DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

Formulae

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_S| \leq \frac{K(b-a)^5}{180n^4}$$

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

Math 1B — First Midterm Exam (LEC 002) — 11:10 am to 12pm

This exam consists of 5 questions. Answer the questions in the spaces provided.

Name and discussion section: ______________________________

GSI's name: ______________________________

1. Compute the following integrals:

   (a) (10 points)

$$\int x^{-2}\ln(x)\,dx$$

Solution: Do integration by parts with $f(x) = \ln(x)$, $g'(x) = x^{-2}$, $g(x) = \frac{-1}{x}$. Hence

$$\int x^{-2}\ln(x)\,dx = \frac{-\ln(x)}{x} - \int \frac{-1}{x}\cdot\frac{1}{x}\,dx$$

$$= -\frac{\ln(x)}{x} + \int x^{-2}\,dx = -\frac{\ln(x)}{x} - \frac{1}{x} + C$$

   (b) (10 points)

$$\int \frac{1}{\sqrt{x^2+4x+5}}\,dx$$

Solution:

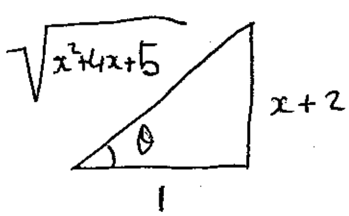
Complete the square : $x^2+4x+5 = (x+2)^2+1$

Hence substitute $x+2 = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$

$\Rightarrow dx = \sec^2\theta\,d\theta$. Hence

$$\int \frac{1}{\sqrt{x^2+4x+5}}\,dx = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$$

$$= \ln\left|\sqrt{x^2+4x+5} + (x+2)\right| + C$$



$\Rightarrow \sec\theta = \sqrt{x^2+4x+5}$

PLEASE TURN OVER

3. (20 points) Find the area of the surface of revolution (about the x-axis) of the curve $y = (x-1)^3$ between $x = 1$ and $x = 2$.

Solution:

$$\text{Surface area} = \int_1^2 2\pi (x-1)^3 \sqrt{1 + \left(3(x-1)^2\right)^2}\, dx$$

$$= 2\pi \int_1^2 (x-1)^3 \sqrt{1 + 9(x-1)^4}\, dx$$

Let $u = (x-1)^4 \Rightarrow \frac{du}{dx} = 4(x-1)^3 \Rightarrow dx = \frac{du}{4(x-1)^3}$

$$\text{Surface area} = \frac{2\pi}{4} \int_0^1 \sqrt{1 + 9u}\, du = \frac{\pi}{2}\left[\frac{2}{27}(1+9u)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{\pi}{27}\left(10\sqrt{10} - 1\right)$$

2. (20 points) Compute the following integral:

$$\int \frac{x^2+3x+3}{(x+1)^3}\,dx$$

Solution:

$$\frac{x^2+3x+3}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$

$$= \frac{Ax^2 + (2A+B)x + (A+B+C)}{(x+1)^3}$$

$$\Rightarrow \begin{array}{l} A = 1 \\ 2A+B = 3 \\ A+B+C = 3 \end{array} \Rightarrow \begin{array}{l} A = 1 \\ B = 1 \\ C = 1 \end{array}$$

$$\int \frac{x^2+3x+3}{(x+1)^3}\,dx = \int \frac{1}{(x+1)}\,dx + \int \frac{1}{(x+1)^2}\,dx + \int \frac{1}{(x+1)^3}\,dx$$

$$= \ln|x+1| - \frac{1}{(x+1)} - \frac{1}{2(x+1)^2} + C$$

4. Evaluate following improper integrals (if divergent, write divergent and explain your reasoning):

(a) (10 points)

$$\int_{-\infty}^{\infty} e^{|x|} dx$$

Solution:

$$\int_0^\infty e^{|x|} dx = \int_0^\infty e^x dx = \lim_{t\to\infty} \left[e^x \right]_0^t$$

$$= \lim_{t\to\infty} \left(e^t - 1 \right) = \infty$$

Hence $\int_{-\infty}^{\infty} e^{|x|} dx$ divergent. //

(b) (10 points)

$$\int_0^2 \frac{4+\cos(x)}{x^5} dx$$

(Hint: use the comparison test)

Solution:

$4 + \cos(x) \geq 3$ for all x in $[0,2]$.

Hence $\frac{4+\cos(x)}{x^5} \geq \frac{3}{x^5}$ for all x in $[0,2]$

But $\int_0^2 \frac{3}{x^5} dx$ diverges, hence $\int_0^2 \frac{4+\cos(x)}{x^5} dx$ diverges. //

5. (a) (10 points) Assume that $f(0) = 3$. Use Simpsons Rule with $n = 6$ to approximate the value of $f(6)$, where $f'(x)$ takes the following values:

x	0	1	2	3	4	5	6
$f'(x)$	0	2	4	3	1	4	5

Solution:

$$\int_0^6 f'(x)\,dx \approx S_6 = \frac{1}{3}\left(0 + 4\cdot 2 + 2\cdot 4 + 4\cdot 3 + 2\cdot 1 + 4\cdot 4 + 5\right)$$

$$= \frac{1}{3}(51) = 17$$

$$\| $$

$$f(6) - f(0)$$

$$\|$$

$$f(6) - 3 \qquad \Rightarrow \quad f(6) \approx 17 + 3 = 20 \quad //$$

(b) (10 points) Assuming that $|f^{(5)}(x)| \leq 1$, for all $0 < x < 6$, how large an n would you need to choose to guarantee that the above estimate is within 0.001 of the true value of $f(6)$? You do not need to give an exact answer, just a rough bound.

Solution:

Estimating $f(6)$ to within 0.001 is equivalent to estimating $\int_0^6 f'(x)\,dx$ to within 0.001.

Observe that $(f')^{(4)}(x) = f^{(5)}(x)$. Hence choose $K=1$ in Simpson's Error Bound. Hence need n such that

$$\frac{6^5}{180\, n^4} < 0.001 \iff \sqrt[4]{\frac{1000 \cdot 6^5}{180}} < n$$

$$\iff \frac{60}{\sqrt[4]{300}} < n \qquad //$$

END OF EXAM