MATH H113 FINAL EXAM
Friday, May 11, 2012

This exam has 6 problems on 14 pages, including this cover sheet and one blank page at the end (which you may use for scratch work, if you desire). The only thing you may have out during the exam is one or more writing utensils. You have 180 minutes to complete the exam. There are 130 points, but it will be rescaled to a 100 point scale before comparison with the midterm.

DIRECTIONS

- Be sure to carefully read the directions for each problem.
- All work must be done on this exam. If you need more space for any problem, feel free to continue your work on the back of the page or on the blank page at the end of the test. Draw an arrow or write a note indicating this, so I know where to look for the rest of your work.
- For the proofs, you may use more shorthand than is accepted in homework, but make sure your arguments are as clear as possible. If you want to use theorems from the homework or reading, you must state the precise result you are using. Exception: for the "big-name" theorems, you may just use the name of the result.
- For multi-part problems, you may use the earlier parts to complete later parts, even if you find you are unable to prove the earlier parts. Do not give up on an entire problem because you can't figure out one of the first parts!
- Good luck; do the best you can!

Problem	Max	Score
1	30	
2	48	
3	10	
4	16	
5	10	
6	16	
Total	130	

1. (3 points each) For each of the following, the answer is worth 1 point, and the justification is worth 2 points. Circle the correct answer, and give a very brief justification of your answer (quote appropriate theorems, show relevant calculations, give a counterexample, etc.).

   (a) There exists a group G of order $8 \cdot 7^{2012}$, and a group homomorphism $\varphi : G \to S_8$ whose image has $8 \cdot 7^2$ elements.

   TRUE *FALSE*

   (b) As fields, $\mathbb{Q}(\sqrt{2}, \sqrt{7}) = \mathbb{Q}(\sqrt{2} + \sqrt{7})$.

   TRUE *FALSE*

   (c) The minimal polynomial of $\frac{1+2i}{\sqrt{5}}$ over $\mathbb{Q}$ has degree 2.

   TRUE *FALSE*

   (d) The ideal generated by x in $\mathbb{Z}[x, y]$ is prime.

   TRUE *FALSE*

   (e) Let R be the ring of continuous functions $f : [0, 1] \to \mathbb{R}$. Then the subset of functions vanishing at $\frac{3}{4}$ is a prime ideal of R.

   TRUE *FALSE*

(f) If I and J are ideals of a ring R, then so is $I \cup J$.

TRUE *FALSE*

(g) Every group of order 42 has subgroups of order 2, 3, and 7.

TRUE *FALSE*

(h) Every group of order pq, where p and q are distinct primes, is abelian.

TRUE *FALSE*

(i) Every group of order $144 = 12^2$ is abelian.

TRUE *FALSE*

(j) The algebraic closure of $\mathbb{R}$ is $\mathbb{C}$.

TRUE *FALSE*

2. (3 points each) For each of the items listed below, either give an example with the property stated, or give a brief reason why no such example exists.

   If you give an example, you do not have to prove that it has the property stated; however, your examples should be specific. If there are many objects of a given sort, you should name a *particular* one. If you give a reason why no example exists, dont worry about giving reasons for your reasons; a simple statement will suffice.

   (a) Let $R = \mathbb{Z}[x, y, z]$ and let $I = (z^2 - xy)$. Give an example of an element in R/I which does not have a unique factorization into primes. (Write down your two non-associate prime factorizations.)

   (b) Give an example of a simple group of order 168 which has precisely 6 elements of order 7.

   (c) Give an example consisting of two non-isomorphic abelian groups each of which contains an identity element, exactly 24 elements of order 5, exactly 100 elements of order 25, and no other elements.

   (d) Give an example of a group G and a subgroup H of G such that the number of conjugates gHg^{-1}, with $g \in G$, is precisely the index $[G : H]$.

(e) Let $R = M_2(Z_6)$ be the ring of 2×2 matrices with entries from the cyclic group Z_6. Let $m = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \in R$. Give an example of a non-diagonal matrix $r \in R$ which commutes with m (i.e. r must have at least one nonzero entry that is not on the diagonal). You may omit equivalence class bars.

(f) Give an example of an integral domain of characteristic 15.

(g) Give an example of a ring R and a nonzero element $r \in R$ such that r is neither a unit nor a zero divisor.

(h) Give an example of a proper subring of the ring of 3×3 matrices with real entries. (Must contain the 3×3 identity matrix.)

(i) Give an example of a number in $\mathbb{C}$ which is algebraic of degree 5 over $\mathbb{Q}$.

(j) Give an example of a group of order 60 that acts transitively on a set of order 20 (also describe the action).

(k) Give an example of an action of a subgroup of D_{12} on a set of 9 elements such that the orbit of every point has size 3.

(l) Give an example of a principal ideal of $\mathbb{Q}[x, y]$ which is not prime.

(m) Give an example of a group G and a characteristic subgroup H of G such that H is not normal in G.

(n) Give an example of a commutative ring which is not an integral domain.

(o) Let G' denote the commutator subgroup of the group G. Give an example of a group G such that G/G' is non-abelian.

(p) Give an example of a prime number p and an ideal I of the ring $R = \mathbb{F}_p[x]$ such that R/I is a field with 9 elements.

3. Let p be an odd prime number.

   (a) (5 points) Consider the polynomial $f(x) = x^3 - 1 \in (\mathbb{Z}/p\mathbb{Z})[x]$. Prove that $f(x)$ has either one or three roots in $\mathbb{Z}/p\mathbb{Z}$. For which primes does $f(x)$ have one root, and for which does it have three?

   (b) (5 points) How many roots does $x^3 - 1$ have in the polynomial ring $(\mathbb{Z}/91\mathbb{Z})[x]$?

4. This problem has multiple parts and spans two pages.

   (a) (5 points) If G is an abelian group of order $2012 = 2^2 \cdot 503$ (note 503 is prime), list all possible isomorphism types for G. Write each type exactly once, and provide justification that this is a complete list.

   (b) (3 points) What information do we know about the Sylow subgroups of G? (E.g. number of such subgroups, their structure, normality.)

(c) (8 points) Find two non-isomorphic, non-abelian groups of order $2012 = 2^2 \cdot 503$. Write these two groups as (nontrivial) semidirect products, explicitly writing down the appropriate homomorphisms. It should seem intuitively clear that your two groups are not isomorphic, but you do not need to rigorously verify it. (Basically, if there is not any clear symmetry you could exploit to get an isomorphism, you will meet the "intuitively clear" criterion.)

5. (10 points) Find the splitting field of $x^5 - x^4 - 3x + 3$ over $\mathbb{Q}$, expressed in the form $\mathbb{Q}(\alpha, \beta)$, where α and β are elements of $\mathbb{C}$ which are algebraic over $\mathbb{Q}$. (Hint: quintics are notoriously hard! See if you can factor first.)

6. This problem has multiple parts and spans two pages.

   Suppose that $R = (\mathbb{Z}/15\mathbb{Z})[x]/(x^2+1)$.

   (a) (4 points) Prove that $(\overline{3})$ is a maximal ideal of R.

   (b) (4 points) Suppose P is a prime ideal of R. Prove that if $\overline{5} \in P$, then $\overline{x+2} \in P$ or $\overline{x+3} \in P$.

(c) (4 points) Prove that $J = (\overline{5}, \overline{x+2})$ is a maximal ideal of R.

(d) (4 points) Using the previous parts, give a complete list of the prime ideals of R, listing each exactly once. (You may use the fact that $(\overline{5}, \overline{x+3})$ is a maximal ideal; its proof is nearly identical to part c.)

This page was intentionally left blank. It may be used for scratch work or extra space. Clearly label any portion which is meant to be graded.