MATH H1B EXAM #1, PART I
Monday, September 26, 2011

Sign below to indicate that you have followed the rules in preparing for Problem #1:

You are allowed to consult your class notes, any handouts from the course website, your textbook, and each other to figure these out, but NO OTHER RESOURCES. You are not permitted to talk to anyone outside the class about the problems, or to consult other books or anything on the internet other than our course website.

Signature: ____

No calculators or notes are permitted on this portion of the exam. If you make a mistake while working, you can either erase cleanly or draw a single line (no crazy scribbling, please) through any parts we should ignore. You may use the back of any page for scratch work.

For the proofs, you may use any of our basic axioms and theorems about arithmetic without specifically quoting them. All proofs should be in paragraph form. If you are doing a nonstandard proof (like contradiction or induction), the first thing you should say is "Proof by contradiction/induction". For induction proofs, make sure you clearly show and label all parts – first the base case, and then the induction step (i.e., showing that $P(n)$ is true implies $P(n+1)$ is true).

This exam has 4 problems on 7 pages, including this cover sheet.

Problem	Max	Score
1	12	
2	10	
3	12	
4	6	
Total	40	

1. Suppose X and Y are nonempty subsets of $\mathbb{R}$ with $Y \subset X$. Assume that X is bounded above. *(You may use earlier parts of this problem to help with later parts.)*

   (a) (2 points) State the axiom about least upper bounds and the corresponding theorem for greatest lower bounds. *(Do not use X or Y as the name of an arbitrary set in this part, since those have already been used to denote some specific sets.)*

   (b) (4 points) Prove that $\sup(X)$ and $\sup(Y)$ both exist.

Recall the statement of the problem: Suppose X and Y are nonempty subsets of $\mathbb{R}$ with $Y \subset X$. Assume that X is bounded above.

(c) (3 points) Fill in the blank to complete the following inequality:

$$\sup(X) \_\_\_\_\_\_ \sup(Y).$$

Prove your assertion.

(d) (3 points) If $\inf(Y)$ exists, prove that $\inf(Y) \leq \sup(Y)$.

2. (10 points) Choose ONE of the following three proofs. (Do not do more than one; there will not be any extra credit for doing so.) **Write only your final proof on page 5**. You may use this page or the back of any page for scratch work.

   (A) Prove that if n is an odd integer, then $n^2 - 1$ is a multiple of 8.

   (In addition to basic arithmetic, you may use any results we proved in class or on homework about even and odd integers.)

   (B) Prove the formula for a finite geometric series. I.e., prove that for every $n \in \mathbb{Z}^+$,

   $$\sum_{k=1}^{n} ar^{k-1} = \frac{a(r^n - 1)}{r - 1},$$

   where $a \in \mathbb{R}$, $r \in \mathbb{R}$, and $r \neq 1$.

   (C) The following statement is false, but one additional condition on x will make it true. Find the least restrictive such condition and prove the modified statement. (For example, don't make your condition "$x = 1$"; that will make it true, but you can do better!)

   If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $xy \notin \mathbb{Q}$. *(Recall that $\mathbb{Q}$ denotes the set of rational numbers.)*

Problem Statement:

Proof:

3. No justification is needed for the following problems, but feel free to do scratchwork in any of the open space. The parts are all independent.

   (a) (6 points) Suppose $x \in \mathbb{R}$ and $0 < x < 1$. Let $S = \{x^n : n \in \mathbb{Z}^+\}$, and let $T = \{25s : s \in S\}$. Consider the statements below, and circle precisely the ones which must be true.

      i. 1 is an upper bound of S

      ii. $\sup(S) = 1$

      iii. T is bounded above

      iv. $\inf(S) = 0$

      v. $S \cap \mathbb{Z} = \emptyset$

      vi. $T \cap \mathbb{Z} = \emptyset$

   (b) (6 points) Suppose that A and B are subsets of $\mathbb{R}$ which are bounded above. Let

$$C = \{a + b : a \in A, b \in B\}.$$

   Circle exactly those statements below which must be true.

      i. $\sup(C) \in C$

      ii. $\sup(C) \geq \sup(A)$

      iii. If $0 \in A$, then $B \subseteq C$.

      iv. If $0 \in B$, then $B \subseteq C$.

      v. $\sup(A \cap B)$ exists

      vi. $\sup(A \cup B) \leq \max\{\sup(A), \sup(B)\}$

4. No explanations are needed for the following problems. The parts are all independent.

   (a) (2 points) Give an example of a subset $P \subset \mathbb{R}$ such that P has infinitely elements, but $P \cap \mathbb{Z}$ has only finitely many elements.

   (b) (2 points) Give an example of a function $f(x)$ such that

   - f is defined on all of $\mathbb{R}$,
   - f is not a constant function, AND,
   - the set $Y = \{y : y = f(x)\} \subset \mathbb{R}$ is bounded above and below.

   (c) (2 points) Let $g(x) = 17x^2$, and let $\epsilon > 0$. Find a real number $\delta > 0$ (possibly depending on ϵ) such that $|x| < \delta$ implies $|g(x)| < \epsilon$. You do not need to give a proof; just show appropriate work to find a suitable δ.