Math 1B Second Midterm Exam (LEC 002) 11:10 am to 12pm

This exam consists of 5 questions. Answer the questions in the spaces provided.

Name and discussion section: ______________________________

GSI's name: ______________________________

1. Determine if the following sequences converge or diverge. Carefully justify your answer.

(a) (10 points)

$$\left\{\frac{(-1)^n \arctan(n)}{n}\right\}_{n=1}^{\infty}$$

Solution:

Observe $\frac{-\pi}{2} < \arctan(n) < \frac{\pi}{2}$ for all $n \geq 1$

$\Rightarrow \quad \frac{-\pi}{2n} < \frac{(-1)^n \arctan(n)}{n} < \frac{\pi}{2n}$ for all $n \geq 1$

But $\frac{-\pi}{2n}, \frac{\pi}{2n} \to 0$ as $n \to \infty$. Hence convergent by squeeze theorem.

(b) (10 points)

$$\left\{\frac{n}{\ln(n+1)}\right\}_{n=1}^{\infty}$$

Solution:

Let $f(x) = \frac{x}{\ln(x+1)}$.

$$\lim_{n\to\infty} \left\{\frac{n}{\ln(n+1)}\right\} = \lim_{x\to\infty} f(x) \underset{\text{l'Hopital's}}{=} \lim_{x\to\infty} \frac{1}{\left(\frac{1}{x+1}\right)} = \lim_{x\to\infty} (x+1)$$

$= \infty$.

Hence sequence divergent.

2. (20 points) Determine whether the following series is absolutely convergent, conditionally convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty}(-1)^n\frac{\ln(n)}{n}$$

Solution:

Let $f(x) = \dfrac{\ln(x)}{x}$ for $x \geq 1$. Observe that

$f'(x) = \dfrac{-\ln(x)}{x^2} + \dfrac{1}{x^2} = \dfrac{1-\ln(x)}{x^2} \Rightarrow f'(x) < 0$ for

$x > e$. Hence $f(x)$ is positive, continuous and eventually decreasing. We may thus use integral test.

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\ln(x)^2\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\ln(t)^2\right) = \infty.$$

Hence $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln(n)}{n}$ not absolutely convergent.

Let $b_n = \dfrac{\ln(n)}{n}$. By above b_n is decreasing after $n > e$. But $\displaystyle\lim_{n\to\infty}\{b_n\} = \lim_{x\to\infty} f(x) \underset{\text{L'Hopital}}{=} \lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{1} = 0$

Hence $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln(n)}{n}$ convergent by alternating series test.

Hence conditionally convergent.

3. (20 points) Determine if the following series is convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty} \frac{(\cos(n)+2)\sqrt{n^2-1}}{n^2+n+1}$$

Solution:

Observe that $\cos(n)+2 \geq 1$ for all $n \geq 1$.

Hence let us do a comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^2+n+1}$.

To determine convergence / divergence of this series do limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\frac{\left(\dfrac{\sqrt{n^2-1}}{n^2+n+1}\right)}{\left(\dfrac{1}{n}\right)} = \frac{n\sqrt{n^2-1}}{n^2+n+1} = \frac{\sqrt{n^4-n^2}}{n^2+n+1} = \frac{\sqrt{1-\frac{n^2}{n^4}}}{1+\frac{1}{n}+\frac{1}{n^2}} \to 1$$

as $n \to \infty$.

Hence by limit comparison test, because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ divergent, $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^2+n+1}$ divergent.

By the usual comparison test we deduce that

$$\sum_{n=1}^{\infty} \frac{(\cos(n)+2)\sqrt{n^2-1}}{n^2+n+1} \text{ divergent.}$$

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4. Determine whether the following series are convergent or divergent. If convergent determine the sum.

(a) (10 points)

$$\sum_{n=1}^{\infty} \cos(\frac{1}{n^2})$$

Solution:

$\frac{1}{n^2} \to 0$ as $n \to \infty$ $\Rightarrow$ $\cos(\frac{1}{n^2}) \to \cos(0) = 1 \neq 0$ as $n \to \infty$. Hence divergent by divergence test.

(b) (10 points)

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$$

Solution:

$0 < \frac{2}{6}, \frac{3}{6} < 1$ $\Rightarrow$ $\sum_{n=1}^{\infty} (\frac{2}{6})^n$ and $\sum_{n=1}^{\infty} (\frac{3}{6})^n$ convergent

$\Rightarrow \sum_{n=1}^{\infty} (\frac{2}{6})^n + \sum_{n=1}^{\infty} (\frac{3}{6})^n = \sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$ convergent

$\sum_{n=1}^{\infty} (\frac{2}{6})^n = \frac{2}{6}\left(\frac{1}{1-\frac{2}{6}}\right) = \frac{1}{2}$

$\sum_{n=1}^{\infty} (\frac{3}{c})^n = \frac{3}{6}\left(\frac{1}{1-\frac{3}{6}}\right) = 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n} = \frac{3}{2}$

5. (20 points) Determine if the following series is convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty} \frac{n!}{\sqrt{n^n}}$$

Solution:

Do ratio test. Let $a_n = \dfrac{n!}{\sqrt{n^n}} = \dfrac{n!}{n^{\frac{n}{2}}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1) \cdot n^{\frac{n}{2}}}{(n+1)^{\frac{n+1}{2}}} = \sqrt{n+1} \cdot \sqrt{\left(\frac{n}{n+1}\right)^n}$$

$\left(\frac{n+1}{n}\right)^n \to e$ as $n \to \infty$ $\Rightarrow \sqrt{\left(\frac{n}{n+1}\right)^n} \to \sqrt{e^{-1}}$

$\sqrt{n+1} \to \infty$ as $n \to \infty$

Hence $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ as $n \to \infty$. Hence divergent by ratio test.

One could also observe that $\sqrt{n^n} \leqslant n^n$ for $n \geqslant 1$

$\Rightarrow \dfrac{n!}{\sqrt{n^n}} \geqslant \dfrac{n!}{n^n}$ for $n \geqslant 1$. Then do ratio test with $\sum_{n \geq 1} \dfrac{n!}{n^n}$.

END OF EXAM

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