This is due Saturday 10/7 by 11:59 pm on Gradescope. Please either neatly write up your solutions or type them up. You can find a .tex template on Canvas. Your proofs should be written in complete sentences and paragraphs, using a combination of words and symbols. They should be **correct, clear, and concise**. You will be graded on all three, especially the first two!

1. There is a one-to-one and onto function $\mathbb{N} \to \mathbb{Q}$ (you can use this fact without proof, but I encourage you to think about what this function is). So, there is a sequence $\{q_n\}_{n=1}^{\infty}$ of rational numbers where every rational number appears in the sequence exactly once.

   Show that for every $x \in \mathbb{R}$ there is a subsequence $\{q_{n_k}\}_{k=1}^{\infty}$ of $\{q_n\}_{n=1}^{\infty}$ converging to x. **Hint:** Use the fact that that rationals are dense in the reals, and the fact that $\lim_{n\to\infty} a_n = a$ if and only if for all $\epsilon > 0$ there are only finitely many n with $|a_n - a| \geq \epsilon$. These facts will help you to inductively construct a subsequence converging to a desired real number. Also make sure that your sequence is actually a sequence, i.e. make sure that $n_1 < n_2 < \dots$.

Solution:

Because of the density of the rational numbers, $\forall$ positive integers k, $\exists$ a rational number q such that q is in the interval $(x - \frac{1}{k}, x + \frac{1}{k})$. Since every rational number appears in $\{q_n\}$ exactly once, there is a least positive number for k (with value 1) called n_1 such that q_{n_1} is in the interval $(x-1, x+1)$.

There is also a least positive integer for k (with value 2) called n_2 such that q_{n_2} is in the interval $(x - \frac{1}{2}, x + \frac{1}{2})$. It is obvious that $n_2 > n_1$.

We can continue this inductively where each positive integer k has a least integer $n_k > n_{k-1}$ such that q_{n_k} is in the interval $(x - \frac{1}{k}, x + \frac{1}{k})$.

We now have a subsequence $\{q_{n_k}\}$ of $\{q_n\}$.

Let $\epsilon > 0$. Based on this choose a value G such that $\frac{1}{G} < \epsilon$. This means that $\forall k \geq G$:

$|q_{n_k} - x| < \frac{1}{k} < \frac{1}{G} < \epsilon$.

So, after some point, all terms of the sequence $\{q_{n_k}\}$ are within ϵ of x.

Since there are only finitely many elements of the subsequence $\{q_{n_k}\}$ that are $\geq \epsilon$, the subsequence converges to x.

So, for every real number x, there is a subsequence of $\{q_n\}$ that converges to x.

2. Suppose that $\{a_n\}$ is a sequence and set $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=1}^{n} |a_i|$. Show that if $\{t_n\}$ is bounded then $\{s_n\}$ converges.

Solution:

Goal: Show that $\{s_n\}$ is a Cauchy sequence.

$\{t_n\}$ is bounded $\implies \exists M > 0$ such that $t_n \leq M \ \forall n \in \mathbb{N}$.

$\forall \epsilon > 0$, choose some value N such that $\forall m, n \geq N$ with $n > m$,

$$|s_n - s_m| = |\sum_{i=m+1}^{n} a_i| \leq \sum_{i=m+1}^{n} |a_i|$$

$$\implies |s_n - s_m| \leq t_n - t_m \leq M - t_m$$

Because of $t_m \leq M \ \forall m$,

$$\implies M - t_m \leq M$$

$$|s_n - s_m| \leq t_n - t_m \leq M - t_m$$

with

$$M - t_m \leq M$$

implies

$$|s_n - s_m| \leq M$$

Since M is fixed and doesn't depend on m or n and because we know that $\{t_n\}$ is bounded, $|s_n - s_m|$ will eventually be smaller than some ϵ. Therefore, $\{s_n\}$ is Cauchy. And since all cauchy sequences converge on the real numbers, $\{s_n\}$ converges.

3. Suppose that $\{a_n\}$ is a sequence and for all n we have that $|a_{n+1} - a_n| < \dfrac{1}{2^n}$. Show that $\{a_n\}$ converges.

Solution:

Goal: Show that $\{a_n\}$ is a Cauchy sequence.

$\forall n \in \mathbb{N}$, let $m > n$. $a_m - a_n$ can be written as:

$$a_m - a_n = (a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)$$

Using the triangle inequality,

$$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n|$$

Since we know that $|a_{k+1} - a_k| < \dfrac{1}{2^k} \ \forall k$, we can rewrite the above as:

$$|a_m - a_n| < \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n}$$

This is part of the geometric series with ratio $\dfrac{1}{2}$.

So, $|a_m - a_n| < \dfrac{1}{2^{n-1}}$.

Given some $\epsilon > 0$, we can choose an N such that $\dfrac{1}{2^{N-1}} < \epsilon$.

Obviously that means that $|a_m - a_n| < \epsilon$, so the sequence is Cauchy, which means it converges.

4. For $\{a_n\}$ a sequence show that $s = \limsup a_n$ is a subsequential limit of $\{a_n\}$, that is show that there is a subsequence of $\{a_n\}$ converging to s. The same is true for $\liminf$ with a very similar proof, but I am just asking you to prove this about $\limsup$. **Note:** This is Theorem 2.6.5 of your text, but the proof in your textbook isn't written very well (and doesn't deal at all with the case where $s = \infty$). The key is to show that for every $\epsilon > 0$ there are infinitely many n with $|a_n - s| < \epsilon$, then you can construct a subsequence converging to s.

Solution:

First, recall the definition of lim sup:

$$s = \limsup a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m$$

Goal: Show that $\forall\epsilon > 0$, there are infinitely many terms a_n such that $|a_n - s| < \epsilon$.

We know that $s-\epsilon$ is not an upper bound of $\{a_n\}$ since s is the least upper bound. In other words, there are infinitely many terms of a_n greater than $s-\epsilon$.

Since s is the least upper bound, $\forall n$, $\exists m \geq n$, such that $a_m > s-\epsilon$ and $a_m \leq s$. So, $|a_m - s| < \epsilon$.

To construct a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to s, we choose some number n_1 such that $|a_{n_1} - s| < \epsilon$. Then, given some number n_k, we choose some $n_{k+1} > n_k$ such that $|a_{n_{k+1}} - s| < \epsilon$. We can continue this process to generate a subsequence.

Since $\forall k, |a_{n_k} - s| < \epsilon$, this subsequence converges to s. This means that $s = \limsup a_n$ is a subsequential limit of $\{a_n\}$.

5. Compute $\liminf (-1)^n + \dfrac{(-1)^n}{2^n}$ and $\limsup (-1)^n + \dfrac{(-1)^n}{2^n}$. **Hint:** Look at example 2.6.3 in your text.

Solution:

We first need to consider the sequence both for odd and even values of n.

If n is even, where $n = 2m$,

$$a_{2m} = (-1)^{2m} + \frac{(-1)^{2m}}{2^{2m}}$$

$$\implies a_{2m} = 1 + \frac{1}{2^{2m}}$$

$$\implies a_{2m} = 1 + \frac{1}{4^m}$$

If n is odd where $n = 2m+1$,

$$a_{2m+1} = (-1)^{2m+1} + \frac{(-1)^{2m+1}}{2^{2m+1}}$$

$$a_{2m+1} = -1 - \frac{1}{2(2^m)}$$

This means that if n is even, the sequence approaches 1 and if n is odd, the sequence approaches -1.

This means that:

$$\liminf a_n = \lim_{m\to\infty} a_{2m+1} = -1$$

$$\limsup a_n = \lim_{m\to\infty} a_{2m} = 1$$

Aka:

$$\liminf (-1)^n + \frac{(-1)^n}{2^n} = -1$$

$$\limsup (-1)^n + \frac{(-1)^n}{2^n} = 1$$