MATH 104-06 MIDTERM 1 SOLUTION

1. (30 points) For a set X and three subsets $A, B, C \subseteq X$, show that

$$(A \setminus B) \cup (A \cap B \cap C) = A \setminus (B \setminus C).$$

Proof. We first show that $(A \setminus B) \cup (A \cap B \cap C) \subseteq A \setminus (B \setminus C)$.

For any $x \in (A \setminus B) \cup (A \cap B \cap C)$, then either $x \in A \setminus B$ or $x \in A \cap B \cap C$.

If $x \in A \setminus B$, so $x \in A$ and $x \notin B$. Since $B \setminus C \subseteq B$, $x \notin B \setminus C$. Since $x \in A$, we have $x \in A \setminus (B \setminus C)$.

If $x \in A \cap B \cap C$, then $x \in A$. Since $x \in C$, $x \notin B \cap C^{\complement} = B \setminus C$. Sine we know that $x \in A$, so $x \in A \setminus (B \setminus C)$.

So we have $(A \setminus B) \cup (A \cap B \cap C) \subseteq A \setminus (B \setminus C)$ holds.

Then we show that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap B \cap C)$.

For any $x \in A \setminus (B \setminus C)$, we have $x \in A$ and $x \notin B \setminus C$.

If $x \notin B$, then we have $x \in A \setminus B \subseteq (A \setminus B) \cup (A \cap B \cap C)$.

If $x \in B$, then we also have $x \in C$, otherwise $x \in B \setminus C$. Since $x \in A$, we have $x \in A \cap B \cap C \subseteq (A \setminus B) \cup (A \cap B \cap C)$.

So we have shown that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap B \cap C)$.

So $(A \setminus B) \cup (A \cap B \cap C) = A \setminus (B \setminus C)$ is true.

$\square$

2. (30 points) Let $E_n \subseteq \mathbb{R}$, $n = 1, 2, \cdots$ be a sequence of subsets of $\mathbb{R}$. Suppose there is a real number $M \in \mathbb{R}$ such that $\sup E_n \leq M$ for each $n \in \mathbb{N}$, show that

$$\sup\left(\cup_{i=1}^{\infty} E_n\right) = \sup\left\{\sup E_n | n \in \mathbb{N}\right\}.$$

Proof. We first show that $\sup\left(\cup_{i=1}^{\infty} E_n\right) \geq \sup\left\{\sup E_n | n \in \mathbb{N}\right\}$.

By definition, for any $x \in \cup_{n=1}^{\infty} E_n$, $\sup\left(\cup_{n=1}^{\infty} E_n\right) \geq x$ holds. Fix an $n \in \mathbb{N}$, and for any $y \in E_n$, $y \in \cup_{n=1}^{\infty} E_n$, so $\sup\left(\cup_{n=1}^{\infty} E_n\right) \geq y$ holds, which implies $\sup\left(\cup_{n=1}^{\infty} E_n\right) \geq \sup E_n$. Since $\sup\left(\cup_{n=1}^{\infty} E_n\right) \geq \sup E_n$ holds for any $n \in \mathbb{N}$, we have $\sup\left(\cup_{n=1}^{\infty} E_n\right) \geq \sup\left\{\sup E_n | n \in \mathbb{N}\right\}$.

Then we show that $\sup\left(\cup_{n=1}^{\infty} E_n\right) \leq \sup\left\{\sup E_n | n \in \mathbb{N}\right\}$.

Since $\sup\left\{\sup E_n | n \in \mathbb{N}\right\} \geq \sup E_n$ for any $n \in \mathbb{N}$, for any $n \in \mathbb{N}$ and any $x \in E_n$, we have $\sup\left\{\sup E_n | n \in \mathbb{N}\right\} \geq x$. Then for any $y \in \cup_{n=1}^{\infty} E_n$, $y \in E_n$ for some $n \in \mathbb{N}$, so we have $\sup\left\{\sup E_n | n \in \mathbb{N}\right\} \geq y$, which implies that $\sup\left\{\sup E_n | n \in \mathbb{N}\right\} \geq \sup\left(\cup_{n=1}^{\infty} E_n\right)$.

So we have shown that $\sup\left(\cup_{n=1}^{\infty} E_n\right) = \sup\left\{\sup E_n | n \in \mathbb{N}\right\}$ holds.

$\square$

3. (30 points) For three sequences of real numbers (t_n), (s_n) and (u_n), if $t_n \leq u_n \leq s_n$ for any $n \in \mathbb{N}$, and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = u$ for some $u \in \mathbb{R}$, then show that

$$\lim_{n\to\infty} u_n^k = u^k$$

for any $k \in \mathbb{N}$. (Hint: show it by induction.)

Proof. We show that $\lim_{n\to\infty} u_n^k = u^k$ holds by induction on k.

So we first show that $\lim_{n\to\infty} u_n = u$. For any $\epsilon > 0$, since $\lim_{n\to\infty} s_n = u$, there exists $N_1 \in \mathbb{N}$, such that for any $n > N_1$, $|s_n - u| < \epsilon$ holds; since $\lim_{n\to\infty} t_n = u$, there exists $N_2 \in \mathbb{N}$, such that for any $n > N_2$, $|t_n - u| < \epsilon$ holds.

So for any $n > \max\{N_1, N_2\}$, we have $|s_n - u| < \epsilon$, so $s_n < u + \epsilon$; also since $|t_n - u| < \epsilon$, we have $t_n > u - \epsilon$. So $u - \epsilon < t_n \leq u_n \leq s_n < u + \epsilon$, which implies that $|u_n - u| < \epsilon$ holds. So we get that $\lim_{n\to\infty} u_n = u$.

Suppose that $\lim_{n\to\infty} u_n^m = u^m$ holds. Then

$$\lim_{n\to\infty} u_n^{m+1} = \lim_{n\to\infty} u_n^m \cdot u_n = (\lim_{n\to\infty} u_n^m) \cdot (\lim_{n\to\infty} u_n) = u^m \cdot u = u^{m+1}.$$

$\square$

4. (10 points) For a sequence of real numbers (a_n), if $\lim_{n\to\infty} a_n = a$ for $a \in \mathbb{R}$, show that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a.$$

Proof. For any any $\epsilon > 0$, we consider $\epsilon' = \frac{\epsilon}{2}$. Since $\lim_{n\to\infty} a_n = a$, there exists $N' \in \mathbb{N}$, such that for any $n > N'$, $|a_n - a| < \epsilon'$ holds.

Let $A = |(a_1 + a_2 + \cdots + a_{N'}) - N'a|$ (we have fixed N' in the previous paragraph), then for $N = \lceil \frac{2A}{\epsilon} \rceil$, we will show that for any $n > N$, $\frac{a_1+a_2+\cdots+a_n}{n} < \epsilon$ holds.

$$\begin{aligned}
&|\frac{a_1 + a_2 + \cdots + a_n}{n} - a| \\
=&|\frac{a_1 + a_2 + \cdots + a_{N'} + a_{N'+1} + \cdots + a_n - na}{n}| \\
=&|\frac{(a_1 + a_2 + \cdots + a_{N'}) - N'a}{n} + \frac{(a_{N'+1} - a) + (a_{N'+2} - a) + \cdots + (a_n - a)}{n}| \\
\leq&\frac{|(a_1 + a_2 + \cdots + a_{N'}) - N'a|}{n} + \frac{|a_{N'+1} - a| + |a_{N'+2} - a| + \cdots + |a_n - a|}{n} \\
<&\frac{A}{n} + \frac{(n - N')\epsilon'}{n} \\
<&\frac{A}{\frac{2A}{\epsilon}} + \frac{\epsilon}{2} \\
=&\epsilon.
\end{aligned}$$

So $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = a$ holds. ∎