Math 3010 Midterm 2 "Cheat Sheet"
Lincoln Sand
- Historical Data:
* al-Khwarizimi; 830 Baghdad; Decimal arith., linear and quadratic alg. eqns.
* Bhaskara II; 1150 Ujjian; Solved Pell's equation using the cyclic process.
* Brahmagupta; 650 Bhinmal; Quadratic eqns., composition formula for Pell's eqn.
* Cardano; 1545 Bologna; First published solution of cubic eqns.
* Descartes; 1637 Holland; Used coords to relate curves to solutions of eqns.
* Desargues; 1639 Paris; Projective geometry
* Leonardo; 1202 Pisa; Introduced Hindu-Arabic notation and algebra to Europe.
* Zhang Cang; 150 BC Chang'an; 246 problems in proportions, 3×3 systems.
* Ptolemy; 145 Alexandria; Astronomy and trig of geocentric universe.
* Qin Juishao; 1247 Hangzhou; Chinese Remainder Theorem, polynomial eqns.
* Copernicus; 1543 Frauenberg; Planet orbits around sun generated by epicycles.
* Kepler; 1609 Prague; Planets orbit sun on elliptical paths of different speeds.
* Lui Hui; 263 Shansi Province; Standard text on systems of eqns and measuring.
* Napier; 1614 Merchison; Tables of logarithms and instructions for their use.
* Viete; 1591 Paris; Solved cubic eqns. using trig identities.
* Archimedes (287 - 212 BC Syracuse);; Apollonius (250 - 175 BC Alexandria);; Heron (25 - 105 AD Alexandria);; Hipparchus (190 - 120 BC Bithynia);; Ptolemy (100 - 178 AD Alexandria)

(b) *Use the algorithm from Nine Chapters or Āryabhaṭīyah to find the square root of* 17,424.

Since the number is less than 1000^2, we look for roots of the form $x = 100a + 10b + c$.

$17,424$	$200^2 = 40,000$ is too big so $a = 1$.
$\underline{-10,000}$	Subtract $(100a)^2$.
$7,424$	Is greater than $2000b$ for $b = 3$ but not $b = 4$.
	Using $(100a + 10b)^2 = 10,000 + 2000b + 100b^2$
$\underline{-6,000}$	we subtract $2000b$
$1,424$	
$\underline{-900}$	and $100b^2$,
524	which is now $17,424 - (100a + 10b)^2$. It is greater than $260c$ for $c = 2$
	but not $c = 3$. Using $(130 + c)^2 = 130^2 + 260c + c^2$
$\underline{-520}$	we subtract $260c$
4	
$\underline{-4}$	and c^2,
0	Thus the square root is $x = 132$.

Brahmagupta's identity:
For given n, the product of two numbers of the form $a^2 + nb^2$ is itself a number of that form. Specifically,

$$(a^2 + nb^2) \cdot (c^2 + nd^2) = (ac - nbd)^2 + n(ad + bc)^2 = (ac + nbd)^2 + n(ad - bc)^2$$

Fermat's Little Theorem: $a^{p-1} \equiv 1(\text{mod p})$.
Chinese Remainder Theorem:
Given $x = a_i \,(\text{mod } m_i)$, compute $M = m_1 \cdot m_2 \ldots m_n$, and for each i, compute $M_i = \frac{M}{m_i}$.
Compute the multiplicative inverses such that $M_i \cdot M_i^{-1} = 1 \,(\text{mod } m_i)$.
The solution is given by $x = \sum_{i=1}^{n} a_i \cdot M_i \cdot M_i^{-1} (\text{mod M})$.
Heron's triangle area: $A^2 = s(s-a)(s-b)(s-c)$.
Hipparchus supplementary angle and half-angle formulas for chords:

$$\text{crd}^2(180° - \beta) = 4R^2 - \text{crd}^2(\beta)$$

$$\text{crd}^2(\beta/2) = R(2R - \text{crd}(180° - \beta))$$

Theorem: Let A, B, C, and D be four points in order on the circle. Then the lengths of sides and diagonals of the quadrilateral $ABCD$ satisfy the equation: $AB \cdot CD + BC \cdot DA = AC \cdot BD$

Archimedes used a marked straightedge to trisect an angle.

Ptolemy's difference formula for chords:

$2R \cdot \text{crd}(\beta - \mu) = \text{crd}(\beta) \cdot \text{crd}(180° - \mu) - \text{crd}(\mu) \cdot \text{crd}(180° - \beta)$

Liu Hui's method to approximate π:

1. Start with hexagon, $P_6 = 6 \cdot 1 = 6$.
2. Doubling number of sides to get dodecagon.
3. Calculate the side length of the dodecagon: For a hexagon inscribed in a unit circle, each triangle formed by two adjacent vertices and the center of the circle is an equilateral triangle. If we add a perpendicular line from the center to the midpoint of one side, we divide the equilateral triangle into two 30-60-90 right triangles.

$s_{12} = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

The side length of the dodecagon is half the side length of the hexagon due to the properties of a 30-60-90 triangle.

$P12 = 12 \cdot s_{12} = 6\sqrt{3}$

This is an approximation for the unit circle circumference, which is 2π, thus, $\pi \approx 3\sqrt{3}$.

Sum of geometric series: $S = \frac{a(1-r^n)}{1-r}$. If $|r| < 1$, then it converges to: $S = \frac{a}{1-r}$.

- Euclidean GCD algorithm/diophantine example:

Find $x, y \in \mathbb{Z}$ such that $gcd(198, 168) = 198x + 168y$.

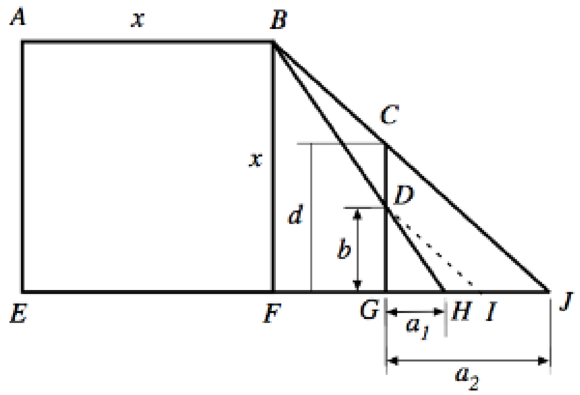
$$198 = 1 \cdot 168 + 30; 168 = 5 \cdot 30 + 18; 30 = 1 \cdot 18 + 12;$$
$$18 = 1 \cdot 12 + 6; 12 = 2 \cdot 6 + 0$$

So, $gcd(198, 168) = 6$.

$$\begin{aligned} 6 &= 18 - 12 = 18 - (30 - 18) = 2 \cdot 18 - 30 = 2 \cdot (168 - 5 \cdot 30) - 30 \\ &= 2 \cdot 168 - 11 \cdot 30 = 2 \cdot 168 - 11 \cdot (198 - 168) = 13 \cdot 168 - 11 \cdot 198 \end{aligned}$$

So, $x = -11, y = 13$.

8. *This is a problem from Liu Hui's 264 book* Sea Island Mathematical Manual. *There is a square, walled city of unknown dimensions. A man erects two poles d feet apart in the north-south direction east of the city and joins them with with a string at eye-level. The southern pole is in a straight line with the southwestern and southeastern corners of the city. By moving eastward a_1 feet from the southern pole, the man's observation with the northeast corner of the city intersects the string at a point b feet from the southern end. He goes again a_2 feet from the pole until the northeastern corner is in line with the northern pole. What is the length of the side of the square city? [Burton, The History of Mathematics 7th ed., 2007, p. 265]*



In the diagram, the poles are located at C and G. Let I be the point on EJ such that DI is parallel to BJ. By the similarity of $\triangle(CGJ)$ and $\triangle(DGI)$ it follows that

$$\frac{GH + HI}{DG} = \frac{GJ}{CG}$$

The the similarity of the triangles $\triangle(BHJ)$ and $\triangle(DHI)$ and of $\triangle(BFH)$ and $\triangle(DGH)$,

$$\frac{HJ}{HI} = \frac{BH}{DH} = \frac{BF}{DG}.$$

Thus

$$x = BF = \frac{DG \cdot HJ}{HI} = \frac{DG \cdot (GJ - GH)}{\frac{GJ \cdot DG}{CG} - GH} = \frac{b \cdot (a_2 - a_1)}{\frac{a_2 \cdot b}{d} - a_1}.$$