DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

In this exam you may assume, without justification the following identity:

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

This exam consists of 5 questions. Answer the questions in the spaces provided.

Name and section: ______________________

GSI's name: ______________________

1. Determine if the following sequences converge or diverge. Carefully justify your answer.

   (a) (10 points)

$$\left\{\frac{e^{-n}}{\sin(\frac{1}{n})}\right\}_{n=1}^{\infty}$$

Solution:

Let $f(x) = \dfrac{e^{-x}}{\sin(\frac{1}{x})}$. By l'Hopitales' Rule

$$\lim_{x\to\infty} \frac{e^{-x}}{\sin(\frac{1}{x})} = \lim_{x\to\infty} \frac{-e^{-x}}{\frac{-1}{x^2}\cos(\frac{1}{x})} = \lim_{x\to\infty} \frac{x^2e^{-x}}{\cos(\frac{1}{x})}$$

$\lim_{x\to\infty}\cos(\frac{1}{x}) = 1$, hence consider $\lim_{x\to\infty} \dfrac{x^2}{e^x} = \lim_{x\to\infty} \dfrac{2}{e^x} = 0$

l'Hopitals Rule.

Hence sequence convergent.

   (b) (10 points)

$$\left\{\frac{1}{2+(-1)^n}\right\}_{n=1}^{\infty}$$

Solution:

$$\left\{\frac{1}{2+(-1)^n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{3}, 1, \frac{1}{3}, \ldots\right\}$$

This sequence is clearly divergent.

2. (20 points) Using the integral test, prove the following series is convergent

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}.$$

Using this, prove that

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^3+1}$$

is convergent.

Solution:

Let $f(x) = \dfrac{e^{\frac{1}{x}}}{x^2}$. Note that $f'(x) = \dfrac{-e^{\frac{1}{x}}}{x^4} - \dfrac{2e^{\frac{1}{x}}}{x^3}$

$= \left(\dfrac{-1-2x}{x^4}\right) e^{\frac{1}{x}}$

Hence $f'(x) < 0$ for all $x \geq 1$

Thus the sequence is positive and decreasing, so we may apply integral test.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to\infty}\left[-e^{\frac{1}{x}}\right]_1^t$$

$$= \lim_{t\to\infty}\left(e - e^{\frac{1}{t}}\right)$$

$$= e - 1$$

By integral test $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ convergent.

Note that $0 < \dfrac{e^{\frac{1}{n}}}{n^3+1} < \dfrac{e^{\frac{1}{n}}}{n^2} \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^3+1}$ convergent

by comparison test.

3. (20 points) Determine if the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{(n-1)2^{\sin(n^2)}}{n^4+3n+1}$$

Solution:

Note that for all $n \geq 1$ $-1 < \sin(n^2) < 1$

$\Rightarrow 2^{-1} < 2^{\sin(n^2)} < 2$

We shall therefore do a comparison with

$$\sum_{n=1}^{\infty} \frac{2(n-1)}{n^4+3n+1}.$$

Note that for all $n \geq 1$ $\frac{2(n-1)}{n^4+3n+1} < \frac{2n}{n^4} = \frac{2}{n^3}$

$\sum_{n=1}^{\infty} \frac{2}{n^3} = 2\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent as $3 > 1$

By comparison test $\sum_{n=1}^{\infty} \frac{2(n+1)}{n^4+3n+1}$ convergent. Hence

$\sum_{n=1}^{\infty} \frac{(n-1)2^{\sin(n^2)}}{n^4+3n+1}$ convergent.

4. Determine whether the following series are convergent or divergent. If convergent determine the sum.

(a) (10 points)

$$\sum_{n=1}^{\infty} n\tan(\frac{1}{n})$$

Solution:

Let $f(x) = x\tan\left(\frac{1}{x}\right) = \dfrac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$

$\lim\limits_{x\to\infty} f(x) \underset{\text{L'Hopitales}}{=} \lim\limits_{x\to\infty} \dfrac{\frac{-1}{x^2}\sec^2\left(\frac{1}{x}\right)}{\frac{-1}{x^2}} = \lim\limits_{x\to\infty} \sec^2\left(\frac{1}{x}\right)$

$= \sec^2(0) = 1 \neq 0.$

Hence series

(b) (10 points)

$$\sum_{n=1}^{\infty} \frac{10^n + 5^n}{6^n + 4^n + 3^n}$$

Solution:

Do limit comparison test with $\sum\limits_{n=1}^{\infty}\left(\frac{10}{6}\right)^n$.

$\dfrac{\left(\frac{10^n+5^n}{6^n+4^n+3^n}\right)}{\left(\frac{10^n}{6^n}\right)} = \dfrac{60^n + 30^n}{60^n+40^n+30^n} = \dfrac{1+\left(\frac{30}{60}\right)^n}{1+\left(\frac{40}{60}\right)^n+\left(\frac{30}{60}\right)^n} \to \frac{1}{1} = 1$

as $n\to\infty$. $1 > 0$. Hence because $\frac{10}{6} > 1$, so $\sum\limits_{n=1}^{\infty}\left(\frac{10}{6}\right)^n$ diverges, so by limit comparison test $\sum\limits_{n=1}^{\infty}\dfrac{10^n+5^n}{6^n+4^n+3^n}$ diverges.

5. (20 points) Determine whether the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^5}{\sin(\frac{1}{n})n!}$$

Solution:

Do ratio test.

Let $a_n = (-1)^n \dfrac{n^5}{\sin(\frac{1}{n})\, n!} \Rightarrow \left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)^5 \sin(\frac{1}{n})\, n!}{n^5 \sin(\frac{1}{n+1})\,(n+1)!}$

$$= \frac{(n+1)^4}{n^5} \cdot \frac{\sin(\frac{1}{n})}{\sin(\frac{1}{n+1})}$$

Consider $\lim\limits_{n\to\infty} \dfrac{(n+1)^4}{n^5} = \lim\limits_{n\to\infty} \dfrac{(1+\frac{1}{n})^4}{n}$

$$= 0$$

Let $f(x) = \dfrac{\sin(\frac{1}{x})}{\sin(\frac{1}{x+1})}$, then $\lim\limits_{x\to\infty} \dfrac{\sin(\frac{1}{x})}{\sin(\frac{1}{x+1})}$

$\underset{\text{l'Hopital's}}{=} \lim\limits_{x\to\infty} \dfrac{\frac{-1}{x^2}\cos(\frac{1}{x})}{\frac{-1}{(x+1)^2}\cos(\frac{1}{x+1})} = \lim\limits_{x\to\infty} \dfrac{(x+1)^2}{x^2} \cdot \dfrac{\cos(\frac{1}{x})}{\cos(\frac{1}{x+1})}$

$$= \lim_{x\to\infty} \frac{(1+\frac{1}{x})^2}{1} \cdot \frac{\cos(\frac{1}{x})}{\cos(\frac{1}{x+1})}$$

$$= 1$$

Hence $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 0 \Rightarrow$ series is absolutely convergent.

END OF EXAM