MATH H53 MIDTERM 2 SOLUTION

1. Let $f(x,y) = \begin{cases} \frac{x^2y+xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(1) Please compute $f_x(0,0)$ and $f_x(x,y)$ for $(x,y) \neq (0,0)$.
(2) Is f_x continuous at $(0,0)$? Justify your answers.

(1) Since $f(x,0) = \frac{x^2\cdot 0 + x\cdot 0^2}{x^2+0^2} = 0$ for any $x \neq 0$, we have

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = 0.$$

For $(x,y) \neq (0,0)$, we can compute $f_x(x,y)$ as

$$\begin{aligned} f_x(x,y) =& \frac{(x^2+y^2)\frac{\partial}{\partial x}(x^2y+xy^2) - (x^2y+xy^2)\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} \\ =& \frac{(x^2+y^2)(2xy+y^2) - (x^2y+xy^2)2x}{(x^2+y^2)^2} \\ =& \frac{-x^2y^2+2xy^3+y^4}{(x^2+y^2)^2}. \end{aligned}$$

(2) f_x is not continuous at $(0,0)$. If we approach $(0,0)$ along the y–axis, i.e. take points $(0,y)$ and let y goes to 0, then

$$\lim_{y\to 0} f_x(0,y) = \lim_{y\to 0} \frac{-0^2\cdot y^2 + 2\cdot 0\cdot y^3 + y^4}{(0^2+y^2)^2} = \lim_{y\to 0}\frac{y^4}{y^4} = 1 \neq 0 = f_x(0,0).$$

2. (5 points) For $z = f(x,y)$ with polar substitution $x = r\cos\theta,\ y = r\sin\theta$, please compute

$$\frac{\partial z}{\partial r},\ \frac{\partial z}{\partial \theta}$$

and

$$\Big(\frac{\partial z}{\partial r}\Big)^2 + \frac{1}{r^2}\Big(\frac{\partial z}{\partial \theta}\Big)^2,$$

in terms of f_x and f_y.

We first compute $\frac{\partial x}{\partial r}$, $\frac{\partial y}{\partial r}$, $\frac{\partial x}{\partial \theta}$ and $\frac{\partial y}{\partial \theta}$:

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta,\ \frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta,$$

and

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta,\ \frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta.$$

By the chain rule, we have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta = \frac{x}{\sqrt{x^2+y^2}}f_x + \frac{y}{\sqrt{x^2+y^2}}f_y$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta = -yf_x + xf_y.$$

So

$$\Big(\frac{\partial z}{\partial r}\Big)^2 + \frac{1}{r^2}\Big(\frac{\partial z}{\partial \theta}\Big)^2 = \big(f_x\cos\theta + f_y\sin\theta\big)^2 + \big(-f_x\sin\theta + f_y\cos\theta\big)^2 = f_x^2 + f_y^2.$$

3. (5 points) Please find all the critical points for

$$f(x,y) = x^3 + 3xy^2 - 15x + y^3 - 15y.$$

Then determine whether these critical points are local maximum, local minimum or saddle points.

To find critical points we need to solve $f_x(x,y) = f_y(x,y) = 0$. That is

$$\begin{cases} f_x(x,y) &=& 3x^2 + 3y^2 - 15 &=& 0 \\ f_y(x,y) &=& 6xy + 3y^2 - 15 &=& 0. \end{cases}$$

Compare two equations, we get $3x^2 = 6xy$, which implies that either $x = 0$ or $x = 2y$.

If $x = 0$, we have $3y^2 - 15 = 0$, i.e. $y = \pm\sqrt{5}$. If $x = 2y$, we have $15y^2 - 15 = 0$, i.e. $y = \pm 1$. So the critical points are $(0, \sqrt{5})$, $(0, -\sqrt{5})$, $(2, 1)$ and $(-2, -1)$.

To determine the maximum or the minimum, we need to compute second derivatives:

$$f_{xx}(x,y) = 6x, \; f_{xy}(x,y) = 6y, \; f_{yy}(x,y) = 6x + 6y.$$

So

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 36(x^2 + xy - y^2).$$

For $(0, \sqrt{5})$ and $(0, -\sqrt{5})$, we have $D(x,y) = -180 < 0$, so these two points are saddle point.

For $(2, 1)$, we have $D(x,y) = 180 > 0$ and $f_{xx}(x,y) = 12 > 0$, so it is a local minimum.

For $(-2, -1)$, we have $D(x,y) = 180 > 0$ and $f_{xx}(x,y) = -12 < 0$, so it is a local maximum.

4. Please evaluate the following integral:

$$\int_0^1 \int_{y^2}^1 e^{x^2} y \, \mathrm{d}x \, \mathrm{d}y.$$

We can rewrite the integration as

$$\int_0^1 \int_{y^2}^1 e^{x^2} y \, \mathrm{d}x \, \mathrm{d}y = \iint_D e^{x^2} y \, \mathrm{d}A,$$

here $D = \{(x, y) \mid 0 \le y \le 1,\ y^2 \le x \le 1\}$.

An alternative description for D is $D = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le \sqrt{x}\}$, so we have

$$\begin{aligned}
&\int_0^1 \int_{y^2}^1 e^{x^2} y \, \mathrm{d}x \, \mathrm{d}y \\
=&\int_0^1 \int_0^{\sqrt{x}} e^{x^2} y \, \mathrm{d}y \, \mathrm{d}x \\
=&\int_0^1 \left[\frac{1}{2} e^{x^2} y^2\right]_{y=0}^{y=\sqrt{x}} \mathrm{d}x \\
=&\int_0^1 \frac{1}{2} x e^{x^2} \, \mathrm{d}x \\
=&\frac{1}{4} e^{x^2}\Big]_{x=0}^{x=1} \\
=&\frac{1}{4}(e-1).
\end{aligned}$$