This is due Wednesday 10/18 by 11:59 pm on Gradescope. Please either neatly write up your solutions or type them up. You can find a .tex template on Canvas. Your proofs should be written in complete sentences and paragraphs, using a combination of words and symbols. They should be **correct, clear, and concise**. You will be graded on all three, especially the first two!

1. (4) Directly from the definition of continuity, show that $f : \mathbb{R}_{>0} \to \mathbb{R}$ defined by $f(x) = 1/x$ is continuous at every $a \in \mathbb{R}_{>0}$.

Solution:
$f(x) = \dfrac{1}{x}$ and we want to show that f is continuous at some arbitrary point. Let's call this point a.
Let $\epsilon > 0$. We need to find a $\delta > 0$ such that, $0 < |x - a| < \delta \implies |\dfrac{1}{x} - \dfrac{1}{a}| < \epsilon$.
Recall:

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{ax}\right| = \frac{|a-x|}{ax}$$

Since it has to be less than ϵ, that means that $\dfrac{|a-x|}{ax} < \epsilon$.
For δ, we can do:

$$|x-a| < \delta \implies \frac{|a-x|}{ax} < \epsilon$$

Now we have to rewrite δ in terms of ϵ. For $|a - x| < \delta$, we can get:

$$\begin{aligned} \frac{|a-x|}{ax} \le \frac{\delta}{a(a-\delta)} &\le \frac{\delta}{a(a - z^2\epsilon/2)} \\ &= \frac{2\delta}{a^2\epsilon} \le \epsilon \end{aligned}$$

We have now shown that for every $\epsilon > 0$, we can find a $\delta > 0$, such that if $|x - a| < \delta$, $|\dfrac{1}{x} - \dfrac{1}{a}| < \epsilon$.
This means that the function $f(x) = \dfrac{1}{x}$ is continous forall of $\mathbb{R}_{>0}$.

2. (4) Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & x = 0 \\ \sin(1/x) & x \neq 0 \end{cases}$ is *not* continuous at $x = 0$. **Hint:** use the sequential formulation of continuity.

Solution:
The function is continuous at 0 if, forall sequences $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ that converges to 0, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to f(0) = 0.
Consider the sequence defined by $x_n = \dfrac{1}{2\pi n}$. This sequence converges to 0 as n goes to infinity. But we have,

$$f(x_n) = sin(2\pi n) = 0$$

So the sequence $f(x_n)$ is constantly 0 and thus trivially converges to 0.

Now consider the sequence $y_n = \dfrac{1}{2\pi n + \frac{\pi}{2}}$. This sequence also converges to 0 as n goes to infinity. However, we have,

$$f(y_n) = sin(2\pi n + \frac{\pi}{2}) = 1$$

Since the numbers aren't both 0, the function is not continuous at x = 0.

3. (4) Show that if $h : \mathbb{R} \to \mathbb{R}$ is bounded then $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = xh(x)$ is continuous at 0. This let's you conclude that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & x = 0 \\ x \sin(1/x) & x \neq 0 \end{cases}$ is continous on all of $\mathbb{R}$.

Solution:

To show that g(x) = xh(x)g(x) = xh(x) is continuous at 0, we need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x| < \delta \implies |g(x) - g(0)| < \epsilon$.

Since g(0) = 0, we want to show that $|xh(x)| < \epsilon$ whenever $|x| < \delta$. Given $\epsilon > 0$, let $\delta = \dfrac{\epsilon}{M}$ where M is the bound of the function. We get:

$$xh(x) \leq |x|M \leq \frac{\epsilon}{M}M = \epsilon$$

Therefore g is continous at 0.

For f(x), we know that $h(x) = sin(\dfrac{1}{x})$ is bounded, so, by the argument above $g(x) = xsin(\dfrac{1}{x})$ is also continous at 0. Since f(x) = 0 when x = 0 and $f(x) = xsin(\dfrac{1}{x})$ otherwise, it is clear that f is continous at 0. And because $f(x) = xsin(\dfrac{1}{x})$ is continous forall other values, it means that f is continuous forall of $\mathbb{R}$.

4. (4) Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and that $f(a) \geq g(a)$, $f(b) \leq g(b)$. Show that there is some $x \in [a, b]$ with $f(x) = g(x)$. **Hint:** Intermediate value theorem.

Solution:

Recall: Let $h(x) = f(x) - g(x)$. If f and g are continous, so is h.

Since $f(a) \geq g(a)$ and $f(b) \leq g(b)$, then $h(a) \geq 0$ and $h(b) \leq 0$.

Now, from the Intermediate Value Theorem, since h is continous on the closed interval [a, b] and changes sign, there must be an $x \in [a, b]$ such that h(x) = 0. i.e. $\exists x \in [a, b]$ such that $f(x) - g(x) = 0$ or $f(x) = g(x)$.

5. (a) (2) Give an example of a continuous function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ and a Cauchy sequence $\{a_n\}$ in D such that $\{f(a_n)\}$ is not Cauchy.

Solution:

Consider $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$. The domain D is a subset of $\mathbb{R}$ and f is continous on D.

Let $\{a_n\}$ be the sequence on D defined by $a_n = \frac{1}{n}$. The sequence is a Cauchy sequence in D since forall D, $\exists N$ such that forall $n, m \geq N$, we get:

$$|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{|n-m}{nm} < \epsilon$$

But $\{f(a_n)\}$ is not Cauchy even though $\{a_n\}$ is and f is continous. The continuity of f is not sufficient to preserve the Cauchy property of the sequence since the function is not uniformly continous on D.

(b) (2) Recall that $f : D \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in D$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$ (You met this concept on the week 7 worksheet).

Show that if $f : D \to \mathbb{R}$ is *uniformly continuous* then if $\{a_n\}$ is a Cauchy sequence in D then $\{f(a_n)\}$ is also Cauchy.

Solution:

Suppose that $\{a_n\}$ is a Cauchy sequence in D. This implies that forall $\epsilon' > 0$, $\exists N$ such that forall $n, m \geq N$, we get $|a_n - a_m| < \epsilon'$.

Since f is uniformly continous, it allows us to select $\epsilon' = \delta$. This means that if $|a_n - a_m| < \delta$, then $|f(a_n) - f(a_m)\epsilon$.

So, for any $\epsilon > 0$, we choose δ as described above and let N be a value such that $|a_n - a_m| < \delta$ when $n, m \geq N$. So, forall $n, m \geq N$, we end up with:

$$|f(a_n) - f(a_m)| < \epsilon$$

We have shown that $\{f(a_n)\}$ is Cauchy.