MATH 142 MIDTERM 2 SOLUTION

1. (10 points) Determine whether the following statements are true of false, no justification is required.

(1) A path-connected component of a topological space may not be a closed subset.

True

(2) The identification space of a Hausdorff space is still Hausdorff.

False

(3) Let X be a topological space and p, q be two points in X, then $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$.

False

(4) Let X, Y be two path-connected topological spaces with isomorphic fundamental groups, then X and Y are homeomorphic to each other.

False

(5) Any contractible topological space is connected.

True

2. (30 points) Let $f : X \to Y$ be an identification map. Suppose that Y is connected, and for each $y \in Y$, $f^{-1}(y) \subseteq X$ is a connected subspace of X. Show that X is a connected space.

Proof. Suppose that $X = U \cup V$ with $U \cap V = \emptyset$, both U and V are open.

For each $y \in Y$, $f^{-1}(y)$ is not empty (f is an onto map). We have $f^{-1}(y) = (f^{-1}(y) \cap U) \cup (f^{-1}(y) \cap V)$ with both $f^{-1}(y) \cap U$ and $f^{-1}(y) \cap V$ are open sets in $f^{-1}(y)$ (under subspace topology), and $(f^{-1}(y) \cap U) \cap (f^{-1}(y) \cap V) = \emptyset$. Since $f^{-1}(y)$ is connected, we have that either $f^{-1}(y) \subseteq U$ or $f^{-1}(y) \subseteq V$ holds and only one of them happens.

Define two subsets U' and V' of Y by $U' = \{y \in Y |\ f^{-1}(y) \subseteq U\}$ and $V' = \{y \in Y |\ f^{-1}(y) \subseteq V\}$. Since for any $y \in Y$, either $f^{-1}(y) \subseteq U$ or $f^{-1}(y) \subseteq V$ and only one of them happens, we have $U' \cup V' = Y$ and $U' \cap V' = \emptyset$. Moreover, since $U = f^{-1}(U')$ and $V = f^{-1}(V')$, U, V are open subsets of X and $f : X \to Y$ is an identification map, U' and V' are open sets in Y.

Since Y is connected, we have that either U' or V' is empty. Since f is onto, $U = f^{-1}(U')$ and $V = f^{-1}(V')$, we have that either U or V is empty. So X is connected. $\square$

3. (30 points) Let G be a path-connected topological group and X be a path-connected topological space, with G acts on X (as a group of homeomorphisms). For each $x \in X$, we can define a continuous function $i_x : G \to X$, with $i_x(g) = g(x)$ for any $g \in G$.

Show that the kernel of $(i_x)_* : \pi_1(G, e) \to \pi_1(X, x)$ is independent of $x \in X$ (i.e. $\ker (i_x)_* = \ker (i_y)_*$ for any $x, y \in X$).

Proof. For $\langle \alpha \rangle \in \pi_1(G, e)$, if it lies in the kernel of $(i_x)_* : \pi_1(G, e) \to \pi_1(X, x)$, then the path $i_x \circ \alpha : I \to X$ defined by $i_x \circ \alpha(s) = \alpha(s)(x)$ satisfies $\langle i_x \circ \alpha \rangle = e \in \pi_1(X, x)$.

Since X is path connected, there exists a path $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma_* : \pi_1(X, x) \to \pi_1(X, y)$ defined by $\gamma_*(\langle \beta \rangle) = \langle \gamma^{-1} \cdot \beta \cdot \gamma \rangle$ for any $\langle \beta \rangle \in \pi_1(X, x)$ is an isomorphism. Since $\langle i_x \circ \alpha \rangle = e \in \pi_1(X, x)$, we have that $e = \gamma_*(e) = \gamma_*(\langle i_x \circ \alpha \rangle) = \langle \gamma^{-1} \cdot i_x \circ \alpha \cdot \gamma \rangle \in \pi_1(X, y)$.

To show that $\langle \alpha \rangle$ lies in the kernel of $(i_y)_* : \pi_1(G, e) \to \pi_1(X, y)$, we need only to show that $i_y \circ \alpha$ is homotopic to $\gamma^{-1} \cdot i_x \circ \alpha \cdot \gamma$ relative to $\{0, 1\}$. The homotopy $F : I \times I \to X$ from $i_y \circ \alpha$ to $\gamma^{-1} \cdot i_x \circ \alpha \cdot \gamma$ is defined by

$$F(s,t) = \begin{cases} \gamma(1-3s) & s \in [0, \frac{t}{3}] \\ \alpha(\frac{3s-t}{3-2t})(\gamma(1-t)) & s \in [\frac{t}{3}, 1-\frac{t}{3}] \\ \gamma(3s-2) & s \in [1-\frac{t}{3}, 1]. \end{cases}$$

Then $F_0(s) = \alpha(s)(\gamma(1)) = \alpha(s)(y) = i_y \circ \alpha(s)$, and

$$F_1(s) = \begin{cases} \gamma(1-3s) & s \in [0, \frac{1}{3}] \\ \alpha(3s-1)(x) & s \in [\frac{1}{3}, \frac{2}{3}] \\ \gamma(3s-2) & s \in [\frac{2}{3}, 1] \end{cases} = (\gamma^{-1} \cdot i_x \circ \alpha \cdot \gamma)(s).$$

So we have that for any element lies in the kernel of $(i_x)_*$, it lies in the kernel of $(i_y)_*$. By switching x and y, we get that any element lies in the kernel of $(i_y)_*$ also lies in the kernel of $(i_x)_*$. So the kernel does not depend on the base point x. $\square$

4. (30 points) For a continuous function $f : S^1 \to S^1$, show that either there exists $e^{i\theta} \in S^1$ such that $f(e^{i\theta}) = -e^{i\theta}$, or there exists $e^{i\phi} \in S^1$ such that $f(e^{i\phi}) = -e^{-i\phi}$.

(Hint: If $f(e^{i\theta}) \neq -e^{i\theta}$ and $f(e^{i\theta}) \neq -e^{-i\theta}$ for any $e^{i\theta} \in S^1$, show that f is homotopic to both $e^{i\theta} \to e^{i\theta}$ and $e^{i\theta} \to e^{-i\theta}$, then try to get a contradiction.)

Proof. Suppose that $f(e^{i\theta}) \neq -e^{i\theta}$ and $f(e^{i\theta}) \neq -e^{-i\theta}$ for any $e^{i\theta} \in S^1$, we can construct two homotopies $F_1, F_2 : S^1 \times I \to S^1$ by

$$F_1(e^{i\theta}, t) = \frac{(1-t)f(e^{i\theta}) + te^{i\theta}}{||(1-t)f(e^{i\theta}) + te^{i\theta}||}$$

and

$$F_2(e^{i\theta}, t) = \frac{(1-t)f(e^{i\theta}) + te^{-i\theta}}{||(1-t)f(e^{i\theta}) + te^{-i\theta}||}.$$

Then F_1 gives a homotopy from f to $e^{i\theta} \to e^{i\theta}$ and F_2 gives a homotopy from f to $e^{i\theta} \to e^{-i\theta}$. Since homotopy is an equivalence relation, we have that $f_1 : S^1 \to S^1$ defined by $f_1(e^{i\theta}) = e^{i\theta}$ and $f_2 : S^1 \to S^1$ defined by $f_2(e^{i\theta}) = e^{-i\theta}$ are homotopic to each other by a homotopy $G : S^1 \times I \to S^1$.

Then $(f_2)_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is conjugate to $(f_1)_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ by the path $\alpha : I \to S^1$ defined by $\alpha(t) = G(1, t)$, i.e. $(f_2)_* = \alpha_* \circ (f_1)_*$. Note that the path $\gamma : I \to S^1$ defined by $\gamma(t) = e^{i2\pi t}$ generates $\pi_1(S^1, 1) \cong \mathbb{Z}$. Since $f_1 \circ \gamma = \gamma$ and $f_2 \circ \gamma = \gamma^{-1}$, the corresponding induced maps on fundamental groups satisfy that $(f_1)_*(\langle\gamma\rangle) = \langle\gamma\rangle$ and $(f_2)_*(\langle\gamma\rangle) = \langle\gamma^{-1}\rangle$.

Since $\pi_1(S^1, 1) \cong \mathbb{Z}$ is abelian, we have $\langle\alpha^{-1}\rangle \cdot \langle\gamma\rangle \cdot \langle\alpha\rangle = \langle\gamma\rangle$. So $\langle\gamma^{-1}\rangle = (f_2)_*(\langle\gamma\rangle) = \alpha_*((f_1)_*(\langle\gamma\rangle)) = \alpha_*(\langle\gamma\rangle) = \langle\alpha^{-1}\rangle \cdot \langle\gamma\rangle \cdot \langle\alpha\rangle = \langle\gamma\rangle$. It implies that $1 = -1$ in the group $\mathbb{Z}$, which is a contradiction.

So either there exists $e^{i\theta} \in S^1$ such that $f(e^{i\theta}) = -e^{i\theta}$, or there exists $e^{i\phi} \in S^1$ such that $f(e^{i\phi}) = -e^{-i\phi}$.

$\square$