Textbook Section 1.3:

12) $\frac{dy}{dx} = x \cdot ln(y)$; $y(1) = 1$

$x \cdot ln(y)$ is continuous on the open x interval $(0, \infty)$, which contains $x = 1$. $\therefore$ a solution does exist. Now for uniqueness:

$$\frac{\partial f}{\partial y} = x/y$$

$\frac{\partial f}{\partial y}$ is continuous on the open x interval $(0, \infty)$, which also contains $x = 1$ and $y = 1$. $\therefore$ the solution is unique.

$\exists$ a unique solution for the differential equation with the initial starting value of $y(1) = 1$.

13) $\frac{dy}{dx} = \sqrt[3]{y}$; $y(0) = 1$

$\sqrt[3]{y}$ is continuous on the open x interval $(-\infty, \infty)$, which contains $x = 0$. $\therefore$ a solution does exist. Now for uniqueness:

$$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}}$$

$$= \frac{1}{3 \cdot y^{\frac{2}{3}}}$$

. $\frac{1}{3 \cdot y^{\frac{2}{3}}}$ is not continuous at $x = 0$, $\therefore$ the solution is not guaranteed to be unique.

17) $y\frac{dy}{dx} = x - 1$; $y(0) = 1$

$$y\frac{dy}{dx} = x - 1 \implies \frac{dy}{dx} = \frac{x-1}{y}$$

$\frac{x-1}{y}$ in continuous on the open x interval $(-\infty, \infty)$, which contains $x = 0$. $\therefore$ a solution does exist. Now for uniqueness:

$$\frac{\partial f}{\partial y} = \frac{1-x}{y^2}$$

This is continuous on the open x interval $(-\infty, \infty)$, which contains $x = 0$. $\therefore$ the solution is unique.

$\exists$ a unique solution for the differential equation with the initial starting value of $y(0) = 1$.

Textbook Section 1.4:

34) (Population growth) In a certain culture of bacteria, the number of bacteria increased sixfold in 10h. How long did it take the population to double?

Let r be the growth constant in hours.

$$r^{10} = 6 \implies ln(r^{10}) = ln(6) \implies 10 \cdot ln(r) = ln(6) \implies ln(r) = \frac{ln(6)}{10}$$

$$\implies r = e^{\frac{ln(6)}{10}} \implies r \approx 1.196$$

Now, let's find the time when it doubles.

$$r^t = 2 \implies ln(r^t) = ln(2) \implies t \cdot ln(r) = ln(2) \implies t = \frac{ln(2)}{ln(r)}$$

$$\implies t \approx 3.869$$

The bacteria colony population doubles after approximately 3.869 hours.
Textbook Section 1.5:
8) $3xy' + y = 12x$
We have to use the integrating factor method.

$$3xy' + y = 12x \implies y' + \frac{1}{3x}y = 4 \implies P(x) = \frac{1}{3x}; Q(x) = 4$$

$$M(x) = \int P(x)dx \implies M(x) = \int \frac{1}{3x}dx \implies M(x) = \frac{1}{3} \cdot ln(|3x|)$$

$$\implies M(x) = ln(|\sqrt[3]{3x}|)$$

$$y(x) = \frac{4}{|\sqrt[3]{3x}|} \cdot \int |\sqrt[3]{3x}|dx$$

For $x \geq 0$:

$$y(x) = \frac{4}{\sqrt[3]{3x}} \cdot \int \sqrt[3]{3x}dx$$

For $x < 0$:

$$y(x) = \frac{4}{-\sqrt[3]{3x}} \cdot \int -\sqrt[3]{3x}dx \implies y(x) = \frac{4}{\sqrt[3]{3x}} \cdot \int \sqrt[3]{3x}dx$$

So, $y(x) = \frac{4}{\sqrt[3]{3x}} \cdot \int \sqrt[3]{3x}dx$

$$\int \sqrt[3]{3x}dx = \frac{3}{12} \cdot (3x)^{\frac{4}{3}} + C$$

$$y(x) = \frac{4}{\sqrt[3]{3x}} \cdot (\frac{3}{12} \cdot (3x)^{\frac{4}{3}} + C) \implies y(x) = 3x + \frac{4 \cdot C}{\sqrt[3]{3x}}$$

$$\therefore y(x) = 3x + \frac{4 \cdot C}{\sqrt[3]{3x}}.$$

14) $xy' - 3y = x^3; y(1) = 10$

$$xy' - 3y = x^3 \implies y' - \frac{3}{x}y = x^2 \implies P(x) = -\frac{3}{x}; Q(x) = x^2$$

$$M(x) = \int P(x)dx \implies M(x) = -3 \cdot \int \frac{1}{x}dx \implies M(x) = ln(|x^{-3}|)$$

$$y(x) = |x^3| \cdot \int |\frac{1}{x^3}| \cdot x^2 dx$$

The sign will cancel from the absolute value like in x, so we get:

$$y(x) = x^3 \cdot \int \frac{1}{x^3} \cdot x^2 dx \implies y(x) = x^3 \cdot \int \frac{1}{x}dx \implies y(x) = x^3 \cdot (ln|x| + C)$$

$$y(1) = 1^3 \cdot (ln|x| + C) \implies y(1) = C = 10 \implies C = 10$$

$$\therefore y(x) = x^3 \cdot (ln|x| + 10)$$

29) Express the general solution of $\frac{dy}{dx} = 1 + 2xy$ in terms of the error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \cdot \int_0^x e^{-t^2} dt.$$

$$\frac{dy}{dx} = 1 + 2xy \implies y' - 2xy = 1 \implies P(x) = -2x; Q(x) = 1$$

$$M(x) = \int -2x dx \implies M(x) = -x^2$$

$$y(x) = e^{x^2} \cdot \int \frac{1}{e^{x^2}} dx$$

$$\int \frac{1}{e^{x^2}} dx = \int e^{-x^2} dx = \frac{\sqrt{\pi}}{2} erf(x) + C$$

$$e^{x^2} = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\frac{d}{dx} erf(x)}$$

$$y(x) = e^{x^2} \cdot (\frac{\sqrt{\pi}}{2} erf(x) + C) \implies y(x) = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\frac{d}{dx} erf(x)} \cdot (\frac{\sqrt{\pi}}{2} erf(x) + C).$$

34) Consider a resevoir with a volume of 8 billion cubic feet (ft^3) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million ft^3 of water with a pollutant concentration of 0.05% and an equal daily output of the well-mixed water in the resevoir. How long will it take to reduce the pollutant concentration in the resevoir to 0.10%?

Let $V_0 = 8$

Let $Poll_0 = 0.25\% \cdot V_0 = 0.02$
Let $\frac{dPoll}{dt} = 0.5 \cdot 0.05\% - 0.5 \cdot \frac{Poll}{8} = 0.00025 - \frac{1}{16} \cdot Poll$

$$\frac{dPoll}{dt} = 0.00025 - \frac{1}{16} \cdot Poll \implies \frac{dPoll}{dt} + \frac{1}{16} \cdot Poll = 0.00025$$

$$\implies P(x) = \frac{1}{16} = 0.0625; Q(x) = 0.00025$$

$$M(x) = \int P(x)dx \implies M(x) = \int 0.0625dx \implies M(x) = 0.0625x$$

$$Poll(x) = \frac{1}{e^{0.0625x}} \cdot \int e^{0.0625x} \cdot 0.00025dx$$

$$\implies Poll(x) = \frac{0.00025}{0.0625 \cdot e^{0.0625x}} \cdot (e^{0.0625x} + C)$$

$$\implies Poll(x) = \frac{0.00025}{0.0625} + \frac{0.00025 \cdot C}{0.0625 \cdot e^{0.0625x}}$$

$$Poll(0) = 0.004 + 0.004 \cdot \frac{C}{e^{0.0625 \cdot 0}} = 0.004 + 0.004 \cdot C = 0.004 \cdot (C+1)$$

$$0.004 \cdot (C+1) = 0.02 \implies C+1 = 5 \implies C = 4$$

$$Poll(x) = 0.004 + \frac{0.016}{e^{0.0625 \cdot x}}$$

$$Poll(x) = 0.10\% \cdot 8 = 0.008$$

$$0.004 + \frac{0.016}{e^{0.0625 \cdot x}} = 0.008 \implies \frac{0.016}{e^{0.0625 \cdot x}} = 0.004 \implies \frac{1}{e^{0.0625 \cdot x}} = 0.25$$

$$\implies e^{-0.0625 \cdot x} = 0.25 \implies -0.0625 \cdot x = ln(0.25) \implies x = \frac{ln(0.25)}{-0.0625} \approx 22.181$$

It will take approximately 22.181 days for the pollutant concentration in the resevoir to reach 0.10%.

36) A tank initially contains 60 gal of pure water. Brine containing 1lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1h.

a) Find the amount of salt in the tank after t minutes.

$\frac{dS}{dt} = 2 - 3 \cdot \frac{S}{60-t}$

$$\frac{dS}{dt} = 2 - 3 \cdot fracS60 - t \implies \frac{dS}{dt} + \frac{3}{60-t}S = 2$$

$$\implies P(t) = \frac{3}{60-t}; Q(t) = 2$$

$$M(t) = 3 \cdot \int \frac{1}{60-t}dt \implies M(t) = ln((60-x)^3)$$

$$S(x) = 2 \cdot (60-x)^{-3} \cdot \int (60-x)^3 dx \implies S(x) = 2 \cdot (60-x)^{-3} \cdot (-\frac{60-x}{4} + C)$$

$$\implies S(x) = -\frac{1}{2}(60-x)^{-2} + 2 \cdot C \cdot (60-x)^{-3}$$

$$S(0) = -\frac{1}{2}(60-x)^{-2} + 2 \cdot C \cdot (60-x)^{-3} = 0 \implies 2 \cdot C \cdot (60)^{-3} = \frac{1}{2}(60)^{-2}$$

$$\implies C = \frac{60}{4} = 15$$

$$S(x) = -\frac{1}{2}(60-x)^{-2} + 2 \cdot 15 \cdot (60-x)^{-3} \implies S(x) = -\frac{1}{2}(60-x)^{-2} + 30 \cdot (60-x)^{-3}$$

$$\therefore S(x) = -\frac{1}{2}(60-x)^{-2} + 30 \cdot (60-x)^{-3}$$

b) What is the maximum amount of salt ever in the tank?

$$\frac{dS}{dt} = 0 \implies 2 - 3 \cdot (-\frac{1}{2}(60-x)^{-2} + 30 \cdot (60-x)^{-3}) = 0$$

$$\implies -\frac{1}{2}(60-x)^{-2} + 30 \cdot (60-x)^{-3} = \frac{2}{3} \implies x \approx 56.513$$

$$S(56.513)$$

is the maximum value of salt in the container.

I think my answer for a is wrong, but I've redone this problem like 4 times and kept finding arithmetic and other nonsense errors, but I can't find anymore, so I'm leaving it as is.

Textbook section 2.1:

1) $\frac{dy}{dt} = x - x^2; x(0) = 2$

12) The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens thereafter?

24) Suppose that a community contains 15,000 people who are susceptible to Michaud's syndrome, a contagious disease. At time $t = 0$ the number $N(t)$ of people who have developed Michaud's syndrome is 5000 and is increasing at the rate of 500 per day. Assume that $N'(t)$ is proportional to the product of the numbers of those who have caught the disease and those who have not. How long will it take for another 5000 people to develop Michaud's syndrome?

Custom problem:

$$\frac{dy}{dx} = y^2; y(1) = 2$$

a)

$$\frac{dy}{dx} = y^2 \implies y^{-2}dy = dx \implies \int y^{-2}dy = \int dx \implies -\frac{1}{y} = x + C$$

$$\implies \frac{1}{y} = C - x \implies y = \frac{1}{C - x}$$

b) It fails because there is no open continuous interval for $y(x)$ where $x = \frac{3}{2}$ is continuous (it is not included in the maximal interval of existence).