Practice and sample problems from past exams

1. Prove that if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, then $\vec{b} = \vec{c}$, provided that $\vec{a} \neq \vec{0}$,

2. Compute the volume of the tetrahedron with the vertices (a, a, a), $(-a, -a, a)$, $(-a, a, -a)$, $(a, -a, -a)$.

3. Find the angle between two disjoint diagonals in two adjacent faces of a cube.

4. Which of the functions can be transformed into each other by linear changes of coordinates:
$$3xy + y^2, \quad x^2 + 3xy + 2y^2, \quad x^2 + 3xy - 2y^2 \ ?$$

5. A scalar-valued function f of a vector argument is called *linear* if
$$f(\lambda \vec{u} + \mu \vec{v}) = \lambda f(\vec{u}) + \mu f(\vec{v})$$
for all vectors $\vec{u}, \vec{v}$ and all scalars λ, μ.

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Which of the following functions of $\vec{r}$ are linear: (a) $f(\vec{r}) = 2 + 3x + 4y + 5z$, (b) $f(\vec{r}) = |\vec{r}|$, (c) $f(\vec{r}) = \vec{r} \cdot \vec{r}$, (d) $f(\vec{r}) = \vec{a} \cdot (\vec{r} \times \vec{b})$, where $\vec{a}, \vec{b}$ are given vectors.

6. Find an equation of the plane passing through the point $(1, 2, 3)$ and parallel to the plane $2x + 3y = 4z + 5$.

7. Express the dot-product $\vec{a} \cdot \vec{b}$ of two vectors through the lengths M and N of their sum and difference: $M = |\vec{a} + \vec{b}|$ and $N = |\vec{a} - \vec{b}|$.

8. Compute the area of the quadrilateral $ABCD$ with the vertices:
$$A = (0, 0, 0), \ B = (1, 2, 3,), \ C = (3, 5, 8), \ D = (2, 3, 5).$$

9. A function f on the plane is said to have a given line as an *axis of symmetry* if $f(P) = f(Q)$ whenever the points P and Q are symmetric to each other about this line. Find all axes of symmetry of the function $f(x, y) = x^2 + y^2 - 4xy$.

10. How many axes of symmetry does the function $x^2 + y^2$ have?

11. Any ellipse can be described as the set of all points in the plane with a fixed sum of distances to two fixed point, called *foci*.

Locate the foci of the ellipse $ax^2 + by^2 = 1$ assuming that $a > b$.

12. Identify graphs of the functions: $A = x^2 + y^2 - 2xy$, $B = x^2 + y^2 - xy$, $C = x^2 + y^2$, $D = x^2 + y^2 + 2xy$, $E = x^2 + y^2 + 3xy$.