Math 1B, Practice material for the First Midterm Examination

Practice exam 1

1.(*15 points*) Evaluate the integral

$$\int (\frac{x}{x+1})^2 dx$$

2.(*20 pnts*) Evaluate the integral

$$\int \sqrt{\frac{1+x}{1-x}} dx$$

3.(*25 pnts*) Determine whether each improper integral is convergent or divergent. Evaluate the integrals which are convergent.

(a)

$$\int_2^{\infty} \frac{dx}{(x^6 - x^2)^{\frac{1}{6}}}$$

(b)

$$\int_1^{2} \frac{dx}{x(\ln\ x)^{\frac{2}{3}}}$$

4.(*20 pnts*) We approximate

$$\int_1^2 10(57x + \frac{1}{x})dx$$

with the Trapezoidal rule, and want to make sure the error is at most 0.0001. Which of the following is true? (Explain your answer)

(a) We can take any $N \leq 50$

(b) We can take any $100 \leq N \leq 400$

c) We can take any $N \geq 50$

d) We can take any $N \geq 5$

5.(*20 pnts*) Consider the curve given by the equation

$$y = \frac{1}{2}(\sin\ x - \int_0^x \frac{dx}{\cos\ y}) = \frac{1}{2}(\sin\ x - \ln(\sec\ x - \tan\ x)) \tag{1}$$

Find the arclength of this curve for $0 \leq x \leq \pi/4$.

Solutions.

1. Using the decomposition into partial fractions we have:

$$\frac{x^2}{(x+1)^2} = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$$

Thus:

$$\int \frac{x^2}{(x+1)^2}dx = \int dx - \int \frac{2}{x+1}dx + \int \frac{1}{(x+1)^2}dx = x - 2\ln|x=1| - \frac{1}{(x+1)^2} + C$$

2.

$$\sqrt{\frac{x-1}{x+1}} = \sqrt{(1-x)^2}1 - x^2$$

After the substitution $x = \sin\theta$, taking into account $1 - \sin^2\theta = \cos^2\theta$ we have

$$\sqrt{\frac{x-1}{x+1}} = \frac{1-\sin\theta}{\cos\theta}$$

Thus, for the integral we have:

$$\int \sqrt{\frac{x-1}{x+1}}dx = \int \frac{1-\sin\theta}{\cos\theta}\cos\theta d\theta = \int d\theta - \int \sin\theta d\theta = \theta + \cos\theta + C$$

3(a).

$$x^6 - x^2 < x^6$$

for $x > 2$. Therefore

$$\frac{1}{(x^6 - x^2)^{1/6}} > \frac{1}{(x^6)^{1/6}} = \frac{1}{x}$$

The integral

$$\int_2^\infty \frac{dx}{x}$$

diverges, therefore by the comparison test the integral in question also diverges.

3(b).

$$\int \frac{dx}{x(\ln x)^{2/3}} = \int \frac{d(\ln x)}{(\ln x)^{2/3}} = \int \frac{dt}{t^2/3}$$

where $t = \ln x$. Thus, the question becomes: Is the integral

$$\int_0^{\ln 2} \frac{dt}{t^2/3}$$

convergent? This is a "p-integral" for $p = 2/3$ between 0 and $\ln 2$. It is convergent.

4. We want the error of the approximation to be not greater then $0.0001 = 1/100^2$.

In our case the error estimate for the Trapezoidal rule is

$$E_T \leq \frac{K}{12N^2}$$

where K is such that $|(10(57x + 1/x)''| < K$ fro all $1 \leq x \leq 2$. After differentiation we see that the second derivative is a monotone function decreasing from 10 to $10/4$ when x varies from 1 to 2.

Thus, we can choose $K = 10$. Taking into account this and the inequality $12 > 10$ we have:

$$E_T \leq \frac{10}{12N^2} < 1/N^2$$

Therefore the we will be within the error 0.0001 if $1/N^2 < 0.0001$, i.e. $N > 100$.

The correct answer is (b).

5.

$$\frac{dy}{dx} = \frac{1}{2}(\cos x - \frac{1}{\cos x})$$

. The following is an elementary identity:

$$1 + (\frac{dy}{dx})^2 = \frac{1}{4}(\cos x + \frac{1}{\cos x})^2$$

Thus, the integral for the arclength becomes

$$\int \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_0^{\pi/4} \frac{1}{2}(\cos x + \frac{1}{\cos x}) dx = \sin x|_0^{\pi/4} + \int_0^{\pi/4} \frac{\cos x}{\cos^2 x} dx =$$
$$\sqrt{2}/2 + \int_0^{\pi/4} \frac{\cos x}{1 - \sin^2 x} dx = \sqrt{2}/2 + \int_0^{\sqrt{2}/2} \frac{ds}{1 - s^2} \quad (2)$$

The last integral can be integrated by decomposition into partial fractions.

Practice exam 2

1.(*15 points*) Evaluate the integral

$$\int \frac{x}{x^2 - x - 6} dx$$

2.(*20 pnts*) Evaluate the integral

$$\int \frac{x^2 dx}{(\sqrt{4 - x^2})^3}$$

3.(*25 pnts*) Determine whether each improper integral is convergent or divergent. Evaluate the integrals which are convergent.

(a)

$$\int_{-1}^{1} \frac{(x+1)}{x^{\frac{1}{5}}} dx$$

(b)

$$\int_{2}^{\infty} \frac{x}{\sqrt{x^3-1}} dx$$

4.(*20 pnts*) We approximate

$$\int_{1}^{2} \ln x dx$$

with the Trapezoidal rule, and want to make sure the error is at most 0.00001. Which of the following is true? (Explain your answer)

(a) We can take any $N \leq 400$

(b) We can take any $N \leq 68$

c) We can take any $N \geq 6$

d) We can take any $N \geq 100$

5.(*20 pnts*) The curve

$$y = \frac{x^2}{2}$$

is rotated about x-axis. Find the area of the resulting surface for $0 \leq x \leq 1$.

Solutions:

1. Taking into account $x^2 - x - 6 = (x-3)(x+2)$ we have the decomposition into partial fractions:

$$\frac{x}{x^2 - x - 6} = \frac{x}{(x-3)(x+2)} = \frac{3/5}{x-3} + \frac{2/5}{x+2}$$

Integrating each term we obtain:

$$\int \frac{x}{x^2 - x - 6} dx = 3/5 \ln|x-3| + 2/5 \ln|x+2| + C$$

2. Use the substitution $x + 2\sin\theta$. Then

$$\frac{x^2}{(\sqrt{4-x^2})^3} dx = \frac{4\sin^2\theta}{(2\cos\theta)^3} 2\cos\theta d\theta = \tan^3\theta d\theta = \sec^2\theta d\theta - d\theta$$

Therefore

$$\int \frac{x^2}{(\sqrt{4-x^2})^3} dx = \int \sec^2\theta d\theta - \int -d\theta = \tan\theta - \theta + C = -\frac{x}{\sqrt{4-x^2}} - \arcsin(\frac{x}{2}) + C$$

3.
$$\int_{-1}^{1} \frac{(x+1)}{x^{1/5}} dx = \int_{-1}^{1} x^{4/5} dx + \int_{-1}^{1} \frac{1}{x^{1/5}} dx$$

First integral is proper. The second integral has a discontinuity point inside of the interval of integration. So, we should investigate $\int_0^1 \frac{1}{x^{1/5}} dx$ and $\int_{-1}^0 \frac{1}{x^{1/5}} dx$ separately. Both of these integrals converge (either argue that they are corresponding "p-integrals" or compute them and find the limits). Thus, the integrals converge and we have:

$$\int_{-1}^{1} x^{4/5} dx + \int_{-1}^{1} \frac{1}{x^{1/5}} dx = \frac{x^{4/5+1}}{4/5+1}|_{-1}^{1} + \frac{x^{-1/5+1}}{-1/5+1}|_{-1}^{1} = \frac{130}{36}$$

4. We have:

$$|(\ln x)''| = \frac{1}{x^2}$$

. Thus, when $1 \leq x \leq 2$, the absolute value of the second derivative is a monotonically decreasing function and therefore $|(\ln x)''| \leq 1$ for these x's.

From the formula for the error bound for the Trapezoidal rule we know that the error will be within 0.00001 if

$$\frac{1}{12N^2} < 0.00001$$

Here we took into account the upper bound for the absolute value of the second derivative obtained above.

Taking into account that $12 > 10$ we conclude that for $N > 100$ the error will be within the given bound.

Thus, (d) is the correct answer.

5. After the substitution $x = tah\theta$ we have:

$$\int_0^1 \sqrt{1+x^2}dx = \int_0^{\pi/4} \frac{1}{\cos^3\theta}d\theta = \int_0^{\pi/4} \frac{\cos\theta d\theta}{(1-\sin^2\theta)^2} = \int_0^{\sqrt{2}/2} \frac{dt}{(1-t^2)^2}$$

Here we used the substitution $t = \sin\theta$. Using a decomposition into partial fractions we have:

$$\frac{1}{(1-t^2)^2} = \frac{1}{4(1-t)^2} + \frac{1}{8(1-t)} + \frac{1}{8(1+t)} + \frac{1}{4(1+t)^2}$$

Integrating this expression term by term we obtain:

$$\int_0^1 \sqrt{1+x^2}dx = (\frac{1}{2(1-t^2)} + \frac{1}{8}\ln|\frac{1+t}{1-t}|)|_0^{\sqrt{2}/2}$$

.

1. Evaluate the following integrals:
(a) $\int x^{1/3} \ln x dx$

(b) $\int \frac{u+3}{(u-1)(u-3)} du$

(c) $\int \frac{dx}{(1+\sin x)}$

(d) $\int (1+\sqrt{x})^{1/2} dx$

2. Determine whether each improper integral is convergent or divergent. Evaluate the integrals which are convergent.
(a) $\int_1^3 \frac{dx}{x-1}$

(b) $\int_0^1 \frac{dx}{1+e^x}$

3. Determine how large do we have to choose n to evaluate the integral $\int_0^1 \sin(x^2)dx$ with an error less than 0.0001 using the Midpoint rule. Write formula for this approximation. Do not evaluate.

4. Determine whether each integral is convergent or divergent. Justify your answer. Do not try to evaluate your integrals!

(a) $\int_0^1 \frac{e^x}{x}dx$

(b). $\int_0^1 \frac{e^x}{\sqrt{x}} dx$

Solutions:

1(a).

$$\int x^{1/3}\ln x dx = \frac{3}{4}\int \ln x d(x^{4/3}) = \frac{3}{4}\ln x x^{1/3} - \frac{3}{4}\int x^{1/3}dx = \frac{3}{4}\ln x x^{1/3} - x^{4/3} + C$$

1(b). Using partial fraction decomposition we obtain:

$$\frac{u+3}{(u-1)(u-3)} = -\frac{2}{u-1} + \frac{3}{u-3}$$

Then integrate this term by term.

1(c). Using trigonometric identities we can transform the integrand as:

$$\frac{1}{(1+\sin x)} = \frac{1+\sin x}{1-\sin^2 x} = \frac{1+\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} = (\tan x)' - (\frac{1}{\cos x})'$$

Thus, the integral is equal to

$$\tan x - \frac{1}{\cos x} + C$$

1(d). Use the substitution $1+\sqrt{x} = y^2$, then

$$\int (1+\sqrt{x})^{1/2}dx = 4\int y(y^2-1)ydy = 4\int y^4dy - 4\int y^2dy = 4/5y^5 - 4/3y^3 + C$$

.

2(a). The integral is divergent:

$$\int_1^3 \frac{dx}{x-1} = \lim_{t\to 1+0} \ln(x-1)|_t^3$$

This limit does not exist.

2(b). $\int_0^1 \frac{dx}{1+e^x}$ It is a proper convergent integral.

3. We have $(\sin(x^2))'' = 2\cos x^2 - 4x^2 \sin x^2$. Taking into account this, the triangle inequality $|a+b| \le |a| + |b|$, and the fact that $|\sin y| \le 1$ and $|\cos y| \le y$ for all y we conclude that

$$|(\sin(x^2))''| \le 2 + 4 = 6$$

.

Thus, in the formula for the error bound we can choose $K = 6$. Which implies that the error will be less then 0.0001 if

$$\frac{6}{24n^2} < 0.0001$$

From here we conclude that $n \ge 25$.

4(a). We have the inequality $\frac{e^x}{x} > \frac{1}{x}$ the integral

$$\int_0^1 \frac{dx}{x}$$

is divergent. Therefore by the comparison test the initial integral also diverges.

4(b). We have the inequality $\frac{e^{\bar{x}}}{\sqrt{x}} < \frac{1}{\sqrt{x}}$. the integral

$$\int_0^1 \frac{1}{\sqrt{x}} dx$$

converges, therefore, by comparison test, the initial integral also converges.