Problem 1. (3210 Review Problem) Make an educated guess what is the limit of the sequence $a_n = \frac{5n^2}{n^3+5}$. Then use the definition of the limit to prove that your guess is correct.

My guess is that the limit of the sequence is 0. This is because the denominator has a higher power than the numerator.

To prove this, we have to show that for $\epsilon > 0$, $\exists N$ such that $\forall n \geq N$, $|a_n - 0| < \epsilon$.

First note that $\frac{5n^2}{n^3+5} < \frac{5n^2}{n^3} = \frac{5}{n}$.

$\frac{5}{n} < \epsilon$, so we choose $N > \frac{5}{\epsilon}$. And since $\frac{5n^2}{n^3+5} < \frac{5}{n}$, this means that the sequence converges to 0.

Problem 2. (3210 Review Problem) Define the sequence $\{a_n\}$ inductively by

$$a_n = 1; a_{n+1} = \frac{n^2+n+1}{n^2+n+2}a_n$$

for $n \geq 1$. Prove that the sequence converges. State all theorems, if any, which you used in the proof.

We need to show that the sequence is both bounded and monotonically increasing. Then we will know it converges by the monotonic convergence theorem.

First, let's show that it is bounded. Note that $\frac{n^2+n+1}{n^2+n+2} < 1$. $\forall n \geq 1$. That means that a_{n+1} is multiplying a_n by a fraction that is less than 1. This implies that the sequence is decreasing. And since all terms are strictly positive, this means that the sequence is bounded below by 0.

For monotonicity, we just take what we showed before: that the sequence is decreasing. This means each a_{n+1} is smaller than the preceding a_n but still positive. So the sequence is monotonically decreasing.

Applying the monotonic convergence theorem tells that the sequence converges since it is monotonically decreasing and bounded below by 0.

Problem 3. For the vectors $x = (1, 0, 2)$ and $y = (-1, 3, 1)$ find

a) $2x + y$;

$2x + y = (2, 0, 4) + (-1, 3, 1) = (1, 3, 5)$.

b) $x \cdot y$;

$x \cdot y = (1 * -1 + 0 * 3 + 2 * 1) = 1$.

c) $||x||$ and $||y||$;

$||x|| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$.

$||y|| = \sqrt{(-1)^2 + 3^2 + 1^2} = \sqrt{1+9+1} = \sqrt{11}$.

d) The cosine of the angle between x and y;

$x \cdot y = ||x||||y|| \cos(\theta)$

$\implies \cos(\theta) = \frac{x \cdot y}{||x||||y||}$

$= \cos(\theta) = \frac{1}{\sqrt{5} \cdot \sqrt{11}} = \frac{\sqrt{55}}{55}$.

e) the distance from x to y.

$d(x, y) = ||y - x|| = ||(-2, 3, -1)|| = \sqrt{(-2)^2 + 3^2 + (-1)^2} = \sqrt{4+9+1} = \sqrt{14}$.

Problem 4. Prove that the equality holds in Cauchy-Schwartz Inequality if and only if one of the vectors u and v is a multiple of the other.

The Cauchy Schwartz Inequality is: $(u \cdot v)^2 \leq (u \cdot u) \cdot (v \cdot v)$.

First direction: If u is a scalar multiple of v, the equality holds.

Suppose that $u = \lambda v$ for some scalar λ. Then, $u \cdot v = (\lambda v) \cdot v = \lambda(v \cdot v)$.

So, $(u \cdot v)^2 = \lambda^2(v \cdot v)^2$. Also, $u \cdot u = \lambda^2(v \cdot v)$.

So, $(u \cdot v)^2 = \lambda^2(v \cdot v)^2 = (u \cdot u) \cdot (v \cdot v)$.

Thus, the equality holds.

Now for the other direction: If the equality holds, u is a scalar multiple of v.

Assume the inequality holds.

Now let $w = u - \frac{u \cdot v}{||v||^2} v$.

Using the dot product, we compute that $w \cdot v = (u \cdot v) - \frac{u \cdot v}{||v||^2}(v \cdot v) = (u \cdot v) - (u \cdot v) = 0$.

That means they are orthogonal.

Now, we use the Pythagorean theorem.

$||w||^2 = ||u||^2 - \frac{(u \cdot v)^2}{||v||^2}$.

But since we have the equality already assumed, this means that $||w||^2 = 0$ (meaning w is the zero-vector), which means that u and v are scalar multiples of each other ($u = \frac{u \cdot v}{||v||^2} v$).

Problem 5. For $x, y \in \mathbb{R}$, define

$d_1(x, y) = \sqrt{|x - y|}$

$d_2(x, y) = |x^2 - y^2|$

$d_3(x, y) = |x - 2y|$

Determine, for each of these, whether it is a metric or not. Justify your answers.

a) For d_1:

Non-negativity: It is non-negative since it's a square root of an absolute value.

Symmetry: $d_1(x, y) = \sqrt{|x - y|} = \sqrt{|y - x|} = d_1(y, x)$.

Triangle inequality: $\sqrt{|x - z|} \leq \sqrt{|x - y|} + \sqrt{|y - z|}$. Using the properties of square roots, it is equivalent to $|x - z| \leq |x - y| + |y - z|$.

Definiteness: This is obvious from the definition.

d_1 is a valid metric.

b) For d_2:

This is not a metric because it fails definiteness. Let $x = -2$ and $y = 2$. You will get that $d_2(x, y) = 0$ even though $x \neq y$.

d_2 is not a valid metric.

c) For d_3:

This is not a metric because it fails symmetry. $d_2(x, y) = |x - 2y|$ is not equal to $d_2(y, x) = |y - 2x|$ in general.

d_3 is not a valid metric.

Problem 6. Using only the definition of the limit of a sequence in the Euclidean space $\mathbb{R}^2$, prove that

$$\lim_{n\to\infty}\left(\frac{2n}{n+3}, \frac{1-n}{n}\right) = (2, -1)$$

We need to show that

$$\sqrt{(a_n - L)^2 + (b_n - M)^2} < \epsilon$$

where $a_n = \frac{2n}{n+3}$, $b_n = \frac{1-n}{n}$, $L = 2$, $M = -1$:

$$\sqrt{\left(\frac{2n}{n+3} - 2\right)^2 + \left(\frac{1-n}{n} + 1\right)^2} < \epsilon$$

Our goal is to show that for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, the above inequality holds.

The expression simplifies to $\sqrt{\frac{36}{(n+3)^2} + \frac{1}{n^2}} < \epsilon \ \forall n \geq N$.

Since both terms inside the square root are positive, we can turn it into two separate inequalities.

We get

$$\frac{36}{(n+3)^2} < \frac{\epsilon^2}{2}$$

$$\frac{1}{n^2} < \frac{\epsilon^2}{2}$$

Solving these yields $n > \sqrt{\frac{72}{\epsilon^2}} - 3$ and $n > \frac{1}{\sqrt{\epsilon/2}}$.

If we choose N to be the larger of these two inequalities, then
$\sqrt{\left(\frac{2n}{n+3} - 2\right)^2 + \left(\frac{1-n}{n} + 1\right)^2} < \epsilon$
$\forall n \geq N$.