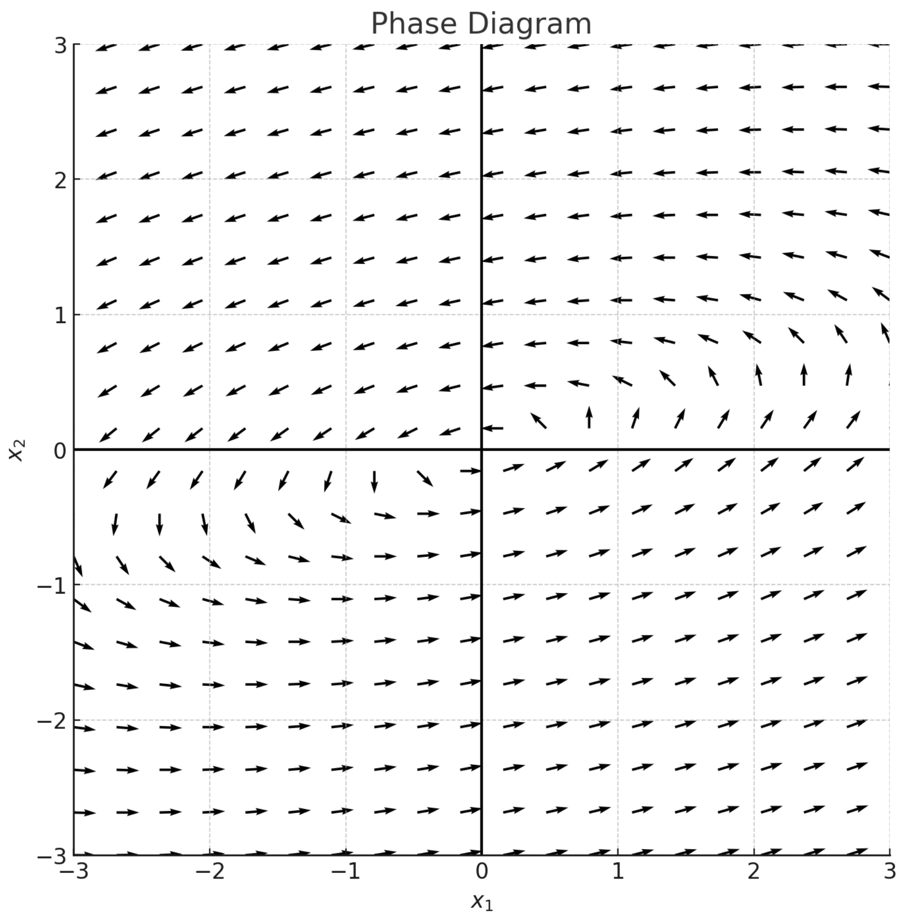
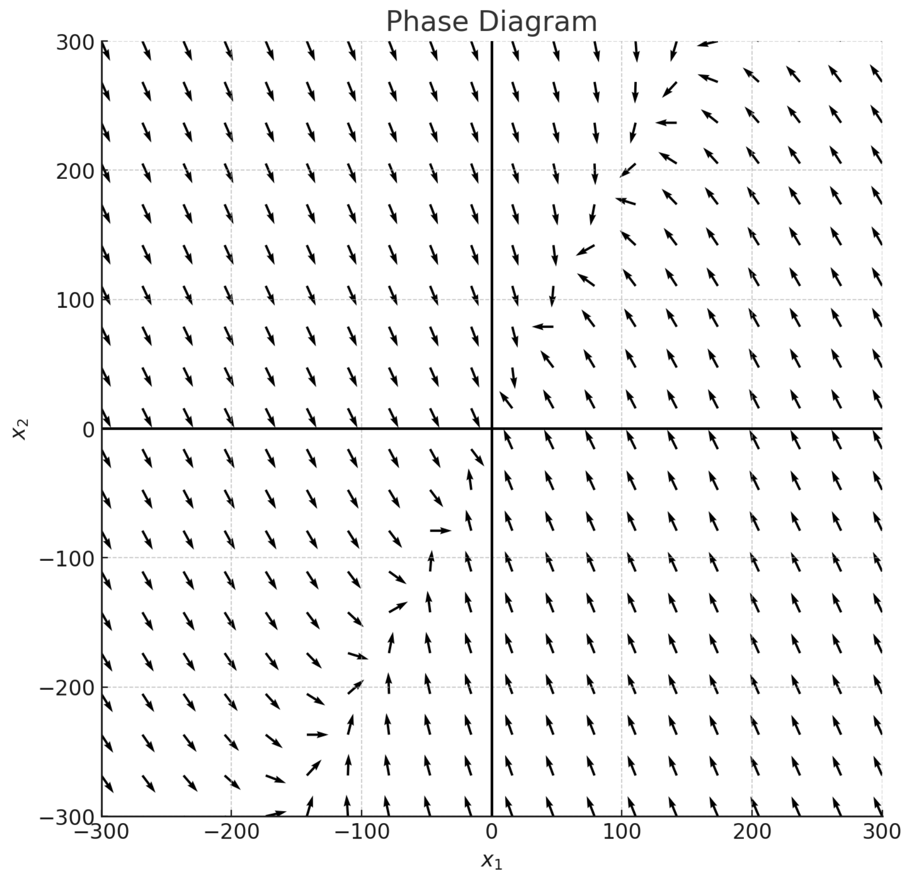
Section 5.3:

8) The eigenvalues and eigenvectors of the coefficient matrix are complex. Since it has complex eigenvalues, it is either a center or spiral. Since the real part is essentially zero, that means the phase diagram is a center.



16) The eigenvalues are -10 and -100. The corresponding eigenvectors are both real and distinct. Therefore, it forms an improper nodal sink.



18) The system has two distinct positive real eigenvalues. This is because it has two lines and is a source.

20) The system has two complex eigenvalues with positive real parts. This is because it's spiral (complex eigenvalues) and a source (positive real part).

21) The system has one repeated positive real eigenvalue with two linearly independent eigenvectors. This is because it's a source (positive eigenvalue) and a proper nodal (repeated non-zero eigenvalue of the same sign).

33) a) Let v be any nonzero vector. We can express v as a linear combination of u_1 and u_2 since they are linearly independent and span the space:

$$v = c_1 v_1 + c_2 v_2$$

where c_1 and c_2 are scalars.

Since u_1 and u_2 are eigenvectors are associated with λ, we have $Au_1 = \lambda u_1$ and $Au_2 = \lambda u_2$.

We then get:

$$Av = A(c_1 u_1 + c_2 u_2) = c_1 A u_1 + c_2 A u_2 = c_1 \lambda u_1 + c_2 \lambda u_2$$

We can rewrite this as:

$$Av = \lambda(c_1u_1 + c_2u_2) = \lambda v$$

This shows that $Av = \lambda v$ for any nonzero vector v. Thus, every nonzero vector v is an eigenvector of A associated with λ.

b) Consider $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These are basis vectors of $\mathbb{R}^2$.

From part a), we know that $Av = \lambda v$.

For v_1, $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This implies that the first column of A is $\begin{bmatrix} \lambda \\ 0 \end{bmatrix}$.

For v_2, $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This implies that the second column of A is $\begin{bmatrix} 0 \\ \lambda \end{bmatrix}$.

Thus, A must be:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

which is exactly the same as equation 22 (a scalar multiple of the identity matrix).

Section 5.5:

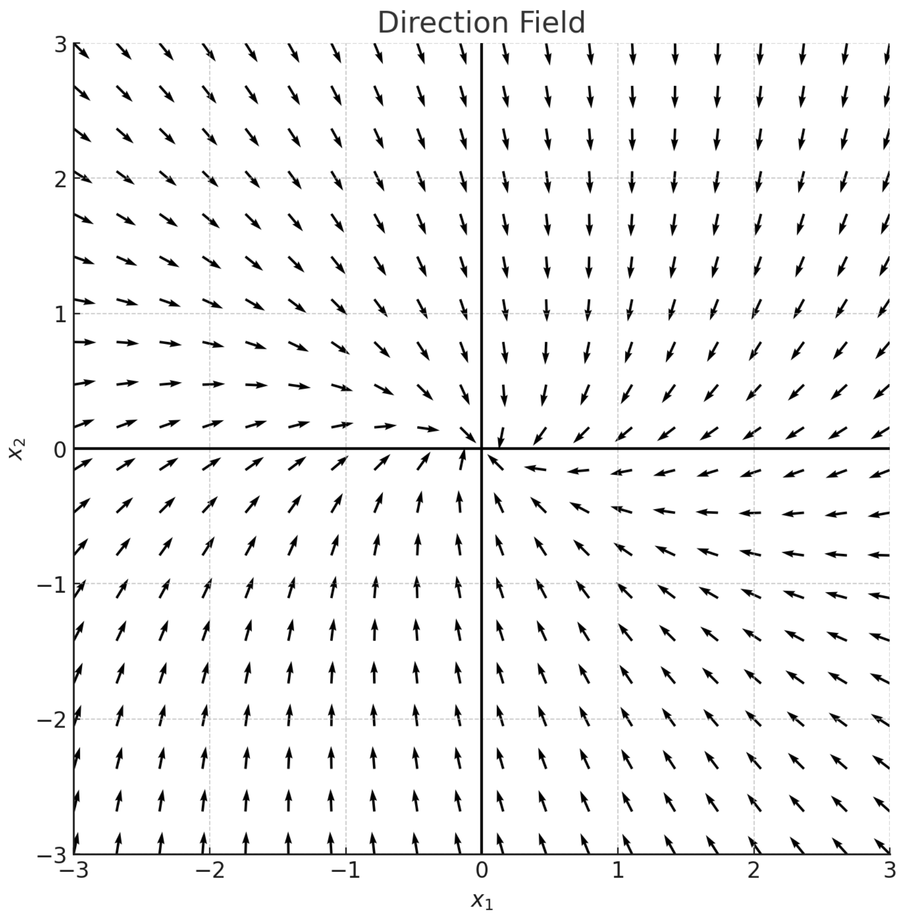
1) Both eigenvalues of the system are -3. The corresponding eigenvectors are: $v_1 = \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$. These eigenvectors are linearly independent.

For a system of differential equations with a repeated eigenvalue and linearly independent eigenvectors, the general solution can be expressed as:

$$x(t) = c_1e^{\lambda t}v_1 + c_2e^{\lambda t}v_2$$

Plugging in the values gives:

$$x(t) = c_1e^{-3t} \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix} + c_2e^{-3t} \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$



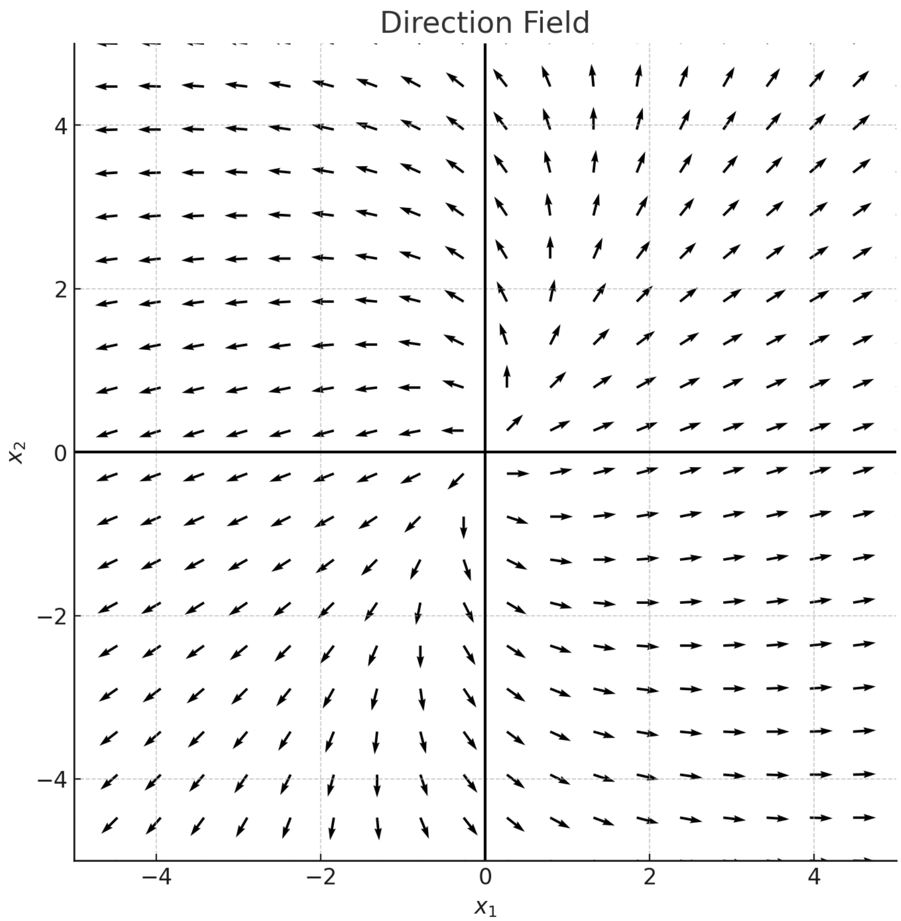
2) The eigenvalue of the system is 2 (repeated) with only one linearly independent eigenvector $v = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$.

In this case, the general solution is given by:

$$x(t) = e^{\lambda t}(c_1 v + c_2(tv))$$

Plugging in our values gives:

$$x(t) = e^{2t}(c_1 \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix} + c_2(t \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}))$$



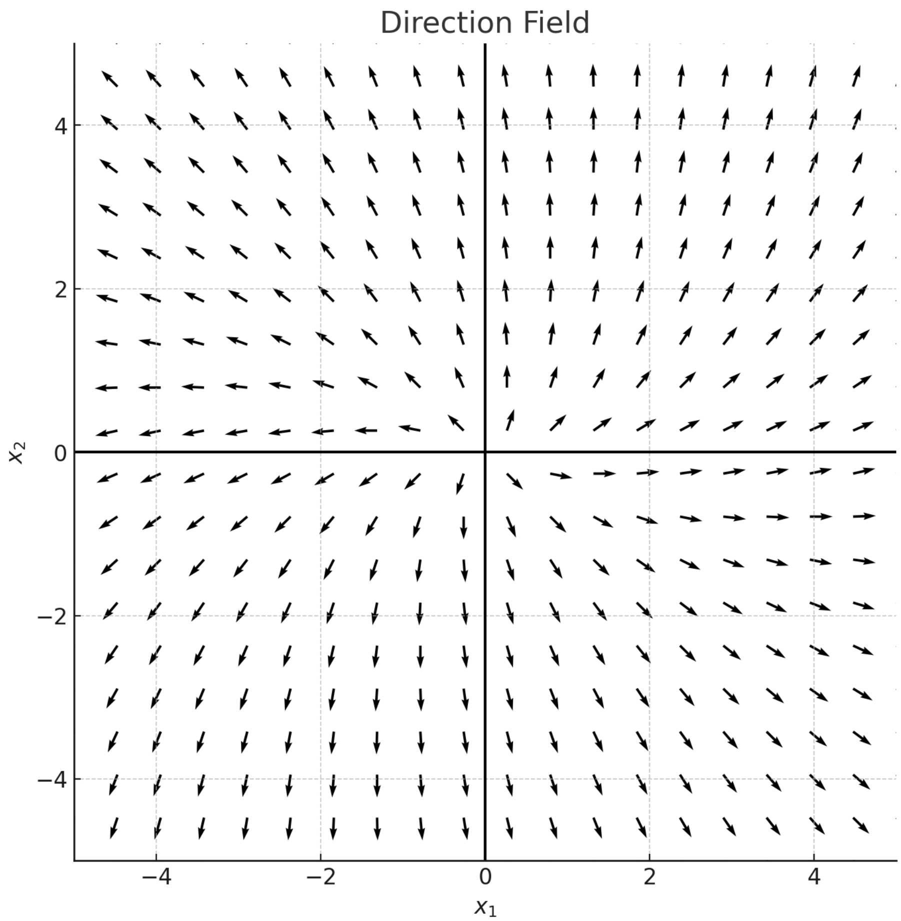
4) The eigenvalues of the system are both 4. There is only one linearly independent eigenvector $v = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$.

In this case, the general solution is given by:

$$x(t) = e^{\lambda t}(c_1 v + c_2(tv))$$

Plugging in our values gives:

$$x(t) = e^{4t}(c_1 \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix} + c_2(t \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}))$$



7) The eigenvalues are 9, 2, andd 2. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0.7071 \\ 0.7071 \\ 0 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 0 \\ -0.7071 \\ 0.7071 \end{bmatrix}$.

The general solution is given by:

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3$$

Plugging in our values we get:

$$x(t) = c_1 e^{9t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0.7071 \\ 0.7071 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ -0.7071 \\ 0.7071 \end{bmatrix}$$

22) The eigenvalues of the system are all 1 (four eigenvalues with value ≈ 1).

Note: I used a computer algebra system to calculate the eigenvalues and eigenvectors for this system (because I don't want to manually calculate them for a 4×4 matrix).

The eigenvectors are:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

I'm not exactly sure how to get the generalized eigenvectors for the general solution.

28) Here's some code to calculate the general solution using sympy:

```
import numpy as np
from scipy.linalg import eig
import sympy as sp

# Define the matrix A
A = np.array([[-15, -7, 4], [34, 16, -11], [17, 7, 5]])

# Compute the eigenvectors and eigenvalues
eigenvalues, eigenvectors = eig(A)

# Identify the repeated eigenvalue
repeated_eigenvalue = 2

# Construct the matrix of generalized eigenvectors
# Since there's only one eigenvector,
#  compute two generalized eigenvectors
P = np.column_stack((eigenvectors[:, 0],
        np.linalg.matrix_power(A -
        repeated_eigenvalue*np.eye(3), 1) @ eigenvectors[:, 0],
        np.linalg.matrix_power(A -
        repeated_eigenvalue*np.eye(3), 2) @ eigenvectors[:, 0]))

# Define the symbolic variables for time and constants
t, c1, c2, c3 = sp.symbols('t c1 c2 c3')

# Define the Jordan matrix
J = sp.Matrix(np.diag([repeated_eigenvalue]*3))
J[0, 1] = J[1, 2] = 1  # Filling the superdiagonal with 1's

# General solution
# x(t) = P * exp(J*t) * c, where c is the vector of constants
c = sp.Matrix([c1, c2, c3])
x_t = sp.Matrix(P) * sp.exp(J * t) * c

print(x_t)
```

34) We need to show that $(A - \lambda I_n)v_2 = v_1$ and $(A - \lambda I_n)v_1 = 0$.

If we actually do the calculations (omited for brevity since it's just some matrix arithmetic), then we will get what we need to show.

Therefore, v_1 and v_2 do form a length 2 chain associated with the eigenvalue $\lambda = 2 + 3i$.

The four independent real-valued solutions are:

1.
$$\begin{bmatrix} e^{2t}\sin(3t) \\ (-3\sin(3t) + 3\cos(3t))e^{2t} \\ 0 \\ -e^{2t}\cos(3t) \end{bmatrix}$$

2.
$$\begin{bmatrix} -e^{2t}\sin(3t) \\ (3\sin(3t) + 3\cos(3t))e^{2t} \\ 0 \\ -e^{2t}\cos(3t) \end{bmatrix}$$

3.
$$\begin{bmatrix} 3e^{2t}\cos(3t) \\ (-9\sin(3t) - 10\cos(3t))e^{2t} \\ e^{2t}\sin(3t) \\ 0 \end{bmatrix}$$

4.
$$\begin{bmatrix} 3e^{2t}\cos(3t) \\ (9\sin(3t) - 10\cos(3t))e^{2t} \\ -e^{2t}\sin(3t) \\ 0 \end{bmatrix}$$

Textbook Section 5.6:

1) The fundamental matrix is:

$$\phi(t) = \begin{bmatrix} 0.5e^t + 0.5e^{3t} & -0.5e^t + 0.5e^{3t} \\ -0.5e^t + 0.5e^{3t} & 0.5e^t + 0.5e^{3t} \end{bmatrix}$$

Applying equation 8 for $x(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ yields the solution:

$$x(t) = \begin{bmatrix} 2.5e^t + 0.5e^{3t} \\ -2.5e^t + 0.5^{3t} \end{bmatrix}$$

22) Let's first try to prove that its nilpotent by computing successive powers of A.

If we compute A^2, we end up with $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is the null matrix, therefore A is nilpotent and powers ≥ 2 give the zero 2×2 matrix.

That means that e^{At} is given by:

$$e^{At} = I_n + At$$

Plugging in A, we get that:

$$e^{At} = \begin{bmatrix} 6t+1 & 4t \\ -9t & 1-6t \end{bmatrix}$$

28) First, notice that $A = D + N$ where $D = 5I_n$ and $N = \begin{bmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{bmatrix}$.

Note that $e^{At} = e^{(D+N)t} = e^{Dt}e^{Nt}$. Finally, $x(t) = e^{At}x(0)$.

$$e^{At} = \begin{bmatrix} e^{5t} & 0 & 0 \\ 10te^{5t} & e^{5t} & 0 \\ (150t^2 + 20t)e^{5t} & 30te^{5t} & e^{5t} \end{bmatrix}$$

That means that $x(t) = \begin{bmatrix} 40e^{5t} \\ 400te^{5t} + 50e^{5t} \\ 1500te^{5t} + 40(150t^2 + 20t)e^{5t} + 60e^{5t} \end{bmatrix}$