MATH 104-06 FINAL SOLUTION

1. (10 points) Determine whether the following statements are true of false, no justification is required.

(1) For a function $f : X \to Y$ and two subsets $A, B \subseteq X$, we always have $f(A) \cap f(B) = f(A \cap B)$.

False

(2) For two sequences of real numbers (s_n) and (t_n), it is possible that

$$\limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n \neq \limsup_{n\to\infty}(s_n + t_n).$$

True

(3) There exists a sequence of real numbers (s_n) such that the set of subsequential limits of (s_n) is $(0, \pi]$.

False

(4) For a function $f : \mathbb{R} \to \mathbb{R}$, if $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$, then f is continuous at $x = 0$.

True

(5) Let (f_n) be a sequence of real–valued functions with $dom(f_n) = [-1, 1]$ for any $n \in \mathbb{N}$. If for some $f : [-1, 1] \to \mathbb{R}$, (f_n) converges pointwise to f on $[-1, 1]$, then (f_n) converges uniformly to f on $[-1, 1]$.

False

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that all the derivatives $f^{(n)}$ of f exist on $\mathbb{R}$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for any $x \in \mathbb{R}$.

False

(7) For a function $f : [0, 1] \to \mathbb{R}$, if $|f|$ is integrable on $[0, 1]$, then so does f.

False

(8) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function, such that $f(x) \geq 1$ for any $x \in [0, 1]$. If $\int_0^1 f = 1$, then $f(x) = 1$ for any $x \in [0, 1]$.

True

(9) Let (S, d) and (S', d') be two metric spaces, for any continuous function $f : S \to S'$ and any closed set $E \subseteq S'$, $f^{-1}(E)$ is always a closed set in S.

True

(10) For a metric space (S, d) and a subset $E \subseteq S$, if E is connected, then it is path–connected.

False

2. (15 points) Let (a_n) be a sequence of real numbers defined by $a_1 = 1$ and

$$a_{n+1} = \frac{a_n + 1}{a_n},$$

show that $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$ by the following steps.

(1) (3 points) Compute the first six terms of (a_n), then show that (a_n) is not a monotone sequence.

(2) (3 points) Show that $a_{2n} > \frac{1+\sqrt{5}}{2}$ and $a_{2n+1} < \frac{1+\sqrt{5}}{2}$ for any $n \in \mathbb{N}$. (Hint: use induction.)

(3) (4 points) Show that both (a_{2n}) and (a_{2n+1}) are both monotone subsequences of (a_n).

(4) (5 points) Show that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = \frac{1+\sqrt{5}}{2}$, then show $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$.

Proof. (1) $a_1 = 1$, $a_2 = 2$, $a_3 = \frac{3}{2}$, $a_4 = \frac{5}{3}$, $a_5 = \frac{8}{5}$, $a_6 = \frac{13}{8}$. Since $a_2 > a_1$ and $a_3 < a_2$, so (a_n) is not monotone.

(2) It is clearly that a_n are all positive real numbers.

We show that $a_{2n} > \frac{1+\sqrt{5}}{2}$ and $a_{2n+1} < \frac{1+\sqrt{5}}{2}$ by induction. The statement clearly holds for $n = 1$ by the above computation.

Suppose that $a_{2k} > \frac{1+\sqrt{5}}{2}$ and $a_{2k+1} < \frac{1+\sqrt{5}}{2}$, then we need to compare a_{2k+2} and a_{2k+3} with $\frac{1+\sqrt{5}}{2}$

$a_{2k+2} = \frac{a_{2k+1}+1}{a_{2k+1}} = 1 + \frac{1}{a_{2k+1}} > 1 + \frac{1}{\frac{1+\sqrt{5}}{2}} = 1 + \frac{-1+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$.

$a_{2k+3} = \frac{a_{2k+2}+1}{a_{2k+2}} = 1 + \frac{1}{a_{2k+2}} < 1 + \frac{1}{\frac{1+\sqrt{5}}{2}} = 1 + \frac{-1+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$.

(3) We first show that (a_{2n+1}) is an increasing sequence.

$$a_{2n+3} = \frac{a_{2n+2} + 1}{a_{2n+2}} = \frac{\frac{a_{2n+1}+1}{a_{2n+1}} + 1}{\frac{a_{2n+1}+1}{a_{2n+1}}} = \frac{2a_{2n+1} + 1}{a_{2n+1} + 1}.$$

Since $a_{2n+1} < \frac{1+\sqrt{5}}{2}$, we have $a_{2n+1} - a_{2n+3} = \frac{a_{2n+1}^2 - a_{2n+1} - 1}{a_{2n+1}} < 0$, which implies that (a_{2n+1}) is an increasing sequence.

Then we show that (a_{2n}) is an increasing sequence.

$$a_{2n+2} = \frac{a_{2n+1} + 1}{a_{2n+1}} = \frac{\frac{a_{2n}+1}{a_{2n}} + 1}{\frac{a_{2n}+1}{a_{2n}}} = \frac{2a_{2n} + 1}{a_{2n} + 1}.$$

Since $a_{2n} > \frac{1+\sqrt{5}}{2}$, we have $a_{2n} - a_{2n+2} = \frac{a_{2n}^2 - a_{2n} - 1}{a_{2n}} > 0$, which implies that (a_{2n}) is a decreasing sequence.

(4) In the previous subquestion, we showed that (a_{2n+1}) is an increasing function, and bounded above by $\frac{1+\sqrt{5}}{2}$; while (a_{2n}) is a decreasing function, and bounded below by $\frac{1+\sqrt{5}}{2}$. So both $\lim_{n\to\infty} a_{2n+1}$ and $\lim_{n\to\infty} a_{2n}$ exists, and suppose that $\lim_{n\to\infty} a_{2n+1} = t_1$ and $\lim_{n\to\infty} a_{2n} t_2$. Since all the numbers a_n are positive, we have $t_1, t_2 \geq 0$.

In the previous subquestion, we got that $a_{2n+3} = \frac{2a_{2n+1}+1}{a_{2n+1}+1}$. By taking the limit and let n goes to infinity, we get

$$t_1 = \lim_{n\to\infty} a_{2n+3} = \lim_{n\to\infty} \frac{2a_{2n+1}+1}{a_{2n+1}+1} = \frac{2t_1+1}{t_1+1}.$$

So we have $t_1^2 - t_1 - 1 = 0$. Moreover, since $t_1 \geq 0$, we have $t_1 = \frac{1+\sqrt{5}}{2}$.

Since $a_{2n+2} = \frac{2a_{2n}+1}{a_{2n}+1}$, by doing the same process as above and let n goes to infinity, we get $t_2 = \frac{1+\sqrt{5}}{2}$. So $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = \frac{1+\sqrt{5}}{2}$ holds.

For any $\epsilon > 0$, since $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = \frac{1+\sqrt{5}}{2}$, there exists N_1, such that for any $n > N_1$, $|a_{2n} - \frac{1+\sqrt{5}}{2}| < \epsilon$ holds; and there also exists N_2 such that for any $n > N_2$, $|a_{2n+1} - \frac{1+\sqrt{5}}{2}| < \epsilon$. So for $N = \max\{2N_1, 2N_2+1\}$, and for any $n > N$, we have $|a_n - \frac{1+\sqrt{5}}{2}| < \epsilon$ holds, so $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$.

$\square$

3. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two uniformly continuous functions. If both f and g are bounded, show that $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = f(x) \cdot g(x)$ is also uniformly continuous.

Proof. Since both f and g are bounded, there exists $M_1 > 0$ such that $|f(x)| < M_1$ for any $x \in \mathbb{R}$; and there exists $M_2 > 0$ such that $|g(x)| < M_2$ for any $x \in \mathbb{R}$. So for $M = \max\{M_1, M_2\}$, we have $|f(x)|, |g(x)| < M$ for any $x \in \mathbb{R}$.

For any $\epsilon > 0$, since f is uniformly continuous on $\mathbb{R}$, there exists $\delta_1 > 0$ such that for any $x, y \in \mathbb{R}$ with $|x - y| < \delta_1$, $|f(x) - f(y)| < \frac{\epsilon}{2M}$ holds. For the same $\epsilon > 0$, since g is uniformly continuous on $\mathbb{R}$, there exists $\delta_2 > 0$ such that for any $x, y \in \mathbb{R}$ with $|x - y| < \delta_2$, $|g(x) - g(y)| < \frac{\epsilon}{2M}$ holds.

Then for $\delta = \min\{\delta_1, \delta_2\}$ and $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have

$$
\begin{aligned}
&|h(x) - h(y)| = |f(x)g(x) - f(y)g(y)| \\
=&\big|\big(f(x)g(x) - f(x)g(y)\big) + \big(f(x)g(y) - f(y)g(y)\big)\big| \\
\le&\big|f(x)\big(g(x) - g(y)\big)\big| + \big|g(y)\big(f(x) - f(y)\big)\big| \\
\le&|f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| \\
<&M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\
=&\epsilon.
\end{aligned}
$$

So $h : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. $\square$

4. (15 points) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of real-valued functions defined by

$$f_n(x) = (-1)^n \frac{x^2}{x^2+n}.$$

(1) (3 points) Find the function $f : \mathbb{R} \to \mathbb{R}$ such that (f_n) converges pointwise to f on $\mathbb{R}$.
(2) (6 points) Show that (f_n) converges uniformly to f on $[-M, M]$ for any $M > 0$.
(3) (6 points) Does (f_n) converges uniformly to f on $\mathbb{R}$? Justify your answers.

Proof. (1) Fix a number $x \in \mathbb{R}$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (-1)^n \frac{x^2}{x^2+n} = \lim_{n\to\infty} (-1)^n \frac{\frac{x^2}{n}}{\frac{x^2}{n}+1} = 0.$$

So (f_n) pointwise converge to $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for any $x \in \mathbb{R}$.

(2) Fix a number $M > 0$, for any $\epsilon > 0$, we need to find N such that for any $n > N$ and any $x \in [-M, M]$, $|f_n(x) - f(x)| = |f_n(x)| < \epsilon$ holds.

Actually, $N = \frac{M^2}{\epsilon}$ works here. This is because for any $n > N$ and $x \in [-M, M]$, we have

$$|f_n(x) - f(x)| = |f_n(x)| = \frac{x^2}{x^2+n} \leq \frac{x^2}{n} < \frac{M^2}{N} = \epsilon.$$

(3) (f_n) does not converge to f uniformly on $\mathbb{R}$. If the convergence is a uniformly convergence, we have that for any $\epsilon > 0$, there exists N such that for any $n > N$ and any $x \in \mathbb{R}$, we have $|f_n(x) - f(x)| = |f(x)| < \epsilon$.

However, we take $\epsilon = \frac{1}{4}$ now. For each n, take $x_n = \sqrt{n} \in \mathbb{R}$, then $|f_n(x_n)| = \frac{x_n^2}{x_n^2+n} = \frac{n}{n+n} = \frac{1}{2} > \epsilon$. So (f_n) does not converge to f uniformly on $\mathbb{R}$.

$\square$

5. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a real–valued function defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} & x > 0 \\ -x & x \le 0. \end{cases}$$

Whether there exists a real–valued function $g : \mathbb{R} \to \mathbb{R}$ such that $g'(x) = f(x)$ for any $x \in \mathbb{R}$? Justify your answers.

Proof. There does not exist a real–valued function $g : \mathbb{R} \to \mathbb{R}$ such that $g'(x) = f(x)$ for any $x \in \mathbb{R}$.

$f = g' : \mathbb{R} \to \mathbb{R}$ should satisfy the intermediate value theorem since it is the derivative of a differentiable function. Now we show that f does not have the intermediate value property.

We first show that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$. This is because that $\lim_{x\to 0^+} \sin x = \lim_{x\to 0^+} x = 0$ and

$$\lim_{x\to 0^+} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0^+} \frac{\cos x}{1} = 1.$$

So the L'Hospital's rule can be applied here.

So for $\epsilon = \frac{1}{2}$, there exists $\delta > 0$, such that for any $0 < x < \delta$, we have $|\frac{\sin x}{x} - 1| < \frac{1}{2}$.

Take $x_1 = 0$ and $x_2 = \frac{\delta}{2}$, we have $f(x_1) = 0$ and $f(x_2) \in (\frac{1}{2}, \frac{3}{2})$, the intermediate value property claims that there exists $x_0 \in (x_1, x_2) = (0, \frac{\delta}{2})$ such that $f(x_0) = \frac{1}{4} \in (f(x_1), f(x_2))$. However, we know that for any $x_0 \in (0, \frac{\delta}{2})$, $f(x_0) \in (\frac{1}{2}, \frac{3}{2})$ holds. So we get a contradiction. $\square$

6. (20 points) Let $f : [-1,1] \to \mathbb{R}$ be a real-valued function defined by

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(1) (4 points) Show that f is discontinuous at $x = 0$.
(2) (10 points) Show that f is integrable on $[-1,1]$.
(3) (6 points) Compute $\int_{-1}^{1} f(x)\mathrm{d}x$.

Proof. (1) For the sequence (x_n) defined by $x_n = \frac{1}{(n+\frac{1}{2})\pi}$, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{(n+\frac{1}{2})\pi} = 0$. However, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin\frac{1}{\frac{1}{(n+\frac{1}{2})\pi}} = \lim_{n\to\infty} \sin(n+\frac{1}{2})\pi = \lim_{n\to\infty}(-1)^n$ and the limit does not exist. So f is discontinuous at $x = 0$.

(2) For any $\epsilon > 0$, we need to find a partition P for $[-1,1]$ such that $U(f,P) - L(f,P) < \epsilon$. Note that $|f(x)| \leq 1$ for any $x \in [-1,1]$.

Since $f(x) = -f(-x)$ for any $x \in [0,1]$, f is an odd function. We will first find a partition Q for $[0,1]$, then $Q \cup (-Q)$ is a partition of $[-1,1]$.

Since f is not continuous at $x = 0$, we need to consider about the point $x = 0$ separately. For our partition $Q = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$, we first take $t_1 = \frac{\epsilon}{10}$. Since f is continuous on $[t_1, 1]$, it is integrable on $[t_1, 1]$. So for $\frac{\epsilon}{4} > 0$, there is a partition $R = \{t_1 < t_2 < \cdots < t_n = 1\}$ of $[t_1, 1]$, such that $U(f,R) - L(f,R) < \frac{\epsilon}{4}$. Then the partition of $[0,1]$ given by $t_0 = 0$ and R is $Q = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$, such that

$$U(f,Q) - L(f,Q) = \big(M(f,[0,t_1]) - m(f,[0,t_1])\big)\cdot t_1 + \big(U(f,R) - L(f,R)\big) < 2\cdot\frac{\epsilon}{10} + \frac{\epsilon}{4} < \frac{\epsilon}{2}.$$

Then $P = \{-1 = -t_n < -t_{n-1} < \cdots < -t_1 < 0 = t_0 < t_1 < \cdots < t_n = 1\}$ is a partition of $[-1,1]$ which is given by $Q \cup (-Q)$. Since $f(x) = -f(-x)$ for any $x \in [0,1]$, we have $M(f,[-t_k,-t_{k-1}]) = -m(f,[t_{k-1},t_k])$ and $m(f,[-t_k,-t_{k-1}]) = -M(f,[t_{k-1},t_k])$ for $k = 1,2,\cdots,n$. So we have

$$\begin{aligned}
&U(f,P) - L(f,P) \\
=&\sum_{k=n}^{1} \big(M(f,[-t_k,-t_{k-1}]) - m(f,[-t_k,-t_{k-1}])\big)\cdot(-t_{k-1} - (-t_k)) + (U(f,Q) - L(f,Q)) \\
=&\sum_{k=n}^{1} \big(-m(f,[t_{k-1},t_k]) + M(f,[t_{k-1},t_k])\big)\cdot(t_k - t_{k-1}) + (U(f,Q) - L(f,Q)) \\
=&2(U(f,Q) - L(f,Q)) \\
<&2\cdot\frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

So f is integrable on $[-1,1]$.

(3) Now we give estimations for $\int_{-1}^{1} f$, and show that it equals 0. For $U(f,P)$, we have

$$\begin{aligned}
&U(f,P)\\
=&\sum_{k=n}^{1} M(f,[-t_k,-t_{k-1}])\cdot(-t_{k-1}-(-t_k))+\sum_{k=1}^{n} M(f,[t_{k-1},t_k])\cdot(t_k-t_{k-1})\\
=&\sum_{k=1}^{n}\big(M(f,[-t_k,-t_{k-1}])+M(f,[t_{k-1},t_k])\big)\cdot(t_k-t_{k-1})\\
=&\sum_{k=1}^{n}\big(M(f,[t_{k-1},t_k]-m(f,[t_{k-1},t_k])\big)\cdot(t_k-t_{k-1})\\
\geq&0.
\end{aligned}$$

For $L(f,P)$, we have

$$\begin{aligned}
&L(f,P)\\
=&\sum_{k=n}^{1} m(f,[-t_k,-t_{k-1}])\cdot(-t_{k-1}-(-t_k))+\sum_{k=1}^{n} m(f,[t_{k-1},t_k])\cdot(t_k-t_{k-1})\\
=&\sum_{k=1}^{n}\big(m(f,[-t_k,-t_{k-1}])+m(f,[t_{k-1},t_k])\big)\cdot(t_k-t_{k-1})\\
=&\sum_{k=1}^{n}\big(m(f,[t_{k-1},t_k]-M(f,[t_{k-1},t_k])\big)\cdot(t_k-t_{k-1})\\
\leq&0.
\end{aligned}$$

So we have that $L(f,P)\leq 0\leq U(f,P)$, and also $L(f,P)\leq L(f)=\int_{-1}^{1} f = U(f)\leq U(f,P)$. Since $U(f,P)-L(f,P)<\epsilon$, we have $|\int_{-1}^{1} f-0|=|\int_{-1}^{1} f|<\epsilon$. Since ϵ can be any arbitrarily small positive number, we have that $\int_{-1}^{1} f=0$ holds.

□

7. (20 points) Let S be the set of all sequences of real numbers $x = (x_1, x_2, \cdots)$ such that $\sum_{k=1}^{\infty} |x_k| < \infty$.

(1) (8 points) For $\mathbf{x}, \mathbf{x}' \in S$, define $d(\mathbf{x}, \mathbf{x}') = \sum_{k=1}^{\infty} |x_k - x_k'|$. Show that d is a metric on S.

(2) (8 points) For any bounded sequence of real numbers $\mathbf{y} = (y_1, y_2, \cdots)$, we can define a real–valued function $f_{\mathbf{y}} : S \to \mathbb{R}$ by $f_{\mathbf{y}}(\mathbf{x}) = \sum_{k=1}^{\infty} y_k \cdot x_k$ for any $\mathbf{x} = (x_1, x_2, \cdots) \in S$. Show that $f_{\mathbf{y}}$ is a continuous function.

(3) (4 points) Let $E \subseteq S$ be the subset consists of all sequences $\mathbf{x} = (x_1, x_2, \cdots)$ such that $\sum_{k=1}^{\infty} |x_k| = 1$. Is E compact? Justify your answers.

Warning: You need to show the series in the problem converge.

Proof. (1) At first, we need to show that for $\mathbf{x}, \mathbf{x}' \in S$, $\sum_{k=1}^{\infty} |x_k - x_k'|$ converges. Otherwise, it is not a real number.

Since $\sum_{k=1}^{\infty} |x_k|$ converges and $\sum_{k=1}^{\infty} |x_k'|$ converges, we have $\sum_{k=1}^{\infty} |x_k| + |x_k'|$ converges. Since $|x_k - x_k'| \leq |x_k| + |x_k'|$ for any k, the comparison test implies that $\sum_{k=1}^{\infty} |x_k - x_k'|$ converges to a real number.

Since each term $|x_k - x_k'| \geq 0$, we have $d(\mathbf{x}, \mathbf{x}') = \sum_{k=1}^{\infty} |x_k - x_k'| \geq 0$. $d(\mathbf{x}, \mathbf{x}') = 0$ if and only if $d(\mathbf{x}, \mathbf{x}') = \sum_{k=1}^{\infty} |x_k - x_k'| = 0$, if and only if $|x_k - x_k'| = 0$ for any k, if and only if $x_k = x_k'$ for any k, if and only if $x = x'$.

Since $|x_k - x_k'| = |x_k' - x_k|$, we have $d(\mathbf{x}, \mathbf{x}') = \sum_{k=1}^{\infty} |x_k - x_k'| = \sum_{k=1}^{\infty} |x_k' - x_k| = d(\mathbf{x}', \mathbf{x})$.

For $\mathbf{x}, \mathbf{x}', \mathbf{x}'' \in S$, since $|x_k - x_k''| \leq |x_k - x_k'| + |x_k' - x_k''|$ for any k, we have $\sum_{k=1}^{\infty} |x_k - x_k''| \leq \sum_{k=1}^{\infty} |x_k - x_k'| + \sum_{k=1}^{\infty} |x_k' - x_k''|$ for any n.

So d is a metric on S, and (S, d) is a metric space.

(2) At first, we need to show that $\sum_{k=1}^{\infty} y_k \cdot x_k$ converges. Since $(y_1, y_2, \cdots)$ is a bounded sequence, there exists $M > 0$ such that $|y_k| \leq M$ for any k. Since $\sum_{k=1}^{\infty} |x_k|$ converges, we have that $\sum_{k=1}^{\infty} M|x_k|$ converges. Since $|y_k \cdot x_k| \leq M|x_k|$ holds for any k, the comparison test implies that $\sum_{k=1}^{\infty} y_k \cdot x_k$ converges to a real number.

For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{M}$, then for any $\mathbf{x}, \mathbf{x}' \in S$ with $d(\mathbf{x}, \mathbf{x}') < \delta$, we will show that $|f_{\mathbf{y}}(\mathbf{x}) - f_{\mathbf{y}}(\mathbf{x}')| < \epsilon$. This is given by the following estimation:

$$|f_{\mathbf{y}}(\mathbf{x}) - f_{\mathbf{y}}(\mathbf{x}')| = |\sum_{k=1}^{\infty} y_k \cdot x_k - \sum_{k=1}^{\infty} y_k \cdot x_k'| = |\sum_{k=1}^{\infty} y_k \cdot (x_k - x_k')|$$

$$\leq \sum_{k=1}^{\infty} |y_k| \cdot |x_k - x_k'| \leq M \cdot \sum_{k=1}^{\infty} \cdot |x_k - x_k'| = M \cdot d(\mathbf{x}, \mathbf{x}') < \epsilon$$

So $f_{\mathbf{y}}$ is a continuous function.

(3) E is not compact. Let $\mathbf{x}^{(\mathbf{n})} \in S$ be the sequence such that the n–th coordinate is 1, and the other coordinates are 0, then $\mathbf{x}^{(\mathbf{n})} \in E$. Let $D_n = \{\mathbf{x} \in S \mid d(\mathbf{x}, \mathbf{x}^{(\mathbf{n})}) < \frac{1}{2}\}$ for any positive integer n, and $C = \{\mathbf{x} \in S \mid d(\mathbf{x}, \mathbf{x}^{(\mathbf{n})}) > \frac{1}{4} \text{ for any } n\}$.

It is easy to see that all the D_ns and C are open sets in S and they form a open cover of E (also cover S). However, since $\mathbf{x}^{(\mathbf{n})} \notin D_m$ for $n \neq m$ and $\mathbf{x}^{(\mathbf{n})} \notin C$, any subcover which covers E have to inclued D_n to cover $\mathbf{x}^{(\mathbf{n})}$. So E is not a compact set.

□