This is due Saturday 11/4 by 11:59 pm on Gradescope. Please either neatly write up your solutions or type them up. You can find a .tex template on Canvas. Your proofs should be written in complete sentences and paragraphs, using a combination of words and symbols. They should be **correct, clear, and concise**. You will be graded on all three, especially the first two!

1. (4) Suppose that $\{b_k\}$ is a sequence of bounded numbers. Show that the sequences of functions $f_n : (-1,1) \to \mathbb{R}$ defined by $f_n(x) = \sum_{k=0}^{n} b_k x^k$ converge (not necessarily uniformly) to a continuous function f on $(-1,1)$. **Hint:** Show that for any $r \in (0,1)$ that the sequence $\{f_n\}$ is uniformly Cauchy on $[-r,r]$. Be sure to explain why showing this suffices to solve the problem. Feel free to use the fact, which follows from the geometric series, that $\sum_{k=0}^{n} |x|^k \leq \frac{1}{1-|x|}$ for $x \in (-1,1)$. **Further Hint:** In class we did this when all the b_k where 1, so you might look at your notes for this.

Solution:

To show that $\{f_n\}$ converges, we have to prove that it is uniformly Cauchy on any closed subinterval [-r, r] for $0 < r < 1$. A sequence of functions is uniformly Cauchy if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$ and all $x \in [-r,r]$, we have:

$$|f_n(x) - f_m(x)| < \epsilon$$

Let's take $m > n$, then:

$$|f_m(x) - f_n(x)| = |\Sigma_{k=n+1}^{m} b_k x^k|$$

From the triangle inequality, this is:

$$\leq \Sigma_{k=n+1}^{m} |b_k||x|^k$$

Since $\{b_k\}$ is bounded, $\exists$ a bound B such that $|b_k| \leq B \ \forall k$. Then we have:

$$\leq B\Sigma_{k=n+1}^{m} |x|^k$$

For $x \in [-r,r]$, $|x|^k \leq r^k$, thus:

$$\leq B\Sigma_{k=n+1}^{m} r^k$$

We can bound the series further by using the geometric series sum formula for $r < 1$:

$$\Sigma_{k=n+1}^{m} r^k \leq \Sigma_{k=n+1}^{\infty} r^k = \frac{r^{n+1}}{1-r}$$

Hence:

$$|f_n(x) - f_m(x)| \leq B\frac{r^{n+1}}{1-r}$$

As $n \to \infty$, the term $r^{n_1} \to 0$ for $r < 1$. This means the right side is arbitrarily small. Thus, $\{f_n\}$ is uniformly Cauchy.

Since every point in $(-1,1)$ is contained in some interval $[-r,r]$, where $0 < r < -1$, we can extend the continuity of f to all of $(-1,1)$ from the continuity of f on every subinterval. This extension is well-defined and continuous because for any two such intervals that overlap, the limit function f will be continuous on their union, which is also a closed interval.

Therefore, by showing that $\{f_n\}$ is uniformly Cauchy on every closed subinterval of $(-1,1)$, we have established that the sequence of functions converges uniformly on these intervals to a continuous function f on $(-1,1)$. This uniform convergence on every compact subinterval implies pointwise convergence on $(-1,1)$ to the same continuous function f.

2. (4) In the previous question, show that if the sequece $\{b_k\}$ is constantly one then the sequence $\{f_n\}$ (defined as in the previous problem) converges pointwise to $f(x) = \frac{1}{1-x}$, but the convergence is *not* uniform.

Solution:
If $\{b_k\}$ is constantly one, then each $f_n(x)$ is the partial sum of the geometric series. In this case, it's given by:

$$f_n(x) = \Sigma_{k=0}^{n} x^k$$

Because of the properties of geometric series, the sequence $\{f_n\}$ converges pointwise to the function:

$$f(x) = \frac{1}{1-x}$$

Now for uniform convergence. For the sequence $\{f_n\}$ to converge uniformly to $f(x)$, we must have that for every $\epsilon > 0$, $\exists$ an N such that $\forall n \geq N$ and $\forall x \in (-1, 1)$, we have:

$$|f_n(x) - \frac{1}{1-x}| < \epsilon$$

To see why the convergence is not uniform, consider what happens as x gets very close to 1. The difference between $f_n(x)$ and $f(x)$ is:

$$|\Sigma_{k=0}^{n} x^k - \frac{1}{1-x}| = |\frac{1-x^{n+1}}{1-x} - \frac{1}{1-x}| = \frac{-x^{n+1}}{1-x}$$

This simplifies to:

$$|x^{n+1}|$$

As x approaches 1, for any fixed n, x^{n+1} approaches 1. Therefore, regardless of how big n is, if x is close enough to 1, the difference $|x^{n+1}|$ will be larger than any fixed $\epsilon > 0$. Hence, it cannot be uniformly convergent.

3. (4) Show that if $\{f_n\}$ is a sequence of uniformly continuous functions $f_n : D \to \mathbb{R}$ that converge uniformly to $f : D \to \mathbb{R}$, then f is also uniformly continuous. **Note:** We proved this in class for *continous functions*, and said that the same proof worked for *uniformly continuous* functions. So, you should appropriately adapt the proof for continuous functions that we gave in class.

Solution:
Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly to f, $\exists$ an $N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in D$, we have:

$$|f_(x) - f(x)| < \frac{\epsilon}{3}$$

Now, because f_N is uniformly continuous, $\exists$ a $\delta > 0$ such that $\forall x, y \in D$ with $|x - y| < \delta$, we get:

$$f_N(x) - f_N(y)| < \frac{\epsilon}{3}$$

Now let's consider any two points $x, y \in D$ such that $|x - y| < \delta$. From the triangle inequality and the above properties, we have:

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Therefore f is uniformly continuous on D.

4. (4) Suppose that $f : (a, b) \to \mathbb{R}$ and that either $u = a^+, u = b^-$, or $u \in (a, b)$. Show that if $\lim_{x\to u} f(x) > 0$, then there is some $\delta > 0$ such that for all $x \in (a, b)$ with $|x - u| < \delta$ we have that $f(x) > 0$.

Solution:

Recall the definition of the limit: For $\lim_{x\to u} f(x) > 0$, it means that for any $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $\forall x$ within the interval (a, b) and $0 < |x - u| < \delta$, it holds that $f(x)$ is within ϵ distance from some positive limit L, where $L > 0$.

Let $\epsilon = \frac{L}{2} > 0$. From the definition of the limit, $\exists$ a corresponding $\delta > 0$ such that if $0 < |x-u| < \delta$ and $x \in (a, b)$, then $|f(x) - L| < \epsilon$.

Since $L > 0$, this means the values of $f(x)$ are contained in the interval $(L - \epsilon, L + \epsilon) = (\frac{L}{2}, \frac{3L}{2})$. All the values of f(x) are positive because $\frac{L}{2}$ is positive (since $L > 0$).

Hence, we have that $\forall x \in (a, b)$ such that $0 < |x - u| < \delta$, $f(x) > \frac{L}{2} > 0$. This means that $f(x)$ is positive in the neighborhood of u, excluding u itself (if $u \notin (a, b)$).

That means there are 3 cases for u: 1. $u = a^+$, then the δ-neighborhood is $(a, a+\delta)$. 2. $u = b^-$, then the δ-neighborhood is $(b - \delta, b)$. 3. $u \in (a, b)$, then the δ-neighborhood is $(u - \delta, u + \delta)$ intersected with (a, b).

In each case, within the appropriate δ-neighborhood, $f(x) > 0$.

5. (4) Proceeding directly from the definition of a limit, show that $\lim_{x\to\infty} \frac{2x^2 + 1}{3x^2 - 1} = \frac{2}{3}$.

Solution:

To show that $\lim_{x\to\infty} \frac{2x^2 + 1}{3x^2 - 1} = \frac{2}{3}$, we need to show that for every $\epsilon > 0$, $\exists$ a number $M > 0$ such that $\forall x > M$, the absolute difference between $\frac{2x^2 + 1}{3x^2 - 1}$ and $\frac{2}{3}$ is less than ϵ.

Let's compute the difference and try to bound it by ϵ:

$$|\frac{2x^2 + 1}{3x^2 - 1} - \frac{2}{3}| = \frac{5}{|9x^2 - 3|}$$

As $x \to \infty$, the deonominator will approach ∞, so the whole expression approaches 0. But we still need to show this happens in a controlled way with respect to ϵ.

We want:

$$\frac{5}{|9x^2 - 3|} < \epsilon$$

To find the appropriate M, we solve for x in terms of ϵ:

$$5 < \epsilon|9x^2 - 3|$$

$$\frac{5}{\epsilon} < |9x^2 - 3|$$

For $x^2 > \frac{1}{3}$, the value will be positive, so we can ignore the absolute value sign.

$$\frac{5}{9x^2 - 3} < \epsilon \implies 9x^2 - 3 > \frac{5}{\epsilon}$$

$$9x^2 > \frac{5}{\epsilon} + 3$$

$$x^2 > \frac{5}{9\epsilon} + \frac{1}{3}$$

Therefore,

$$x > \sqrt{\frac{5}{9\epsilon} + \frac{1}{3}}$$

If we choose M to be:

$$M > \max\{\frac{1}{\sqrt{3}}, \sqrt{\frac{5}{9\epsilon} + \frac{1}{3}}\}$$

we ensure that $9x^2 - 3$ is both positive and the fraction is less than ϵ $\forall x > M$. Thus, with the above value of M, we get:

$$|\frac{2x^2 + 1}{3x^2 - 1} - \frac{2}{3}| < \epsilon, \forall x > M.$$

Which proves that $\lim_{x \to \infty} \frac{2x^2 + 1}{3x^2 - 1} = \frac{2}{3}$.