MATH 104-06 MIDTERM 2 SOLUTION

1. (10 points) Determine whether the following statements are true of false, no justification is required.

(1) For a sequence of real numbers (a_n), if $\limsup_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

True

(2) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

False

(3) If a continuous function $f : \mathbb{R} \to \mathbb{R}$ is bounded, then f is uniformly continuous on $\mathbb{R}$.

False

(4) For a power series $\sum_{n=1}^{\infty} a_n x^n$, if the radius of convergence equals 1, and the power series converges at $x = 1$, then it also converges at $x = -1$.

False

(5) Let (f_n) be a sequence of continuous functions, with the domain of each f_n being $\mathbb{R}$. Suppose that (f_n) uniformly converges to $f : \mathbb{R} \to \mathbb{R}$, then f is also continuous.

True

2. (30 points) Let s be a real number and (s_n) be a sequence of real numbers. If for any subsequence (s_{n_k}) of (s_n), (s_{n_k}) has a subsequence $(s_{n_{k_l}})$ satisfying $\lim_{l\to\infty} s_{n_{k_l}} = s$, show that $\lim_{n\to\infty} s_n = s$ holds.

Proof. Method 1: For the sequence (s_n), there is a subsequence (s_{n_k}) such that $\lim_{k\to\infty} s_{n_k} = \limsup_{n\to\infty} s_n$. Since the limit of any subsequence of (s_{n_k}) equals $\lim_{k\to\infty} s_{n_k}$, which also equals s by the assumption. So we have $\limsup_{n\to\infty} s_n = s$.

The same argument shows that $\liminf_{n\to\infty} s_n = s$. So $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$, which implies $\lim_{n\to\infty} s_n = s$.

Method 2: Suppose that $\lim_{n\to\infty} s_n \neq s$, then there exists $\epsilon > 0$, such that for any $N \in \mathbb{N}$, there exists $n > N$ such that $|s_n - s| \geq \epsilon$.

First take an arbitrary $n_1 \in \mathbb{N}$ such that $|s_{n_1} - s| \geq \epsilon$. For $n_1 \in \mathbb{N}$, there exists $n_2 > n_1$ such that $|s_{n_2} - s| \geq \epsilon$. Then for $n_2 \in \mathbb{N}$, there exists $n_3 > n_2$ such that $|s_{n_3} - s| \geq \epsilon$. By doing this process inductively, we get a subsequence (s_{n_k}) of (s_n) such that $|s_{n_k} - s| \geq \epsilon$ for any $k \in \mathbb{N}$.

So for any subsequence $(s_{n_{k_l}})$ of (s_{n_k}), we have $|s_{n_{k_l}} - s| \geq \epsilon$ for any $l \in \mathbb{N}$. This implies that $\lim_{l\to\infty} s_{n_{k_l}} \neq s$, and we get a contradiction here.

$\square$

3. (30 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by the following rule. If x is an irrational number, then we define $f(x) = 0$. If x is a rational number, then x is uniquely expressed as $x = \frac{p}{q}$ with $q \in \mathbb{N}$ and p, q are relative prime integers (they do not have common factors). In this case we define $f(x) = \frac{1}{q}$ (for example $f(0) = f(\frac{0}{1}) = \frac{1}{1} = 1$, $f(1) = f(\frac{1}{1}) = \frac{1}{1} = f(\frac{-1}{1}) = f(-1)$ and $f(\frac{1}{2}) = f(-\frac{1}{2}) = \frac{1}{2}$).

Show that f is continuous at all irrational numbers and discontinuous at all rational numbers.

Proof. For any rational point $x_0 = \frac{p}{q}$, we have $f(x_0) = f(\frac{p}{q}) = \frac{1}{q}$. We can choose a sequence of irrational numbers (x_n) with $\lim_{n\to\infty} x_n = x_0$ (by taking an arbitrary irrational number $x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$). Then since all the x_n are irrational, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \neq \frac{1}{q} = f(x_0)$. So f is not continuous at x_0, and f is discontinuous at all rational numbers.

For any irrational number x_0, we will show that for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $x \in \mathbb{R}$ with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$. Take a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, and consider all the rational numbers with denominator smaller than n.

For any natural number $m < n$, there are at most $2m$ rational numbers in $(x_0 - 1, x_0 + 1)$ with denominator equals m. Since for $2m + 1$ different rational numbers with denominator equals m, the maximum minus the minimum is greater or equal to 2. So there are finitely many rational numbers (at most $n(n-1)$ such numbers) in $(x_0 - 1, x_0 + 1)$ with denominator smaller than n. Since x_0 is an irrational number, there exists $\delta \in (0, 1)$ such that there is not a rational number with denominator smaller than n lying in $(x_0 - \delta, x_0 + \delta)$.

Then for any $|x - x_0| < \delta$, if x is irrational, then $|f(x) - f(x_0)| = |0 - 0| = 0 < \delta$. If x is irrational, then $x = \frac{p}{q}$ with $p \in \mathbb{N}$ and $p \geq n$. So $|f(x) - f(x_0)| = |\frac{1}{q} - 0| = \frac{1}{q} \leq \frac{1}{n} < \delta$. So f is continuous at all irrational points.

$\square$

4. (30 points) Suppose that $f : [2, \infty) \to \mathbb{R}$ is a uniformly continuous function, show that $g : [2, \infty) \to \mathbb{R}$ defined by $g(x) = \frac{f(x)}{x}$ is a bounded function.

Then show that $g : [2, \infty) \to \mathbb{R}$ is a uniformly continuous function.

(Note that f may not be differentiable on $(2, \infty)$.)

Bonus (5 points): Does $\lim_{x\to+\infty} g(x)$ always exist? Prove your claim or give a counterexample.

Proof. Since f is uniformly continuous on $[2, \infty)$, for $\epsilon = 1$, there exists $\delta > 0$, such that for any $x, y \in [2, \infty)$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < 1$.

Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. So if $|x - 2| \leq \frac{1}{n}$ and $x \in [2, \infty)$, then $|f(x) - f(2)| < 1$. An induction argument shows that if $|x - 2| \leq \frac{m}{n}$ and $x \in [2, \infty)$, then $|f(x) - f(2)| < m$. So for any $x \in [2 + \frac{k-1}{n}, 2 + \frac{k}{n}]$ with $k \geq 1$, we have

$$\frac{|f(x)|}{x} \leq \frac{|f(x) - f(2)| + |f(2)|}{x} \leq \frac{k + |f(2)|}{\frac{k-1}{n} + 2} \leq \frac{k + |f(2)|}{\frac{k}{n}} = n\frac{k + |f(2)|}{k} \leq n(1+|f(2)|).$$

So $g : [2, \infty) \to \mathbb{R}$ is bounded by $M = n(1 + |f(2)|)$.

For any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [2, \infty)$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Then for any $x, y \in [2, \infty)$ with $|x - y| < \min\{\delta, \frac{\epsilon}{M}\}$, we have

$$|g(x)-g(y)| = |\frac{f(x)}{x} - \frac{f(y)}{y}| = \frac{|yf(x) - xf(y)|}{xy} \leq \frac{|yf(x) - xf(x)| + |xf(x) - xf(y)|}{xy}$$

$$\leq \frac{|y - x|}{y} \cdot \frac{|f(x)|}{x} + \frac{|f(x) - f(y)|}{y} \leq \frac{\epsilon}{2M} \cdot M + \frac{\epsilon}{2} = \epsilon.$$

So g is uniformly continuous on $[2, \infty)$

$\lim_{x\to\infty} g(x)$ may not exist. For example, let $f : [2, \infty) \to \mathbb{R}$ be defined by $f(x) = x \cdot \sin(\log x)$.

Then

$$f'(x) = \sin(\log x) + x \cdot \frac{1}{x} \cdot \cos(\log x) = \sin(\log x) + \cos(\log x),$$

and $|f'(x)| \leq 2$ for any $x \in [2, \infty)$. Since f' exists and bounded in $[2, \infty)$, f is uniformly continuous on $[2, \infty)$.

However $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \sin(\log x)$ does not exists. For example, take $x_n = e^{(n+\frac{1}{2})\pi}$, then $\lim_{n\to\infty} x_n = \infty$. However, $g(x_n) = \sin((n + \frac{1}{2})\pi) = (-1)^n$ so $\lim_{n\to\infty} g(x_n)$ does not exist. □