Practice Midterm Exam #3 Solutions

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1. Find the partial fraction decomposition for the function:

$$\frac{x+1}{x^2(x^2+1)}$$

Solution: Since x^2 is a repeated factor, we will have to have one term for each of its powers, and since x^2+1 is an irreducible quadratic we will have to have a linear factor over it. Thus the form of the decomposition will be:

$$\frac{x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

Multiplying through by the denominator of the LHS:

$$x+1 = A(x)(x^2+1) + B(x^2+1) + Cx(x^2) + D(x^2)$$

Gathering coefficients of the various powers of x:

$$x+1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

Thus, seperating this into one equation for each power of x.

$$\begin{cases} 0 = A+C & \text{from the } x^3 \text{ equation} \\ 0 = B+D & \text{from the } x^2 \text{ equation} \\ 1 = A & \text{from the } x \text{ equation} \\ 1 = B & \text{from 1 equation} \end{cases}$$

Hence

$$A = B = 1, \quad C = D = -1$$

Thus the partial fractions decomposition is:

$$\frac{x+1}{x^2(x^2+1)} = \frac{1}{x} + \frac{1}{x^2} + \frac{-x-1}{x^2+1}$$

2. Evaluate

$$\int x^{-3} e^{\frac{1}{x}} dx.$$

Solution: Let the integral we are trying to calculate be denoted by I. We may rewrite this as

$$I = -\int \left(\frac{-1}{x^2}\right)\left(\frac{1}{x}\right) e^{\frac{1}{x}} dx.$$

Then we make the subsitution $u = \frac{1}{x}$, so $du = \frac{-1}{x^2}dx$, to yield

$$I = \int ue^u du.$$

Then we integrate by parts, letting $f = u$ and $dg = e^u$, which makes $df = 1$, and $g = e^u$, giving

$$I = ue^u - \int e^u du = (u-1)e^u + C$$

where $C \in \mathbb{R}$ is the constant of integration. Substituting back to our original variable yields

$$I = \left(\frac{1}{x} - 1\right) e^{\frac{1}{x}}$$

3. Evaluate whether the integral is convergent or divergent, and evaluate it if it is convergent.

$$\int_e^\infty \frac{dx}{x(\ln x)^2}$$

Solution: There does not appear to be an obvious comparison to prove the integral is convergent or divergent, so we will attempt to decide by integrating it. First, we make the substitution $u = \ln x$, so $du = \frac{1}{x}dx$. To find the new bounds of integration, we want the lower bound to be $\ln e = 1$, and the upper bound to be $\lim_{t\to\infty} \ln t = \inf$. This gives:

$$\int_1^\infty \frac{du}{u^2} = \lim_{t\to\infty} \int_1^t \frac{du}{u^2} \lim_{t\to\infty} \left[-\frac{1}{u}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1$$

Thus the integral is convergent, specifically it converges to 1.

4. Evaluate the integral:

$$\int (\cos^4 x - \sin^4 x)dx$$

Solution: We observe that the integrand factors

$$\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$$

Applying the identities

$$\cos^2 x - \sin^2 x = \cos(2x) \qquad \text{and} \qquad cos^2 x + sin^2 x = 1$$

gives us:

$$\int (\cos^4 x - \sin^4 x)dx = \int \cos(2x)dx$$

doing the substitution $u = 2x$, $du = 2dx$,

$$\frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(2x) + C$$

5. Find the surface area of the surface of revolution of $y = \sin x$ about the x-axis, $0 \le x \le \pi$.

Solution: Applying the formula for rotation about the x-axis:

$$A = \int_a^b 2\pi y\sqrt{1+y'^2}dx$$

we get

$$A = \int_0^\pi 2\pi \sin x\sqrt{1+\left(\frac{d}{dx}(\sin x)\right)^2}dx = 2\pi\int_0^\pi \sin x\sqrt{1+(\cos x)^2}dx$$

We substitute $u = \cos x$, $du = -\sin x dx$, giving:

$$-2\pi\int_1^{-1}\sqrt{1+u^2}du = 2\pi\int_{-1}^1\sqrt{1+u^2}du$$

This looks like a trig substitution, so we let $u = \tan\theta$, $du = \sec^2\theta$, giving:

$$2\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta = 2\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,d\theta$$

We evaluate the indefinite integral by integration by parts, letting $f = \sec\theta$ and $dg = \sec^2\theta$:

$$\begin{aligned} I &= \int\sec^3\theta\,d\theta \\ &= \sec\theta\tan\theta - \int\sec\theta\tan^2\theta\,d\theta \\ &= \sec\theta\tan\theta - \int\sec\theta(\sec^2\theta-1)\,d\theta \\ &= \sec\theta\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta \\ &= \sec\theta\tan\theta - I + \int\sec\theta\,d\theta \end{aligned}$$

Thus

$$I = \frac{1}{2}\left(\sec\theta\tan\theta + \int\sec\theta\,d\theta\right) = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) + C$$

Evaluating the original integral then gives:

$$\begin{aligned} 2\pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,d\theta &= \pi\left(\left[\sec\tfrac{\pi}{4}\tan\tfrac{\pi}{4} + \ln\left|\sec\tfrac{\pi}{4}+\tan\tfrac{\pi}{4}\right|\right] - \left[\sec\tfrac{-\pi}{4}\tan\tfrac{-\pi}{4} + \ln\left|\sec\tfrac{-\pi}{4}+\tan\tfrac{-\pi}{4}\right|\right]\right) \\ &= \pi\left(\left[\sqrt{2}+\ln\left|\sqrt{2}+1\right|\right] - \left[-\sqrt{2}+\ln\left|\sqrt{2}-1\right|\right]\right) \\ &= 2\pi\left(\sqrt{2}+\ln\left|\sqrt{2}+1\right|\right) \end{aligned}$$