MATH 142: ELEMENTARY ALGEBRAIC TOPOLOGY

REVIEW OF IMPORTANT CONCEPTS AND THEOREMS

Short Version

Chapter 2: Topological space and continuity.

- **Topological space**: defined by open sets. Neighborhoods, closed sets, interior, closure, limit point, denseness, base. Subspace topology; half–interval topology and finite complement topology on $\mathbb{R}$.
- **Continuous functions**: preimage of open sets are open. Composition of continuous functions is continuous, **homeomorphism**, inclusion.
- Space filling curve: continuous surjective map from I to $I \times I$.
- Metric space: metric topology, Hausdorff space, Tietze extension theorem.

Chapter 3: Compactness and connectedness.

- **Compact space**: definition. Closed and bounded subsets of $\mathbb{E}^n$ are compact.
- Properties of compace spaces: continuous image of compact sets, closed sets versus compact sets, homeomorphisms from compact spaces to Hausdorff spaces, Bolzano–Weierstrass property of compact spaces, Lebesgue's lemma.
- Product space: product topology, product of compact spaces is compact.
- **Connectedness**: no nontrivial open decomposition. Intervals are connected, continuous image of connected spaces, closure and product of connected spaces, connected component.
- Path–connectedness: path–connected spaces are connected, connected open sets in $\mathbb{E}^n$ are path–connected, path–component. Connected but not path–connected examples.

Chapter 4: Identification spaces.

- **Identification topology**: U is open if and only if $\pi^{-1}(U)$ is open. Identification topology versus identification map, open maps and closed maps are identification maps.
- Möbius strip, torus, Klein bottle, projective spaces, attaching maps.
- Topological groups: definition. Homomorphism, isomorphism, subgroup, left and right translation. Examples of topological groups: $\mathbb{R}$, S^1, product of topological groups, $GL(n)$, $SL(n)$, $SO(n)$, $O(n)$.
- Topological groups action on topological spaces, orbit spaces.

Chapter 5: Fundamental group.

- **Homotopic maps**: deformation between continuous maps $f, g : X \to Y$. Homotopic relative to a subset, homotopy is a equivalence relation. Straight–line homotopy, homotopy between maps to spheres, construct homotopy by breaking I to pieces.
- **Definition of fundamental groups**: homotopy classes of paths in X with end points mapped to the base point, multiplication given by concatenation of paths.

- Change base points of fundamental groups, induced homomorphism on fundamental groups by a continuous map.
- Simply connected space. $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(S^n) \cong \{e\}$ for $n > 1$.
- Homotopy type: definition of **homotopic equivalence** and examples. Homotopic equivalent spaces have isomorphic fundamental groups, **homotopic maps induce the same homomorphism on fundamental groups**. Contractible spaces.
- Brouwer fixed–point theorem (for dimension 2), retract of a space.

Chapter 6: Triangulations.

- q–simplex, **simplicial complexes**. Triangulation of a topological space, specific triangulation of S^1, S^2, T^2, Klein bottle. Isomorphism between simplicial complexes, basic topological properties of polyhedrons.
- Barycentric subdivision: make the size of simplexes smaller. (proof not required)
- Simplicial map, **simplicial approximation theorem**. (proof not required, but application is important)
- Computation of $\pi_1(|K|, v)$: edge group $E(K, v)$, $G(K, L)$ for a subcomplex L with contractible polyhedron, write $\pi_1(|K|, v)$ as a generator–relator presentation.
- **Van Kampen's theorem**: compute $\pi_1(|K_1 \cup K_2|)$ by $\pi_1(|K_1|)$, $\pi_1(|K_2|)$ and $\pi_1(|K_1 \cap K_2|)$, require path–connectedness.
- Triangulating orbit spaces: may need to take barycentric subdivision.
- Infinite simplicial complexes: definition, local finiteness.

Chapter 7: Surfaces.

- n–dimensional manifold, closed n–dimensional manifold.
- **Classification theorem of closed surfaces**: S^2, $H(p)$ and $M(q)$. Klein bottle is homeomorphic to $M(2)$, mixing handle and Möbius operations.
- Triangulation of closed surfaces: combinatorial surface, orientable surface versus orientable combinatorial surface, thickening of 1–dimensional subgraphs.
- **Euler characteristic of** K: $\chi(K) = \sum_{i=0}^{n} (-1)^i \alpha_i$. For a combinatorial surface K, $\chi(K) \leq 2$, $\chi(K) = 2$ if and only if $|K|$ is homeomorphic to S^2.
- Surgery: remove the thickening of a separating polygonal simple closed curve, then add discs, Euler characteristic increases by 1 or 2.
- Cut along curves in closed surfaces to get a polygon, with pair of edges identified. Surface symbol, computation of fundamental groups for closed surfaces.

Chapter 8: Simplicial homology.

- Orientations on a q–simplex. **Definition of simplicial homology groups**: $C_q(K)$, $\partial : C_q(K) \to C_{q-1}(K)$ satisfies $\partial^2 = 0$, $Z_q(K)$, $B_q(K)$, $H_q(K) = Z_q(K)/B_q(K)$.
- Examples of simplicial homology groups: $H_0(K)$ versus the number of components of $|K|$; $H_1(K)$ versus $\pi_1(K)$; $H_2(K)$ versus orientability of K when K is a combinatorial surface; simplicial homology groups of a cone, $H_n(\Delta^{m+1})$, $H_n(\Sigma^m)$.
- A simplicial map $s : |K| \to |L|$ induces $s_q : C_q(K) \to C_q(L)$, and $s_{q*} : H_q(K) \to H_q(L)$. A general chain map also induces $\phi_{q*} : H_q(K) \to H_q(L)$.
- Composition of stellar subdivisions give the barycentric subdivision. Stellar subdivision gives the isomorphism $\chi_* : H_q(K) \to H_q(K^m)$ for barycentric subdivision. (proof not required)
- For a continuous map $f : |K| \to |L|$, define $f_* : H_q(K) \to H_q(L)$ by taking a simplicial approximation $s : |K^m| \to |L|$. f_* is independent of the simplicial approximation. (proof not required)

- For two continuous maps $f : |K| \to |L|$ and $g : |L| \to |M|$, $g_* \circ f_* = (g \circ f)_* : H_q(K) \to H_q(M)$. For two homotopic maps $f, g : |K| \to |L|$, $f_* = g_* : H_q(K) \to H_q(M)$. **Homotopic equivalent spaces and homeomorphic spaces have the same simplicial homology group.** (proof not required)
- S^n and S^m are not homeomorphic spaces, $\mathbb{E}^n$ and $\mathbb{E}^m$ are not homeomorphic spaces, Brouwer fixed–point theorem.

Chapter 9: Degree and Lefschetz number (9.1, 9.2, 9.4)

- Degree of $f : S^n \to S^n$: definition. Reflections on S^n have degree -1, the antipodal map on S^n has degree $(-1)^{n+1}$. Degree of self–maps on S^n versus fixed–point.
- **Euler–Poincaré formula**: $\chi(K) = \sum_{q=0}^{n}(-1)^q \beta_q$, here β_q is the rank of $H_q(K)$. Simplicial homology group with rational coefficients.
- Lefschetz fixed–point theorem: for a self–map $f : |K| \to |K|$, if $\Lambda_f \neq 0$, then f has a fixed–point. Here $\Lambda_f = \sum_{q=0}^{n}(-1)^q trace f_{q*}$.

Full Version

Chapter 2: Topological space and continuity.

- **Topological space**: defined by open sets: $\emptyset$ and X are open, finite intersection and any union of open sets are open. Neighborhoods, closed sets, interior, closure, limit point, denseness, base (countable base).
- Examples of topological spaces: $\mathbb{E}^n$, subspace topology, discrete topology, trivial topology; half–interval topology and finite complement topology on $\mathbb{R}$.
- **Continuous functions**: preimage of open sets are open. Check continuity by checking on a base, composition of continuous functions is continuous, **homeomorphism**, inclusion. $\mathbb{E}^n$ is homeomorphic with S^n with one point removed.
- Space filling curve: continuous surjective map from I to $I \times I$, (limit of a sequence of continuous functions).
- Metric space: metric topology, Hausdorff space, distance functions are continuous, Tietze extension theorem.

Chapter 3: Compactness and connectedness.

- **Compact space**: any open cover has a finite subcover. A subset in $\mathbb{E}^n$ is compact if and only if it is closed and bounded.
- Properties of compace spaces: continuous images of compact sets are compact sets, closed set versus compact set (Hausdorff condition), homeomorphic criterion of functions from compact spaces to Hausdorff spaces, Bolzano–Weierstrass property of compact spaces, Lebesgue's lemma.
- Product space: product of open sets gives the base of product topology, a function to a product space is continuous if and only if the projection to each component is continuous, product of compact spaces is compact.
- **Connectedness**: does not have a nontrivial open decomposition. Intervals are connected, continuous image of connected spaces are connected, closure of connected spaces are connected, product of connected spaces are connected. Connected component, connected components are closed sets.
- **Path–connectedness**: path–connected spaces are connected, connected open sets in $\mathbb{E}^n$ are path–connected, path–component. The closure of $\{(x, \sin\frac{1}{x}) \mid s \in (0,1]\}$ is connected, but not path–connected.

Chapter 4: Identification spaces.

- **Identification topology**: $\pi : X \to Y$ identifying equivalent classes in X, $U \subseteq Y$ is open if and only if $\pi^{-1}(U)$ is open. Identification topology versus identification map, open maps and closed maps are identification maps (the reverse is not true), criterion of identification maps form compact spaces to Hausdorff spaces, gluing lemma.
- Examples of identification space: Möbius strip, torus, Klein bottle, (topological) cone, projective spaces, attaching maps.
- Topological groups: topological space structure and group structure are compatible, also Hausdorff. Homomorphism, isomorphism, subgroup, left and right translation.
- Examples of topological groups: $\mathbb{R}$, S^1, product of topological groups, $GL(n)$, $SL(n)$, $SO(n)$, $O(n)$.
- Topological groups action on topological spaces: group actions preserve the topological structures, orbit spaces.

- Examples of topological groups action on topological spaces: subgroups action on a topological group by left (right) translation, cyclic groups ($\mathbb{Z}_2$) action on T^2, $\mathbb{Z}_p$ action on S^3, $\mathbb{R}$ action on T^2 with irrational slope.

Chapter 5: Fundamental group.

- **Homotopic maps**: deformation between continuous maps $f, g : X \to Y$, i.e. $F : X \times I \to Y$. Homotopic relative to a subset, homotopy is a equivalence relation, compositions of homotopic maps are homotopic.
- Examples of homotopies: straight–line homotopy, homotopy between maps to spheres (no antipodal image of the same point), construct homotopy by breaking I to pieces.
- **Definition of fundamental groups**: homotopy classes of paths in X with end points mapped to the base point, multiplication given by concatenation of paths.
- Change base point of fundamental groups by the isomorphism induced by paths ($\pi_1(X)$ can be defined for path–connected spaces without base point), continuous maps induce homomorphisms on fundamental groups.
- Simply connected space. $\pi_1(S^1) \cong \mathbb{Z}$, similarly $\pi_1(X/G) \cong G$ if $X \to X/G$ is a "covering map" and X is simply connected. $\pi_1(S^n) \cong \{e\}$ for $n > 1$, $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.
- Homotopy type: definition of **homotopic equivalence** and examples, deformation retract. Homotopic equivalent spaces have isomorphic fundamental groups, **homotopic maps induce the same homomorphism on fundamental groups** (up to base point changing). Contractible spaces.
- Brouwer fixed–point theorem (for dimension 2), retract of a space. Separation of the plane (just the statement). Interior points versus boundary points of surfaces.

Chapter 6: Triangulations.

- q–simplex: the set of all convex linear combinations of $q + 1$ points in $\mathbb{E}^n$ which are in general position.
- **Simplicial complexes**: collection of simplexes in $\mathbb{E}^n$ that fit together nicely, the polyhedron of a simplicial complex. Triangulation of a topological space, specific triangulation for S^1, S^2, T^2, Klein bottle. Isomorphism of simplicial complexes, basic topological properties of polyhedrons, connected polyhedrons are path–connected.
- Barycentric subdivision: make the size of simplexes smaller. (proof not required)
- Simplicial map. **Simplicial approximation theorem**: approximate a continuous map between polyhedrons by a simplicial map, with taking certain barycentric subdivision of the domain. (proof not required, but application is important)
- Computation of the fundamental groups of a simplicial complex (using simplicial approximation theorem): edge group $E(K, v)$, $G(K, L)$ for a subcomplex L with contractible polyhedron, express fundamental groups by generator–relator presentations.
- **Van Kampen's theorem**: compute $\pi_1(|K_1 \cup K_2|)$ by $\pi_1(|K_1|)$, $\pi_1(|K_2|)$ and $\pi_1(|K_1 \cap K_2|)$, require path–connectedness.
- Triangulating orbit spaces: $|K/G|$ might be different from $|K|/G$, may need to take barycentric subdivision.
- Infinite simplicial complexes: the polyhedron is a closed set in $\mathbb{E}^n$, identification topology agrees with subspace topology, local finiteness, fundamental groups of infinite simplicial complexes.

Chapter 7: Surfaces.

- n–dimensional manifold: Hausdorff space with a countable base, locally homeomorphic to $\mathbb{E}^n$ or $\mathbb{E}^n_+$. Closed n–dimensional manifold: n–dimensional manifold which is connected, compact and has no boundary.
- **Classification theorem of closed surfaces**: all closed surfaces are S^2, $H(p)$ (S^2 with p handles added) or $M(q)$ (S^2 with q discs replaced by Möbius strips). Klein bottle is homeomorphic to $M(2)$, mixing p handle and q Möbius strip operations equals $2p+q$ Möbius strip operations ($q>0$).
- Triangulation of closed surfaces: combinatorial surface. Orientable surface versus orientable combinatorial surface: any triangulation of an orientable surface gives an orientable combinatorial surface. Thickening of 1–dimensional subcomplexes: thickening of a tree is a disc, thickening of a simple closed curve is either a cylinder or a Möbius band.
- **Euler characteristic of a simplicial complex** K: $\chi(K)=\sum_{i=0}^{n}(-1)^i\alpha_i$, α_i is the number of i–simplexes in K. For a combinatorial surface K, $\chi(K)\leq 2$, $\chi(K)=2$ if and only if $|K|$ is homeomorphic to S^2 if and only if any polygonal simple closed curve in $|K^1|$ separates $|K|$.
- Surgery: remove the thickening of a separating polygonal simple closed curve, then add one or two discs, Euler characteristic increases by 1 or 2. Do such surgery for finitely many times to get S^2.
- Cut along curves in closed surfaces to get a polygon, with pair of edges identified, surface symbol. $H(p): a_1b_1a_1^{-1}b_1^{-1}\cdots a_pb_pa_p^{-1}b_p^{-1}$, $M(q): a_1^2a_2^2\cdots a_q^2$, use the surface symbol to compute π_1 and their abelianization.

Chapter 8: Simplicial homology.

- Orientations on a q–simplex: ordering of $q+1$ vertices modulo even permutations. Two distinct orientations for $q>0$ and only one orientation for $q=0$.
- **Definition of simplicial homology groups**:
  - $C_q(K)$ is the free abelian group generated by oriented q-simplexes in K, with the same simplex with opposite orientations are negative to each other.
  - $\partial: C_q(K)\to C_{q-1}(K)$ defined by $\partial(v_0,v_1,\cdots,v_q)=\sum_{i=0}^{q}(-1)^i(v_0,\cdots,\hat{v}_i,\cdots,v_q)$, then $\partial^2=0$.
  - $Z_q(K)=ker\{\partial: C_q(K)\to C_{q-1}(K)\}$ set of q–cycles, $B_q(K)=im\{\partial: C_{q+1}(K)\to C_q(K)\}$ set of bounding q–cycles. $B_q(K)\subseteq Z_q(K)$, and the qth homology group is defined by $H_q(K)=Z_q(K)/B_q(K)$.
- Examples of simplicial homology groups:
  - $H_0(K)\cong\mathbb{Z}^{\beta_0}$, β_0 is the number of components of $|K|$.
  - If K is connected, $H_1(K)$ is isomorphic to the abelianization of $\pi_1(K)$.
  - If K is a combinatorial surface, $H_2(K)\cong\mathbb{Z}$ if K is orientable, and $H_2(K)\cong 0$ if K is nonorientable.
  - $H_n(H(q))\cong\begin{cases}\mathbb{Z} & n=0\\ \mathbb{Z}^{2p} & n=1\\ \mathbb{Z} & n=2,\end{cases}$ $H_n(M(q))\cong\begin{cases}\mathbb{Z} & n=0\\ \mathbb{Z}^q\oplus\mathbb{Z}_2 & n=1\\ 0 & n=2.\end{cases}$
  - Simplicial homology groups of a cone is isomorphic to the simplicial homology group of a point. $H_n(\Delta^{m+1})\cong\begin{cases}\mathbb{Z} & n=0\\ 0 & n\neq 0,\end{cases}$ $H_n(\Sigma^m)\cong\begin{cases}\mathbb{Z} & n=0,m\\ 0 & n\neq 0,m.\end{cases}$
- A simplicial map $s:|K|\to|L|$ induces $s_q: C_q(K)\to C_q(L)$, with $\partial s_q=s_{q-1}\partial$. s **induces** $s_{q*}: H_q(K)\to H_q(L)$. For a sequence of homomorphisms $\phi_q: C_q(K)\to C_q(L)$, if $\partial\phi_q=\phi_{q-1}\partial$, then $\{\phi_q\}$ is called a chain map and induces $\phi_{q*}: H_q(K)\to H_q(L)$.

- Stellar subdivision: add the barycenter of one simplex in K to get a new simplicial complex, compositions of stellar subdivision give the barycentric subdivision. Such a composition gives isomorphisms $\chi_* : H_q(K) \to H_q(K^m)$ for the mth barycentric subdivision. (proof not required)
- For a continuous map $f : |K| \to |L|$, define $f_* : H_q(K) \to H_q(L)$ by the following way. Take a simplicial approximation $s : |K^m| \to |L|$, then f_* is defined to be $s_* \circ \chi_* : H_q(K) \to H_q(K^m) \to H_q(L)$. f_* is independent of the simplicial approximation. (For our exam, we will only compute f_* for simplicial maps. Proof not required)
- For two continuous maps $f : |K| \to |L|$ and $g : |L| \to |M|$, $g_* \circ f_* = (g \circ f)_* : H_q(K) \to H_q(M)$ holds. For two homotopic maps $f, g : |K| \to |L|$, we have $f_* = g_* : H_q(K) \to H_q(L)$. **Homotopic equivalent spaces and homeomorphic spaces have the same simplicial homology group.** (proof not required)
- S^n and S^m are not homeomorphic to each other, $\mathbb{E}^n$ and $\mathbb{E}^m$ are not homeomorphic to each other, Brouwer fixed–point theorem.

Chapter 9: Degree and Lefschetz number (9.1, 9.2, 9.4)

- Define the degree of $f : S^n \to S^n$ by $f_* : H_n(S^n) \cong \mathbb{Z} \to H_n(S^n) \cong \mathbb{Z}$. Reflections on S^n have degree -1, the antipodal map on S^n has degree $(-1)^{n+1}$. Degree of self–maps of S^n versus fixed–point or antipodal point.
- **Euler–Poincaré formula**: $\chi(K) = \sum_{q=0}^{n} (-1)^q \beta_q$, here β_q is the rank of $H_q(K)$. Simplicial homology group with rational coefficients, Euler characteristic is a topological invariant.
- Lefschetz fixed–point theorem: for a self–map $f : |K| \to |K|$, if $\Lambda_f \neq 0$, then f has a fixed–point. Here $\Lambda_f = \sum_{q=0}^{n} (-1)^q trace f_{q*}$ for $f_{q*} : H_q(K; \mathbb{Q}) \to H_q(K; \mathbb{Q})$.