 = -\frac{5}{2}; C_2 = 5$$

So, the final solution to the initial value problem is:

$$x(t) = \frac{5}{2}cos(3t) + e^{-2t}(-\frac{5}{2}cos(t) + 5sin(t))$$

Plot: https://www.desmos.com/calculator/vtcnlhhsqd

15) This problem gives the parameters for a forced mass-spring-dashpot system with equation $mx'' + cx' + kx = F_0 cos\omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady periodic forced oscillations with frequency ω. Sketch the graph of $C(\omega)$ and find the practical resonance frequency ω (if any).

m = 1; c = 2; k = 2; $F_0 = 2$

We have:

$$x'' + 2x' + 2x = 2cos\omega t$$

We can find the amplitude of the steady periodic forced oscillations if we assume a particular solution of the form:

$$x_{sp}(t) = C(\omega)cos(\omega t - \alpha)$$

where $C(\omega)$ is the amplitude we want to find.

Plugging the above steady state form back into the differential equation gives us:

$$-C(\omega)\omega^2 cos(\omega t - \alpha) - 2C(\omega)\omega sin(\omega t - \alpha) + 2C(\omega)cos(\omega t - \alpha) = 2cos(\omega t)$$

Solving that equation gives us:

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

If we substitute in the given values, we have:

$$C(\omega) = \frac{2}{\sqrt{(2 - \omega^2)^2 + (2\omega)^2}}$$

To find the practical resonance frequency, we have to get the maximum of $C(\omega)$. So we have to find when the derivative of $C(\omega)$ with respect to ω is 0.

$$\frac{dC(\omega)}{d\omega} = \frac{-4\omega(2 - \omega^2) + 8\omega}{((2 - \omega^2)^2 + 4\omega^2)^{\frac{3}{2}}}$$

So,

$$-4\omega(2 - \omega^2) + 8\omega = 0$$

$$\omega^3 - 2\omega = 0$$

The roots are $\omega = 0, \pm\sqrt{2}$. But ω must be non-negative, so we will discard $-\sqrt{2}$ as a valid root.

$$C(\sqrt{2}) = \frac{1}{\sqrt{2}}$$

$$C(0) = 1$$

Therefore, the maximum amplitude occurs at $\omega = 0$ and the practical resonance frequency is $\omega = 0$.

Sketch: https://www.desmos.com/calculator/aevlaufb5n

24)

For small angles, the restoring force from the pendulum is:

$$F_p = mg\theta$$

The restoring force from the spring is:

$$F_s = kx$$

The equation of motion for the pendulum:

$$m\frac{d^2\theta}{dt^2} = -mg\theta - kx$$

The equation of motion for the spring:

$$m\frac{d^2x}{dt^2} = -kx - mg\theta$$

Because the oscillations are small, x and θ can be related via arclength:

$$x = L\theta$$

If we substitute these back into the above equations of motion, we get:

$$m\frac{d^2\theta}{dt^2} = -mg\theta - kL\theta$$

$$m\frac{d^2x}{dt^2} = -kL\theta - mg\theta$$

The general form for the equation for simple harmonic motion is given by:

$$\frac{d^2y}{dt^2} = -\omega_0^2 y$$

So, we have:

$$-\omega_0^2 m\theta = -mg - kL\theta$$

$$\omega_0^2 m\theta = g\theta + \frac{kL\theta}{m}$$

If we solve for ω_0^2, we have:

$$\omega_0^2 = \frac{g}{L} + \frac{k}{m}$$

Therefore, the natural circular frequency of the mass's motion is:

$$\omega_0 = \sqrt{\frac{g}{L} + \frac{k}{m}}$$