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**Abstract Algebra. Midterm Exam. March 21, 2013**

**1.** How many conjugacy classes of elements of order 6 are there in the group  $S_9$  of permutations of 9 symbols?

Permutations lie in the same conjugacy class if and only if they have the same cycle structure. The order of a permutation is the least common multiple of the lengths of the cycles. Thus, we are looking for the number of partitions of 9 with the least common multiple of the parts equal to 6. Such partitions are:  $6 + 3 = 6 + 2 + 1 = 6 + 1 + 1 + 1 = 3 + 3 + 2 + 1 = 3 + 2 + 2 + 2 = 3 + 2 + 2 + 1 + 1 = 3 + 2 + 1 + 1 + 1 + 1$ . Thus there are 7 such conjugacy classes.

**2.** In the group  $S_4$ , find a non-normal subgroup isomorphic to  $\mathbf{Z}_2^2$ ?

Elements of order 2 in  $S_4$  must have the cycle structure corresponding to the partitions of 4:  $2 + 2$  or  $2 + 1 + 1$ . There are 3 elements of the cycle structure  $2 + 2$ , which together with the identity element form a group isomorphic to  $\mathbf{Z}_2^2$ . However this subgroup is *the* Klein subgroup, and it is normal.

Elements with the cycle structure  $2 + 1 + 1$  are transpositions. The subgroup consisting of two commuting transpositions (e.g. (12) and (34)), as well as of their product and the identity is isomorphic to  $\mathbf{Z}_2^2$ . It is not normal, since not all transpositions (which form one conjugacy class) lie in it.

**3.** Let  $\mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \cdots \oplus \mathbf{Z}_{m_r}$  be a direct sum of cyclic groups,  $m_1 \geq m_2 \geq \cdots \geq m_r > 1$ . For how many different arrays  $(m_1, \dots, m_r)$  is such a group isomorphic to  $\mathbf{Z}_{60}$ ?

In  $\mathbf{Z}_{m_1} \oplus \cdots \oplus \mathbf{Z}_{m_r}$ , the maximal order of elements is equal to the least common multiple of  $m_1, \dots, m_r$ . To be cyclic, the group must contain an element of the order equal to the order  $m_1 \cdots m_r$  of the group. This happens if and only if the factors  $m_i$  are pairwise relatively prime. Thus the problem reduces to listing all factorings of  $60 = 2^2 \cdot 3 \cdot 5$  into the product of pairwise relatively prime factors. Such factorings are:  $60 = 5 \times 4 \times 3 = 12 \times 5 = 20 \times 3 = 15 \times 4$ , and the total number of them is 5.

4. In the symmetry group  $D_6$  of the regular hexagon, describe all normal subgroups, and list all, up to isomorphism, homomorphic images of  $D_6$ .

The group  $D_6$  contains rotations  $\rho^k$ ,  $k = 0, \pm 1, \pm 2, 3$ , through the angles  $60^\circ \times k$ . The conjugation of  $\rho^k$  by a reflection is equal to  $\rho^{-k}$ . Therefore rotations form 4 conjugacy classes:  $\{\rho^0\}$ ,  $\{\rho^3\}$ ,  $\{\rho^{\pm 1}\}$ , and  $\{\rho^{\pm 2}\}$ .

Reflections in the group  $D_6$  form two conjugacy classes: those about the lines passing through the vertices, and those passing through the midpoints of the sides. If a normal subgroup  $H$  contains reflections of both kinds, then their products give all rotations, i.e.  $H = D_6$ , and the quotient group is trivial. Vice versa, if a normal subgroup  $H$  contains all rotations and at least one reflection, then it contains all reflections (which are obtained from one by conjugating it by rotations) and thus  $H = D_6$ .

If  $H$  contains reflections of only one kind, then their products form rotations  $\rho^{\pm 2}$  (and  $\rho^0$ ), and the subgroup containing nothing else but these 6 elements is normal. The quotient group has 2 elements and thus is isomorphic to  $\mathbf{Z}_2$ .

If  $H$  contains no reflections, then it is a subgroup of the cyclic group of order 6 consisting of rotations. Each such subgroup is normal in  $D_6$  (because conjugacy classes of rotations in dihedral groups consist of pairs of inverse rotations). There is only one such subgroup of each of the orders 1, 2, 3, 6 dividing 6, with the quotient groups isomorphic to  $D_6$ ,  $D_3$ ,  $D_2 \simeq \mathbf{Z}_2^2$ , and  $\mathbf{Z}_2$ .

By the epimorphism theorem, every homomorphic image of a group is isomorphic to the quotient group by the kernel of the homomorphism. Therefore the list of possible, up to isomorphism, homomorphic images of  $D_6$  is:  $\{e\}$ ,  $\mathbf{Z}_2$ ,  $\mathbf{Z}_2^2$ ,  $D_3$ ,  $D_2$ ,  $D_6$ .

5. How many geometrically different cubes can be made by painting each of the 8 vertices into one of two colors (black or white), assuming two cubes with painted vertices geometrically the same, whenever one can be identified with the other by rotation of the cube?

The group of rotations of the cube acts on the set of  $2^8$  colorings of the 8 vertices in the two colors. The number of geometrically different colored cubes coincides with the number of orbits of the action, which we compute, following Cauchy's counting principle, as the average number of fixed points of the group elements. Out of 24 group elements, there are 8 rotations through  $\pm 120^\circ$  about the 4 diagonals, having  $2^4$  fixed coloring each; 6 ro-

tations through  $180^\circ$  about the lines connecting the midpoints of opposite edges, having  $2^4$  fixed colorings each; 6 rotations through  $\pm 90^\circ$  about the axes through the centers of the cube's faces, having  $2^2$  fixed colorings each; 3 rotations through  $180^\circ$  about such axes, having  $2^4$  fixed colorings each; and the identity, fixing all the  $2^8$  colorings. Thus, for the number  $N$  of the orbits, we have

$$N = \frac{1}{24} [8 \cdot 2^4 + 6 \cdot 2^4 + 6 \cdot 2^2 + 3 \cdot 2^4 + 1 \cdot 2^8] = 23.$$