Research Module

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1 Solving the Kernel Function, an example

In this paper, we study in the RKHS $\mathcal{H} = \mathcal{H}_{\omega,\delta}$ consisting of differentiable functions $h : [0, \infty) \to \mathbb{R}$ of the form $h(x) = \int_0^x h'(t)dt$ with continuous derivatives, $h'(x) = h'(0) + \int_0^x h''(t)dt$ for integrable h'', and with finite norm

$$\langle h, h\rangle = \| h \|_{\omega,\delta} = (\int_0^\infty (\delta h'(x)^2 + (1-\delta)h''(x)^2)\omega(x)dx)^{\frac{1}{2}} \tag{1}$$

for some measurable weight function $\omega : [0, \infty) \to [1, \infty)$ and shape parameter $\delta \in (0, 1)$. With additional assumption in research paper's appendix A.2, we can extend it to the case $\delta \in \{0, 1\}$.

The Lemma 3 assumes that for any fixed $y \geq 0$, exists a solution ϕ of the linear differential equation

$$\delta\phi\omega - (1-\delta)(\phi'\omega)' = 1_{[0,y]} \tag{2}$$

and for $\psi \in \mathcal{H}_{\omega,\delta}, \psi(x) = \int_0^x \phi(t)dt$, then for $h \in \mathcal{H}_{\omega,\delta}$ with $h'(x) = 0$ for $x > n$ for some finite n, we can write

$$\langle \psi, h\rangle_{\omega,\delta} = \int_0^\infty (\delta\psi'(x)h'(x) + (1-\delta)\psi''h''(x))\omega(x)dx \tag{3}$$

according to the definition for any $h \in \mathcal{H}_{\omega,\delta}$. The assumption of sloution exists and Lemma 4 could give us that $\langle \psi, h\rangle = h(y)$ which implies that $k(\cdot, y) = \psi$ by the reproducing property. Then $k(x, y) = \psi(x)$, and remind that $\psi(x) = \int_0^x \phi(t)dt$, then we can find the form of $k(x, y)$ if we know the form of ϕ, and we could solve ϕ by giving different value of δ and ω. Below is an example of how to solve the research paper's equation 8.

In the research paper, the weight function $\omega(x) = e^{\alpha x}$, if $\alpha = 0, \delta = 1$, then $\phi = 1_{[0,y]}$, and $k(x, y) = \psi(x) = \int_0^x 1_{[0,y]}dt = min\{x, y\}$.