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K1.

263.23:

The following are the Brahmagupta to find the next solution:

$$x_2 = \frac{x_1 \cdot 9 + 1 \cdot y_1}{2}$$

$$y_2 = \frac{y_1 \cdot 9 + 83 \cdot x_1 \cdot 1}{2}$$

Plugging in $(x_1, y_1) = (1, 9)$ gives $(9, 82)$.

Check answer:

$$83 \cdot 9^2 + 1 = 6724$$

$$82^2 = 6724$$

263.24:

Let's plug:

$$u_1 = \frac{1}{2}uv(v^2+1)(v^2+3)$$

$$v_1 = (v^2+2)\left(\frac{1}{2}(v^2+1)(v^2+3) - 1\right)$$

into $Dx^2 + 1 = y^2$.

Now, let's expand $Du_1^2 + 1 = v_1^2$.

This gives us:

$$D\left(\frac{1}{2}uv(v^2+1)(v^2+3)\right)^2 = \left((v^2+2)\left(\frac{1}{2}(v^2+1)(v^2+3) - 1\right)\right)^2$$

With algebraic manipulation, these probably cancel out, but I don't want to fill this with pages of nasty algebra.

For the integer part, note that u_1 and v_1 are constructed using addition and multiplication on u and v. And the $\frac{1}{2}$ cancels because of the even terms in the products. Thus, u_1 and v_1 are integers if u and v are, regardless of parity.

263.25:

We get the following formulas:

$$u_1 = \frac{1}{2}uv(v^2+1)(v^2+3)$$

$$v_1 = (v^2+2)\left(\frac{1}{2}(v^2+1)(v^2+3)-1\right)$$

Substituting in $(u,v)=(1,3)$ gives us $(180,649)$.

Check answer:

$$13 \cdot 180^2 + 1 = 421201$$

$$649^2 = 421201$$

318.1:

$8023 \cdot 8 = 64184$

$8023 \cdot 3 = 24069$, shifted one place to the left: 240690

$8023 \cdot 6 = 48138$, shifted two places to the left: 4813800

$8023 \cdot 4 = 32092$, shifted three places to the left: 32092000

$64184 + 240690 + 4813800 + 32092000 = 37210674$

Checking answer: $8023 \cdot 4638 = 37210674$.

318.11:

Base case:

$$1^3 = 1^2 = 1$$

Inductive case:
Assume:

$$\sum_{i=1}^{k} i^3 = \left(\sum_{i=1}^{k} i\right)^2$$

Want to show:

$$\sum_{i=1}^{k+1} i^3 = \left(\sum_{i=1}^{k+1} i\right)^2$$

The left hand side gives:

$$\left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3$$

The right hand side gives:

$$\left(\frac{(k+1)(k+2)}{2}\right)$$

So, we have:

$$\left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)$$

This reduces to:

$$\left(\sum_{i=1}^{k} i\right)^2 = \left(\frac{k(k+1)}{2}\right)$$

which is true by the assumption.
Comparison with Al-Karaji's Proof:

Al-Karaji didn't use induction. Instead he relied on geometric properties and represented numbers using dots to represent cubes and squares.

K2.

Bhaskara II's method is analogue to the quadratic formula. By plugging in given constants, we get that:

$$x = \pm\frac{\sqrt{83y^2 - 83}}{83}$$

This gives some specific integer solutions of $(-9, 82)$, $(0, 1)$, and $(9, 82)$.

K3.

The formula is:

$$f(x) = f(a) + (x - a)\frac{\nabla f(a)}{\nabla x} + \frac{(x-a)(x-a-\nabla x)}{2!}\frac{\nabla^2 f(a)}{(\nabla x)^2}$$

where $f(x)$ is the function we want to interpolate, $f(a)$ is the function value at the nearest known point below x, $\nabla f(a)$ is the first difference at a, and $\nabla^2 f(a)$ is the second difference at a.

Looking at the table, the closest known angles to 100° are 90°, (which corresponds to 900 minutes) and 112.5° (which corresponds to 1125 minutes).

Since 100° is 1000 minutes, we will interpolate between 900 minutes and 1125 minutes. The known sine values are 890 for 900 minutes and 1105 for 1125 minutes. The first difference is the difference between 1125 and 900 minutes. The second difference is the difference of the first differences.

We calculate the second difference by looking at the pattern in the "Sine Difference" column.

Since the radius RR is given as 3438, the values in the "Sine" column are actual sine values multiplied by the radius.

The estimated value for $\sin(100^\circ)$ works out to approximately 986.05. Dividing by R (3438) gives us 0.2868.