DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

Formulae

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \int \sec(x)\ dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$e^x = 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \cdots = \textstyle\sum_{n=0}^{\infty} \tfrac{x^n}{n!}$$

$$\sin x = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \tfrac{x^9}{9!} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \tfrac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \tfrac{x^8}{8!} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \tfrac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \tfrac{x^7}{7} + \tfrac{x^9}{9} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \tfrac{x^{2n+1}}{2n+1}$$

CALCULATORS ARE NOT PERMITTED

Math 1B Practice Final Exam

This exam consists of 10 questions. Answer the questions in the spaces provided.

Name and section: ______________________________

GSI's name: ______________________________

1. Compute the following integrals:

   (a) (5 points)

$$\int x \ln(x+1)\, dx$$

Solution:

Let $u = x+1 \Rightarrow dx = du \Rightarrow \int x \ln(x+1)\, dx = \int (u-1)\ln(u)\, du$

Do integration by parts with $f(u) = \ln(u)$, $g(u) = \frac{1}{2}u^2 - u \Rightarrow$

$$\int x\ln(x+1)\, dx = \left(\tfrac{1}{2}u^2 - u\right)\ln(u) - \int \tfrac{1}{2}u - 1\, du = \left(\tfrac{1}{2}u^2 - u\right)\ln(u) - \tfrac{1}{4}u^2 + u + C$$

$$= \left(\tfrac{1}{2}(x+1)^2 - (x+1)\right)\ln(x+1) - \tfrac{1}{4}(x+1)^2 + (x+1) + C$$

   (b) (5 points)

$$\int x^3\sqrt{x^2+4}\, dx$$

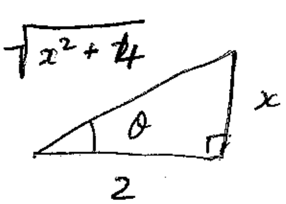
Solution:

Let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta \Rightarrow \int x^3\sqrt{x^2+4}\, dx = 32\int \tan^3\theta \sec^3\theta\, d\theta$

$= 32\int \sin^3\theta \cos^{-6}\theta\, d\theta$. Let $u = \cos\theta \Rightarrow d\theta = \frac{du}{-\sin\theta}$

$$\Rightarrow 32\int \sin^3\theta\cos^{-6}\theta\, d\theta = -32\int (1-u^2)u^{-6}\, du = \frac{32}{5}u^{-5} - \frac{32}{3}u^{-3} + C$$

$$= \frac{32}{5}\cos^{-5}\theta - \frac{32}{3}\cos^{-3}\theta + C = \frac{32}{5}\cdot\frac{(x^2+4)^{\frac{5}{2}}}{2^5} - \frac{32}{3}\frac{(x^2+4)^{\frac{3}{2}}}{2^3} + C$$

(

$\Rightarrow \cos\theta = \frac{2}{\sqrt{x^2+4}}$)

PLEAE TURN OVER

2. (10 points) Determine if the following series are absolutely convergent, conditionally convergent or divergent. You do not need to show your working.

(a)

$$\sum_{n=1}^{\infty}(\arctan(n+1)-\arctan(n))$$

Solution: ABSOLUTELY CONVERGENT

(b)

$$\sum_{n=1}^{\infty}\frac{n}{n^3+2}$$

Solution: ABSOLUTELY CONVERGENT

(c)

$$\sum_{n=1}^{\infty}\frac{(2n)!}{(3n)!}$$

Solution: ABSOLUTELY CONVERGENT

(d)

$$\sum_{n=1}^{\infty}\frac{1+4^n}{2^n+3^n+4^n}$$

Solution: DIVERGENT

(e)

$$\sum_{n=1}^{\infty}(-1)^n\sin\left(\frac{1}{n}\right)$$

Solution: CONDITIONALLY CONVERGENT.

3. Determine the radius and interval of convergent of the following power series:

(a) (5 points)

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(x-4)^{2n}$$

Solution:

Let $a_n = \frac{2^n}{n}(x-4)^{2n} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{2n}{(n+1)} \cdot |x-4|^2$

$= \frac{2}{1+\frac{1}{n}}|x-4|^2 \to 2|x-4|^2$. Hence convergent if $2|x-4|^2 < 1$ as $n \to \infty$. Divergent if $2|x-4|^2 > 1$,

$2|x-4|^2 < 1 \iff |x-4| < \frac{1}{\sqrt{2}}$. Hence $R = \frac{1}{\sqrt{2}}$ and Interval of convergence is $(4-\frac{1}{\sqrt{2}}, 4+\frac{1}{\sqrt{2}})$.

(b) (5 points)

$$\sum_{n=1}^{\infty} \frac{n!}{(4n)!}x^n$$

Solution:

Let $a_n = \frac{n!}{(4n)!}x^n \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\,|x|}{(4n+1)(4n+2)(4n+3)(4n+4)}$

$= \frac{(\frac{1}{n^3}+\frac{1}{n^4})|x|}{(4+\frac{1}{n})(4+\frac{2}{n})(4+\frac{3}{n})(4+\frac{4}{n})} \to 0$ as $n \to \infty$.

Hence $R = \infty$ and and interval of convergence is the whole of the real numbers, i.e. $(-\infty, \infty)$.

4. (10 points) What is the domain of the function $f(x)$ given by the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{12^n n}$$

What is the value of $f^{(3)}(3)$?

Solution:

First we determine the interval of convergence.

$$a_n = \frac{(x-3)^n}{12^n n} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|\,n}{12\,(n+1)} = \frac{|x-3|}{12} \cdot \frac{1}{1+\frac{1}{n}}$$

$$\rightarrow \frac{|x-3|}{12} \text{ as } n \to \infty.$$ Hence $R = 12$ and

$(-9, 15)$ is the interval of convergence. Need to check end points.

$x = -9$ gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ← convergent by alternating series test.

$x = 15$ gives $\sum_{n=1}^{\infty} \frac{1}{n}$ ← divergent harmonic series.

Hence domain is $[-9, 15)$.

$$\frac{f^{(3)}(3)}{3!} = \frac{1}{12^3 \cdot 3} \Rightarrow f^{(3)}(3) = \frac{2}{12^3} \text{ //}$$

5. (10 points) Determine the Taylor series of the function

$$\frac{1}{(1-x)^2},$$

about the point $x = \frac{1}{2}$.

Solution:

$$f(x) = \frac{1}{(1-x)^2} = (1-x)^{-2} \Rightarrow f(\tfrac{1}{2}) = (\tfrac{1}{2})^{-2} = 2^2$$

$$f'(x) = 2(1-x)^{-3} \Rightarrow f'(\tfrac{1}{2}) = 2\cdot(\tfrac{1}{2})^{-3} = 2\cdot 2^3$$

$$f''(x) = 3!\,(1-x)^{-4} \Rightarrow f''(\tfrac{1}{2}) = 3!\,2^4$$

$$f'''(x) = 4!\,(1-x)^{-5} \Rightarrow f'''(\tfrac{1}{2}) = 4!\,2^5$$

$$\vdots$$

$$f^{(n)}(x) = (n+1)!\,2^{n+2}$$

The Taylor series for $\frac{1}{(1-x)^2}$ is therefore

$$2^2 + \frac{2\cdot 2^3}{1!}(x-\tfrac{1}{2}) + \frac{3!\cdot 2^4}{2!}(x-\tfrac{1}{2})^2 + \frac{4!\cdot 2^5}{3!}(x-\tfrac{1}{2})^3 + \dots$$

$$\dots + \frac{(n+1)!}{n!}\cdot 2^{n+2}(x-\tfrac{1}{2})^n + \dots$$

$$= \sum_{n=0}^{\infty} (n+1)\,2^{n+2}(x-\tfrac{1}{2})^n$$

6. (10 points) Find the general solution to the following differential equation

$$x\frac{dy}{dx} = y - x.$$

Solution:

This is a linear 1^{st} order differential equation.

$$x\frac{dy}{dx} = y - x \Rightarrow \frac{dy}{dx} - \left(\frac{1}{x}\right)\cdot y = -1$$

Let $I(x) = e^{\int \frac{-1}{x}dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1}$

$$\Rightarrow \frac{d(x^{-1}y)}{dx} = \left(\frac{1}{x}\right)\frac{dy}{dx} - \left(\frac{1}{x^2}\right)y = \frac{-1}{x}$$

$$\Rightarrow x^{-1}y = -\int \frac{1}{x}dx = -\ln|x| + C$$

$$\Rightarrow y = -x\ln|x| + Cx$$

//

7. (10 points) Find the equation of the curve which passes through (1,0) and is orthogonal to the family of curves given by $y^2 = kx^3$, for k a constant.

Solution:

Chain Rule

$$y^2 = kx^3 \Rightarrow \frac{dy^2}{dx} = \frac{d\,kx^3}{dx} \Rightarrow 2y\frac{dy}{dx} = 3kx^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3kx^2}{2y} = \frac{3\left(\frac{y^2}{x^3}\right)x^2}{2y} = \frac{3y}{2x}$$

Thus need to solve

$$\frac{dy}{dx} = \frac{-2x}{3y}$$

Equation is separable, hence must solve

$$\int 3y\,dy = -\int 2x\,dx$$

$$\Rightarrow \frac{3}{2}y^2 = -x^2 + C$$

$$\Rightarrow x^2 + \frac{3}{2}y^2 = C$$

$$1^2 + \frac{3}{2}\cdot 0^2 = C \Rightarrow 1 = C.$$

Hence equation is $x^2 + \frac{3}{2}y^2 = 1$ //

8. (10 points) Find a solution to the initial-value problem

$$\sec(x)\frac{dy}{dx} - xe^{-y} = 0, \quad y(0) = 2$$

Solution:

This is separable.

$$\sec(x)\frac{dy}{dx} - xe^{-y} = 0 \Rightarrow e^{y}\,dy = x\cos(x)\,dx$$

Hence solve

$$\int e^{y}\,dy = \int x\cos(x)\,dx \Rightarrow$$

Integration by parts

$$e^{y} = x\sin(x) + \cos(x) + C$$

$$\Rightarrow \quad y = \ln\left(x\sin(x) + \cos(x) + C\right)$$

$$y(0) = 2 \Rightarrow 2 = \ln(1 + C) \Rightarrow C = e^{2} - 1$$

$$\Rightarrow y(x) = \ln\left(x\sin(x) + \cos(x) + e^{2} - 1\right) \quad //$$

9. (10 points) Find the general solution to the following differential equation

$$y'' + 2y' + 2y = x\sin(2x) + x^2$$

Solution:

First solve $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0 \Rightarrow r_1 = -1 + i,\ r_2 = -1 - i \Rightarrow \alpha = -1,\ \beta = 1$

$\Rightarrow y_c(x) = c_1 e^{-x}\cos(x) + c_2 e^{-x}\sin(x)$

Let $y_{p_1}(x) = Ax\cos(2x) + Bx\sin(2x) + C\cos(2x) + D\sin(2x) \Rightarrow$

$y_{p_1}''(x) + 2y_{p_1}'(x) + 2y_{p_1}(x) = (4B-2A)x\cos(2x) + (-4A-2B)x\sin(2x) + (2A+4B-2C+4D)\cos(2x) + (-4A+2B-4C-2D)\sin(2x)$

Need to solve $4B - 2A = 0$, $-4A - 2B = 1$ $\Rightarrow A = 2B \Rightarrow -10B = 1 \Rightarrow B = \frac{-1}{10},\ A = \frac{-2}{10}$

Now solve $2A + 4B - 2C + 4D = 0$, $-4A + 2B - 4C - 2D = 0$ $\Rightarrow -2C + 4D = -8B = \frac{8}{10}$, $-4C - 2D = 6B = \frac{-6}{10}$ $\Rightarrow -10C = \frac{8}{10} + 2\left(\frac{-6}{10}\right) = \frac{-4}{10}$ $\Rightarrow C = \frac{4}{100} \Rightarrow D = \frac{22}{100}$

$\Rightarrow y_{p_1}(x) = \frac{-2}{10}x\cos(2x) + \frac{-1}{10}x\sin(2x) + \frac{4}{100}\cos(2x) + \frac{22}{100}\sin(2x).$

Let $y_{p_2}(x) = Ax^2 + Bx + C \Rightarrow y_{p_2}'' + 2y_{p_2}' + 2y_{p_2} = 2A + 4Ax + 2B + 2Ax^2 + 2Bx + 2C$

Now solve $2A = 1$, $4A + 2B = 0$, $2A + 2B + 2C = 0$ $\Rightarrow A = \frac{1}{2},\ B = -1,\ C = \frac{1}{2}$ $\Rightarrow y_{p_2}(x) = \frac{1}{2}x^2 - x + \frac{1}{2}$

Thus general solution is

$y(x) = \frac{-2}{10}x\cos(2x) + \frac{-1}{10}x\sin(2x) + \frac{4}{100}\cos(2x) + \frac{22}{100}\sin(2x) + \frac{1}{2}x^2 - x + \frac{1}{2} + c_1 e^{-x}\cos(x) + c_2 e^{-x}\sin(x)$ //

10. (10 points) Find a non-zero power series solution to the following differential equation

$$y'' - xy = 0.$$

You do not need to show convergence.

Solution:

Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$

and $xy = \sum_{n=1}^{\infty} c_{n-1} x^n \Rightarrow$

$$y'' - xy = 2c_2 + \sum_{n=1}^{\infty} \left((n+2)(n+1) c_{n+2} - c_{n-1}\right) x^n$$

Hence need to find c_n such that

$2c_2 = 0$ and

$(n+2)(n+1) c_{n+2} - c_{n-1} = 0$ for all $n \geq 1$

$\Rightarrow c_2 = 0$ and $c_{n+3} = \dfrac{c_n}{(n+3)(n+2)}$ for all $n \geq 0$

Seeing as we are only try to find a single power series solution assume $c_1 = 0$ and $c_0 = 1$.

$c_2 = 0 \Rightarrow c_{3n+2} = 0$ for all $n \geq 0$, $c_1 = 0 \Rightarrow c_{3n+1} = 0$ for all $n \geq 0$.

$c_3 = \dfrac{c_0}{3 \cdot 2}$, $c_6 = \dfrac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}$ ... $c_{3n} = \dfrac{c_0}{3n \cdot (3n-1) \cdots 6 \cdot 5 \cdot 3 \cdot 2}$

Thus a power series solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{3n \cdot (3n-1) \cdots 6 \cdot 5 \cdot 3 \cdot 2}$$

END OF EXAM