MATH 142 MIDTERM 1 SOLUTION

1. (30 points) For a topological space X and two subsets $A, B \subseteq X$, show that

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

and

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

hold, then give an example to show that

$$\overline{A \cap B} = \overline{A} \cap \overline{B}$$

may not hold.

Proof. We first show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ holds.

The first step is to show $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

If $x \in A \cup B$, then $x \in A$ or $x \in B$, so $x \in \overline{A} \cup \overline{B}$ since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$.

If x is a limit point of $A \cup B$, then for any open set U containing x, we have $(U \setminus \{x\}) \cap (A \cup B) \neq \emptyset$. Then we claim that x lies in either $\overline{A}$ or $\overline{B}$. Otherwise there are open sets U_1 and U_2 containing x, such that $(U_1 \setminus \{x\}) \cap A = \emptyset$ and $(U_2 \setminus \{x\}) \cap B = \emptyset$. Then $U = U_1 \cap U_2$ is an open set containing x, with $(U \setminus \{x\}) \cap (A \cup B) = \emptyset$, and we get a contradiction here. So $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ holds.

Then we show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

For any $x \in \overline{A} \cup \overline{B}$, we only prove $x \in \overline{A \cup B}$ for the case $x \in \overline{A}$. The proof for the case $x \in \overline{B}$ is exactly the same.

For $x \in \overline{A}$, if $x \in A$, then $x \in A \cup B \subseteq \overline{A \cup B}$.

If x is a limit point of A, then for any open set U containing x, $(U \setminus \{x\}) \cap A \neq \emptyset$. So $(U \setminus \{x\}) \cap (A \cup B) \neq \emptyset$, and x is a limit point of $A \cup B$. So we have shown that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

So $\overline{A \cup B} = \overline{A} \cup \overline{B}$ is true.

Now we show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ holds.

For any $x \in \overline{A \cap B}$, if $x \in A \cap B$, then $x \in A \subseteq \overline{A}$ and $x \in B \subseteq \overline{B}$. So we have $x \in \overline{A} \cap \overline{B}$.

If x is a limit point of $A \cap B$, then for any open set U containing x, $(U \setminus \{x\}) \cap (A \cap B) \neq \emptyset$. So we have $(U \setminus \{x\}) \cap A \neq \emptyset$, which implies that x is a limit point of A; we also have $(U \setminus \{x\}) \cap B \neq \emptyset$, which implies that x is a limit point of B. So we have that $x \in \overline{A}$ and $x \in \overline{B}$, then $x \in \overline{A} \cap \overline{B}$. So $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ holds.

Let $X = \mathbb{E}^1$, $A = (-1, 0)$ and $B = (0, 1)$. $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$. However, $\overline{A} = [-1, 0]$ and $\overline{B} = [0, 1]$, so $\overline{A} \cap \overline{B} = \{0\}$. In this case $\overline{A \cap B} = \emptyset \neq \{0\} = \overline{A} \cap \overline{B}$. $\square$

2. (30 points) Let X be the topological space with the set being $\mathbb{R}$ and a subset $U \subseteq \mathbb{R}$ is a neighborhood of $x \in \mathbb{R}$ if $[x, x+\epsilon) \subseteq U$ for some $\epsilon > 0$. Let $\mathbb{E}^1$ be the set of real numbers with the standard topology, and $f : \mathbb{E}^1 \to X$ defined by $f(x) = x$ for any $x \in \mathbb{R}$, then f is clearly a bijection.

Is f a continuous function? Is f^{-1} a continuous function? Justify your answers.

Answer: f is not a continuous function, while f^{-1} is continuous.

Proof. $[0, 1)$ is an open set in X, we will show that $f^{-1}([0, 1)) = [0, 1) \subseteq \mathbb{E}^1$ is not an open set in $\mathbb{E}^1$. For $0 \in [0, 1) \subseteq \mathbb{E}^1$ and any open neighborhood U of 0, U contains an open interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. However, $-\frac{\epsilon}{2} \in U$ but $-\frac{\epsilon}{2} \notin [0, 1)$, so $U \nsubseteq [0, 1)$. So we have that $f^{-1}([0, 1)) = [0, 1) \subseteq \mathbb{E}^1$ is not open, and f is not a continuous function.

For any open set $U \subset \mathbb{E}^1$, we need to show that $(f^{-1})^{-1}(U) = f(U) \subseteq X$ is an open set in $\mathbb{E}^1$. For any $x \in f(U) \subseteq X$, we also have $x \in U \subseteq \mathbb{E}^1$ (consider U as both subsets of X and $\mathbb{E}^1$). Since $U \subseteq \mathbb{E}^1$ is open, there is some $\epsilon > 0$, such that $(x - \epsilon, x + \epsilon) \subseteq U$. So for a neighborhood $[x, x+\epsilon)$ of x in X, $[x, x+\epsilon) \subset f(U)$ holds. Since x is an arbitrary point in $f(U)$, we know that $f(U) \subseteq X$ is open, and f^{-1} is continuous. $\square$

3. (20 points) For two topological spaces X and Y, suppose that $O \subseteq X$ is a dense subset of X and $f : X \to Y$ is a continuous function, show that $f(O) \subseteq f(X)$ is also dense.

Proof. We need to show that, under the subspace topology of $f(X) \subseteq Y$, $\overline{f(O)} = f(X)$ holds. In the following, we will reserve the notation $\overline{f(O)}$ to be the closure of $f(O)$ in $f(X)$, instead of the closure in Y.

So we need to show that for any $y \in f(X) \setminus f(O)$, y is a limit point of $f(O)$ (in $f(X)$). Since $y \in f(X) \setminus f(O)$, there is a point $x \in X \setminus O$ such that $f(x) = y$.

For any open set U in $f(X)$ containing y, by the definition of subspace topology, $U = V \cap f(X)$ for an open set $V \subseteq Y$. Since f is a continuous function, $f^{-1}(V)$ is an open set in X with $x \in f^{-1}(V)$. Since O is dense in X and $x \notin O$, x is a limit point of O. So $(f^{-1}(V) \setminus \{x\}) \cap O$ is not empty, and take any point $z \in (f^{-1}(V) \setminus \{x\}) \cap O$.

Then we claim that $f(z) \in (U \setminus \{y\}) \cap f(O)$. Since $z \in O$, $f(z) \in f(O)$ holds. Since $z \in f^{-1}(V)$ and $z \in X$, $f(Z) \in V \cap f(X) = U$. Since $f(z) \in O$ and $y \notin O$, $f(z) \neq y$ also holds. So we have that $f(z) \in (U \setminus \{y\}) \cap f(O)$, and in particular $(U \setminus \{y\}) \cap f(O) \neq \emptyset$.

So y is a limit point of $f(O)$ (in $f(X)$), which implies that O is dense in $f(X)$.

$\square$

4. (20 points) Show that a compact Hausdorff space is both regular and normal.

Recall that a topological space X is regular if it is Hausdorff, and for any point $x \in X$ and any closed subset $B \subseteq X$ with $x \notin B$, there are disjoint open sets U and V of X such that $x \in U$ and $B \subseteq V$.

Also recall that a topological space X is normal if it is Hausdorff, and for any two disjoint closed sets $A, B \subseteq X$, there are disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

Proof. For a compact Hausdorff space X, we first show that X is regular. X is Hausdorff by assumption.

For any point $x \in X$ and a closed subset $B \subseteq X$ with $x \notin B$, we will construct two disjoint open sets U and V of X such that $x \in U$ and $B \subseteq V$. Since X is a compact space, and $B \subseteq X$ is a closed subset, B is also compact.

For any point $b \in B$, we know that$b \neq x$. Since X is Hausdorff, there are disjoint open sets U_b, V_b such that $x \in U_b$ and $b \in V_b$. Then $\{V_b\}_{b \in B}$ is an open cover of B. Since B is compact, the open cover $\{V_b\}_{b \in B}$ has a finite subcover. We denote this finite subcover by $\{V_1, V_2, \cdots, V_n\}$, and the corresponding opens sets containing x are denoted by $U_1, U_2, \cdots, U_n$.

So $U = \cap_{i=1}^n U_i$ is an open set containing x, and $V = \cup_{i=1}^n V_i$ is an open set containing B. Since $U_i \cap V_i = \emptyset$ for any $1 \leq i \leq n$, $U = \cap_{i=1}^n U_i$ and $V = \cup_{i=1}^n V_i$ are disjoint.

So X is regular.

To show that X is normal, we need only to show that for any two disjoint closed sets $A, B \subseteq X$, there are disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$. Since X is compact and A is a closed subset of X, A is compact.

For any $a \in A$, since X is regular, there exist two disjoint open sets U_a and V_a with $a \in U_a$ and $B \subseteq V_a$. Then $\{U_a\}_{a \in A}$ is an open cover of A. Since A is compact, the open cover $\{U_a\}_{a \in A}$ has a finite subcover. We denote this finite subcover by $\{U_1, U_2, \cdots, U_m\}$, and the corresponding opens sets containing B are denoted by $V_1, V_2, \cdots, V_m$.

So $U = \cup_{i=1}^m U_i$ is an open set containing A, and $V = \cap_{i=1}^m V_i$ is an open set containing B. Since $U_i \cap V_i = \emptyset$ for any $1 \leq i \leq m$, $U = \cap_{i=1}^m U_i$ and $V = \cup_{i=1}^m V_i$ are disjoint.

So X is normal. $\square$