

ASSIGNMENT 1 - MVP'S

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§ 1 INTRUDCTION

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§ 2 REPRODUCING KERNEL HILBERT SPACE

§ 2.1 HILBERT SPACE

Recall that an inner product $\langle a, b \rangle$ can be

- a usual dot product: $\langle a, b \rangle = a'b = \sum_i a_i b_i$.
- a kernel product: $\langle a, b \rangle = k(a, b) = \psi(a)' \psi(b)$, where $\psi(a)$ may have infinite dimensions.

A normed space is a vector space N on which a norm is defined. A nonnegative function $\|\cdot\|$ is a norm if and only if $\forall f, g \in N$ and $\alpha \in \mathbb{R}$:

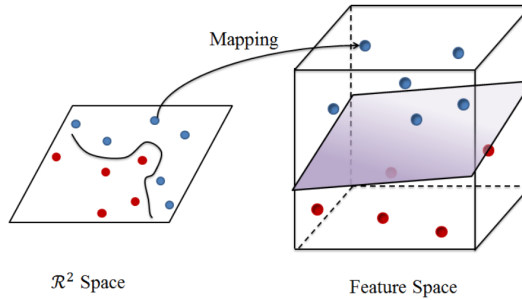
- $\|f\| \geq 0$ and $\|f\| = 0$ iff $f = 0$;
- $\|f + g\| \leq \|f\| + \|g\|$;
- $\|\alpha f\| = |\alpha| \|f\|$

Now we can define a Hilbert space an inner product space that is complete and separable with respect to the norm defined by inner product. An example of a defined norm in Hilbert space (i.e, the space L_2 of square integrable functions) can be

$$\|f\| = \left(\int_a^b f^2(t) dx \right)^{\frac{1}{2}}. \quad (1)$$

§ 2.2 KERNEL METHOD

The motivation of kernel method is simple. Imagine there are some blue dots and red dots on a vector space \mathbb{R}^n and we want to separate them by color. As it shows in the left hand side figure, it is difficult to divide them through a straight line. However, we may be able to separate them easily by mapping each dot into a high-dimension feature space. The figure below shows how feature map works:



Let's now use a simple example to illustrate the idea of kernel method. we set two vectors $\mathbf{x} = [x_1 \ x_2]$ and $\mathbf{y} = [y_1 \ y_2]$ in a two-dimension space. Two functions $\phi(x)$ and $\phi(y)$ are defined as:

$$\phi(x) = [x_1 x_1 \quad x_1 x_2 \quad x_2 x_1 \quad x_2 x_2]$$

$$\phi(y) = \begin{bmatrix} y_1 y_1 & y_1 y_2 & y_2 y_1 & y_2 y_2 \end{bmatrix}$$

We now successfully mapping them into a four-dimension feature space. To write the above example in a general form, we first set an equation where $\phi(\cdot) \in \mathbb{R}^m$ and $\phi(x)$ is defined as the mapping function. Note that we assume there is a linear relation between y and $\phi(x)$:

$$\begin{aligned} y &= \phi(x)^\top w \\ &= \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_m(x) \end{bmatrix} w \\ &= [\phi_1(x) \cdots \phi_m(x)] w \end{aligned} \quad (2)$$

Y and Φ in generalization is defined by:

$$Y = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^\top \quad (3)$$

$$\begin{aligned} \Phi &= [\phi(x_1) \quad \cdots \quad \phi(x_n)]^\top \\ &= \begin{bmatrix} \phi_1(x_1) & \cdots & \phi_m(x_1) \\ \vdots & \vdots & \vdots \\ \phi_1(x_n) & \cdots & \phi_m(x_n) \end{bmatrix} \end{aligned} \quad (4)$$

Ridge In most of the cases, we want to avoid computing $\phi(x)$ in a explicit way, especially when m is very large. Therefore, we define a kernel function:

$$[\Phi \Phi^\top]_{i,j} = \phi(x_i)^\top \phi(x_j) = K(x_i, x_j) \quad (5)$$

$\phi(x_i)^\top \phi(x_j)$ is usually very difficult to compute, and thus we use $K(x_i, x_j)$ to replace.

For example, the Gaussian kernel is

$$k(x_i, x_j) = e^{-\frac{\|x_i - x_j\|^2}{\sigma^2}}, \quad (6)$$

(Gaussian kernel meaning the similarity between two points) where $\|x_i - x_j\|$ is the Euclidean distance between x_i and x_j , and $\sigma^2 \in \mathbb{R}^+$ is the bandwidth of the kernel function, and it satisfies the following properties:

for $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is a kernel if

- k is symmetric: $k(x, y) = k(y, x)$.
- k is positive semi-definite, meaning that $\sum_i \sum_j \alpha_i \alpha_j k(x_i, x_j) \geq 0, \forall \alpha_i, \alpha_j \in \mathbb{R}, x \in \mathbb{R}^D, D \in \mathbb{Z}^+$.
- We define the corresponding kernel matrix as the matrix K with entries $k_{ij} = k(x_i, x_j)$, the second property of k is equivalent to saying that $c^\top K c \geq 0$.

Recall the simple example above, instead of computing the inner product of $\langle \phi(x), \phi(y) \rangle$, we can define a corresponding kernel function $K(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle^2$. It can be easily proofed that $K(\mathbf{x}, \mathbf{y})$ is the same as $\langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle$:

$$\begin{aligned} K(\mathbf{x}, \mathbf{y}) &= \langle \mathbf{x}, \mathbf{y} \rangle^2 \\ &= (x_1 y_1 + x_2 y_2)^2 \\ &= x_1^2 y_1^2 + 2x_1 y_2 x_2 y_1 + x_2^2 y_2^2 \\ &= \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle \end{aligned}$$

(representer theorem) From the similarity-based point of view, the use of kernels for regression can be described in two stages. We set $y_i \in \mathbb{R}$ as dependent variable, and x_i as a $1 \times D$ vector $x_i \in \mathbb{R}^D$ of covariate values. Assume that (y_i, x_i) where $i = 1, \dots, N$ is i.i.d. We first define a target function $y = f(x)$ and assume that in a space of functions, there exists a function that can estimate $y = f(x)$ well. The target function could be represented by

$$f(x) = \sum_{i=1}^N w_i k(x, x_i), \quad (7)$$

A perfect solution to estimate N parameters for N observations would be choosing $\hat{w} = K^{-1}y$. However, such a fit could result in extremely high variance. Therefore, we impose an additional assumption that a smoother curve with less oscillations is preferred. We then utilize regularization to simplify the function and satisfy the additional assumption in the second stage. To achieve this purpose, Hilbert space and reproducing kernel Hilbert space will be introduced below.

§ 3 GAUSSIAN PROCESS AND BAYESIAN PERSPECTIVE

After estimating the discount function $g(\mathbf{x})$, we want to do inference with this function, and because of the nonparametric estimation, we cannot calculate the statistical distribution of parameters, hence here we use Gaussian Process to estimate the distribution of discount function and construct confidence interval.

§ 3.1 DEFINITION

We have data $\mathbf{D} = \{(\mathbf{x}_i, y_i)\}_{i=1}^M$, and assume that mean of y is 0.

Task: find the distribution of $f^*(x)$.

Assume that the true form of prediction function is: $y_i = f(\mathbf{x}_i) + \epsilon_i$ and $\epsilon_i \sim \mathcal{N}(0, \sigma_i^2)$. Here we have a M dimensional dependent variable \mathbf{y} , and a $M \times N$ dimensional independent variable \mathbf{X} , where M is the number of observations, and N is the dimension of \mathbf{x} , i.e. $\mathbf{x}_i \in \mathbb{R}^N$. The function $f(\mathbf{x}_i) : \mathbb{R}^N \rightarrow \mathbb{R}$ takes vector $\mathbf{x}_i \in \mathbb{R}^N$. Let $\mathbf{K}_{X,X} = k(\mathbf{x}, \mathbf{x}^T)$ which is the matrix of $k(\mathbf{x}_i, \mathbf{x}_j)$. Thus, \mathbf{K} is a $M \times M$ matrix.

The assumption of Gaussian Process is as following:

For a given vector \mathbf{y} , and its corresponding data \mathbf{X} , where vector $\mathbf{y} \in \mathbb{R}^M$ and \mathbf{X} is $M \times N$ matrix. In addition, for \mathbf{y} and \mathbf{X} data, the error term $\epsilon \sim \mathcal{N}(\mathbf{0}, \Sigma^\epsilon)$, and $\Sigma^\epsilon = \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2, \dots, \sigma_M^2)$. Meanwhile we have arbitrary $n \times N$ matrix \mathbf{Z} and predicted value $f^*(\mathbf{z}) \in \mathbb{R}^n$, where $\mathbf{z} = (z_1, z_2, z_3, \dots, z_n)^T$.

Then we assume \mathbf{y} and $f^*(\mathbf{z})$ follow a $(M+n)$ multivariate normal distribution(MVN):

$$\begin{bmatrix} f^*(\mathbf{z}) \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \mu_{f^*(\mathbf{z})} \\ \mu_{\mathbf{y}} \end{bmatrix}, \begin{bmatrix} \mathbf{K}_{Z,Z} & \mathbf{K}_{Z,X} \\ \mathbf{K}_{X,Z} & \hat{\mathbf{K}}_{X,X} \end{bmatrix} \right) \quad (8)$$

where $\hat{\mathbf{K}}_{X,X} = \mathbf{K}_{X,X} + \Sigma^\epsilon$.

Then given data \mathbf{y} , \mathbf{X} and \mathbf{Z} , according to the conditional distributions of the multivariate normal distribution¹, we have the posterior distribution

$$f^*(\mathbf{z}) | \mathbf{y}, \mathbf{X}, \mathbf{Z} \sim \mathcal{N}(\mu_{f^*(\mathbf{z})} + \mathbf{K}_{Z,X} \hat{\mathbf{K}}_{X,X}^{-1} (\mathbf{y} - \mu_{\mathbf{y}}), \mathbf{K}_{Z,Z} - \mathbf{K}_{Z,X} \hat{\mathbf{K}}_{X,X}^{-1} \mathbf{K}_{X,Z}) \quad (9)$$

§ 3.2 INTUITION BEHIND GAUSSIAN PROCESS

The idea behind this process is that, assume our interested function is $f(\mathbf{x})$, $f(\mathbf{x}) : \mathbb{R}^N \rightarrow \mathbb{R}$, and we have an arbitrary vector of independent variable $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M)^T$, and for each $\mathbf{x}_i, i = 1, 2, \dots, M, \mathbf{x}_i \in \mathbb{R}^N$, then we can obtain a series of $f(\mathbf{x}) = (f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \dots, f(\mathbf{x}_M))^T$. We assume that the series of $f(\mathbf{x})$ follows a multivariate normal distribution which is:

$$f(\mathbf{x}) \sim \mathcal{N}(\mu(\mathbf{x}), k(\mathbf{x}, \mathbf{x}^T)) \quad (10)$$

¹<https://statproofbook.github.io/P/mvn-cond>

This is the prior distribution of our function $f(x)$, here we have a set of infinitely functions that follow this distribution, their mean is the function $\mu(x_i)$, and the variance of them is $k(x_i, x_i^T)$. This makes the distribution of $f(x)$ to be called Gaussian Process (GP). Note that if we add a noise term $\epsilon \sim \mathcal{N}(\mathbf{0}, \Sigma^\epsilon)$, then our prior distribution of $y = f(x) + \epsilon \sim \mathcal{N}(\mu(x), k(x, x^T) + \Sigma^\epsilon)$ is also a Gaussian Process. Here we use kernel matrix to denote variance-covariance matrix because kernel value represents how near two data points in the space are, with this property we can obtain a smooth function.

Remind that our goal is to estimate the distribution of $f(x^*)$ given observed training data set $D = \{x_i, y_i\}_{i=1}^M$ and test data set $\{x_j^*\}_{j=1}^n$. Firstly we compare our nonparametric case to a parametric case. In a parametric case, assume the parameter θ determines the form of $f_\theta(\cdot)$, according to the Bayesian rule, $p(y^*|x^*, x, y) = \int_\theta p(y^*, \theta|x^*, x, y) d\theta = \int_\theta p(y^*|\theta, x^*) p(\theta|x, y) d\theta$, where y^* is the prediction of given data x^* , and its form of model is determined by parameter θ . Estimated θ value is determined by training data D . This is to say that we update our parameter θ by given D , and use $p(\theta|x, y)$ as a new prior probability, and based on this to predict posterior of y^* .

Therefore, back to our GP nonparametric case, θ could be substituted by function $f(\cdot)$. One can show that the joint distribution of $(f(x^*), y)^T$ follows a multivariate normal distribution as in the definition before, because of the assumption of GP and the property of MVN. With the joint distribution, we want to find posterior probability: $p(f(x^*)|x^*, x, y) = \int p(f(x^*)|f, x^*) p(f|x, y) df$, where $p(f|x, y)$ is the posterior of $f(\cdot)$ given D , and is regarded as prior when estimating $p(f(x^*)|x^*, D)$, this process is called Bayesian updating. Fortunately, we do not need to take any integral in GP, because the posterior of $f(x^*)$ could be calculated by formula of conditional distribution in MVN as mentioned in former section.

§ 3.3 GAUSSIAN PROCESS IN RESEARCH PAPER

In this research paper, authors assumed discount function $g(z)$ given a vector of different maturities $z = (z_1, z_2, \dots, z_n)$ follows a MVN distribution $\mathcal{N}(m(z), k(z, z^T))$. Then this is a Gaussian Process, and by Bayesian updating for given price P , corresponding cash flow matrix C and time to maturities x , we can obtain the posterior mean and variance function in research paper's equation (12) and equation (13). Therefore, the variance function of MVN could give us the confidence interval of $g(z)$ i.e. for each maturity time z we calculate $k^{post}(z, z)$ as its normal variance, that could help us to evaluate the precision of our prediction. Furthermore, with the posterior distribution of $g(x)$, it is implied that the coupon bond price $Cg(x) \sim \mathcal{N}(Cm^{post}(x), Ck^{post}(x, x^T)C^T)$.

Note that authors assume the variance covariance matrix of error term $\Sigma^\epsilon = \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2, \dots, \sigma_M^2)$ where diagonal elements all satisfy $\omega_i = \frac{\lambda}{\sigma_i^2}$, this implies that we give a higher weight for a bond price which has less noise. In addition, we assume that the prior mean function is constant $m(x) = 1$ which assumes no time value of money. With these assumptions, the posterior mean function coincides with estimated $\hat{g}(x)$ as in research paper's equation (5).

§ 4 EMPIRICAL STUDY

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