MATH 142 FINAL SOLUTION

1. (10 points) Determine whether the following statements are true of false, no justification is required.

(1) For any continuous map $f : X \to Y$, and any subset $E \subseteq X$, $f(\overline{E}) \subseteq \overline{f(E)}$ always holds.

True

(2) If $A \subseteq \mathbb{E}^3$ is a closed bounded subset, then A is compact.

True

(3) If $f : X \to Y$ is an identification map, and $A \subseteq X$ is an open subset of X, then $f(A)$ is an open subset of Y.

False

(4) For a topological space X and a path $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = x \in X$, $\gamma_* : \pi_1(X, x) \to \pi_1(X, x)$ always equals the identity isomorphism of $\pi_1(X, x)$.

False

(5) Any compact subset of $\mathbb{E}^2$ is triangulable.

False

(6) Connected surfaces are all path–connected.

True

(7) For two path–connected topological spaces X, Y, if $\pi_1(X) \cong \pi_1(Y)$, then X and Y are homotopic equivalent to each other.

False

(8) For a simplicial complex K, if the dimensions of the simplexes in K are all smaller or equal to n, then $H_q(K)$ is trivial for any integer $q > n$.

True

(9) For a simplicial complex K, if $H_1(K)$ is trivial and $|K|$ is connected, then $\pi_1(|K|)$ is trivial.

False

(10) The only closed surfaces which have positive Euler characteristics are S^2 and the projective plane.

True

2. (20 points) Let X be the set $\mathbb{R}$ with the half–open interval topology ($U \subseteq X$ is a neighborhood of $x \in X$ if $[x, x+\epsilon) \subseteq U$ for some $\epsilon > 0$), show that X is not metrizable by the following steps.

We say that X is not metrizable if X is not homeomorphic to any metric space (with the metric topology).

(1) (6 points) Show that X has a countable dense subset.
(2) (6 points) Show that X does not have a countable base.
(3) (6 points) If a metric space has a countable dense subset, show that it has a countable base.
(4) (2 points) Show that X is not metrizable.

Proof. (1) We show that $\mathbb{Q} \subseteq X$ is a dense subset.

For any $x \in X \setminus \mathbb{Q}$ and any open set U containing x, there exists $\epsilon > 0$ such that $[x, x+\epsilon) \subseteq U$. Since there always exists rational numbers in $(x, x+\epsilon)$, we have $(U \setminus \{x\}) \cap \mathbb{Q} \neq \emptyset$. So any point in $X \setminus \mathbb{Q}$ is a limit point of $\mathbb{Q}$, which implies that the closure of $\mathbb{Q}$ is X. So $\mathbb{Q}$ is a countable dense subset of X.

(2) For any base $\mathcal{B}$ of X, we show that $\mathcal{B}$ is not countable.

For any open set $x \in X$, $[x, x+1)$ is an open set in X. So there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subseteq [x, x+1)$, which implies that $\inf U_x = x$. So for any two distinct real numbers $x, y \in X$, we have that $U_x \neq U_y$ since they have distinct infimums. So we get an injection $\mathbb{R} \to \mathcal{B}$ which maps $x \in X$ to U_x. Since $\mathbb{R}$ is not countable, we get that $\mathcal{B}$ is not countable.

(3) Let (S, d) be a metric space, and $\{s_i\}_{i=1}^{\infty}$ be a countable dense set in S. We will show that $\mathcal{B} = \{B(x_i, \frac{1}{j}) \mid i, j = 1, 2, \cdots\}$ is a base of S, which is clearly countable.

For any open set $U \subseteq S$ and $x \in S$, there exists $n \in \mathbb{N}$ such that $B(x, \frac{1}{n}) \subseteq U$. Since $\{s_i\}_{i=1}^{\infty}$ is dense in S, there exists some s_i lies in $B(x, \frac{1}{2n})$. So we have that $x \in B(s_i, \frac{1}{2n}) \subseteq B(x, \frac{1}{n}) \subseteq U$, which shows that $\mathcal{B}$ is a base of (S, d).

(4) Suppose that X is homeomorphic to a metric space (S, d) by a homeomorphism $f : X \to S$. Then $f(\mathbb{Q})$ is a countable dense set in S, which implies that S has a countable base $\mathcal{B}$. Then $f^{-1}(\mathcal{B})$ is a countable base of X, which contradicts with the result of the second subquestion: X does not have countable base. $\square$

3. (20 points) Let $I_2 \in SL(2)$ be the 2×2 identity matrix, show that $\pi_1(SL(2), I_2) \cong \mathbb{Z}$ by the following steps.

Let $T(2)$ be the subgroup of $SL(2)$ (as topological groups) consists of all upper triangular matrices with positive diagonal entries, i.e.

$$T(2) = \left\{ \begin{pmatrix} r & s \\ 0 & r^{-1} \end{pmatrix} : r > 0, s \in \mathbb{R} \right\}.$$

(1) (3 points) Show that $f : SO(2) \times T(2) \to SL(2)$ defined by $f(A, B) = A \cdot B$ is a continuous and injective function.

(f is actually a homeomorphism, and we will use it as a fact in the following subquestions. The surjectivity of f and continuity of f^{-1} follows by the QR decomposition in linear algebra.)

(2) (3 points) Show that $f : SO(2) \times T(2) \to SL(2)$ is not an isomorphism (as topological groups).

(3) (7 points) Show that $T(2)$ is contractible.

(4) (7 points) Show that $\pi_1(SL(2), I_2) \cong \mathbb{Z}$.

Proof. (1) Since the topology of $SO(2)$ and $T(2)$ are just the subspace topology induced from $SL(2)$ and the multiplication on $SL(2)$ is continuous ($SL(2)$ is a topological group), we have that $f : SO(2) \times T(2) \to SL(2)$ is continuous.

Suppose that $f(A, B) = f(A', B')$, then $A \cdot B = A' \cdot B'$. So we have that $(A')^{-1} \cdot A = B' \cdot (B)^{-1}$. Since $A, A' \in SO(2)$ and $B, B' \in T(2)$, we have that $(A')^{-1} \cdot A \in SO(2)$ and $B' \cdot (B)^{-1} \in T(2)$. Since elements in $SO(2)$ are in the form of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and the elements in $T(2)$ are in the form of $\begin{pmatrix} r & t \\ 0 & r^{-1} \end{pmatrix}$, we have that $SO(2) \cap T(2) = \{I_2\}$. So $A = A'$ and $B = B'$, thus f is injective.

(2) $f(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $f(\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \sqrt{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$.

However, $f(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = f(\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$. It is not equal to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}}{2} & \sqrt{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$.

So f is not a group homomorphism.

(3) We need to define a homotopy from $id : T(2) \to T(2)$ to the constant map from $T(2)$ to $I_2 \in T(2)$. The homotopy is given by first deformation retract to the set of diagonal matrixes in $T(2)$, then to I_2. Here is the definition of the homotopy

$F : T(2) \times I \to T(2)$:

$$F(\begin{pmatrix} r & s \\ 0 & r^{-1} \end{pmatrix}, t) = \begin{cases} \begin{pmatrix} r & s(1-2t) \\ 0 & r^{-1} \end{pmatrix} & t \in [0, \frac{1}{2}] \\ \begin{pmatrix} r^{2-2t} & 0 \\ 0 & r^{2t-2} \end{pmatrix} & t \in [\frac{1}{2}, 1] \end{cases}$$

It is easy to see that $F(A, 0) = A$ and $F(A, 1) = I_2$, so $T(2)$ is contractible.

(4) Since $SL(2)$ is homeomorphic to $SO(2) \times T(2)$, $\pi_1(SL(2), I_2)$ is isomorphic with $\pi_1(SO(2), I_2) \times \pi_1(T(2), I_2)$. Since $T(2)$ is contractible, $\pi_1(T(2), I_2)$ is the trivial group. $SO(2)$ is homeomorphic to S^1, which is given by $g : S^1 \to SO(2)$ defined by $g(e^{i\theta}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. So $\pi_1(SO(2), I_2)$ is isomorphic to $\mathbb{Z}$.

So $\pi_1(SL(2), I_2)$ is isomorphic to $\mathbb{Z}$. $\square$

4. (15 points) Compute the fundamental groups of the following spaces.

(1) (7 points) $\mathbb{E}^3$ with the x–, y– and z–axes removed.

(2) (8 points) The union of two copies of $S^1 \times S^1$, with $S^1 \times \{1\}$ in the first copy identified with $\{1\} \times S^1$ in the second copy, by a homeomorphism $f : S^1 \times \{1\} \to \{1\} \times S^1$ defined by $f(e^{i\phi}, 1) = (1, e^{i\phi})$.

Proof. (1) Let X be the space of $\mathbb{E}^3$ with the x–, y– and z–axes removed.

$S^2 \subseteq \mathbb{E}^3$ intersects with the three axes at six points, and we denote the set of intersection points by P. Then $S^2 \setminus P$ is a deformation retract of X, by project each point in X to the standard 2-dimensional sphere. Since S^2 with one point removed is homeomorphic to $\mathbb{E}^2$, $S^2 \setminus P$ is homeomorphic to $\mathbb{E}^2$ with five points removed.

It is easy to see that $\mathbb{E}^2$ with five points has a deformation retract which is the one–point union of five S^1, so $\pi_1(X)$ is isomorphic to the fundamental group of the one–point union of five S^1, which is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

(2) Let T_1 be the first copy of $S^1 \times S^1$ and T_2 be the second copy, and let X be the space given by the union of T_1 and T_2. Take a base point $x_0 = (1,1) \in T_1 \cap T_2$.

Since $T_1 \cap T_2 = S^1$, we need to study $(j_1)_* : \pi_1(S^1, x_0) \to \pi_1(T_1, x_0)$ and $(j_2)_* : \pi_1(S^1, x_0) \to \pi_1(T_2, x_0)$. Since T_1 and T_2 are both tori, their fundamental groups are both isomorphic with $\mathbb{Z}^2$. Moreover, $a = \langle (1, e^{2\pi i t}) \rangle \in \pi_1(T_1, x_0)$ and $c = \langle (e^{2\pi i t}, 1) \rangle \in \pi_1(T_1, x_0)$ generate $\pi_1(T_1, x_0) \cong \mathbb{Z}^2$; while $b = \langle (e^{2\pi i t}, 1) \rangle \in \pi_1(T_2, x_0)$ and $d = \langle (1, e^{2\pi i t}) \rangle \in \pi_1(T_2, x_0)$ generate $\pi_1(T_2, x_0) \cong \mathbb{Z}^2$.

Let e be the generator of $\pi_1(T_1 \cap T_2, x_0)$ given by $(e^{2\pi i t}, 1)$ in T_1, then we have that $(j_1)_*(e) = c$ and $(j_2)_*(e) = d$. By Van Kampen's theorem, we have that

$$\begin{aligned} \pi_1(X, x_0) &= \mathbb{Z}^2 * \mathbb{Z}^2 / (c \sim d) \\ &\cong \langle a, b, c, d \mid aca^{-1}c^{-1} = 1, bdb^{-1}d^{-1} = 1, c = d \rangle \\ &\cong \langle a, b, c \mid aca^{-1}c^{-1} = 1, bcb^{-1}c^{-1} = 1 \rangle. \end{aligned}$$

□

5. (15 points) Please answer the following questions about closed surfaces.

(1) (4 points) Please list all closed surfaces.
(2) (4 points) If we add a handle to a Klein bottle, which closed surface do we get? What is its fundamental group?
(3) (7 points) For the double torus $H(2)$, find a subspace $A \subseteq H(2)$ which is homeomorphic to S^1 such that A is a retract of $H(2)$. Then find another subspace $B \subseteq H(2)$ which is homeomorphic to S^1 such that B is not a retract of $H(2)$.
   ($A \subseteq X$ is a retract of X if there exists a continuous map $f : X \to A$ such that $f(a) = a$ for any $a \in A$.)

Proof. (1) All the closed surfaces are: S^2, S^2 with p handles added ($H(p)$) and S^2 with q discs replaced by q Möbius strips ($M(q)$). Here $p, q \geq 1$.

(2) A Klein bottle is homeomorphic to $M(2)$: S^2 with 2 discs replaced by 2 Möbius strips. Since $M(2)$ contains a Möbius strip, adding a handle to a Möbius strip in the "correct" way and the "wrong" way make no difference. Since adding a handle in the "wrong" way is same with have 2 discs replaced by 2 Möbius strips. We have that the result surface is S^2 with 4 discs replaced by 4 Möbius strips, which is exactly $M(4)$.

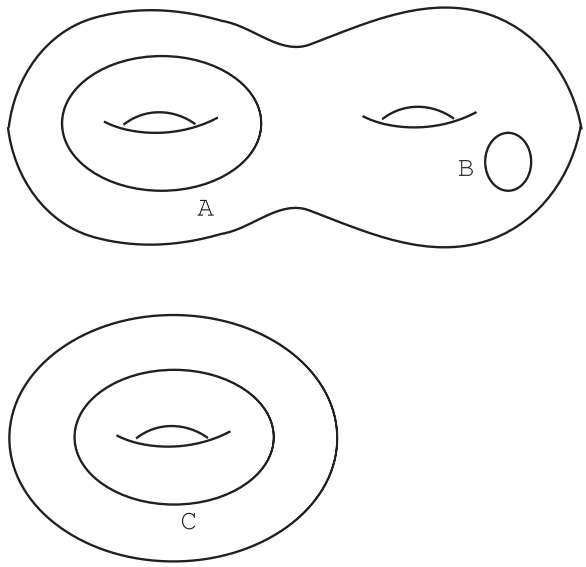
By cutting along simple closed curves in $M(4)$, we know that the surface symbol of $M(4)$ is $a_1^2a_2^2a_3^2a_4^2$, and its fundamental group is

$$\langle a_1, a_2, a_3, a_4 \mid a_1^2a_2^2a_3^2a_4^2 = 1\rangle.$$

(3) The subspaces A and B of $H(2)$ is as shown in the following picture.

A is a retract of $H(2)$. We can first pinch the right torus in $H(2)$ to get a map $f : H(2) \to T^2$ (as shown in the picture bellow $H(2)$), with $f|_A : A \to C$ is a homeomorphism. Then identify C as $S^1 \times \{1\} \subseteq S^1 \times S^1 = T^2$. Then we can define a map $g : T^2 \to A$ by $g(e^{i\theta}, e^{i\phi}) = (f|_A)^{-1}(e^{i\theta}, 1)$. Then $g \circ f : H(2) \to A$ is a continuous map such that $g \circ f(x) = x$ for any $x \in A$.

B is not a retract of $H(2)$ since B is the boundary of a disc $D \subseteq H(2)$, and B is not a retract of D. Suppose that there exists a retract $r : D \to B$, then for the embedding $i : B \to D$, after fixing a base point, we have that $r_* \circ i_* : \pi_1(B) \to \pi_1(B)$ is the identity homomorphism. However, $\pi_1(B) \cong \mathbb{Z}$ and $\pi_1(D) \cong 0$, such a composition $\mathbb{Z} \to 0 \to \mathbb{Z}$ can not give the identity homomorphism.



$\square$

6. (20 points) Please answer the following questions about the torus.

(1) (4 points) Compute $H_i(S^1 \times S^1)$ for $i = 0, 1, 2$.

(2) (8 points) Let $f : S^1 \times S^1 \to S^1 \times S^1$ be defined by $f(e^{i\phi}, e^{i\theta}) = (e^{i(\phi+\theta)}, e^{-i\theta})$, and $g : S^1 \times S^1 \to S^1 \times S^1$ be defined by $g(e^{i\phi}, e^{i\theta}) = (e^{-i\phi}, e^{i(\phi+\theta)})$. Compute $f_* : H_i(S^1 \times S^1) \to H_i(S^1 \times S^1)$ and $g_* : H_i(S^1 \times S^1) \to H_i(S^1 \times S^1)$ for $i = 0, 1, 2$. (Hint: use one of the simplest triangulation of $S^1 \times S^1$, then f and g are simplicial.)

(3) (4 points) Show that there exists a map $f' : S^1 \times S^1 \to S^1 \times S^1$ which is homotopic to f and does not have any fixed–point.

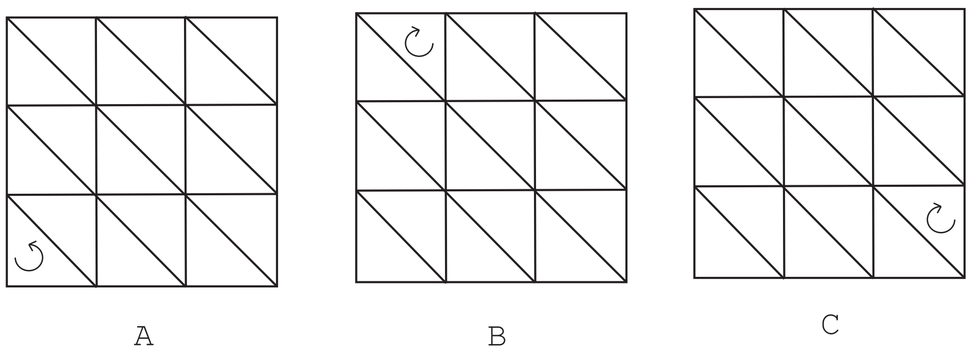
(4) (4 points) For any map $h : S^1 \times S^1 \to S^1 \times S^1$ which is homotopic to $f \circ g$, show that h always has at least one fixed–point.

Proof. (1) Since $S^1 \times S^1 = T^2$ is connected, $H_0(S^1 \times S^1) \cong \mathbb{Z}$.

Since $\pi_1(S^1) \cong \mathbb{Z}$, we have $\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$. Since H_1 is the abelianization of π_1, and $\mathbb{Z} \times \mathbb{Z}$ is a abelian group, we have $H_1(S^1 \times S^1) \cong \pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

Since T^2 is an orientable surface, all the triangulations for T^2 are orientable, so $H_2(S^1 \times S^1) \cong \mathbb{Z}$. (See also the following triangulation of T^2).

(2) Consider the triangulation of T^2 shown as A in the following picture, with opposite edges of the square identified with each other. The horizontal direction corresponds with the ϕ direction (first S^1), and the vertical direction corresponds with the θ direction (second S^1). Then the maps f and g are both simplicial with respect to this triangulation, and map the oriented lower left triangle to the oriented triangle shown in B and C respectively.



For $f : S^1 \times S^1 \to S^1 \times S_1$, we always have $f_* : H_0(S^1 \times S^1) \to H_0(S^1 \times S^1)$ is just $id : \mathbb{Z} \to \mathbb{Z}$.

Since $\pi_1(S^1 \times S^1) \cong H_1(S^1 \times S^1)$, we need only to compute $f_* : \pi_1(S^1 \times S^1, x_0) \to \pi_1(S^1 \times S^1, x_0)$ (here $x_0 = (1, 1)$), which also gives the induced map on H_1. Fix a basis a, b of $\pi_1(S^1 \times S^1)$ by $a = \langle (e^{2\pi it}, 1) \rangle$ and $b = \langle (1, e^{2\pi it}) \rangle$. They also correspond with a basis for $H_1(S^1 \times S^1)$, and we abuse notation and still use a, b to denote them. Since $f(e^{2\pi it}, 1) = (e^{2\pi it}, 1)$, we have $f_*(a) = a$; since $f(1, e^{2\pi it}) = (e^{2\pi it}, e^{-2\pi it})$, we have $f_*(b) = a - b$.

For $f_* : H_2(S^1 \times S^1) \to H_2(S^1 \times S^1)$, since it maps the counterclockwise oriented triangle in picture A to the clockwise oriented triangle in figure B and the induced map on the set of triangles is a bijection, f_* maps the generator of $H_2(S^1 \times S^1)$ to the negative of itself. So $f_* : H_2(S^1 \times S^1) \to H_2(S^1 \times S^1)$ corresponds with $\mathbb{Z} \to \mathbb{Z}$ with 1 mapped to -1.

The same argument for f also works for g. So we have $g_* : H_0(S^1 \times S^1) \to H_0(S^1 \times S^1)$ is just $id : \mathbb{Z} \to \mathbb{Z}$. $g_* : H_1(S^1 \times S^1) \to H_1(S^1 \times S^1)$ is given by $g_*(a) = -a + b$ and $g_*(b) = b$. $g_* : H_2(S^1 \times S^1) \to H_2(S^1 \times S^1)$ corresponds with $\mathbb{Z} \to \mathbb{Z}$ with 1 mapped to -1.

(3) $F : T^2 \times I \to T^2$ defined by $F(e^{i\phi}, e^{i\theta}, t) = (e^{i(\phi+\theta)}, e^{-i(\theta+t)})$ is a homotopy from f to $(e^{i\phi}, e^{i\theta}) \to (e^{i(\phi+\theta)}, e^{-i(\theta+1)})$. $(e^{i\phi}, e^{i\theta}) \to (e^{i(\phi+\theta)}, e^{-i(\theta+1)})$ does not have any fixed–point, since $e^{i\theta} = e^{-i(\theta+1)}$ never holds.

(4) Since h is homotopic to $f \circ g$, we have $h_* = f_* \circ g_* : H_i(S^1 \times S^1) \to H_i(S^1 \times S^1)$ for $i = 0, 1, 2$. So we can compute h_* by the above computation of f_* and g_*.

By the above computation, we have that $h_* : H_0(S^1 \times S^1) \to H_0(S^1 \times S^1)$ is just $id : \mathbb{Z} \to \mathbb{Z}$. $h_* : H_2(S^1 \times S^1) \to H_2(S^1 \times S^1)$ is also $id : \mathbb{Z} \to \mathbb{Z}$, since it is the composition of two $1 \to -1$. For $h_* : H_1(S^1 \times S^1) \to H_1(S^1 \times S^1)$, we have $h_*(a) = f_*(g_*(a)) = f_*(-a + b) = -a + a - b = -b$ and $h_*(b) = f_*(g_*(b)) = f_*(b) = a - b$.

So we have $\mathrm{trace}(h_{0*}) = \mathrm{trace}(h_{2*}) = 1$ and $\mathrm{trace}(h_{1*}) = -1$. Then the Lefschetz number is $1 - (-1) + 1 = 3$, which implies that h has at least one fixed–point. $\square$