This exam consists of 5 questions. Answer the questions in the spaces provided.

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1. Compute the following integrals:

(a) (10 points)

$$\int \ln(x)^2 \, dx$$

Solution:

Do integration by parts using $f(x) = \ln(x)^2$ and $g'(x) = 1$.
Choose $g(x) = x$. Then

$$\int \ln(x)^2 dx = x\ln(x)^2 - \int x \cdot \frac{1}{x} \cdot 2\ln(x)\, dx = x\ln(x)^2 - 2\int \ln(x) dx$$

Do integration by parts with $f(x) = \ln(x)$, $g'(x) = 1$.
Choose $g(x) = x$. Then

$$\int \ln(x)^2 dx = x\ln(x)^2 - 2x\ln(x) + 2x + C \quad //$$

(b) (10 points)

$$\int \tan^5(x)\sec^{-3}(x)\, dx$$

Solution:

$\tan^5(x)\sec^{-3}(x) = \sin^5(x)\cos^{-2}(x)$. Substitute $u = \cos(x)$

$$\Rightarrow \frac{du}{dx} = -\sin(x) \Rightarrow dx = \frac{-1}{\sin(x)} du.$$

Hence $\int \tan^5(x)\sec^{-3}(x)\,dx = -\int (1-u^2)^2 u^{-2} du$

$$= -\int (u^{-2} - 2 + u^2)\, du = \frac{1}{u} + 2u - \frac{1}{3}u^3 + C$$

$$= \cos^{-1}(x) + 2\cos(x) - \frac{1}{3}\cos^3(x) + C \quad //$$

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2. (20 points) Find the arc length of the the curve $y = \ln(\cos(x))$ between 0 and $\frac{\pi}{3}$.

Solution:

$$f(x) = \ln(\cos(x)) \Rightarrow f'(x) = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

Hence

$$\text{arc length} = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2(x)}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec(x)\, dx = \Big[\ln|\sec(x)+\tan(x)|\Big]_0^{\frac{\pi}{3}}$$

$$= \ln\left(\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right)$$

$$= \ln(2+\sqrt{3})$$

3. (20 points) Compute the following integral:

$$\int \frac{x^3 + x^2 - x + 1}{(x-1)^2(x^2+1)}\, dx$$

Solution:

$$\frac{x^3 + x^2 - x + 1}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

$$= \frac{Ax^3 - Ax^2 + Ax - A + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D}{(x-1)^2(x^2+1)}$$

$$= \frac{(A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D)}{(x-1)^2(x^2+1)}$$

$$\Rightarrow \left.\begin{array}{l} A+C = 1 \\ -A+B-2C+D = 1 \\ A+C-2D = -1 \\ -A+B+D = 1 \end{array}\right\} \begin{array}{l} C = 1-A, \text{ so} \\ -A+B-2(1-A)+D = 1 \\ 1-2D = -1 \\ -A+B+D = 1 \end{array} \left.\right\} \begin{array}{l} D = 1 \\ +A+B = 2 \\ -A+B = 0 \end{array} \left.\right\} \begin{array}{l} A = B = 1 \\ C = 0 \\ D = 1 \end{array}$$

Hence

$$\int \frac{x^3 + x^2 - x + 1}{(x-1)^2(x^2+1)}\, dx = \int \frac{1}{(x-1)} + \frac{1}{(x-1)^2} + \frac{1}{x^2+1}\, dx$$

$$= \ln|x-1| - \frac{1}{(x-1)} + \arctan(x) + C \quad //$$

4. (a) (10 points) Use the Trapizoidal Rule with n=4 to approximate the definite integral

$$\int_0^8 f(x)\,dx,$$

where $f(x)$ takes the following values:

x	0	1	2	3	4	5	6	7	8
$f(x)$	0	2	4	3	1	4	5	5	3

Solution:

$$T_4 = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$$

$$= \frac{2}{2}\cdot\left(0 + 2\cdot 4 + 2\cdot 1 + 2\cdot 5 + 3\right)$$

$$= 8 + 2 + 10 + 13 = 23 \;//$$

(b) (10 points) Assuming that $|f''(x)| \leq 2$, for all $0 < x < 8$, how large an n would we need to choose to guarantee that

$$|E_T| \leq 0.01$$

Solution: Choose $K=2$ in error bound. Hence

$|E_T| \leq \dfrac{2\cdot 8^3}{12\cdot n^2}$. Need n such that

$$\frac{2\cdot 8^3}{12\cdot n^2} \leq \frac{1}{100} \iff \frac{320}{\sqrt{12}} < n \;//$$

5. (20 points) Evaluate following improper integral:

$$\int_{-1}^{0} \frac{(x+1)^5}{\sqrt{(-x^2-2x)}}\,dx$$

If it is divergent, write divergent and explain your reasoning.

Solution:

$-x^2-2x = 1-(x+1)^2$. Do trigonometric substitution

$x+1 = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta\, d\theta$

$(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$

Hence $\displaystyle\int_{-1}^{0} \frac{(x+1)^5}{\sqrt{1-(x+1)^2}}\,dx = \int_{0}^{\frac{\pi}{2}} \sin^5\theta\, d\theta \leftarrow$ convergent as $\sin^5\theta$ continuous on $[0, \frac{\pi}{2}]$

Do substitution $u = \cos\theta \Rightarrow \frac{du}{d\theta} = -\sin(\theta) \Rightarrow d\theta = \frac{-1}{\sin\theta}du$

Hence $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^5(\theta)\, d\theta = -\int_{1}^{0} \sin^4\theta\, du = -\int_{1}^{0} (1-u^2)^2\, du$

$\displaystyle= \int_{0}^{1} (1-u^2)^2\, du = \int_{0}^{1} (1-2u^2+u^4)\, du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1$

$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$ //

END OF EXAM