This is due Saturday 11/11 by 11:59 pm on Gradescope. Please either neatly write up your solutions or type them up. You can find a .tex template on Canvas. Your proofs should be written in complete sentences and paragraphs, using a combination of words and symbols. They should be **correct, clear, and concise**. You will be graded on all three, especially the first two!

1. (a) (2) Consider $f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

   Is f differentiable at 0?

   Solution:
   The limit definition of the derivative at $x = 0$ gives:

   $$\lim_{h\to 0} \sin(\frac{1}{h})$$

   This does not give us a finite number as we take the limit, it just oscilates faster as we approach 0, therefore the limit does not exist.
   Therefore, no, f is not differentiable at 0.

   (b) (2) Consider $f(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

   Is f differentiable at 0?

   Solution:
   The limit definition of the derivative at $x = 0$ gives:

   $$\lim_{h\to 0} h\sin(\frac{1}{h})$$

   Evaluating this limit gives us a finite value of 0 because we are multiplying 0 times some arbitrary number.
   Since the limit converges to a finite fixed number, it exists.
   Therefore, yes, f is differentiable at 0.

2. (4) Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ and that $f'(c) > 0$. Show that there is some $\delta > 0$ such that for all $x, y \in (a, b)$ with $x < c < y$ and $|x - c| < \delta$ and $|y - c| < \delta$ we have $f(x) < f(c) < f(y)$. **Note**: We are not assuming that f is differentiable on all of (a, b), we are only assuming differentiability at c.

   Solution:
   Because f is differentiable at c, that means that for any $\epsilon > 0$, $\exists\delta > 0$ such that $\forall x$ with $|x - c| < \delta$, $\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \epsilon$.

   Since $f'(c) > 0$, we can choose a small ϵ so that $\frac{f(x) - f(c)}{x - c}$ is positive. This implies that f(x) and f(c) are either both increasing or both decreasing as $x \to c$.

   Now, for any two points x, y such that $x < c < y$ and $|x - c| < \delta$, $|y - c| < \delta$, $\exists$ a point z and x and y such that $f'(z) = \frac{f(y) - f(x)}{y - x}$. We want to show that $f(x) < f(c) < f(y)$ for these x and y.

Combining the above, we get that $f(y) > f(x)$.

Differentiability at c implies continuity at c. Therefore, there is an interval around c where f(x) is increasing.

Thus, for $x < c < y$, if x, y are within δ of c, then $f(x) < f(c) < f(y)$.

□

3. (4) Suppose that $f : (a, b) \to \mathbb{R}$ differentiable on (a, b), and that for $x, y \in (a, b)$ with $x < y$ we have that $f'(x)$ and $f'(y)$ have different signs (i.e. one is positive and one is negative). Show that there is some c with $x < c < y$ and $f'(c) = 0$. **Hint:** This would follow from the IVT if we knew that f' was continuous, but we don't know that f' is continuous! Instead use the extreme value theorem and the previous problem.

Solution:

Note: EVT = Extreme Value Theorem

Since f is differentiable on (a, b), it is also continuous on (a, b). By the EVT, f attains its maximum and minimum on the closed interval [x, y]. Furthermore, f must attain its minimum and maximum either at a critical point or at the endpoints: x and/or y.

Recall from the previous problem that if f is differentiable at some point c in an interval and $f'(c) > 0$, then $\exists$ a $\delta > 0$ such that $f(x) < f(c) < f(y)$ $\forall x, y$ in the interval where $x < c < y$ and $|x - c| < \delta$, $|y - c| < \delta$. Also, if $f'(c) < 0$, then $f(y) < f(c) < f(x)$.

First, let's note that f'(c) and f'(y) have different signs. Let's assume $f'(x) < 0$ and $f'(y) > 0$.

That means that it is increasing near x and decreasing near y. That means that it cannot attain its maximum and minimum points at both x and y.

That means there must be at least one critical point $c \in (x, y)$ (i.e. $f'(c) = 0$).

□

4. (4) Suppose that $f : (a, b) \to \mathbb{R}$ is continuous and differentiable on (a, b), and that for $x, y \in (a, b)$ with $x < y$ there is some z between $f'(x)$ and $f'(y)$. Show that there is a c with $x \leq c \leq y$ with $f'(c) = z$. **Remark and Hint:** This says that f' satisfies the conclusion of the intermediate value theorem, even though f' is not necessarily continuous. To show this, use the previous problem and a technique similar to how we used Rolle's theorem to prove the mean value theorem.

Solution:

Let us construct a new function g on the interval [x, y] such that g is continuous on [x, y] and differentiable on (x, y). For the sake of this proof, we can choose g to be $g(t) = f(t) - zt$

Now, let's compute the derivative of g. We have $g'(t) = f'(t) - z$

Goal: Show $\exists c \in (x, y)$ such that $g'(c) = 0$, i.e. $f'(c) = z$.

Since f'(x) and f'(y) are such that z lies between them, either $f'(x) < z < f'(y)$ or $f'(y) < z < f'(x)$. This means that g'(x) and g'(y) have different signs.

Recall from the previous problem that if a function is differentiable on an interval and the endpoints of a subinterval have different signs, then $\exists$ at least one point within that subinterval where the derivative equals 0.

For g, this means that $\exists c \in (x, y)$ where $g'(c) = 0$, i.e. $f'(c) = z$.

□

5. (4) Give an example of a function $f : (a, b) \to \mathbb{R}$ that is not the derivative of any function. **Hint:** By the previous problem, if you have a function that doesn't satisfy the conclusion of the intermediate value theorem it is not the derivative of any function.

Solution:

To provide an example of a function $f : (a, b) \to \mathbb{R}$ that is not the derivative of any function, we need to construct a function that does not satisfy the conclusion of the IVT. This means that f must not take on all intermediate values between any two of its values.

A function that fulfills these requirements is a function with a jump discontinuity.

For instance,

$$f(x) = \begin{cases} 1 & x < c \\ -1 & x > c \end{cases}$$

where $c \in (a, b)$.

This function doesn't satisfy IVT because there isn't a point in (a, b) where f(x) has a value between (not inclusive) -1 and 1. Since it doesn't satisfy IVT, it isn't the derivative of any function on (a, b).