Math 53 Practice Midterm 1 B – Solutions

Problem 1.

a) $P = (1,0,0)$, $Q = (0,2,0)$ and $R = (0,0,3)$. Therefore $\overrightarrow{QP} = \hat{\imath} - 2\hat{\jmath}$ and $\overrightarrow{QR} = -2\hat{\jmath} + 3\hat{k}$.

b) $\cos\theta = \dfrac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QP}\right|} = \dfrac{\langle 1,-2,0\rangle\cdot\langle 0,-2,3\rangle}{\sqrt{1^2+2^2}\sqrt{2^2+3^2}} = \dfrac{4}{\sqrt{65}}$

Problem 2.

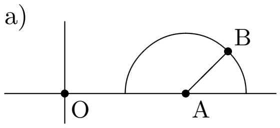
a) $\overrightarrow{PQ} = \langle -1,2,0\rangle$, $\overrightarrow{PR} = \langle -1,0,3\rangle$. So $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$.

Then $area(\Delta) = \dfrac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \dfrac{1}{2}\sqrt{6^2+3^2+2^2} = \dfrac{1}{2}\sqrt{49} = \dfrac{7}{2}$.

b) A normal to the plane is given by $\overrightarrow{N} = \overrightarrow{PQ}\times\overrightarrow{PR} = \langle 6,3,2\rangle$. Hence the equation has the form $6x+3y+2z = d$. Since P is on the plane, $d = 6\cdot 1 + 3\cdot 1 + 2\cdot 1 = 11$. So the equation of the plane is $6x+3y+2z = 11$.

c) The line is parallel to $\langle 2-1, 2-2, 0-3\rangle = \langle 1,0,-3\rangle$. Since $\overrightarrow{N}\cdot\langle 1,0,-3\rangle = 6-6 = 0$, the line is parallel to the plane.

Problem 3.

a) 

$\overrightarrow{OA} = \langle 10t, 0\rangle$ and $\overrightarrow{AB} = \langle \cos t, \sin t\rangle$, hence

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \langle 10t + \cos t, \sin t\rangle.$$

The rear bumper is reached at time $t = \pi$ and the position of B is $(10\pi - 1, 0)$.

b) $\overrightarrow{V} = \langle 10 - \sin t, \cos t\rangle$, thus $|\overrightarrow{V}|^2 = (10-\sin t)^2 + \cos^2 t = 100 - 20\sin t + \sin^2 t + \cos^2 t = 101 - 20\sin t$.

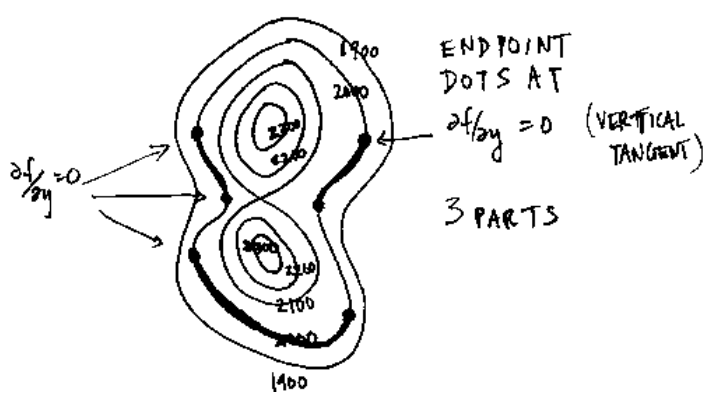
The speed is then given by $\sqrt{101 - 20\sin t}$. The speed is smallest when $\sin t$ is largest i.e. $\sin t = 1$. It occurs when $t = \pi/2$. At this time, the position of the bug is $(5\pi, 1)$. The speed is largest when $\sin t$ is smallest; that happens at the times $t = 0$ or π for which the position is then $(0,0)$ and $(10\pi - 1, 0)$.

Problem 4. a) $\overrightarrow{N}\cdot\overrightarrow{r}(t) = 6$, where $\overrightarrow{N} = \langle 4,-3,-2\rangle$.

b) We differentiate $\overrightarrow{N}\cdot\overrightarrow{r}(t) = 6$:

$$0 = \frac{d}{dt}\left(\overrightarrow{N}\cdot\overrightarrow{r}(t)\right) = \frac{d}{dt}\overrightarrow{N}\cdot\overrightarrow{r}(t) + \overrightarrow{N}\cdot\frac{d}{dt}\overrightarrow{r}(t) = \overrightarrow{0}\cdot\overrightarrow{r}(t) + \overrightarrow{N}\cdot\frac{d}{dt}\overrightarrow{r}(t) \quad \text{and hence } \overrightarrow{N}\perp\tfrac{d}{dt}\overrightarrow{r}(t).$$

Problem 5.



Problem 6. a) $\nabla f = \langle 2xy^2 - 1, 2x^2y\rangle = \langle 3,8\rangle = 3\hat{\imath} + 8\hat{\jmath}$.

b) $z - 2 = 3(x-2) + 8(y-1)$ or $z = 3x + 8y - 12$.

c) $\Delta x = 1.9 - 2 = -1/10$ and $\Delta y = 1.1 - 1 = 1/10$. So $z \simeq 2 + 3\Delta x + 8\Delta y = 2 - 3/10 + 8/10 = 2.5$

d) $D_{\hat{u}} f = \nabla f \cdot \hat{u} = \langle 3, 8 \rangle \cdot \dfrac{\langle -1, 1 \rangle}{\sqrt{2}} = \dfrac{-3+8}{\sqrt{2}} = \dfrac{5}{\sqrt{2}}$

Problem 7. a) $\left.\begin{array}{l} w_x = -6x - 4y + 16 = 0 \quad \Rightarrow \quad -3x - 2y + 8 = 0 \\ w_y = -4x - 2y - 12 = 0 \quad \Rightarrow \quad 4x + 2y + 12 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = -20 \\ y = 34 \end{array}\right.$

Therefore there is just one critical point at $(-20, 34)$. Since

$$w_{xx}w_{yy} - w_{xy}^2 = (-6)(-2) - (-4)^2 = 12 - 16 = -4 < 0,$$

the critical point is a saddle point.

b) There is no critical point in the first quadrant, hence the maximum must be at infinity or on the boundary of the first quadrant.

The boundary is composed of two half-lines:

- $x = 0$ and $y \geq 0$ on which $w = -y^2 - 12y$. It has a maximum ($w = 0$) at $y = 0$.
- $y = 0$ and $x \geq 0$, where $w = -3x^2 + 16x$. (The graph is a parabola pointing downwards). Maximum: $w_x = -6x + 16 = 0 \quad \Rightarrow x = 8/3$. Hence w has a maximum at $(8/3, 0)$ and $w = -3(8/3)^2 + 16 \cdot 8/3 = 64/3 > 0$.

We now check that the maximum of w is not at infinity:

- If $y \geq 0$ and $x \to +\infty$ then $w \leq -3x^2 + 16x$, which tends to $-\infty$ as $x \to +\infty$.
- If $0 \leq x \leq C$ and $y \to +\infty$, then $w \leq -y^2 + 16C$, which tends to $-\infty$ as $y \to +\infty$.

We conclude that the maximum of w in the first quadrant is at $(8/3, 0)$.

Problem 8. a) $\left\{\begin{array}{lll} w_x & = u_x w_u + v_x w_v & = -\frac{y}{x^2} w_u + 2x w_v \\ w_y & = u_y w_u + v_y w_v & = \frac{1}{x} w_u + 2y w_v \end{array}\right.$

b) $x w_x + y w_y = x(-\dfrac{y}{x^2} w_u + 2x w_v) + y(\dfrac{1}{x} w_u + 2y w_v) = (-\dfrac{y}{x} + \dfrac{y}{x}) w_u + (2x^2 + 2y^2) w_v = 2v w_v$.

c) $x w_x + y w_y = 2v w_v = 2v \cdot 5v^4 = 10v^5$.

Problem 9. a) $f(x, y, z) = x$; the constraint is $g(x, y, z) = x^4 + y^4 + z^4 + xy + yz + zx = 6$. The Lagrange multiplier equation is:

$$\nabla f = \lambda \nabla g \quad \Leftrightarrow \quad \left\{\begin{array}{lll} 1 & = & \lambda(4x^3 + y + z) \\ 0 & = & \lambda(4y^3 + x + z) \\ 0 & = & \lambda(4z^3 + x + y) \end{array}\right.$$

b) The level surfaces of f and g are tangent at (x_0, y_0, z_0), so they have the same tangent plane. The level surface of f is the plane $x = x_0$; hence this is also the tangent plane to the surface $g = 6$ at (x_0, y_0, z_0).

Second method: at (x_0, y_0, z_0), we have $\left.\begin{array}{rl} 1 = & \lambda g_x \\ 0 = & \lambda g_y \\ 0 = & \lambda g_z \end{array}\right\} \quad \Rightarrow \quad \lambda \neq 0$ and $\langle g_x, g_y, g_z \rangle = \langle \dfrac{1}{\lambda}, 0, 0 \rangle$.

So $\langle \frac{1}{\lambda}, 0, 0 \rangle$ is perpendicular to the tangent plane at (x_0, y_0, z_0); the equation of the tangent plane is then $\frac{1}{\lambda}(x - x_0) = 0$, or equivalently $x = x_0$.

Problem 10.

a) Taking the total differential of $x^2 + y^3 - z^4 = 1$, we get: $2x\,dx + 3y^2\,dy - 4z^3\,dz = 0$. Similarly, from $z^3 + zx + xy = 3$, we get: $(y + z)\,dx + x\,dy + (3z^2 + x)\,dz = 0$.

b) At $(1, 1, 1)$ we have: $2\,dx + 3\,dy - 4\,dz = 0$ and $2\,dx + dy + 4\,dz = 0$. We eliminate dz (by adding these two equations): $4\,dx + 4\,dy = 0$, so $dy = -dx$, and hence $dy/dx = -1$.