Math 3220-1: Homework 5, due 02/21/2024

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Problem 1. Let f be the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by

$$f(x) = \sin(1/x)$$

Show that $\lim_{x\to 0} f(x)$ does not exist.

Does f have a continuous extension to $\mathbb{R}$? Justify your answer.

Comment: This is a review problem from Math 3210.

To demonstrate that $f(x) = \sin(1/x)$ has no limit as x approaches 0, we must show that the function values do not approach a single finite value as x gets arbitrarily close to 0.

The function is defined on $\mathbb{R} \setminus \{0\}$, so x can be any value except 0 itself.

Let us first note the fact that $\sin(1/x)$ oscillates infinitely as x approaches 0. This is because as x gets closer to 0, 1/x grows without any bound, which causes the sine function to oscillate between -1 and 1 infinitely many times in any neighborhood of 0. This means there is no value approached by f within any neighborhood of 0.

Let us consider two sequences that approach 0: $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+1)\pi}$, where n is a positive integer. Notice that $f(x_n) = \sin(2n\pi) = 0$ and $f(y_n) = \sin((2n+1)\pi) = 0$. However, if we choose other sequences that approach 0, we can get f to oscillate between -1 and 1, showing that the limit is not stable.

From above, for any proposed limit L as x approaches 0, we can always find values of x arbitrarily close to 0 for which f(x) is not arbitrarily close to L (due to the oscillatory nature of sine).

For a function to have a continuous extension to $\mathbb{R}$, it must be possible to define the function at the points where it is currently undefined in a way such that the extended function is continuous at those points. For the function $f(x) = \sin(1/x)$, this would mean defining $f(0)$.

However, since $\lim_{x\to 0} f(x)$ does not exist, there is no single real number that we could assign to $f(0)$ to make f continuous at 0. This means that we can not define a continuous extension to $\mathbb{R}$ for the function $f(x) = \sin(1/x)$.

Problem 2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Is this function continuous at $(0,0)$? Justify your answer.

To determine if $f : \mathbb{R}^2 \to R$ is continuous at $(0,0)$, we need to check if $\lim_{(x,y)\to(0,0)} = f(0,0)$.

A function is continuous at a point if the limit of the function as it approaches that point is equal to the function's value at that point.

Here, $f(0,0) = 0$, so we need to verify if:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$$

We can solve this with direct substitution techniques. Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. In polar coordinates, the limit $(x,y) \to (0,0)$ is equivalent to $r \to 0$ regardless of θ.

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} &= \lim_{r\to 0} \frac{r\cos(\theta)r^2\sin^2(\theta)}{r^2(\cos^2(\theta)+\sin^2(\theta))} \\ = \lim_{r\to 0} \frac{r^3\cos(\theta)\sin^2(\theta)}{r^2} &= \lim_{r\to 0} r\cos(\theta)\sin^2(\theta) \end{aligned}$$

Note that $|\cos(\theta)\sin^2(\theta)| \leq 1\ \forall\theta$, so the above limit approaches 0 as r approaches 0. This suggests that the function $f(x,y)$ approaches 0 as (x,y) approaches $(0,0)$, which matches $f(0,0)$. Thus, f is continuous at $(0,0)$.

Problem 3. Does the function $f : \mathbb{R}^2 - \{0, 0\} \to \mathbb{R}$ defined by

$$f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$$

have a limit as (x, y) approaches $(0, 0)$? Justify your answer.

We can use a similar technique as the previous problem. With $r = \cos(\theta)$ and $r = \sin(\theta)$, and the resulting equality: $r = \sqrt{x^2 + y^2}$, we have:

$$\frac{x}{\sqrt{x^2 + y^2}} = \frac{r\cos(\theta)}{r} = \cos(\theta)$$

This expression depends solely on θ and not r. This means the behavior of f as (x, y) approaches $(0, 0)$ depends on the angle of the path approaching it.

We can show this further by using path testing. Let's consider approaching $(0, 0)$ along different paths:

Along the x-axis: If $y = 0$, then $\theta = 0$ or π and $\cos(\theta) = 1$ or -1. So, $f(x, 0) = 1$ as x approaches 0 from the positive side and $f(x, 0) = -1$ as x approaches 0 from the negative side.

Along the y-axis: If $x = 0$, then $\theta = \pi/2$ or $-\pi/2$, and $\cos(\theta) = 0$. So, $f(0, y) = 0$.

Since the limit along different paths differ as (x, y) approach $(0, 0)$ ranges from -1 to 1 (depending on the path), this means the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ does not exist.

Problem 4. Let X be a metric space, and let $c \in X$. Show that the function $f : X \to \mathbb{R}$ defined by

$$f(x) = d(c, x)$$

is continuous. (hint: use the inequality $|d(x, z) - d(y, z)| \leq d(x, y)$ for any $x, y, z \in X$.

Let $x, y \in X$ and $\epsilon > 0$ be given. We want to show that $\exists \delta > 0$ such that if $d(x, y) < \delta$, then $|f(x) - f(y)| < \epsilon$.

The function f is defined as $f(x) = d(c, x)$ $\forall x \in X$. Therefore, the difference $|f(x) - f(y)|$ can be written as:

$$|f(x) - f(y)| = |d(c, x) - d(c, y)|$$

Using the given inequality, we have:

$$|d(c, x) - d(c, y)| \leq d(x, y)$$

For continuity, we want $|d(c, x) - d(c, y)| < \epsilon$. From the above inequality, this will be satisfied if $d(x, y) < \epsilon$. Thus, we can choose $\delta = \epsilon$.

Thus, for any $\epsilon > 0$, if we set $\delta = \epsilon$, then whenever $d(x, y) < \delta$, it follows that $|f(x) - f(y)| < \epsilon$. This shows that $f(x) = d(c, x)$ is continuous $\forall x \in X$ since the choice of δ does not depend on the specific points x and y, but only on the distance between them and the fixed point c.