MATH H53: HONORS MULTIVARIABLE CALCULUS

REVIEW OF CONCEPTS AND FORMULAE: CHAPTER 14, SECTION 15.1 - 15.6

Chapter 14: Partial derivatives.

- Function of two (or more) variables: two (or more) independent variables, one dependent variable. Domain and range for multivariable functions: domain lies in the plane, 3-dimensional space, or higher dimensional space. Level curves and level surfaces of two and three variables functions.
- Limits and continuity of multivariable functions.
  - $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that for any (x, y) lying in the domain with $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$.
  - Limit along a path versus the limit itself. Limit along paths can only determine the non–existence of the limit.
  - f is continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. f is continuous on D if it is continuous at each point. For example, rational functions are continuous on the domain.
- Partial derivatives: fix one variable, and treat the function as a single variable function.
  - $f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$. It is the slope of the tangent line of the intersection of the graph of f with the plane with fixed second coordinate.
  - Higher derivatives: $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$, etc.
  - Clairaut's theorem: if both f_{xy} and f_{yx} are continuous at (a, b), they they are equal at (a, b).
- Tangent planes and linear approximations.
  - The tangent plane of $z = f(x, y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

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  - f is differentiable at (x_0, y_0), if

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta_y$$

  and $\epsilon_1, \epsilon_2 \to 0$ when $(\Delta x, \Delta y) \to (0, 0)$. It is the linear approximation of f at (x_0, y_0).
  - If f_x and f_y are continuous at (a, b), then f is differentiable at (a, b).
  - If f is differentiable at (a, b), the differential of f at (a, b) is

$$\mathrm{d}z = f_x(a,b)\mathrm{d}x + f_y(a,b)\mathrm{d}y$$

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- The chain rule.
  - Suppose u is a differentiable function with n (intermediate) variables $x_1, \cdots, x_n$ and each x_j is a differentiable function of m (independent) variables $t_1, \cdots, t_m$.

Then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}.$$

- If the implicit function $z = f(x, y)$ is given by $F(x, y, z) = 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

- Directional derivatives and gradient vectors.
  - The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$\mathrm{D}_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

  - The gradient of f (of two variables) is

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = f_x\mathbf{i} + f_y\mathbf{j}.$$

  - If f is differentiable at (x_0, y_0), then

$$\mathrm{D}_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \nabla f(x_0, y_0)\cdot \mathbf{u}.$$

  - The gradient direction is the direction for which the directional derivative takes the maximum value. It is also a normal vector of the tangent line of the level curve (if f is a function of two variables) or the tangent plane of the level surface (if f has three variables).
- Maximum and minimum.
  - Local maximum (local minimum): larger (smaller) or equal to the value of points near it.
  - Local maximum or minimum at (a, b): if derivatives exits, $f_x(a, b) = f_y(a, b) = 0$ (critical point).
  - Second derivatives test: if $f_x(a, b) = f_y(a, b) = 0$ and second partial derivatives are continuous, consider $f_{xx}(a, b)$ and $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$:
    * $D > 0$ and $f_{xx} > 0$: local minimum;
    * $D > 0$ and $f_{xx} < 0$: local maximum;
    * $D < 0$: neither maximum nor minimum.
  - Absolute maximum and minimum: always exists if the function is continuous and defined on a closed bounded set.
  - To find the maximum/minimum of a continuous function f on a closed bounded set D:
    (1) Find the values of f at critical points.
    (2) Find the extreme values of f on the boundary of D.
    (3) Compare all the values you get in the previous two steps.
- Lagrange multipliers.
  - To find the maximum/minimum of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (suppose $\nabla g(x, y, z) \neq \mathbf{0}$): first solve $\nabla f(x_0, y_0, z_0) = \lambda\nabla g(x_0, y_0, z_0)$ and $g(x_0, y_0, z_0) = k$, then compare all the values of f you get.
  - To find the maximum/minimum of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ and $h(x, y, z) = c$ (suppose $\nabla g(x, y, z), \nabla h(x, y, z) \neq \mathbf{0}$): first solve $\nabla f(x_0, y_0, z_0) = \lambda\nabla g(x_0, y_0, z_0) + \mu\nabla h(x_0, y_0, z_0)$, $g(x_0, y_0, z_0) = k$ and $h(x_0, y_0, z_0) = c$, then compare all the values of f you get.

Chapter 15: Multiple integrals.

- Double integrals over rectangles.
  - For a rectangle $R = [a, b] \times [c, d]$ and a function f defined on R, the integral of f on R, denoted by
  $$\iint_R f(x, y)\mathrm{d}A,$$
  is defined as the following.
  Divide $[a, b]$ to m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{m}$ and divide $[c, d]$ to n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \frac{d-c}{n}$. Take any $(x_{ij}^*, y_{ij}^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, then
  $$\iint_R f(x, y)\mathrm{d}A = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y,$$
  if the limit exists.
  - Average value: $f_{\text{ave}} = \frac{1}{\text{area}(R)} \iint_R f(x, y)\mathrm{d}A$.
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  $$\iint_R [f + g]\mathrm{d}A = \iint_R f\mathrm{d}A + \iint_R g\mathrm{d}A, \ \iint_R cf\mathrm{d}A = c\iint_R f\mathrm{d}A,$$
  if $f(x, y) \geq g(x, y)$ for any $(x, y) \in D$, then
  $$\iint_R f\mathrm{d}A \geq \iint_R g\mathrm{d}A.$$
- Iterated integrals.
  - Iterated integral:
  $$\int_a^b \int_c^d f(x, y)\mathrm{d}y\mathrm{d}x = \int_a^b \Big[\int_c^d f(x, y)\mathrm{d}y\Big]\mathrm{d}x.$$
  - Fubini's theorem: if f is continuous on $R = [a, b] \times [c, d]$, then
  $$\iint_R f(x, y)\mathrm{d}A = \int_a^b \int_c^d f(x, y)\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x, y)\mathrm{d}x\mathrm{d}y.$$
  The theorem is still true under some weaker assumption, say, piecewise continuous.
- Double integrals over general regions.
  - For a function $f(x, y)$ defined on a region D, take a rectangle R containing D and let
  $$F(x, y) = \begin{cases} f(x, y) & \text{if} \quad (x, y) \in D \\ 0 & \text{if} \quad (x, y) \notin D, \end{cases}$$
  then define
  $$\iint_D f(x, y)\mathrm{d}A = \iint_R F(x, y)\mathrm{d}A.$$
  - For regions of type I or type II, compute double integrals by iterated integrals: e.g. if $D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ and f is continuous on D, then
  $$\iint_D f(x, y)\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\mathrm{d}y\mathrm{d}x.$$
  - If $D = D_1 \cup D_2$, D_1 and D_2 overlap at most along their boundaries, then
  $$\iint_D f\mathrm{d}A = \iint_{D_1} f\mathrm{d}A + \iint_{D_2} f\mathrm{d}A.$$
- Double integrals in polar coordinates.

- Polar rectangle: $R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$.
- If f is continuous on a polar region $D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then
$$\iint_D f(x,y)\mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r \mathrm{d}r \mathrm{d}\theta.$$

- Applications of double integrals.
  - Density and mass: for a laminar over region D with density $\rho(x,y)$, the mass is
$$m = \iint_D \rho(x,y)\mathrm{d}A.$$
  - Moments and center of mass: for the same laminar, the moment about the x–axis is
$$M_x = \iint_D y\rho(x,y)\mathrm{d}A,$$
while the moment about the y–axis is defined similarly. The center of mass is $(\bar{x},\bar{y}) = (\frac{M_x}{m}, \frac{M_y}{m})$.
  - Moment of inertia: the moment of inertia about the x–axis is
$$I_x = \iint_D y^2\rho(x,y)\mathrm{d}A,$$
while the moment of inertia about the y–axis is defined similarly. The moment of inertia about the origin is
$$I_0 = I_x + I_y = \iint_D (x^2+y^2)\rho(x,y)\mathrm{d}A.$$
  - Probability: joint density function $f(x,y) \ge 0$, with
$$P((X,Y) \in D) = \iint_D f(x,y)\mathrm{d}A.$$
  - Expected values: X–mean (expected value of X) is
$$\mu_1 = \iint_{\mathbb{R}^2} x f(x,y)\mathrm{d}A,$$
while the Y–mean (expected value of Y) is defined similarly.
- Surface area.
  - Surface area is approximated by the sum of the area of small parallelograms.
  - The area of the graph of $z = f(x,y)$ ($(x,y) \in D$), when f_x and f_y are both continuous, is
$$\mathrm{A}(S) = \iint_D \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\mathrm{d}A.$$