Math 3220-1: Homework 2, due 01/24/2024

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Unless stated otherwise, $\mathbb{R}^d$ is the Euclidean space with the "usual" inner product, norm and metric.

Problem 1. Let $\{\boldsymbol{x}_n\}$ be a bounded sequence in $\mathbb{R}^d$ and let $\{a_n\}$ be a sequence of scalars converging to 0. Prove that the sequence $\{a_n \boldsymbol{x}_n\}$ converges to $\mathbf{0}$ in $\mathbb{R}^d$.

Using properties of norms, we have:

$$||a_n x_n|| = |a_n| \cdot ||x_n|| \leq |a_n| \cdot M$$

Since $\{a_n\}$ converges to 0, for any given $\epsilon > 0$, we can choose $\epsilon' = \frac{\epsilon}{m}$. Then $\exists$ $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n| < \frac{\epsilon}{m}$.

Thus, for any $n \geq N$:

$$||a_n x_n|| \leq |a_n| \cdot M < \frac{\epsilon}{M} \cdot M = \epsilon$$

Thus $\{a_n x_n\}$ converges to 0.

Problem 2. Let $\mathcal{C}([0,1])$ be the space of all real valued continuous functions regarded as a metric space with the distance

$$d(f,g) = \|f-g\|_\infty$$

Find $d(f,g)$ where

(a) $f(x) = x$ and $g(x) = x^2$
(b) $f(x) = x$ and $g(x) = 2x$

$$||h||_\infty = \sup_{x\in[0,1]} |h(x)|$$

a) $|x-x^2|$ attains its maximum at $x = \frac{1}{2}$ on the interval $[0,1]$. Plugging it in gives a value of $|\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Thus, $d(x,x^2) = \frac{1}{4}$.

b) $|x-2x| = |x|$, which obviously attains its maximum at $x = 1$ on the interval $[0,1]$ with a value of 1. Thus, $d(x,2x) = 1$.

Problem 3. Let $D = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 4,\ 0 < x_2 < 4\}$. Find the largest radius r such that the open ball $B_r(\boldsymbol{c})$ is contained in D, where

(1) $\boldsymbol{c} = (1, 2)$
(2) $\boldsymbol{c} = (2, 2)$
(3) $\boldsymbol{c} = (2, 7/2)$
(4) $\boldsymbol{c} = (c_1, c_2)$ be an arbitrary point in D.

Use (4) to prove that D is an open set in $\mathbb{R}^2$.

1) For the x_1-axis, it's $\min(1, 3) = 1$. For the x_2-axis, its $\min(2, 2) = 2$. So the answer is $r = \min(x_1, x_2) = 1$.

2) $r = \min(2, 2) = 2$.

3) Similar to in part a, we get that
$x_1 = \min(2, 2) = 2$
$x_2 = \min(\frac{7}{2}, \frac{1}{2}) = \frac{1}{2}$
$r = \min(x_1, x_2) = \frac{1}{2}$

4) $x_1 = \min(c_1, 4 - c_1)$.
$x_2 = \min(c_2, 4 - c_2)$
$r = \min(x_1, x_2) = \min(\min(c_1, 4 - c_1), \min(c_2, 4 - c_2))$.

To show that D is an open set, we need to show that for every point $c \in D$, $\exists$ a radius $r > 0$ such that $B_r(c) \subset D$. Since c is an arbitrary point in D and the above calculation for part 4) shows we can always find a radius r, then D is indeed an open set in $\mathbb{R}^2$.

Problem 4. Let X be a metric space. Prove that every finite subset of X is closed (in X).

Let F be a finite subset of X where $F = \{x_1, x_2, \ldots, x_n\}$ where n is a finite number and $x_1 \in X$ for $i = 1, 2, \ldots, n$.

We need to show that $X \setminus F$ is open.

For any point in $y \in X \setminus F$, since y is not in F, the distance $d(y, x_i)$ is positive/non-zero for $i = 1, 2, \ldots, n$.

Now, let $r = \min(d(y, x_1), d(y, x_2), \ldots, d(y, x_n))$. Since all these distances are positive, then r is also positive.

Consider $B_r(y)$. From our choice of r, none of the points in F are in this open ball. Thus, $B_r(y) \subset X \setminus F$.

Since y is an arbitrary point in $X \setminus F$, then $X \setminus F$ is open.

Therefore, X is closed.

Problem 5. Let
$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{Q}\}$$
Find

(a) The interior D° of D

(b) The closure $\overline{D}$ of D

(c) The boundary $\partial(D)$ od D

Justify your answers.

a) Because of the density of the rational numbers in the reals, any open set in $\mathbb{R}^2$ will contain irrational numbers for x_2.

Thus, there are no open balls in D that are entirely contained in D.

Therefore, $D^\circ = \emptyset$.

b) In $\mathbb{R}^2$, for any point (a, b) where b is irrational, we can find a sequence of points in D with x_2 as rational numbers that converges to (a, b) because the rationals are dense in the reals.

Since for any point in $\mathbb{R}^2$ we can find a sequence in D that converges to it, the closure of D is $\mathbb{R}^2$.

$\overline{D} = \mathbb{R}^2$.

c) $\partial(D) = \bar{(}D) \setminus D^\circ = \mathbb{R}^2 \setminus \emptyset = \mathbb{R}^2$.

Problem 6. Let D be a subset of X and c an element of $\overline{D} - D$. Prove that every neighborhood of c contains infinitely many points of D.

Let's consider a neighborhood N of c in X. Since c is a limit point of D, N intersects D. Goal: Show that this intersection contains infinitely many points of D.

Let's use proof by contradiction. Suppose that N only contains finitely many points of D. Let those points be $\{d_1, d_2, \dots, d_n\}$. Since c is not in D, c is different from each d_i. Therefore, $d(c, d_i) > 0$.

Let $\epsilon = \min(d(c, d_1), d(c, d_2), \dots, d(c, d_n))$. Now, consider the neighborhood $B_{\frac{\epsilon}{2}}(c)$. This neighborhood is a subset of N. However, by our choice of ϵ, $B_{\frac{\epsilon}{2}}$ doesn't contain any points of D. This contradicts the fact that c is a limit point of c (defined as every open ball of a limit point contains at least on point of D).

Thus, our assumption that N contains only finitely many points of D cannot be true. Therefore, we have shown that every neighborhood of c contains infinitely many points of D.