NOTES FOR MATH 142

SEPARATION AXIOMS (09/15/2014)

Recall that the proof of the Tietze extension theorem only need the topological space X to satisfy the following property:

For any two disjoint closed set $A, B \subset X$, there is a continuous map $f : X \to \mathbb{E}^1$ such that $f(A) = \{0\}$, $f(B) = \{1\}$ and $f(X) \subseteq [0, 1]$.

We will find a more general condition on topological spaces to satisfy this property.

Separation Axioms:

T1. First Separation Axiom. For any two distinct points $x, y \in X$, there is an open set U with $x \in U$ and $y \notin U$, and also an open set V with $y \in V$ and $x \notin V$.

Lemma 1. *If X satisfies the first separation axiom, then one point sets are closed.*

Proof. For a one point set $\{x\}$, and any $y \neq x$, by the first separation axiom, there is an open set U_y with $y \in U_y$ and $x \notin U_y$.

So $X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} U_y$. □

T2. Second Separation Axiom, Hausdorff space. For any two distinct points $x, y \in Y$, there are two disjoint open sets U and V such that $x \in U$ and $y \in V$.

T3. Third Separation Axiom, regular space. A topological space X is regular if it is Hausdorff, and for any closed set $B \subseteq X$ and $x \notin B$, there are two disjoint open sets U and V such that $x \in U$ and $B \subseteq V$.

T4. Fourth Separation Axiom, normal space. A topological space X is normal if it is Hausdorff, and for any two disjoint closed sets $A, B \subseteq X$, there are two disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

The desired property implies the fourth separation Axiom, by taking $U = f^{-1}([0, 1/3))$, $V = f^{-1}((2/3, 1])$.

Lemma 2. *Assume X is a Hausdorff space.*

1. *X is regular if and only if for any $x \in X$, and open set W with $x \in W$, there is an open set U satisfying $x \in U \subseteq \overline{U} \subseteq W$.*
2. *X is normal if and only if for any closed set $A \subseteq X$, and open set W with $A \subseteq W$, there is an open set U satisfying $A \subseteq U \subseteq \overline{U} \subseteq W$.*

Proof. We only prove the second statement.

If X is normal, and $A \subseteq W$ with A closed and W open. Let $B = X \setminus W$, then B is closed and disjoint from A. By the definition of normal space, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$. Since $X \setminus V$ is a closed set containing U, it contains $\overline{U}$. So $\overline{U} \subseteq X \setminus V \subseteq X \setminus B = W$, and $A \subseteq U \subseteq \overline{U} \subseteq W$.

If for any closed set $A \subseteq X$, and open set W with $A \subseteq W$, there is an open set U satisfying $A \subseteq U \subseteq \overline{U} \subseteq W$. Then for any two disjoint closed sets A, B, $W = X \setminus B$ is closed with $A \subseteq W$. So there is an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq W = X \setminus B$, then $V = X \setminus \overline{U}$ is an open set containing B and disjoint from U. □

Theorem 1. Urysohn Lemma. *Suppose X is a normal space, let A and B be two disjoint closed sets in X, then there exists a continuous map $f : X \to [0,1]$ such that $f(A) = \{0\}$, $f(B) = \{1\}$.*

Proof. Let $P = \{\frac{k}{2^m} | m \in \mathbb{N}, 1 \leq k \leq 2^m - 1 \textit{ odd number}\} \cup \{0,1\}$.

For any $r \in P$, we will construct an open set U_r such that $\overline{U_r} \subseteq U_s$ if $r < s$.

Let $U_1 = X \setminus B$. Since $A \subseteq U_1$ with A open and U_1 closed, and X is normal, there is an open set U_0 such that $A \subseteq U_0 \subseteq \overline{U_0} \subset U_1$.

Since $\overline{U_0} \subseteq U_1$ with $\overline{U_0}$ open and U_1 closed, and X is normal, there is an open set $U_{1/2}$ such that $\overline{U_0} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subset U_1$.

Since $\overline{U_0} \subseteq U_{1/2}$ with $\overline{U_0}$ open and $U_{1/2}$ closed, and X is normal, there is an open set $U_{1/4}$ such that $\overline{U_0} \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subset U_{1/2}$.

Since $\overline{U_{1/2}} \subseteq U_1$ with $\overline{U_{1/2}}$ open and U_1 closed, and X is normal, there is an open set $U_{3/4}$ such that $\overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subset U_1$.

By doing this inductive argument for m in $\frac{k}{2^m}$, we can construct open sets U_p for $p \in P$ and $\overline{U_p} \subseteq U_q$ if $p < q$.

Define $V_1 = X$ and $V_p = U_p$ for $p \in P \setminus \{1\}$, then $\overline{V_p} \subseteq V_q$ if $p < q$ still holds.

Define $f : X \to [0,1]$ by $f(x) = \inf \{p \in P | x \in V_p\}$. $\{p \in P | x \in V_p\}$ is not empty since $V_1 = X$.

It is clearly that $f(A) = \{0\}$ and $f(B) = \{1\}$, since $U_p \subseteq X \setminus B$ for any $p < 1$. So it suffices to show that f is continuous.

The continuity of f follows by $f(U_q \setminus \overline{U_p}) \subseteq [p,q]$ and $f^{-1}((p,q)) \subseteq U_q \setminus \overline{U_p}$ for $p < q$ with $p, q \in P$.

Corollary 1. *Tietez extension theorem holds for normal spaces.*

Definition 1. If a topological space has a basis with countably many open sets, then it is *second countable.*

Proposition 1. *If a second countable space has a open cover, it has a countable subcover.*

Proposition 2. *If a regular space is also second countable, it is normal.*

Theorem 2. *If a regular space is also second countable, it is metrizable (homeomorphic to a metric space).*

□