MATH 104-06: INTRODUCTION TO ANALYSIS

REVIEW OF IMPORTANT CONCEPTS AND THEOREMS

Short Version

Chapter 0: Sets and functions.

- Sets: subsets, intersection, union, difference, product, equivalence relation.
- Functions: injection, surjection, bijection, inverse function, composition of functions.
- Induction.

Chapter 1: Real numbers.

- Real numbers: axioms of ordered fields, absolute value.
- Completeness Axiom: upper bound and lower bound, **supremum** and **infimum**, **completeness Axiom**.
- Symbols of $+\infty$ and $-\infty$.

Chapter 2: Sequences.

- **Limits of sequences**: ϵ–N definition.
- Limit theorems of sequences: limit of sum, difference, product, quotient of two convergent sequences, limit to infinity.
- Monotone sequences always have limits.
- Definition of **limsup** and **liminf**, use $\limsup s_n = \liminf s_n$ to show convergence, **Cauchy sequence**.
- Subsequences: **Bolzano–Weierstrass theorem**, set of subsequential limits S.
- Series: Cauchy criterion of series, absolutely convergence, comparison test, ratio test, root test, alternating series, integral test.

Chapter 3: Continuity.

- **Continuity**: definition by sequences and by ϵ–δ. Sum, difference, product, quotient, composition of continuous functions are continuous.
- Properties of continuous functions: supremum and infimum on closed intervals are realized, intermediate value theorem.
- **Uniformly continuous**. Definition, continuous functions on closed intervals are uniformly continuous.
- Limit of functions: definition by sequences, and by ϵ–δ or ϵ–N.

Chapter 4: Sequence and series of functions.

- Power series: radius of convergence $R = \frac{1}{\beta}$ with $\beta = \limsup |a_n|^{\frac{1}{n}}$.
- Pointwise convergence: definition, pointwise limit of continuous functions may not be continuous.
- **Uniform convergence**: definition, uniform limit of continuous functions is continuous.

- Uniformly Cauchy for sequences, Weierstrass M–test, limit of the integration of uniformly convergent sequences of function.
- Power series uniformly converges in $[-R_1, R_1]$ for any $0 < R_1 < R$, so we can compute the integration and derivative by taking integration and derivative for each term.

Chapter 5: Differentiation.

- **Differentiable (derivative)**: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$. Derivative of sum, difference, product, quotient, composition of functions.
- **Mean value theorem**: $f'(x_0) = \frac{f(b)-f(a)}{b-a}$ for some $x_0 \in (a, b)$. Intermediate value theorem for derivatives.
- L'Hospital's Rule: compute $\lim_{x \to s} \frac{f(x)}{g(x)}$ by computing $\lim_{x \to s} \frac{f'(x)}{g'(x)}$.
- Taylor's series: estimate the remainder $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$, $R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$ for some y lies between x and c.

Chapter 6: Integration.

- **Definition of integration**: upper Darboux sum and lower Darboux sum for a partition, upper Darboux integral $U(f)$ and lower Darboux integral $L(f)$. f is integrable on $[a, b]$ if $U(f) = L(f)$, then $\int_a^b f = U(f) = L(f)$.
- **Theorem**: f is integrable on $[a, b]$ if and only if for any $\epsilon > 0$, there exists a partition P of $[a, b]$, such that $U(f, P) - L(f, P) < \epsilon$.
- Properties of Riemann integral.
  - Monotone functions, continuous functions, are integrable.
  - Sum, product, absolute value of integrable functions are integrable.
  - Intermediate value theorem for integration.
- Fundamental theorem for calculus: integration of derivatives and derivative of integrations.
- Improper integral.

Chapter 7: Metric spaces (Section 13, 21, 22)

- **Metric space**: definition for a metric (distance function).
- Convergence of sequences and Cauchy sequences in metric spaces, completeness for metric spaces.
- Open sets, closed sets in metric spaces; interior, closure; Cantor set; compactness.
- Continuous functions between metric spaces: ϵ–δ definition and definition by open sets. Continuous functions on compact spaces are uniformly continuous, and the images are compact.
- Baire category theorem: Intersection of open dense sets in a complete metric space is still dense.
- Connectedness and path–connectedness of metric spaces: disjoint open decomposition versus connection by paths. Intervals are connected, path–connected spaces are connected (the reverse is not true).

Full Version

Chapter 0: Sets and functions.

- Sets: subsets, intersection, union, difference, product, equivalence relation.
  To show two sets A and B are equal: show $A \subseteq B$ and $B \subseteq A$.
- Functions: injection, surjection, bijection, inverse function, composition of functions.
- Induction.

Chapter 1: Real numbers.

- Real numbers: Dedekind cuts (not required), axioms of ordered fields, absolute value.
- Completeness Axiom: upper bound and lower bound, **supremum** (smallest upper bound) and **infimum** (greatest lower bound), Archimedean property, denseness of $\mathbb{Q}$ (so does $\mathbb{R} \setminus \mathbb{Q}$).
  Completeness Axiom: for every nonempty subset of $\mathbb{R}$, if it is bounded above, then the supremum exists.
  To show that $M \geq \sup S$, need only to show that $M \geq s$ for any $s \in S$, i.e. M is an upper bound of S.
- Symbols of $+\infty$ and $-\infty$.

Chapter 2: Sequences.

- **Limits of sequences**: $\lim_{n\to\infty} s_n = s$: for any $\epsilon > 0$, there exists N, such that for any $n > N$, $|s_n - s| < \epsilon$.
  $\lim_{n\to\infty} s_n \neq s$: there exists $\epsilon > 0$, such that for any N, there exists $n > N$, such that $|s_n - s| \geq \epsilon$.
- Limit theorems of sequences: limit of sum, difference, product, quotient of two convergent sequences, basic examples of limit of sequences (Theorem 9.7), limit to infinity.
- Monotone sequences: monotone sequences always have limits, bounded monotone sequences always converge.
- **limsup** and **liminf**: $\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n > N\}$.
  $\lim_{n\to\infty} s_n = s$ if and only if $\limsup s_n = \liminf s_n = s$.
- **Cauchy sequence**: for any $\epsilon > 0$, there exists N, such that for any $m, n > N$, $|s_m - s_n| < \epsilon$.
  (s_n) is Cauchy if and only if (s_n) converges.
- Subsequences: criterion of subsequential limit (Theorem 11.2), set of subsequential limits S.
  Bolzano–Weierstrass theorem: every bounded sequence has a convergent subsequence.
  $\sup S = \limsup_{n\to\infty} s_n$, $\inf S = \liminf_{n\to\infty} s_n$, S is a closed set.
- Series: series as limit of partial sums, Cauchy criterion of series, absolutely convergence, comparison test, ratio test, root test, alternating series, integral test.

Chapter 3: Continuity.

- **Continuity**: $f : dom(f) \to \mathbb{R}$ is continuous at $x_0 \in dom(f)$ if for any sequence (x_n) in $dom(f)$, $\lim_{n\to\infty} x_n = x_0$ implies $\lim_{n\to\infty} f(x_n) = f(x_0)$.
  Equivalent definition of continuity at $x_0 \in dom(f)$: for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in dom(f)$ with $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$ holds.

Sum, difference, product, quotient, composition of continuous functions are continuous.

- Properties of continuous functions:
  - Continuous functions on closed intervals realize the supremum and infimum.
  - Intermediate value theorem.
  - Inversions of continuous strictly increasing functions are continuous.
- **Uniformly continuous**: for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in dom(f)$ with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$ holds.
  Theorem: Continuous functions on closed intervals are uniformly continuous.
  Uniformly functions map Cauchy sequences to Cauchy sequences.
- Limit of functions: $\lim_{x \to a^S}$, $\lim_{x\to a}$, $\lim_{x\to a^-}$, $\lim_{x\to a^+}$, $\lim_{x\to -\infty}$, $\lim_{x\to+\infty}$, definition by sequences, and by ϵ–δ or ϵ–N (possibly $a \notin dom(f)$).
  $\lim_{x\to a} f(x)$ exists if and only if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and equal; f is continuous at a if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$.

Chapter 4: Sequence and series of functions.

- Power series: For $\sum a_n x^n$, the radius of convergence $R = \frac{1}{\beta}$ with $\beta = \limsup |a_n|^{\frac{1}{n}}$, the power series converges for $|x| < R$ and diverges for $|x| > R$.
- Pointwise convergence: (f_n) converges pointwise to f on S if for any $x \in S$, $\lim f_n(x) = f(x)$ (fix $x \in S$, for any $\epsilon > 0$, there exists N, such that for any $n > N$, $|f_n(x) - f(x)| < \epsilon$). Pointwise limit of continuous functions may not be continuous.
- **Uniform convergence**: (f_n) converges uniformy to f on S if for any $\epsilon > 0$, there exists N, such that for any $n > N$, $|f_n(x) - f(x)| < \epsilon$ for any $x \in S$.
  Theorem: Uniform limit of continuous functions is continuous.
  Uniform limit must be the pointwise limit. If the pointwise limit f is known, calculus might be applied to determine whether (f_n) converges to f uniformly.
- Uniformly Cauchy for sequences (useful for series of functions), Weierstrass M–test.
  For a sequences of continuous functions which uniformly converges, the integrations of this sequence also converge to the integration of the limit function.
- We can compute the integration and derivative of a power series by taking integration and derivative for each term (in $(-R, R)$).
  Power series uniformly converges in $[-R_1, R_1]$ for any $0 < R_1 < R$, and uniformly converges in $[0, R]$ if it converges at $x = R$.

Chapter 5: Differentiation.

- **Differentiable (derivative)**: f is differentiable at $a \in dom(f)$ if $dom(f)$ contains an open interval containing a, and $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.
  Derivative of sum, difference, product, quotient, composition (chain rule) of functions. Differentiable implies continuous.
- The mean value theorem.
  - f take maximum at $x_0 \in (a, b)$ and also differentiable at x_0, then $f'(x_0) = 0$.
  - Rolle's theorem: f continuous on $[a, b]$, differentiable on (a, b), if $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.
  - **Mean value theorem**: f continuous on $[a, b]$, differentiable on (a, b), then there exists $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.
  - $f' = 0, \geq 0$ or > 0 implies f is constant, increasing, or strictly increasing.
  - Intermediate value theorem for derivatives.

- L'Hospital's Rule: If $\lim_{x\to s} f(x) = \lim_{x\to s} g(x) = 0$ or $\lim_{x\to s} |g(x)| = +\infty$, and $\lim_{x\to s} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to s} \frac{f(x)}{g(x)} = L$. (Tricks for changing the expressions.)
Generalized mean value theorem.
- Taylor's series: estimate the remainder $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$, the Taylor's series converges to $f(x)$ at x if and only if $\lim_{n\to\infty} R_n(x) = 0$.
$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$ for some y lies between x and c (which shows the Taylor's series of a few functions converge to the original functions).
$R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\mathrm{d}t = (x-c) \cdot \frac{(x-y)^{n-1}}{(n-1)!} \cdot f^{(n)}(y)$ for some y lies between x and c.
- Newton's method and secant method for numerical computation of roots of functions.

Chapter 6: Integration.

- **Definition of integration**: $f : [a,b] \to \mathbb{R}$ bounded, $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ partition of $[a,b]$; for any $S \subseteq [a,b]$, $M(f,S) = \sup\{f(x) \mid x \in S\}$ and $m(f,S) = \inf\{f(x) \mid x \in S\}$.
  - Upper Darboux sum: $U(f,P) = \sum_{k=1}^n M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$.
  - Lower Darboux sum: $L(f,P) = \sum_{k=1}^n m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$.
  - Upper Darboux integral: $U(f) = \sup\{U(f,P) \mid P \textit{ is any partition of } [a,b]\}$.
  - Lower Darboux integral: $L(f) = \inf\{L(f,P) \mid P \textit{ is any partition of } [a,b]\}$.

  f is integrable on $[a,b]$ if $U(f) = L(f)$, then $\int_a^b f = \int_a^b f(x)\mathrm{d}x = U(f) = L(f)$.
- For any two partitions P, Q of $[a,b]$, $L(f,P) \le L(f, P\cup Q) \le U(f, P\cup Q) \le U(f,Q)$, so $L(f) \le U(f)$, and $L(f,P) \le L(f) \le U(f) \le U(f,P)$.
Theorem: f is integrable on $[a,b]$ if and only if for any $\epsilon > 0$, there exists a partition P of $[a,b]$, such that $U(f,P) - L(f,P) < \epsilon$.
- Properties of Riemann integral.
  - Monotone functions, continuous functions, piecewise bounded monotone functions, piecewise continuous functions are integrable.
  - Sum, product, absolute value of integrable functions are integrable.
  - Continuous nonnegative functions have positive integral, except the functions is the constant function that equals 0.
  - Intermediate value theorem for integration.
- Fundamental theorem for calculus: integration of derivatives and derivative of integrations. Application: integration by parts, change of variables.
- Improper integral.

Chapter 7: Metric spaces (Section 13, 21, 22)

- **Metric space** For a set S, $d : S \times S \to \mathbb{R}$ is a metric if
  - $d(x,y) \ge 0$ and $d(x,y) = 0$ if and only if $x = y$.
  - $d(x,y) = d(y,x)$.
  - $d(x,z) \le d(x,y) + d(y,z)$.

  Then (S,d) is a metric space.
- Convergence of sequences in metric spaces, Cauchy sequences in metric spaces, completeness for metric spaces. (Sequences in $\mathbb{R}^n$, completeness of $\mathbb{R}^n$.)
- Open sets, closed sets in metric spaces; interior, closure; Cantor set. Compactness (equivalent with closed and bounded for subsets of $\mathbb{R}^n$).

- Continuous functions between metric spaces: replace absolute values by metrics in the real–valued function case, equivalent definition by only using open sets. Continuous functions on a compact space is uniformly continuous, and the image is compact, such real–valued functions reach the supremum and infimum.
- Baire category theorem: Intersection of open dense sets in a complete metric space is still dense, and a few equivalent statements.
- Connectedness and path–connectedness of metric spaces: disjoint open decomposition versus connection by paths. Intervals are connected, path–connected spaces are connected (the reverse is not true).