MATH H1B FINAL EXAM, PART II

Wednesday, December 14, 2011

Sign below to indicate that you have followed the rules in preparing for the preview proofs:

You are allowed to consult your class notes, any handouts from the course website, your textbook, and your classmates to figure these out, but NO OTHER RESOURCES. You are not permitted to talk to anyone outside the class about the problems, or to consult other books or anything on the internet other than our course website.

Signature: ____

No calculators or notes are permitted on this portion of the exam. If you make a mistake while working, you can either erase cleanly or draw a single line (no crazy scribbling, please) through any parts we should ignore. You may use the back of any page for scratch work.

For the proofs, you may use any of our basic axioms and theorems about arithmetic without specifically quoting them. All proofs should be in paragraph form.

This exam has 4 problems on 6 pages, including this cover sheet. You have 60 minutes to complete this portion of the exam.

Problem	Max	Score
1	12	
2	10	
3	8	
4	20	
Total	55	

1. (a) State Rolle's Theorem.

   (b) State and prove Cauchy's Mean Value Theorem.

2. Using the formal definition of definite integrals (i.e. the one involving limits of Riemann sums), prove the following.

   Suppose f and g are integrable functions on the interval $[a, b]$. If $f(x) \geq g(x)$ for all $x \in [a, b]$, prove that $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

3. Suppose y_1 is a solution of the differential equation $f(x)y'' + g(x)y' + h(x)y = A(x)$, and y_2 is a solution of the differential equation $f(x)y'' + g(x)y' + h(x)y = B(x)$, where f, g, h, A, and B are functions of x. Prove that $y_1 + y_2$ is a solution of the differential equation
$$f(x)y'' + g(x)y' + h(x)y = A(x) + B(x).$$
(You may use theorems we know about derivatives.)

4. In this problem, we will prove a very nice theorem that is frequently used by real life differential equationists[1], called Gronwall's Inequality. The various parts of the problem will provide an outline of the proof for you to fill in. You may use anything we have learned this semester, including "big theorems" (like the FTC or product rule), but make sure you clearly explain what you are using.

   Suppose that u and f are real-valued continuous functions on a closed interval $[a, b]$ with $a < b$. Also suppose that f is differentiable on the open interval (a, b). Gronwall's Inequality says that if
   $$u'(t) \leq u(t)f(t) \text{ for all } t \in (a, b),$$
   then
   $$u(t) \leq u(0)e^{\int_0^t f(s)ds}.$$

   (a) Consider the function $g(t) = e^{\int_0^t f(s)ds}$, defined on the interval $[a, b]$. Show that for all $t \in (a, b)$, we have
   $$g'(t) = g(t)f(t).$$

(Continued on the next page.)

[1] Totally made up term. They exist, but nobody will tell me what to call them.

(b) Note that $g(t) \geq 0$ for all $t \in [a, b]$. Using what you know about derivatives, plus information from earlier parts, prove that

$$\frac{d}{dt}\left(\frac{u(t)}{g}(t)\right) \leq 0 \text{ for all } t \in (a, b).$$

(c) Using part (b), compare $\frac{u(t)}{g(t)}$ and $\frac{u(0)}{g(0)}$, and use this to complete the proof.