Textbook Section 5.6:

2) The eigenvalues are $\lambda_1 = 0, \lambda_2 = 4$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$.

That means the fundamental matrix is

$$\phi(t) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}e^{4t} \\ 1 & e^{4t} \end{bmatrix}$$

$$\phi^{-1}(0) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

Thus,

$$x(t) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}e^{4t} \\ 1 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

6) The eigenvalues are $\lambda_1 = 5 - 4i, \lambda_2 = 5 + 4i$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} \frac{1}{2} - i \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} \frac{1}{2} + i \\ 1 \end{bmatrix}$.

That means the fundamental matrix is

$$\phi(t) = \begin{bmatrix} e^{5t}(\frac{1}{2}\cos(4t) + \sin(4t)) & e^{5t}(\frac{1}{2}\cos(4t) - \sin(4t)) \\ e^{5t}(\frac{1}{2}\sin(4t) - \cos(4t)) & e^{5t}(\frac{1}{2}\sin(4t) + \cos(4t)) \end{bmatrix}$$

$$\phi^{-1}(0) = \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{2}{3} \end{bmatrix}$$

Thus,

$$x(t) = \begin{bmatrix} e^{5t}(\frac{1}{2}\cos(4t) + \sin(4t)) & e^{5t}(\frac{1}{2}\cos(4t) - \sin(4t)) \\ e^{5t}(\frac{1}{2}\sin(4t) - \cos(4t)) & e^{5t}(\frac{1}{2}\sin(4t) + \cos(4t)) \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

8) The eigenvalues are $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 3$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

That means the fundamental matrix is

$$\phi(t) = \begin{bmatrix} 0 & -e^{t} & e^{3t} \\ -e^{-2t} & e^{t} & -e^{3t} \\ e^{-2t} & 0 & e^{3t} \end{bmatrix}$$

$$\phi^{-1}(0) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus,

$$x(t) = \begin{bmatrix} 0 & -e^{t} & e^{3t} \\ -e^{-2t} & e^{t} & -e^{3t} \\ e^{-2t} & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

10) The matrix A is $\begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix}$.
The eigenvalues are $\lambda_1 = 2, \lambda_2 = 0$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$.
$e^{At} = Pe^{Dt}P{-}1$.
$P = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$
$e^{Dt} = \begin{bmatrix} e^{2t} & 0 \\ 0 & 1 \end{bmatrix}$
$P^{-1} = \begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix}$$

17) The matrix A is $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.
The eigenvalues are $\lambda_1 = 4, \lambda_2 = 2$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
$e^{At} = Pe^{Dt}P{-}1$.
$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
$e^{Dt} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix}$
$P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$e^{At} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

24) $A = \begin{bmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
This means that $e^{At} = I_n + At + \frac{A^2t^2}{2}$.
Textbook Section 5.7:
1) Substituting $x_p(t) = A, y_p(t) = B$ into the original equation gives:

$$A + 2B + 3 = 0$$

$$2A + B - 2 - 0$$

Solving this gives $A = \frac{7}{3}$ and $B = -\frac{8}{3}$.
That means our particular solution is:

$$x_p(t) = \frac{7}{3}$$

$$y_p(t) = -\frac{8}{3}$$

This solution is valid for any initial conditions since it represents a constant solution to the non-homogeneous system.

5) We guess the particular solution is of the form $x_p(t) = A + Cte^{-t}$ and $y_p(t) = B + Dte^{-t}$.

Through arithmetic after Substituting into the original equation, we know that $A = -4$ and $B = -2$.

We can then solve the system and get that:

$$C = \frac{-10te^t - 14t - 10e^t}{6t - 1}$$

and

$$D = \frac{-10te^t - 14t + 2}{6t - 1}$$

13) Assume the particular solution is of the form:

$$x_p(t) = Ae^t$$

$$y_p(t) = Be^t$$

Substituting back into the original equation gives:

$$Ae^t = 2Ae^t + Be^t + 2e^t$$

$$Be^t = Ae^t + 2Be^t - 3e^t$$

After some solving, we get:

$$0 = Ae^t + Be^t + 2e^t$$

$$0 = Ae^t - Be^t - 3e^t$$

Solving this gives us that:

$$A = \frac{1}{2}$$

and

$$B = -\frac{5}{2}$$

Thus, the particular solution to the system is:

$$x_p(t) = \frac{1}{2}e^t$$

$$y_p(t) = -\frac{5}{2}e^t$$

17) The homogeneous solution is given by

$$x_h(t) = e^{At}x_0$$

. That means that the homogeneous solution is the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The particular solution is given by

$$x_p(t) = \int_0^t e^{A(t-s)} f(s) ds$$

.

$$x_p(t) = \begin{bmatrix} -7e^{5t} + 102 - 95e^{-t} \\ -e^{5t} + 96 - 95e^{-t} \end{bmatrix}$$

Thus,

$$x(t) = x_p(t) = \begin{bmatrix} -7e^{5t} + 102 - 95e^{-t} \\ -e^{5t} + 96 - 95e^{-t} \end{bmatrix}$$