Solutions to Practice Midterm #1

1. (a) $\int x^{1/3} \ln x \, dx = \int \ln x d\left(\frac{3}{4}x^{4/3}\right) = \frac{3}{4}x^{4/3} \ln x - \int \frac{3}{4}x^{4/3} d \ln x$
$= \frac{3}{4}x^{4/3} \ln x - \frac{3}{4}\int x^{1/3}dx = \frac{3}{4}x^{4/3} \ln x - \frac{9}{16}x^{4/3} + C;$

(b) $\int \frac{u+3}{(u-1)(u-3)}du = \int \left(\frac{3}{u-3} - \frac{2}{u-1}\right) du = 3 \ln|u-3| - 2 \ln|u-1| + c;$

(c) $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{(1-\sin x)(1+\sin x)}dx = \int \frac{(1-\sin x)}{(1-\sin^2 x)}dx$
$= \int \frac{(1-\sin x)}{\cos^2 x}dx = \int \frac{dx}{\cos^2 x} - \int \frac{\sin x dx}{\cos^2 x} = \tan x - \sec x + C.$

(d) Make the substitution $u = (1+\sqrt{x})^{1/2}$. Then

$$x = (u^2-1)^2, \ dx = 4u(u^2-1);$$

$$\int (1+\sqrt{x})^{1/2}dx = \int 4u^2(u^2-1)du = \frac{4}{5}u^5 - \frac{4}{3}u^3 + C = \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + C.$$

2. (a) $\int_1^3 \frac{dx}{x-1} = \lim_{t\to 1^+} \int_t^3 \frac{dx}{x-1} = \lim_{t\to 1^+} (\ln 2 - \ln(t-1)) = \infty$. Integral diverges.

(b) First make the substitution $u = e^x$. Then

$$\begin{aligned}
\int \frac{dx}{1+e^x} &= \int \frac{du}{u(u+1)} = \int \frac{du}{u} - \int \frac{du}{u+1} \\
&= \ln|u| - \ln|u+1| + C = \ln\left|\frac{u}{u+1}\right| + C = \ln \frac{e^x}{1+e^x} + C. \\
\int_0^\infty \frac{dx}{1+e^x} &= \lim_{t\to\infty} \int_0^t \frac{dx}{1+e^x} = \lim_{t\to\infty} \left(\ln \frac{e^x}{1+e^t} - \ln \frac{e^0}{1+e^0}\right) \\
&= \ln\left(\lim_{t\to\infty} \frac{e^t}{1+e^t}\right) - \ln \frac{1}{2} = \ln 1 + \ln 2 = \ln 2.
\end{aligned}$$

(c) We use the formula $|S_n| \le \frac{(b-a)^5 K}{180n^4}$ if $|f^{(4)}(x)| \le K$ for $a \le x \le b$. First let us estimate K.

$$(\sin(x^2))^{(4)} = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2).$$

One can choose $K = 12 + 48 = 60$ since $|(16x^4 - 12)\sin(x^2)| \le 12$ and $|48x^2\cos(x^2)| \le 48$. Then $|S_n| \le \dfrac{60}{180n^4}$. Apply the condition $|S_n| \le 0.001$. Since $\dfrac{60}{180n^4} \le 0.001$ implies that $n^4 \ge 60000/180 \approx 333$. Thu $n = 5$ is sufficiently large but since n is even in Simpson rule take $n = 6$. The formula for calculation of the integral is

$$S_6 = \frac{1}{18}\left(0 + 4\sin\frac{1}{36} + 2\sin\frac{1}{9} + 4\sin\frac{1}{4} + 2\sin\frac{4}{9} + 4\sin\frac{25}{36} + \sin 1\right).$$

4. The first integral is divergent because

$$\frac{e^x}{x} \ge \frac{1}{x} \text{ for } 0 \le x \le 1$$

and

$$\int_0^1 \frac{dx}{x}$$

is divergent.

The second integral converges because

$$\frac{e^x}{\sqrt{x}} \le \frac{e}{\sqrt{x}} \text{ for } 0 \le x \le 1$$

and the integral

$$\int_0^1 \frac{e}{\sqrt{x}}dx = e\int_0^1 \frac{dx}{\sqrt{x}}$$

is convergent.