MATH H53 MIDTERM 1 SOLUTION

1. (5 points) Find the length of the curve $x = e^{at}\cos bt$, $y = e^{at}\sin bt$, $0 \le t \le 2\pi$. Here both a and b are positive numbers.

We use the formula $L = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t$.

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left((e^{at}\cos bt)'\right)^2 + \left((e^{at}\sin bt)'\right)^2}\mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{(ae^{at}\cos bt - be^{at}\sin bt)^2 + (ae^{at}\sin bt + be^{at}\cos bt)^2}\mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{a^2e^{2at} + b^2e^{2at}}\mathrm{d}t \\
&= \int_0^{2\pi} \sqrt{a^2+b^2}e^{at}\mathrm{d}t \\
&= \frac{\sqrt{a^2+b^2}}{a}(e^{2\pi a} - 1).
\end{aligned}$$

2. (5 points)

(1) (2 points) Find a polar equation for the curve represented by Cartesian equation $x^2 + y^2 = |x| + |y|$ and sketch the curve.

(2) (3 points) Find the area of the region that lies inside this curve.

(1) Use $x = r\cos\theta$ and $y = r\sin\theta$, we get that the polar equation is $r^2 = |r\cos\theta| + |\sin\theta|$. So we have $|r| = |\cos\theta| + |\sin\theta|$, which is the same curve as $r = |\cos\theta| + |\sin\theta|$. This curve is symmetric with respect to the reflection along x– and y–axes.

(2) We need only to compute the area in the first quadrant, since the curve has the symmetric property. By using the formula $A = \int_a^b \frac{1}{2} r^2 \mathrm{d}\theta$. The area is

$$
\begin{aligned}
A =& 4\int_0^{\frac{\pi}{2}} \frac{1}{2} r^2(\theta)\mathrm{d}\theta \\
=& 2\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)^2 \mathrm{d}\theta \\
=& 2\int_0^{\frac{\pi}{2}} \big(\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta\big)\mathrm{d}\theta \\
=& 2\int_0^{\frac{\pi}{2}} (1 + \sin 2\theta)\mathrm{d}\theta \\
=& 2(\theta - \frac{\cos 2\theta}{2})|_{\theta=0}^{\frac{\pi}{2}} \\
=& \pi + 2.
\end{aligned}
$$

3. (5 points) Let L_1 be the line of the intersection of two planes: $x + y + z = 3$ and $x + 2y + 3z = 6$, and L_2 be the line passing through $(1, 0, -1)$ and perpendicular with the plane $3x + 4y + 5z = 0$.

Find the distance between L_1 and L_2.

The direction of L_1 is given by $\langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 1, -2, 1\rangle$, and an intersection point of these two planes is $\langle 1, 1, 1\rangle$. So the equation of L_1 is $\langle 1, 1, 1\rangle + t\langle 1, -2, 1\rangle$.

The direction of L_2 is $\langle 3, 4, 5\rangle$ and L_2 passes through $\langle 1, 0, -1\rangle$. So the equation of L_2 is $\langle 1, 0, -1\rangle + t\langle 3, 4, 5\rangle$.

A vector which is perpendicular to both of L_1 and L_2 is $\langle 1, -2, 1\rangle \times \langle 3, 4, 5\rangle = \langle -14, -2, 10\rangle$. Then the distance between L_1 and L_2 is the absolute value of the scalar project of $\langle 1, 1, 1\rangle - \langle 1, 0, -1\rangle = \langle 0, 1, 2\rangle$ on $\langle -14, -2, 10\rangle$. It is equal to

$$\left|\frac{\langle 0, 1, 2\rangle \cdot \langle -14, -2, 10\rangle}{|\langle -14, -2, 10\rangle|}\right| = \left|\frac{18}{\sqrt{300}}\right| = \frac{3\sqrt{3}}{5}.$$

4. (5 points) For two vector–valued functions $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(s) = \langle 1 + 2s, 1 + 6s, 2 + 12s \rangle$,

(1) (1 points) Find the intersection point of these two space curves.

(2) (4 points) Let P be the plane that contains the tangent lines of $\mathbf{r}_1$ and $\mathbf{r}_2$ at the intersection point. Find an equation of P.

(1) We need to solve the equation

$$\begin{cases} t &= 1 + 2s \\ t^2 &= 1 + 6s \\ t^3 &= 2 + 12s \end{cases}$$

The first two equations implies $1 + 6s = t^2 = (1 + 2s)^2 = 1 + 4s + 4s^2$. So $s = 0, t = 1$ or $s = \frac{1}{2}, t = 2$. The first solution does not satisfy $t^3 = 2 + 12s$, while the second solution does. So the coordinate of the intersection point is $(2, 4, 8)$. For this point $t = 2$ and $s = \frac{1}{2}$.

(2) The tangent line of $\mathbf{r}_1$ at the intersection point has direction $\mathbf{r}_1'(2) = \mathbf{r}_1'(t)|_{t=2} = \langle 1, 2t, 3t^2 \rangle|_{t=2} = \langle 1, 4, 12 \rangle$.

The tangent line of $\mathbf{r}_2$ at the intersection point has direction $\mathbf{r}_2'(\frac{1}{2}) = \mathbf{r}_2'(s)|_{s=\frac{1}{2}} = \langle 2, 6, 12 \rangle|_{s=\frac{1}{2}} = \langle 2, 6, 12 \rangle$.

Since both of these tangent lines lie in P, the normal vector of P is $\langle 1, 4, 12 \rangle \times \langle 2, 6, 12 \rangle = \langle -24, 12, -2 \rangle$.

So the equation of P is $-24(x-2)+12(y-4)-2(z-8) = 0$, i.e. $12x-6y+z-8 = 0$.