

Abstract Algebra. Final Exam. May 17, 2013

Solve 6 of the following 11 problems.

1. How many cyclic subgroups are there in the group \mathbf{Z}_5^3 ?

In \mathbf{Z}_5^3 , there is 1 element of order 1 and $5^3 - 1 = 124$ elements of order 5, each generating a cyclic subgroup isomorphic to \mathbf{Z}_5 . In each \mathbf{Z}_5 there are 4 generators, therefore the total number of cyclic subgroups is $124/4 + 1 = 31 + 1 = 32$.

2. How many different isomorphisms $\mathbf{Z}_{120} \rightarrow \mathbf{Z}_{120}$ are there?

An isomorphism maps the generator 1 to a generator, i.e. a residue class relatively prime to 120. Thus the number of isomorphisms in question is equal to $\phi(120) = \phi(2^3 \cdot 3 \cdot 5) = 2^2(2-1)(3-1)(5-1) = 32$.

3. Is the group \mathbf{Z}_{1000}^\times (of units of the ring \mathbf{Z}_{1000}) cyclic? Why?

Since $1000 = 2^3 \times 5^3$, $\mathbf{Z}_{1000}^\times \simeq \mathbf{Z}_8^\times \times \mathbf{Z}_{125}^\times$, where \mathbf{Z}_8^\times is not cyclic (for $\pm 1^2 \equiv \pm 3^2 \equiv 1 \pmod{8}$). Therefore the whole group is not cyclic.

4. Let I be the principal ideal in $\mathbf{Z}_5[x]$ generated by the polynomial $x^3 + x^2 + x + 3$. Prove that $\mathbf{Z}_5[x]/I$ is a field and find the number of elements in it.

The polynomial has no roots in \mathbf{Z}_5 and, being of degree 3, is irreducible in $\mathbf{Z}_5[x]$. Since this ring is a PID, the prime ideal generated by this polynomial is maximal, and hence the quotient is a field. Each polynomial in the ring can be divided by this polynomial with a remainder, which can be any polynomial of degree < 3 . The remainder uniquely determines the coset modulo I . Thus the number of cosets is equal to the number of polynomials of degree < 3 with coefficients in \mathbf{Z}_5 . Thus the answer is $5^3 = 125$.

5. How many roots does the polynomial $x^{100000} - 1$ have in the field \mathbf{Z}_{65537} , where $65537 = 2^{24} + 1$ is the fourth Fermat number F_4 ?

We are looking for the number of units x in \mathbf{Z}_{F_4} whose order divides 100000. Since F_4 is prime, the group $\mathbf{Z}_{F_4}^\times$ is cyclic of order 2^{16} . Thus the order of x , which is a power of 2 not exceeding the 16th, must be a divisor of $100000 = 2^5 \cdot 5^5$. Thus the roots are those elements x which satisfy $x^{2^5} = 1$. In the cyclic group of order 2^{16} , there are $2^5 = 32$ such elements.

6. How many permutations of order 3 are there in the group S_6 of permutations of 6 objects?

Such permutations must have the cycle structure either $3 + 1 + 1 + 1$, or $3 + 3$. There are $6!/3 \cdot 3!$ of them of the first type, and $6!/3 \cdot 3 \cdot 2!$ of them of the second type. Both numbers are equal to 40, and the total number is 80.

7. Let I be the ideal in $\mathbf{C}[x, y]$ generated by two polynomials: $x^9 - 1$ and $y^{11} - x$. How many maximal ideals are in the quotient ring $\mathbf{C}[x, y]/I$? Why?

There are 9 distinct complex solutions to the equation $x^9 = 1$ (all non-zero!), and for each of them, there are 11 distinct complex solutions to the equation $y^{11} = x$. Thus $\mathbf{C}[x, y]/I$ is the ring of complex valued function on the discrete set of 99 solutions to the system $x^9 = 1, y^{11} = x$. Maximal ideals in this ring consist of functions vanishing at one of the points, and thus the number of such ideals is equal to 99.

8. Find the greatest common divisor of $f = x^7 + 6x^3 - 4x - 12$ and $g = x^5 - 3x^4 + 2x^2 - 18$, and determine whether it is irreducible in $\mathbf{Q}[x]$.

To find a GCD, one can use the Euclidean algorithm: divide f by g with a remainder of degree ≤ 4 , then divide g by the remainder. Doing this one discovers that the first division yields a remainder of degree 4, and the second division yields zero remainder. Therefore the first remainder is the GCD. It will turn out to be $x^4 + 2x + 6$. (In fact $f = (x^4 + 2x + 6)(x^3 - 2)$ and $g = (x^4 + 2x + 6)(x - 3)$.) Since $x^4 + 2x + 6$ is monic, with other coefficients divisible by 2 but the free term not divisible by 4, it is irreducible in $\mathbf{Z}[x]$ (and hence in $\mathbf{Q}[x]$) by Eisenstein's criterion.

9. Find a prime factorization of $3 + 7i$ in $\mathbf{Z}[i]$.

We have: $(3+7i)(3-7i) = 9+49 = 58 = 2 \cdot 29 = (-i)(1+i)^2(5+2i)(5-2i)$. Note that 2 and 29 are prime, and hence $1+i$, $5+2i$, and $5-2i$ are irreducible in $\mathbf{Z}[i]$. Divide $3 + 7i$ by $(1 + i)$:

$$\frac{3 + 7i}{1 + i} = \frac{(3 + 7i)(1 - i)}{2} = \frac{10 + 4i}{2} = 5 + 2i.$$

Thus $3 + 7i = (1 + i)(5 + 2i)$ is a prime factorization.

10. A *benzene* molecule C_6H_6 has the shape of a regular hexagon formed by 6 carbon atoms with one hydrogen atom attached to each of them. There are two stable isotopes of carbon: ^{12}C and ^{13}C , and two of hydrogen: 1H and 2H . How many stereo-isotopes (i.e. molecules geometrically different due to presence of isotopes) of benzene can be found?

The pair $C - H$ at each vertex of the hexagon can be of one of $2 \times 2 = 4$ isotope types. Thus, the number of stable stereo-isotopes of benzene is equal to the number of orbits of the dihedral group D_6 of symmetries of the hexagon

on the set of 4^6 combinations of the isotope types of the vertices. We use Cauchy's counting principle to find the number of orbits as the average over the group number of fixed points. The identity has 4^6 fixed points, rotations through $\pm 60^\circ$ have 4 fixed points each, through $\pm 120^\circ$ have 4^2 fixed points each, the rotation through 180° has 4^3 fixed points, three reflections about lines passing through the midpoints of opposite sides have 4^3 fixed points each, and three reflections about the lines passing through opposite vertices have 4^4 fixed points each. Thus the number of orbits is:

$$\frac{4^6 + 2 \cdot 4 + 2 \cdot 4^2 + 4^3 + 3 \cdot 4^3 + 3 \cdot 4^4}{12} = \frac{1024 + 2 + 8 + 16 + 48 + 192}{3} = \frac{1290}{3} = 430.$$

11. The group $G = GL_2(\mathbf{Z}_3)$ of invertible 2×2 matrices over \mathbf{Z}_3 acts on the set of four 1-dimensional subspaces (namely, lines $y = kx$ with slopes $k = 0, 1, -1, \infty$ on the xy -plane \mathbf{Z}_3^2). This action defines a homomorphism $G \rightarrow S_4$ to the permutation group of four objects. Find the kernel H of this homomorphism, and determine how many normal subgroups in G containing H are there.

The whole group G consists of matrices whose first column can be any non-zero vector from \mathbf{Z}_3^2 , and the second column any vector except those 3 which are proportional to the first column. Thus the order of the group is $(3^2 - 1)(3^2 - 3) = 8 \cdot 6 = 48$. Matrices which preserve the x -axis $y = 0$ are the upper-triangular ones. Those which preserve the y -axis $x = 0$ are the lower-triangular ones. Thus, invertible diagonal matrices are those which preserve both, and of them only the scalar matrices $\pm I$ preserve each of the lines $y = \pm x$. Thus the kernel of the homomorphism consists of the two invertible scalar matrices $\pm I$. By the epimorphism theorem, the image has the order $48/2 = 24 = 4!$, and is therefore the whole group S_4 . Thus normal subgroups in G containing the kernel are exactly the inverse images of normal subgroups in S_4 . The answer is 4, as there are 4 normal subgroups in S_4 . They are: the identity group, whose inverse image is the kernel $\{\pm I\}$, the whole group S_4 , whose inverse image is the whole group G , the alternating group A_4 (its inverse image consists of matrices with the determinant equal to 1, but not -1), and the Klein group, whose inverse image is some normal subgroup of order 8.