This is due Saturday 11/25 by 11:59 pm on Gradescope. Please either neatly write up your solutions or type them up. You can find a .tex template on Canvas. Your proofs should be written in complete sentences and paragraphs, using a combination of words and symbols. They should be **correct, clear, and concise**. You will be graded on all three, especially the first two!

1. (4) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and that $\lim_{x\to\infty} f'(x) = \infty$. Show that for all $M \in \mathbb{R}$ there is some $N \in \mathbb{R}$ so that for all x and y greater than N we have that $|f(x) - f(y)| > M|x - y|$.

Solution:

Let's first apply the MVT. For any two numbers x and y with $x < y$, $\exists$ a c between x and y such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

i.e.

$$f(y) - f(x) = f'(c)(y - x)$$

We know that $\lim_{x\to\infty} f(x) = \infty$. This means that for any given M, $\exists$ some N such that $\forall$ $x > N$, f'(x) ¿ M.

Choose any $x, y > N$ with $x < y$. From MVT, there is some c between x and y such that $f(y) - f(x) = f'(c)(y - x)$. Since $c > x > N$, $f'(c) > M$. Therefore, $f(y) - f(x) = f'(c)(y - x) > M(y - x)$.

Notice that

$$|f(y) - f(x)| = |f'(c)||y - x| > M|y - x|$$

$f'(c) > M \implies |f'(c)| > M$.

Thus, we have shown that $\forall M \in \mathbb{R}$, $\exists$ an $N \in \mathbb{R}$ such that $\forall x, y > N$, $|f(x) - f(y)| > M|x - y|$.

2. (4) Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim_{x\to\infty} f'(x) = \infty$ then f is *not* uniformly continuous on $\mathbb{R}$.

Solution:

We will show this by using a proof by contradiction.

Assume f is uniformly continuous on $\mathbb{R}$. Then, for $\epsilon = 1$, $\exists \delta > 0$ such that $\forall x, y \in \mathbb{R}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < 1$.

Since $\lim_{x\to\infty} f'(x) = \infty$, $\exists$ some X such that $\forall x > X$, $f'(x) > \frac{1}{\delta}$.

Choose $x, y > X$ such that $y - x < \delta$. This is always possible because $\delta > 0$. According to our assumption of uniform continuity, $|f(x) - f(y)| < 1$.

This creates a contradiction because by MVT, $\exists$ a c between x and y such that $f(y) - f(x) = f'(c)(y - x)$. Since $c > X$, $f'(c) > \frac{1}{\delta}$, so

$$|f(y) - f(x)| = |f'(c)||y - x| > \frac{1}{\delta}|y - x|$$

. But $|y - x| < \delta$, so $|f(y) - f(x)| > 1$, which contradicts the assumption of uniform continuity that $|f(y) - f(x)| < 1$.

3. (4) Find $\lim_{x\to 0} \frac{\cos(x) - 1}{x^2}$.

Solution:
Since $\dfrac{\cos(x)-1}{x^2}$ at $x=0$ is $\dfrac{0}{0}$, we can use L'Hopital's rule. So we get $\lim_{x\to 0}\dfrac{-\sin(x)}{2x}$. This new limit at $x=0$ is $\dfrac{0}{0}$. Thus, we can apply L'Hopital's rule again. So we get $\lim_{x\to 0}\dfrac{-\cos(x)}{2}$. Evaluating this limit at $x=0$ gives us $-\dfrac{1}{2}$.

Therefore, $\lim_{x\to 0}\dfrac{\cos(x)-1}{x^2} = -\dfrac{1}{2}$

4. (a) (2) Show that $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution:
We will use proof by induction.
Base case:
For $n=1$, the left side is $\displaystyle\sum_{i=1}^{1} i^2 = 1^2 = 1$. The rhs is $\dfrac{1(1+1)(2\cdot 1+1)}{6} = \dfrac{1\cdot 2\cdot 3}{6} = 1$.
Induction case:
Assume that $\displaystyle\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ is true.
Goal: Show that $\displaystyle\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ is true.

$$\sum_{i=1}^{k+1} i^2 = (k+1)^2 + \sum_{i=1}^{k} i^2$$

From the inductive hypothesis, we get:

$$\sum_{i=1}^{k+1} i^2 = (k+1)^2 + \frac{k(k+1)(2k+1)}{6}$$

Expanding and simplifying this gives us $\dfrac{(k+1)(k+2)(2k+3)}{6}$.
Since both the base case and inductive cases hold, the proof is complete.

(b) (2) Show that $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Solution:
We will use proof by induction.
Base case:
For $n=1$, the left side is $\displaystyle\sum_{i=1}^{1} i^3 = 1^3 = 1$. The rhs is equal to $\left(\dfrac{1(1+1)}{2}\right)^2 = \left(\dfrac{2}{2}\right)^2 = 1^2 = 1$.
Induction case:
Assume that $\displaystyle\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$.

Goal: Show that $\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+2)}{2}\right)$ is true.

$$\sum_{i=1}^{k+1} i^3 = (k+1)^3 + \sum_{i=1}^{k} i^3$$

From the inductive hypothesis, we get:

$$\sum_{i=1}^{k+1} i^2 = (k+1)^2 + \left(\frac{k(k+1)}{2}\right)^2$$

Expanding and simplifying this gives us $\left(\frac{(k+1)(k+2)}{2}\right)$.

Since both the base case and inductive cases hold, the proof is complete.

You can show both of these by induction, and that is how I suggest you proceed. You might also try to come up with a counting proof or a proof by picture.

5. (4) Compute, using only things from section 5.1 of your text, $\int_0^2 x^2 - 1 \, \mathrm{d}x$.

Solution:

First partition the interval. Divide $[0, 2]$ into n subintervals, each of length $\Delta x = \frac{2}{n}$. The points of division will be $x_i = 0, \frac{2}{n}, \frac{4}{n}, \ldots, 2$.

The infimum is $f(x_{i-1}) = (x_{i-1})^2 - 1$. The supremum is $f(x_i) = (x_i)^2 - 1$.

$$L = \sum_{i=1}^{n} ((x_{i-1})^2 - 1)\Delta x$$

$$U = \sum_{i=1}^{n} ((x_i)^2 - 1)\Delta x$$

Substituting $x_i = \frac{2i}{n}$ and $\Delta x = \frac{2}{n}$, we get:

$$L = \sum_{i=1}^{n} \left(\left(\frac{2(i-1)}{n}\right) - 1\right) \frac{2}{n}$$

$$U = \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right) - 1\right) \frac{2}{n}$$

Evaluating either U or L as $n \to \infty$ gives us $\frac{2}{3}$.

Thus, $\int_0^2 x^2 - 1 \, \mathrm{d}x = \frac{2}{3}$.