MORE MATH 55 PRACTICE PROBLEMS (FROM CHAPTERS 8-10), SPRING 2014

(1) Use generating functions to solve $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$. (From Section 8.4 # 33.)

Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$

$a_k = 3a_{k-1} + 2 \Rightarrow$

$a_k x^k = 3a_{k-1}x^k + 2x^k \Rightarrow$

$\sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} 3a_{k-1}x^k + \sum_{k=1}^{\infty} 2x^k \Rightarrow$

$A(x) - a_0 = 3x \sum_{k=1}^{\infty} a_{k-1}x^{k-1} + 2(x + x^2 + x^3 + \ldots) \Rightarrow$

$A(x) - 1 = 3x\, A(x) + 2x(1 + x + x^2 + \ldots) \Rightarrow$

$A(x)[1-3x] = 1 + \frac{2x}{1-x} = \frac{1-x+2x}{1-x} = \frac{1+x}{1-x} \Rightarrow$

$A(x) = \frac{1+x}{(1-x)(1-3x)}$.

Partial Fractions: Can write $\frac{1+x}{(1-x)(1-3x)} = \frac{a}{1-x} + \frac{b}{1-3x}$.

$\frac{a(1-3x) + b(1-x)}{(1-x)(1-3x)} = \frac{1+x}{(1-x)(1-3x)} \Rightarrow a+b=1$ and $-3a-b=1$.

$\Rightarrow a=-1,\ b=2$

$\therefore A(x) = \frac{-1}{1-x} + \frac{2}{1-3x} = \underbrace{-1(1+x+x^2+\ldots) + 2(1+3x+3^2x^2+\ldots)}$.

The coeff of x^k here is $-1 + 2\cdot 3^k$

$\therefore\ a_k = -1 + 2\cdot 3^k$

(2) Suppose that a relation R is symmetric. Show that R^* is symmetric. (From Section 9.4 #23).

Let R be relation on set A.
R symmetric means whenever $(a,b) \in R$, also $(b,a) \in R$.
$R^* = R \cup R^2 \cup R^3 \cup \dots$

Suppose that $(c,d) \in R^*$.
Then $(c,d) \in R^k$ for some k.
By def of R^k, $\exists\ x_1, x_2, \dots, x_{k-1} \in A$ s.t.
$(c,x_1) \in R$, $(x_1,x_2) \in R$, $(x_2,x_3) \in R$, $\dots$ $(x_{k-1}, d) \in R$.
But then since R is symmetric,
$(d,x_{k-1}) \in R$, $\dots$, $(x_3,x_2) \in R$, $(x_2,x_1) \in R$, $(x_1,c) \in R$.
$\therefore$ by def of R^k, $(d,c) \in R^k$, and so
$(d,c) \in R^*$.

$\blacksquare$

(3) Show that the property that a graph is bipartite is a graph invariant. (From Section 10.3 #66).

Suppose $f: G_1 \to G_2$ is a graph isomorphism, and G_1 is bipartite. We need to show G_2 is bipartite. Let V_1 and V_2 be the vertex sets of G_1 and G_2.

G_1 is bipartite means we can partition vertex set V_1 into 2 disjoint sets X and Y, s.t. $V_1 = X \cup Y$, and such that every edge of G_1 connects a vertex x of X with a vertex y of Y.

Now consider the subsets $f(X)$ and $f(Y)$ of V_2. Since f is a bijection from V_1 to V_2, $f(X)$ and $f(Y)$ are disjoint and $f(X) \cup f(Y) = V_2$.

Now consider 2 vertices w, z of V_2. If w and z both lie in $f(X)$, then $f^{-1}(w)$ and $f^{-1}(z)$ both lie in X. Since G_1 is bipartite, there's no edge between $f^{-1}(w)$ and $f^{-1}(z)$, & since f preserves adjacencies, there's no edge between w and z in G_2.

Similarly, if w and z both lie in $f(Y)$, there is no edge between them.

$\therefore$ every edge of G_2 connects a vertex of $f(X)$ w/ a vertex of $f(Y)$, so G_2 is bipartite, w/ bipartition $(f(X), f(Y))$.

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