MATH H53: HONORS MULTIVARIABLE CALCULUS

REVIEW OF CONCEPTS AND FORMULAE: CHAPTER 10, 12, 13

Section 10.6, 13.3, 13.4 will not be covered in the exam.

Chapter 10: Parametric Equations and Polar Coordinates.

- Parametric curve: $t \to (x(t), y(t))$. Difference between curves and parametric curves.
- Calculus of parametric curves:
  - Slope of the tangent line: if $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$, then
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$
  - Area under the curve: if $x'(t) > 0$, then
$$\int_\alpha^\beta y(t)x'(t)\mathrm{d}t.$$
  - Arc length:
$$\int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t.$$
  - Area of the rotational surface about x–axis: if $y(t) \geq 0$,
$$\int_\alpha^\beta 2\pi y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t.$$
- Polar coordinate: (r, θ). (r, θ), $(r, \theta + 2n\pi)$, $(-r, \theta + (2n+1)\pi)$ correspond to the same point.
- $\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x}. \end{cases}$
- Calculus of polar curves $r = r(\theta)$.
  - Slope of the tangent line:
$$\frac{(r(\theta)\sin\theta)'}{(r(\theta)\cos\theta)'} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}.$$
  - Arc length:
$$\int_a^b \sqrt{\big((r(\theta)\sin\theta)'\big)^2 + \big((r(\theta)\cos\theta)'\big)^2} = \int_a^b \sqrt{(r(\theta))^2 + (r'(\theta))^2}\mathrm{d}\theta.$$
  - Area of the region bound by $\theta = a$, $\theta = b$ and $r = r(\theta)$:
$$\int_a^b \frac{1}{2}r^2(\theta)\mathrm{d}\theta.$$
- Conic sections:
  - Parabola: points which have the same distance from a point $((0, p))$ and from a line $(y = -p)$:
$$x^2 = 4py.$$

- Ellipse: points which have fixed sum ($2a$) of distances from two points ($(c, 0)$ and $(-c, 0)$):
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ b^2 = a^2 - c^2.$$
- Hyperbola: points which have fixed difference ($2a$) of distances from two points ($(c, 0)$ and $(-c, 0)$):
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ b^2 = c^2 - a^2.$$

- Polar coordinate of conic sections: the points which have a fixed quotient (e) of distances from a point (the pole) and from a line ($r\cos\theta = d$):
$$r = \frac{ed}{1 + e\cos\theta}.$$
It is an ellipse if $e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$.

Chapter 12: Vectors and the Geometry of Space.

- Three dimensional coordinate system: visualizing. Distance between two points:
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
Equation of a sphere:
$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$
- Vectors: describing a quantity with both magnitude and direction (the initial point is not important).
  - Addition of vectors: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Scalar multiplication, difference.
  - Component of vectors and standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.
  - Length of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- Dot product:
  - For $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$,
$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$
  - Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$, then
$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\cos\theta.$$
  - Scalar projection and vector projection from one vector to another.
- Cross product:
  - For $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$:
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$
  - Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$, then
$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}|\sin\theta.$$
  - $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. But $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ and
$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}.$$
Its absolute value is the volume of the parallelepiped determined by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.
- Equation of lines:
  - Vector equation: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.
  - Parametric equation: $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$.

  - Symmetric equation: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.
- Equation of plans:
  - Vector equation: $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$ ($\mathbf{n}$ is perpendicular to the plane).
  - Scalar equation: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.
  - Linear equation: $ax + by + cz + d = 0$.
  - The distance from (x_1, y_1, z_1) to $ax + by + cz + d = 0$ is
$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$
- Quadratic surfaces:
  - Ellipsoid:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
  - Cone:
$$\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$
  - Elliptic Paraboloid:
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$
  - Hyperboloid of one sheet:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$
  - Hyperboloid of two sheets:
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
  - Hyperbolic paraboloid:
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Chapter 13: Vector Functions

- Vector–valued function: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.
- Limit, derivative, integral of vector–valued functions.
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t).$$
- Arc length: similar to the two dimensional formula. Parametrize a curve with respect to arc length: $|\mathbf{r}'(t)| = 1$.
- Curvature:
  - Unit tangent vector: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.
  - Curvature:
$$\kappa = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$
- Motion in space:
  - Velocity vector: $\mathbf{v}(t) = \mathbf{r}'(t)$. Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.
  - Kepler's first law: a planet revolves around the sun in an elliptical orbit with the sun at one focus.