H113 – Final Exam Preview

NO DISCUSSION OR COLLABORATION WHATSOEVER is permitted for these problems. The only resources you may use are our textbook (Dummit and Foote) and our course website. You may not talk to any person other than me about them (and I will only answer clarification-type questions). You may not consult any website other than our course website. You may not consult any books other than our textbook. You may not use any other resources you manage to dream up.

Please keep in mind that the university honor code requires you to not only follow these rules yourself, but also to immediately report to me any violations you become aware of. Do not put yourself, your friends, or me into an unpleasant situation.

A few of these will turn out to be very short, possibly easy for you; these might show up in the TF section. Of the more complicated ones (i.e. not one-liners), aapproximately one third of the problems (or closely related variants) will appear on the final exam.

Unless otherwise stated, you may use results from the book (including homework problems that were assigned) assuming a) doing so doesn't trivialize the problem, and b) you precisely state the relevant result. Be sure you can also carefully state any relevant definitions. All questions require proof/justification, even if they don't explicitly say so.

1. Explicitly give a description of the dihedral group D_{20} as a semidirect product of two of its proper subgroups. Prove that your construction is correct.

2. Suppose G is a group of order $2012 = 2^2 \cdot 503$ (note 503 is prime). List all possible isomorphism types of G, both abelian and non-abelian. Be sure to list each group *exactly* once, and prove that you have listed all groups. (For a non-abelian group which is a semidirect product, it is enough to write its semidirect product decomposition.)

3. Find all prime ideals of the quotient ring $R = \mathbb{Z}[x]/(15, x^2+1)$ and determine which are also maximal ideals. List each ideal exactly once, and prove that your answer is complete. Use bar notation to distinguish elements in the quotient ring from elements in $\mathbb{Z}[x]$.

4. Let G be a group of order $8 \cdot 7^{2012}$. If $\varphi : G \to S_8$ is a group homomorphism, prove that the image of φ has fewer than 60 elements.

5. Let p be an odd prime number.

   (a) Consider the polynomial $f(x) = x^3 - 1 \in (\mathbb{Z}/p\mathbb{Z})[x]$. For which primes does $f(x)$ have 0 roots in $\mathbb{Z}/p\mathbb{Z}$? Exactly 1 root? Exactly 2 roots? Exactly 3 roots? More roots? Justify your answer.

   (b) How many roots does $x^3 - 1$ have in the polynomial ring $(\mathbb{Z}/77\mathbb{Z})[x]$?

6. Let $R = M_2(Z_6)$ be the ring of 2×2 matrices with entries from the cyclic group Z_6. Let $m = \begin{pmatrix} \overline{1} & 0 \\ 0 & \overline{5} \end{pmatrix} \in R$. Determine, with proof, the number of elements of R which commute with m, i.e. how many $r \in R$ satisfy $rm = mr$, using typical matrix multiplication?

7. Find the minimal polynomial of $\frac{1+2i}{\sqrt{5}}$ over $\mathbb{Q}$.

8. Find the splitting field of $x^4 - 3$ over $\mathbb{Q}$, expressed in the form $\mathbb{Q}(\alpha, \beta)$, where α and β are elements of $\mathbb{C}$ which are algebraic over $\mathbb{Q}$.

9. How many elements of order 7 are there in a simple group of order 168?

10. Let G be an abelian group that contains the identity, exactly 8 elements of order three, exactly 18 elements of order nine, and no other elements. List all possibilities for G, by giving explicit decompositions into cyclic groups, up to isomorphism.

11. Let $R = \mathbb{Z}[x, y, z]$ and $f(x, y, z) = z^2 - xy$. Is R a UFD? Is $R/(f)$ a UFD?

12. Let H be a subgroup of a finite group G.

    (a) Prove that the number of conjugates gHg^{-1}, with $g \in G$, is at most the index $[G : H]$. Give one example where it is precisely $[G : H]$ and one example where it is strictly smaller.

    (b) If H is a proper subgroup of G, show that $G \neq \bigcup_{g \in G} gHg^{-1}$.

13. Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{7}) = \mathbb{Q}(\sqrt{2} + \sqrt{7})$.

14. Let G be a group of order p^k, where p is prime, and k is a positive integer. Suppose G acts on a finite set X. Prove that the number of fixed points of the action is congruent to the order of X mod p.

15. In the group S_7, let $\sigma = (1, 2, 3)(4, 5, 6, 7)$, written as a product of disjoint cycles.

    (a) Let H denote the subgroup of S_7 generated by σ. What is the order of H?

    (b) How many elements of S_7 are conjugate to σ?

    (c) How many subgroups of S_7 are conjugate to H, i.e. are of the form gHg^{-1} for some $g \in S_7$?

    (d) What is the order of the centralizer of σ in S_7?