MATH H1B FINAL EXAM, PART I
Wednesday, December 13, 2011

No calculators are permitted on this portion of the exam. You may use one sheet of paper filled with notes. Some formulas are provided on the next page.

You must show your work for all problems, unless specifically indicated otherwise. Answers with insufficient or incorrect work will be given little (if any) credit.

You are not required to write all your explanations in complete sentences, but your work should be organized and easy to read. You may use the back of any page to do scratch work that you don't want graded. If you make a mistake, you can cleanly erase or put a large X or a single line through anything we should ignore. If you need more space, draw an arrow to indicate that you will continue on the back.

Some advice: don't stay stuck on a problem for long unless you have already solved the other problems. I arranged the problems roughly in order of increasing difficulty, and I don't expect that everyone will finish everything.

This exam has 10 problems on 9 pages, including this cover sheet. You have 100 minutes to complete this portion of the exam.

Problem	Max	Score
1	10	
2	6	
3	6	
4	8	
5	10	
6	12	
7	10	
8	12	
9	14	
10	12	
Total	100	

1. (2 points each) Determine whether each of the statements below is true or false. Choose TRUE only if the statement is always true under the given circumstances. No justification is needed.

   (a) Using Euler's method with 20 steps gives a more accurate answer than using Euler's method with 4 steps.

   TRUE *FALSE*

   (b) All solutions of the differential equation $y'' = -x^2 - \frac{20}{x^4}$ are concave down functions.

   TRUE *FALSE*

   (c) The differential equation $y' = 6xy - 15x + 2y - 5$ is separable.

   TRUE *FALSE*

   (d) The second-order equation $y'' + 5xy = \sqrt{y}$ is linear.

   TRUE *FALSE*

   (e) If y_1 and y_2 are solutions of the differential equation $y''+y'+y = x$, then $3y_1+7y_2$ is a solution of the equation $y'' + y' + y = 10x$.

   TRUE *FALSE*

2. (2 points each) Tell whether each of the following systems of different equations represents a pair of populations with a *predator-prey* relationship, *competition* for common resources, or *cooperation* for mutual benefit. (No justification necessary.)

   (a) $\frac{dx}{dt} = 0.12x - 0.0006x^2 + 0.000001xy$
   $\frac{dy}{dt} = 0.08x + 0.00004xy$

   (b) $\frac{dx}{dt} = 0.2x - 0.0002x^2 - 0.0006xy$
   $\frac{dy}{dt} = -0.015y + 0.00008xy$

   (c) $\frac{dx}{dt} = 0.15x - 0.0002x^2 - 0.0006xy$
   $\frac{dy}{dt} = 0.2y - 0.00008y^2 - 0.00007xy$

3. (6 points) Suppose you are using the method of undetermined coefficients to solve the differential equation

$$y'' + 2y' + 10y = \frac{x^2 \cos(5x)}{e^x}.$$

What would you expect a particular solution to look like? (Use capital letters to indicate any unknown constants.)

4. (8 points) Solve the following boundary value problem, or explain why it has no solution.

$$y'' + 100y = 0, \quad y(0) = 3, \quad y(\pi) = 7$$

5. The direction field for a first-order differential equation $\frac{dy}{dx} = f(x,y)$ is shown below. (Note that f is a function involving both x and y, but you are not given a formula for f.)

   (a) (4 points) On the graph, sketch the solution to the differential equation which satisfies the initial condition $y(-2.5) = 0.5$. Label this curve A.

   (b) (6 points) On the graph, use Euler's method with five equally spaced steps to approximate $y(2.5)$ if we are given the initial condition $y(-2.5) = 0.5$. Use the provided straight edge, and label your work B for this part. Note, you have to do this graphically, as you do not have a formula to do algebraic computations. Make sure that what you draw indicates you understand how Euler's method works.

   $y(2.5) \approx$ ____________

6. Find the fourth degree Taylor polynomial which approximates the series solution to the differential equation
$$y'' + x^2y' + xy = 0.$$

7. (12 points) Find the equation of the curve which passes through the point $(1, 1)$ and which is orthogonal to the family of curves $y^2 = kx^3$. (An implicit expression is fine.)

8. A force of 16 N is needed to keep a spring with a 2-kg mass stretched 0.5 m beyond its natural length.

   (a) (4 points) What is the spring constant? What damping constant would produce critical damping in this system?

   (b) (8 points) Assume we have critical damping, as we found in part (a). If the mass starts stretched 0.5 m beyond its natural length, with an initial velocity of 1 m/s, find a formula for its position at time t.

9. (14 points) The following Bernoulli differential equation is not linear, but you can transform it into a first-order linear equation using a substitution. Find the general solution.

$$\frac{dy}{dx} + \left(\frac{2}{x}\right) y = \left(\frac{1}{x^2}\right) y^3$$

10. (12 points) Solve the following differential equation using the method of variation of parameters:

$$y'' - 2y' + y = e^{2x}.$$

(No credit will be given for using the method of undetermined coefficients instead.)