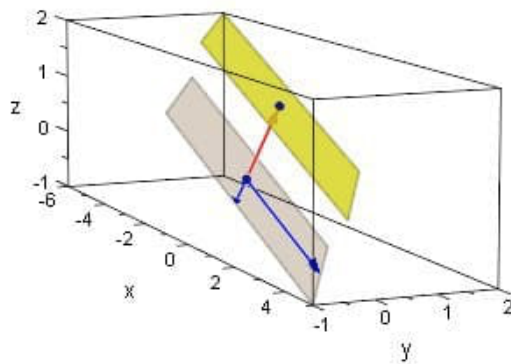Exploring Math $\frac{\Sigma}{math}$ with EIGENMATH

Linear Algebra *Interactive!* with Eigenmath

Part 3

Singular Linear Systems • Analytic Geometry

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2020

Contents

Preface

These small scripts want to introduce the reader or student to the elementary Linear Algebra and at the same time into the use of CAS EIGENMATH.

In this part 3 of a series of interactive learning parcourses we treat singular linear systems via the GAUSS-JORDAN algorithm resp. the `RREF` with the help of EIGENMATH. We focus on inhomogeneous linear systems $A * X = B$ and the associated homogeneous linear system $A * X = O$ to study their solution sets, giving in parallel their geometric interpretation as lines and planes and introduce the concept of the linearly in/dependence of vectors.
The scalar product is introduced as a special matrix product and is present in EIGENMATHs ubiquitous `dot()`–function. We use it to define in EIGENMATH formulae for the length of vectors, the distance of points and the angle between two vectors.
The cross and box (triple) products are used to derive and implement some main formulae of Analytical Geometry and to automate calculations and decisions: dependence checks, incidence tests, to compute the intersection of lines and planes and some volumina.

To hold the presentation elementary at college niveau, we limit ourselves to work in 2/3–dimensional space $\mathbb{R}^2/\mathbb{R}^3$. It allows for geometrical insights using appropriate plot utilities. Most plots in these notes are set with CAS MuPAD and a few older ones with DERIVE. For user created ad hoc plots we recommend to use the free online software CALCPLOT3D, especially in dealing with the exercises, see [10].

The EIGENMATH routines, which are written for this Part 3, are collected in the toolbox `geoBox.txt` for the convenience of the user and are invoked by the command `run("geoBox.txt")` in a running EIGENMATH Online[1] session.

These booklets grew out from a series of lessons that I developed between 2001 and 2006 at Mercator University of Duisburg and at FernUniversität Hagen in Germany, but they are revised, renewed and adapted to EIGENMATH. The material was repeatedly tested at a German hight school resp. college. These learning parcourses were originally developed using notebooks compiled with CAS MuPAD 3.1 and in a very early status using the free CAS MuPAD^{Light} 2.5.3 with accompanying learning materials in PDF booklets.

The pure data material of some exercises are borrowed from the German college books [2] and [1]. So the interested reader may compare the traditional computations by paper & pencil with a CAS oriented interactive presentation.

What may be the benefit of such a presentation with the prominent and continuous use of the two central tools EIGENMATH and CALCPLOT3D?

---

[1]Running the EIGENMATH app on the iMac this command has to be substituted trough `run("downloads/geoBox.txt")`. The file `geoBox.txt` has therefore to be copied to the 'downloads' folder.

○ *First, the reader does not need to install any software* to follow this text interactively! The CAS Eigenmath works directly out of this text, on any operating system, on every hardware (Smartphone, iPhone, tablet, PC, etc.), at any place: you only must be online and click on a link like ▷*Click here to invoke* Eigenmath (◁ please click here! Really!) with success. From this point on you can run a given script or fork with own computations.

○ *Second* and comparable, *the reader does not need to install any software to produce quality plots interactively*! You only must be online to press a link like `CalcPlot3D` (◁ **please click here! Really!**) in this script with success. At this point you can make a 2D/3D–plot to visualize a concept or to make a calculation visually evident.

With these both online computational tools we are well equipped to follow this text dynamically.

I thank George Weigt for his friendly support, hints and help regarding his Eigenmath. So it was a real pleasure to write down these notes.

Any feedback from the user is very welcome.

PS: Being retired and no native speaker, I have no support from colleges at high school or university, therefore the reader may excuse me for my grammatical and spelling mistakes.

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Leichlingen, Germany
January 2021

7   The 3rd model problem

So far we have studied (almost) exclusively quadratic *regular* linear systems of equations $A*X = B$, for which the system matrix A has an inverse matrix A^{-1} and the corresponding unique solution $X = A^{-1} * B$ represents geometrically one point. In addition to the explicit calculation with the help of the inverse matrix A^{-1}, we were also able to obtain this unambiguous solution with the GAUSS-JORDAN method, realized in our routine RREF. With *singular* linear systems there is no inverse of the matrix of coefficients, therefore there are richer solution phenomena. Both the geometric *shape of the solution set* and suitable *solution methods* are the subject of this section. The GAUSS-JORDAN method will develop to its full bloom.

7.1   *Project*: Management in Metallurgy

A metallurgical company has found out during a brainstorming session that an innovatively hard and relatively light alloy can be produced with commercially available aluminum, if this alloy contains exactly 4 % titanium (Ti) and 2 % chromium (Cr). A name for the new product has already been found: *Ticro*.[2]

Since pure titanium and pure chromium are expensive, one would like to produce the appropriate mixture from cheaper aluminum-titanium-chromium alloys. Four different alloys A, B, C, D are available on the market, whose Ti or Cr content has been compiled by the management in the following table:

Alloy	A	B	C	D
Titan (Ti)	6 %	1 %	4 %	3 %
Chromium (Cr)	1 %	3 %	0 %	4 %
Price (per ton in US$)	2000	1000	3000	2000

The management now has the following questions:

a. Can you get the alloys available in such quantities, for example "buy x_1 tons from A , x_2 from B , x_3 from C and x_4 from D" so that exactly 1 ton of the new alloy can be melted from them?

b. If that is possible: what is the cheapest way to procure it?

---

[2][1, pp. 7–8, 72–74]

Step **1** : **Make a plan**

for the problem-solving process itself.
∘ In which steps should you proceed?
∘ Think long enough before you read any further.

Here is a possible plan

which takes place in a sequence of steps.

1. set up a model consisting of linear equations

2. write the linear system in matrix form $A * X = B$

3. solve this system of equations

4. are there alternative solutions to do an independent controlling calculation

5. solve the *optimization* problem

6. make the management decision

♡ Take your time and try this hard enough before reading further.
Stop after every step - and think, how you can go further.

Step **2** : **set up a linear model**

The conditions are in words . . .

1. A total of 1 ton is to be produced.

2. The alloy produced is said to contain 4% Ti.

3. The alloy produced is said to contain 2% Cr.

. . . and in formulas:

$$x_1 + x_2 + x_3 + x_4 \quad = \quad 1 \tag{7.1}$$
$$0.06 \cdot x_1 + 0.01 \cdot x_2 + 0.04 \cdot x_3 + 0.03 \cdot x_4 \quad = \quad 0.04 \tag{7.2}$$
$$= \tag{7.3}$$

Fill in the gap.

Step **3** : **Set up the linear system in matrix notation** $A * X = B$

$$
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.06 & 0.01 & 0.04 & 0.03 \\ 0.01 & 0.03 & 0 & 0.04 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.04 \\ 0.02 \end{bmatrix}
$$

○   How should we solve the system? Why is there an obstacle?
○   Can we solve with the help of the inverse matrix A^{-1}? If yes: why? not: why not?
Here are the data for trying your idea - rewritten in precise fractions, so that EIGENMATH
also calculates internally with them:

———————— EIGENMATH ————————

```
A = ((     1,      1,      1,      1),
     (6/100, 1/100, 4/100, 3/100),
     (1/100, 3/100,     0, 4/100))


B = (1, 4/100, 2/100)


--          x1     x2      x3      x4      RHS
AB = ((      1,      1,      1,      1,      1),
      (6/100, 1/100, 4/100, 3/100, 4/100),
      (1/100, 3/100,     0, 4/100, 2/100))
```

▷   *Click here to run the script.*
Exercise: What happens, if you enter A and B in decimal form (the original formulation).

Step **4** : **"Solve" the system of equations using** `RREF`

We use the linear system (short: LS) in its notation as augmented system matrix AB and
call the routine GAUSS-JORDAN alias `RREF` to read off the solution: ———————— EIGENMATH ————

```
run("gjBox.txt")     -- on the iMac write run("downloads/gjBox.txt")
RREF(AB)
```

$$
\begin{bmatrix} 1 & 0 & 0 & 1 & \frac{2}{3} \\ 0 & 1 & 0 & 1 & \frac{4}{9} \\ 0 & 0 & 1 & -1 & -\frac{1}{9} \end{bmatrix}
$$

Output:

○ Try to understand or interpret the result. What does it reflect?

○ How many equations or unknowns does the LS have?
   Describe the situation from the point of view of the GAUSS-JORDAN alogoritm.

Step **5** : **Check the** `RREF`–**status**

The Row Reduced Echelon Form final status is:

$$x_1 \ x_2 \ x_3 \ x_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2/3 \\ 0 & 1 & 0 & 1 & 4/9 \\ 0 & 0 & 1 & -1 & -1/9 \end{bmatrix}$$

$$\uparrow \ \uparrow \ \uparrow \ \Uparrow$$

Pivots   free variable

$$\uparrow \ \uparrow \ \uparrow$$

basic variables

○ The positions of the $\boxed{1}$ are called *pivotpositions* or *pivots* for short.

○ The associated variables x_1, x_2, x_3 are called *base variables.*

○ The variable x_4 is called a *free variable.*

How can we interpret the result using the new vocabulary?
Why were these terms invented? What do you want to express with them?

Step **6** : **Read off an interpretation of the** `RREF` **result**

Therefore we interpret back the Row Reduced Echelon Form status into classical equations making the unknowns visible again:

$$x_1 \ x_2 \ x_3 \ x_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2/3 \\ 0 & 1 & 0 & 1 & 4/9 \\ 0 & 0 & 1 & -1 & -1/9 \end{bmatrix}$$

classic $\downarrow$ system of equations

$$
\begin{aligned}
x_1 \qquad\quad + x_4 &= \frac{2}{3} \\
x_2 \quad + x_4 &= \frac{4}{9} \\
x_3 - x_4 &= -\frac{1}{9}
\end{aligned}
$$

○ What, if the fourth alloy, D i.e. x_4, is currently not available on the market?

○ Why is this "theoretically possible" solution of the LS pointless for the management?

We see: *would* e.g. the fourth alloy D be not available on the market, we would have $D = x_4 = 0$ and we would have the *corresponding* particular solution $x_1 = \frac{2}{3}$ and $x_2 = \frac{4}{9}$ and $x_3 = -\frac{1}{9}$ of the LS. But it is no solution for the company, because we cannot give a negative "-1/9" portion of the third allow into the mix.

∘   But we see more: with an *arbitrary* quantity $x_4 = d$ of alloy D, the LS can always be solved! If we insert a *arbitrarily chosen* number d for the unknown x_4 in the LS, the corresponding solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} - d \\ \frac{4}{9} - d \\ -\frac{1}{9} + d \\ d \end{bmatrix}$$

Since we are *free*[3] to choose a value for d, there are *infinitely many solutions* – namely for each choice of d a different one. This is a new phenomenon.

Step 7 : alternative solution idea – 'Lifting'!

By adding a 4th *dummy line* $x_4 = d$ to the original LS or the Row Reduced Echelon Form we "*lift*" the original 3×4 – LS to a uniquely solvable 3×4 – LS and read off the solution (quantity d remains free!) directly:

———————   EIGENMATH   ———————

```
A = ((     1,      1,      1,      1),
      (6/100, 1/100, 4/100, 3/100),
      (1/100, 3/100,      0, 4/100),
      (0     ,      0,      0,      1))   -- dummy line 1*x4 = ...

B = (1, 4/100, 2/100, d)               -- insert dummy variable

AB = ((     1,      1,      1,      1,       1),
       (6/100, 1/100, 4/100, 3/100,   4/100),
        (1/100, 3/100,      0, 4/100,   2/100),
          (0     ,      0,      0,      1,       d))   -- insert dummy equation x4=d

run("gjBox.txt")       -- on the iMac write run("downloads/gjBox.txt")
RREF(AB)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -d + \frac{2}{3} \\ 0 & 1 & 0 & 0 & -d + \frac{4}{9} \\ 0 & 0 & 1 & 0 & d - \frac{1}{9} \\ 0 & 0 & 0 & 1 & d \end{bmatrix}$$

Output:

---

[3]Therefore x_4 is called a **free variable** i.e. we have complete freedom or arbitrariness to choose a value for x_4. Such *free variables* are often called **parameters**. The letters r, s, t are often used for parameters – just as one like to use the letters x, y, z for the unknowns. The word *parameter* itself is derived from the Greek terms *para* (= beside) and *metron* (= measure).

▷   *Click here to run the script.*
At this `RREF` status we read off the solution in the last column.

Step **8** : **Nevertheless – an alternative solution of the LS with the inverse**

In step 7 the `RREF` call is successful in producing the solutions, *if we 'lift' the LS to a quadratic one.* Because the inverse of A was constructed via the Row Reduced Echelon Form, it should be possible to calculate the solution via `inv(A)`$= A^{-1}$, too:

──────────────   EIGENMATH   ──────────────

```
A = ((     1,       1,       1,       1),
      (6/100, 1/100, 4/100, 3/100),
      (1/100, 3/100,      0, 4/100),
      (0     ,      0,       0,      1))   -- dummy line

B = (1, 4/100, 2/100, d)                  -- insert dummy variable

inv(A)

X=dot( inv(A), B)
X
```

$$X = \begin{bmatrix} -d + \frac{2}{3} \\[4pt] -d + \frac{4}{9} \\[4pt] d - \frac{1}{9} \\[4pt] d \end{bmatrix}$$

Output:

▷   *Click here to run the script.*   This technically ends the solution determination. But the management decision is still pending ..

Step **9** : **Solving the optimization problem**

From the infinitely many solution vectors are only those admissible, where *each entry is non-negative*, i.e.:

$$(x_1, x_2, x_3, x_4) = (\frac{2}{3} - d, \frac{4}{9} - d, -\frac{1}{9} + d, d) \geq 0 = (18d + 13)/9$$

Therefore we have:

$$\begin{aligned} 2/3 - d \geq 0 &\Rightarrow d \leq 2/3 \\ 4/9 - d \geq 0 &\Rightarrow d \leq 4/9 \\ -1/9 + d \geq 0 &\Rightarrow 1/9 \leq d \\ d \geq 0 &\Rightarrow 0 \leq d \end{aligned}$$

i.e. we have: $\frac{1}{9} \leq d \leq \frac{2}{3}$. It follows that the cost function is

$$\mathrm{cost}(d) = (2/3 - d) \cdot 2 + (4/9 - d) \cdot 1 + (d - 1/9) \cdot 3 + d \cdot 2 = \frac{18 \cdot d + 13}{9}$$

This function is monoton increasing, taking its minimum at $d = 1/9$.
Result: The company buys $5/9$ tons of alloy A, $5/9$ tons of alloy B, none of alloy C and
$1/9$ tons of alloy D for the total price of **1666.66 US\$**.

7.2   *Exercise*: **Review and preview**

7.2.1   Review

○   Remember the *core ideas* for solving the previous problem.
○   How many solutions did the LS have and why?
○   How did you recognize this?

a.   What happens if you omit the third column with unknown x_3 in the problem *metal-
lurgy*? How is this omission to be interpreted?

```
A = ((    1,     1,      1),
      (6/100, 1/100, 4/100),
      (1/100, 3/100,     0))

B = (1, 4/100, 2/100)
```

▷ *Click here to invoke* EIGENMATH and then mark–copy–paste these data into the
EIGENMATH window.

○   How many solutions does this new LS have and why? Guess before you calc!

○   How do you recognize this? How could you alternatively solve?

b.   And if you leave out a line, e.g. the last? Again, before studying EIGENMATH's answer,
guess!

7.2.2   Preview

Here is another system of 3 linear equations given in its augmented matrix shape:

```
AB =((6,2,4, 10),
      (2,4,2,  2),
      (4,-2,2, 8))
```

a.   Show, that the `RREF` tool does not work for this LS AB. Why? Use the `gjBox.txt` for
your investigation and to reach the following final status:

$$
\begin{bmatrix}
6 & 2 & 4 & 10 \\
0 & \frac{10}{3} & \frac{2}{3} & -\frac{4}{3} \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

**b.** 'Lift" the intermediate matrix in a. to get the following solution:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{5}t + \frac{9}{5} \\ 0 & 1 & 0 & -\frac{1}{5}t - \frac{2}{5} \\ 0 & 0 & 1 & t \end{bmatrix}$$

We conclude: the solution set consists of all points $(-3/5 \cdot t + 9/5, -1/5 \cdot t - 2/5, t)$ with parameter $t \in \mathbb{R}$ arbitrary.

Let's plot this "*parametric*" solution set of infinitely many points:
▷   *Click here to call WolframAlpha.*

Then copy the text "`3d parametric plot (-3/5*t+9/5, -1/5*t-2/5,t)`" in the input frame and press ▭. You should get the plot:
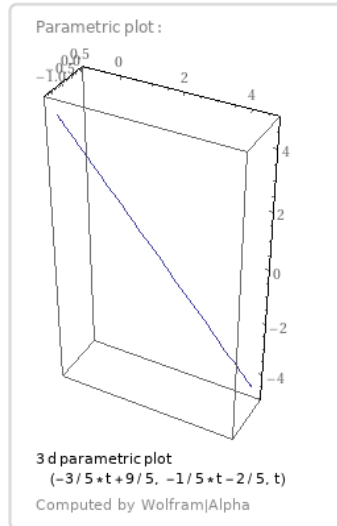

We see, that the solution points are not scattered around randomly in space, but obviously fill a straight line. To investigate this phenomenon in detail is topic in the next section.

8 $A * X = O$

The study of linear systems of type $A * X = E$ with E the quadratic $\square$ identity matrix, lead to the concept of the inverse matrix, i.e. to the solution $X = \begin{smallmatrix} \text{Math: } A^{-1} \\ \text{EIGENMATH: } \text{inv(A)} \end{smallmatrix}$.

Now we discuss another special case of a linear system: the matrix equation $A * X = O$ with O the zero matrix matching in type with matrix A, i.e. if $\overset{m \times n}{A}$ then $\overset{m \times 1}{O}$. We begin our exploration with a game to get a feeling for the importance of this 'trivial' equation.

P55. Linear combination game I.
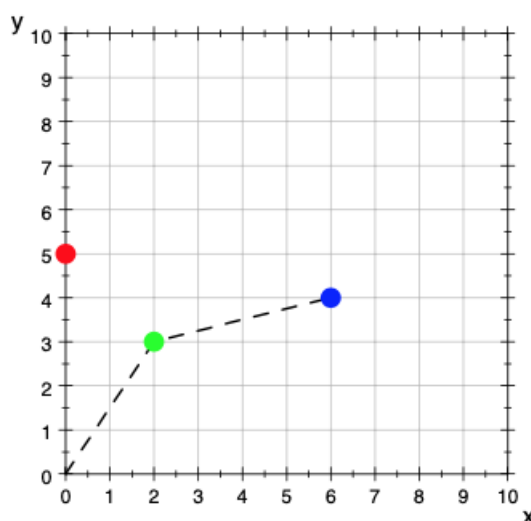
We play a steering game. It is called "linear combination" (with the columns of a matrix). We think e.g. at a robot in a warehouse that can be steered in only the two directions $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and that he is to be remotely directed to the goal – the red shelf $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$. We model the situation as a *'linear combination'*, i.e. as a sum of the two stretched direction vectors:

$$ x \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \qquad (\star) $$

a. Guess in your head, what parameter pair (x, y) sums the direction vectors up to result in the red target point? This should train your imagination.

b. Use the EIGENMATH function $Lk(x, y)$[4] to drive the robot to the target by trying several pairs (x, y).

```
Lk(x,y)= x*(2,3)+y*(4,1)
Lk(x,y)
```
$$ \begin{bmatrix} 2x + 4y \\ 3x + y \end{bmatrix} $$

```
Lk(1,1)   -- go 1 times in direction (2,3) and
          -- from there 1 times in direction (4,1)
```
$$ \begin{bmatrix} 6 \\ 4 \end{bmatrix} $$

---

[4]$Lk \overset{def}{=}$ **L**inear **k**ombination

c. Formulate equation $(\star)$ as a matrix equation and compute the correct coefficient pair (x, y) using EIGENMATH. Check the solution in the figure above. Why is only *one* solution possible? ▷ *Click here to invoke* EIGENMATH

d. Choose $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as new red target position. What are now the coefficients alias the stretch factors x and y?

P56. Linear combination game II.
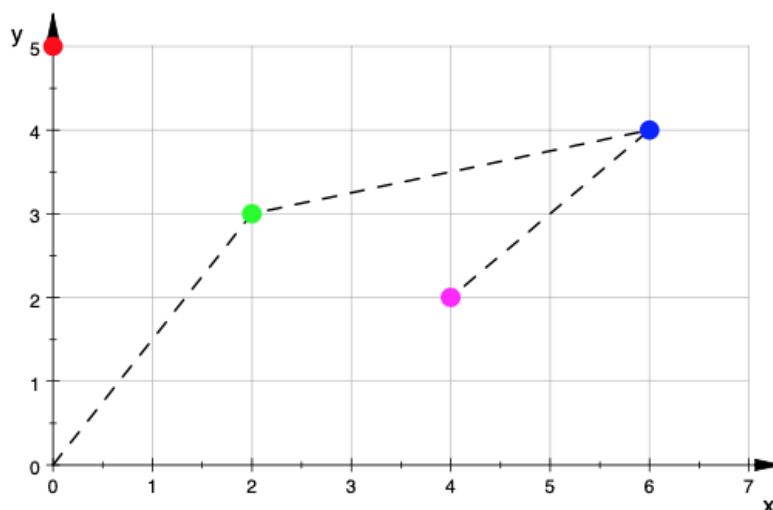
EIGENMATH–function `Lk3(x,y,z)` has to linear combine 3 columns of a 2×4 matrix to their 4th column. Here is a start:

_____ EIGENMATH _____

```
M = ((2,4,-2, 0),
     (3,1,-2, 5))

Mt = transpose(M)

Lk3(x,y,z) = x*Mt[1]+y*Mt[2]+z*Mt[2]
Lk3
```

▷   *Click here to run the script.*

a. Guess in your head, what parameter triple (x, y, z) leads to the red target point?

b. Use the EIGENMATH function $Lk3(x, y, z)$ to drive the robot to the target by trying several triples (x, y, z).

c. Compute a correct coefficient triple (x, y, z) using EIGENMATH. Check the solution in the figure above. How many solution(s) are possible? Why?

d. Chose $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as new red target position. What are now possible coefficients x and y?

Solution to c.:

────────────── EIGENMATH ──────────────

```
M = ((2,4,-2, 0),
     (3,1,-2, 5))

run("gjBox.txt")
RREF(M)

M1 = ((2,4,-2, 0),
      (3,1,-2, 5),
      (0,0, 1, t))      -- added dummy line, i.e. 'lift" to 3 x 4 matrix

RREF(M1)
```

▷   *Click here to run the script.*

EIGENMATH output:

$$\texttt{RREF}(M)=\begin{bmatrix} 1 & 0 & -\frac{3}{5} & 2 \\ 0 & 1 & -\frac{1}{5} & -1 \end{bmatrix} \text{ and } \texttt{RREF}(M1)=\begin{bmatrix} 1 & 0 & 0 & \frac{3}{5}t+2 \\ 0 & 1 & 0 & \frac{1}{5}t-1 \\ 0 & 0 & 1 & t \end{bmatrix}.$$

Conclusion: we found infinitely many possibilities to combine the 3 column vectors to the goal $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$, e.g. for $z = t = 5$ we have $x = 5$ and $y = 0$ and for $z = t = 0$ we have $x = 2$ and $y = -1$.

We can check our conclusion in EIGENMATH with a tiny helper function `solSet` as follows:

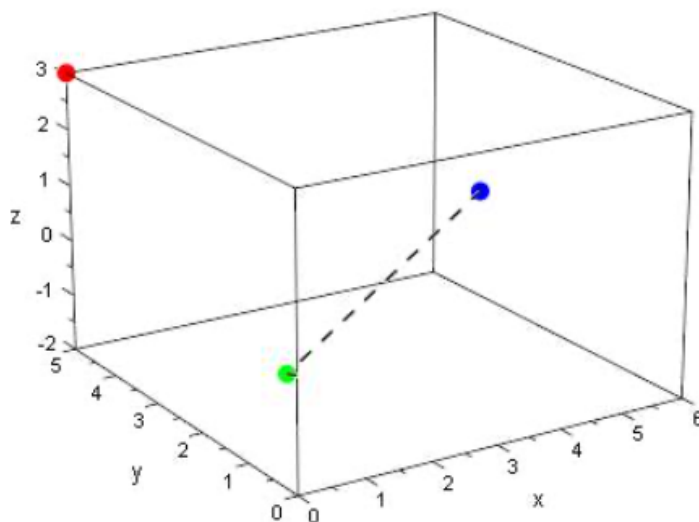────────────── EIGENMATH ──────────────

```
solSet(t) = (3/5*t+2, 1/5*t-1, t)
solSet(5)     -- returns (5,0,5)=(x,y,z)
solSet(0)     -- returns (2,-1,0)

Lk3(5,0,5)
Lk3(2,-1,0)
```

▷   *Click here to run the script.*

P57. Linear combination game III.

The green, blue and red points of the figure are put in a 3×3 matrix:

$$M = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 5 \\ -2 & 2 & 3 \end{pmatrix}$$

a. Write a function $Lk3D$, which is supposed to linearly combine the first 2 columns of matrix M to the 3rd, so our game becomes 3-dimensional and now takes place in space.

Now playfully test the new command $Lk3D$ in EIGENMATH. ▷*Invoke* EIGENMATH

b. What are the "winning coefficients"?

c. What, if we chose $(0, 0, 0)$ as new red target position?
What are now possible coefficients x and y?

d. Also experiment with other matrices M, e.g. with a 3×4 matrix.

8.1   Linear independence alias $A * X = O$ only for $X = O$

Hereinafter we examine a relationship between a special property of the columns
(vectors) of a matrix A and the number of solutions of the associated *"homogeneous"*
matrix equation $A * X = O$ or equivalently the augmented equation matrix $[A\ O]$
with the zero vector O as RHS.

8.1.1   Exercise

a. Consider the following linear combination of three columns, which should result in O.
Determine numbers x, y, z which fulfill this condition.

$$x \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} + z \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (\dagger)$$

b. The system in a. is equivalent to the following matrix equation:

$$\begin{pmatrix} 2 & 4 & -1 \\ 3 & 1 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\ddagger)$$

$$A * X = O$$

Show that this *'homogeneous'* LS $A * X = O$ is only ambiguously solvable, i.e. that
several different solutions X can be given. Give 3 different solutions X.

▷ *To invoke* EIGENMATH

c. Experiment: replace the right hand side O of this LS $A * X = O$ with any right side
$B \neq O$. What's about the number of solutions of your new LS $A * X = B$?

d. Keep experimenting: delete any column of the left hand side system matrix A.
What's about the number of solutions of your new system $A * X = B$?

Verify, that the associated homogeneous system $A * X = O$ is *only solved uniquely*.
Determine *this* unique solution X.

What is the relationship between the number of solutions of an *homogeneous* LS $A*X = O$ and an associated *inhomogeneous* system $A * X = B$ with the same matrix A?

8.1.2   *Definition*: linearly in|dependent column vectors

> The column vectors of matrix A are $\begin{array}{c} \textit{linearly dependent} \\ \textit{linearly independent} \end{array} \overset{def}{\Longleftrightarrow} A * X = O \begin{array}{c} \text{with an } X \neq O \\ \text{only for } X = O \end{array}$.

○   i.e. the column vectors of matrix A are *linearly independent* if the zero vector O is the
only (trivial) solution of the equation $A * X = O$.
○   i.e. if the zero vector O is only uniquely combinable with the columns of matrix A,
then the columns of A are called *linearly–independent*.

P58. Again – The balance of unchangeable components.
Sulfur dioxide is e.g. used as a bleaching agent for wool or as a preservative for fruit. In the extraction of sulfur dioxide, the ore sulfur pyrites (FeS_2) is roasted in an oxygen stream (O_2). This creates sulfur dioxide (SO_2) and gravel burn-off (Fe_2O_3), which can be further melted into iron. *What are the exact proportions of the substances involved in this reaction?*

$$a FeS_2 + b O_2 \longrightarrow c Fe_2O_3 + d SO_2$$
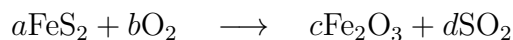
We know, that the corresponding linear system is

$$
\begin{array}{rccc}
 & before & = & after \\
\text{Fe:} & a & = & 2c \\
\text{S:} & 2a & = & d \\
\text{O:} & 2b & = & 3c + 2d \\
\end{array}
$$

a. Solve the LS with a 'lift' and give the complete solution set.

b. Plot the solution set, see 7.2.2.c.

c. Interpret the solutions in the chemical context, i.e. find the *smallest integer* solution of the LS and use it to give the reaction equation.

Solution: $4FeS_2 + 11O_2 \longrightarrow 2Fe_2O_3 + 8SO_2$
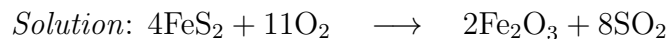
P59. Vector combinations.
Are there different possibilities for the vector $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ to be combined *as the sum of multiples* of the vectors $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

a. Justify the approach $x_1 \begin{bmatrix} 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

b. The approach in a. ask for the solution of the 'underdetermined' LS

$$
\begin{bmatrix}
4 & 1 & -2 & 7 \\
0 & 2 & 1 & 5
\end{bmatrix}
$$

Determine the complete solution set.

▷ *Click here to invoke* EIGENMATH

8.1.3   Remark: *Underdetermined* alias *singular* linear systems.

The problems studied in this section led to associated linear system `A.X = B`, for which the matrix of coefficients A had *no* inverse matrix and the corresponding solution was no longer unique. The matrix of coefficients of the respective linear system was rectangular, i.e. not square. With such *singular* linear systems, where there is no inverse, there are more interesting phenomena in regard to the geometric *shape of the solution set* and the suitable *solution methods*. We distinguish two cases of singular LS:

1. *underdetermined* LS: we have *less* equations (m) than unknowns (n), i.e. $m < n$. Usually there is an infinite number of solutions. Such LS occur frequently in applications and are studied in this section.

2. *overdetermined* LS: we have *more* equations than unknowns, i.e. $m > n$. They are not solvable (by means of this section); nevertheless, in scientific practice, they are very important and are studied in the next part 4.

To sum up:

A *regular* LS ...   ... and 2 *singular* LS

$$
\begin{aligned}
2x - 3y &= 1 \\
5x + 1y &= 14
\end{aligned}
\qquad\qquad
2x - 3y = 1
\qquad\qquad
\begin{aligned}
2x - 3y &= 1 \\
5x + 1y &= 14 \\
3x - 2y &= 7
\end{aligned}
$$

$$
\boxed{m=n} \cdot \Box = \Box
\qquad
\boxed{m<n} \cdot \Box = \Box
\qquad
\boxed{m>n} \cdot \Box = \Box
$$

$A = \Box \ \&\ \det(A) \neq 0$ ... *underdetermined*   ... *overdetermined*

P60. First experiences with *singular* LS.
Represent the following LS in matrix form $A * X = B$. Plot the solution set if possible.

a. Calculate the only solution of the 2×2 – LS $\left[\begin{smallmatrix} 2x-3y= 1 \\ 5x+1y=14 \end{smallmatrix}\right]$.

b. Give three different solutions of the 1×2 – LGS $\left[2x - 3y = 1\right]$.

c. Are there solutions for the following 3×2 – LGS?

$$
\begin{aligned}
2x - 3y &= 1 \\
5x + 1y &= 14 \\
3x - 2y &= 7
\end{aligned}
$$

▷ *Click here to invoke* EIGENMATH

Use `CalcPlot3D` for the plot.

8.2   Solution sets of underdetermined linear systems I

♡ The following task is central. Process it with the greatest care. Think through it and read it several times. After that we study a shorter example of the exercise.

8.2.1   Solution recipe for a singular LS

$$2x - y + z = 2 \tag{8.1}$$
$$2y - 6z = 0 \tag{8.2}$$

Study here how to obtain the solution set $\mathbb{L}$ of the LS *without* EIGENMATH.

1st step: write the LS as one matrix
$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 2 & -6 & 0 \end{bmatrix}$$

2nd step: transfer the LS to rref
$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 0 \end{bmatrix}$$
$$\uparrow \ \uparrow$$
pivot columns

3rd step: mark the pivot positions with ($\uparrow$). These are the **base** variables.
The remaining variables are the **free** variables. The free's are designated with parameters, here t for z. Therefore x, y are the base variables and z is **freely** selectable.

$$x \quad - t = 1$$
$$y - 3t = 0$$

$$\uparrow \ \uparrow$$

4th step: solve for the base variables x and y:

$$y = 3t$$
$$x = 1 + t$$

5th step: describe the solution set using the parameter t instead of z

$[x, y, z] = [1 + t, 3t, t]$ with $t \in \mathbb{R}$ arbitrary; $z = t$ according to 3rd step

Exercise:
○   Solve the LS for comparison with EIGENMATH. Use `gjBox.txt`. ▷EIGENMATH
○   Plot the solution set $\mathbb{L}$, i.e the set of all *points* P in the three-dimensional visual space, which are of the form $P = (x, y, z) = (1 + t, 3t, t)$ with $t \in \mathbb{R}$ arbitrary. ▷CALCPLOT3D

Result: the solutions of the underdetermined linear system of equations

$$2x - y + z = 2$$
$$2y - 6z = 0$$

are not randomly distributed in the space $\mathbb{R}^3$, *but obviously form a* **straight line**.
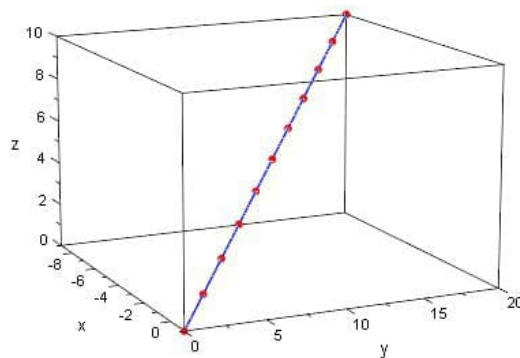
Figure 1: The red points are produced with parameter values $t = 0$ until $t = 10$ step 1 in the description $P(t) = (1+t, 3t, t) \in \mathbb{L}$.

P61. Linear system with an infinite number of solutions.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 - 4x_2 + x_3 &= 6 \end{aligned}$$

a. Show that all points of the set $\mathbb{L} = \left\{ (7 + \frac{1}{2}t; 2 + \frac{1}{2}t; t) | t \in \mathbb{R} \right\}$ are solutions of the LS.

b. Determine the full solution set $\mathbb{L}$ given in **a.** by manual calculation.

c. Solve the LS with EIGENMATH.

d. Plot the solutions set. Use

    ○   the Wolfram'Widget with the input x(t)=7+0.5t, y(t)=2+0.5t, z(t)=t

    ○   or a query at Wolfram'Alpha with input 3d parametric plot (7+0.5t, 2+0.5t, t)

    ○   or use ▷CALCPLOT3D.

8.3   Straight lines

The solution sets $\mathbb{L}$ of the *underdetermined* linear systems of the last problem with *one* parameter t can be written as *parametrized row* vector

$$[x_1, x_2, x_3] = [7 + \frac{1}{2}t; 2 + \frac{1}{2}t; t] = [7, 2, 0] + [\frac{1}{2}, \frac{1}{2}, 1]$$

or reshaped as *parametrized column* vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

$$\uparrow \qquad\qquad \uparrow$$

reference point P   direction vector R

= partial solution of the LS

or written down in EIGENMATH notation

```
L(t)=(7,2,0) + t * (0.5,0.5,1)
```

and represents geometrically a straight line. We abstract this exploration to the

8.3.1   Definition:

Every straight line ℓ can be described by an equation of the form

$$X = P + t \cdot R \qquad (t \in \mathbb{R})$$

P *is called a* **reference point** *or position vector and R a* **direction vector** *of ℓ.*

○ One writes briefly:

$$\ell \colon X = P + tR$$

and calls this equation a *linear equation in parametric form* with the parameter t or shorter a *parametric equation* for ℓ.

○ P is sometimes called *support vector*, because the arrow $\overrightarrow{OP}$ from $O = (0, 0, 0)$ to $P \in \ell$ is poising the straight line in the point P.

○ P is sometimes also called *reference point*, because P *lies on* ℓ, i.e. its coordinates *fulfill* the associated linear system.

○ R is called *direction vector*, because R points from P in the "direction of the straight line".

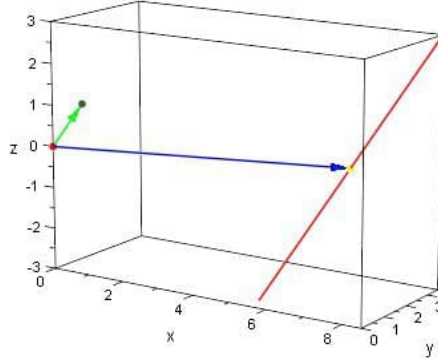○ To understand these new terms, consider and analyze the following graphic scene:

Figure 2:
Red: the straight line ℓ.
Blue: the position vector OP.
Green: direction vector R, which is parallel to ℓ.
Yellow: point $P \in \ell$.

8.3.2   Attention

- *Usually R is not in ℓ,*
  because e.g. the above *direction* $R = [\frac{1}{2}, \frac{1}{2}, 0]$ is *no solution* of the LS: $\begin{smallmatrix} x_1 - 3x_2 + x_3 = 1 \\ 2x_1 - 4x_2 + x_3 = 6 \end{smallmatrix}$!
  E.g. there is no suitable t, such that $R = [7 + \frac{1}{2}t; 2 + \frac{1}{2}t; t]$.

○ *Instead*: if one has 2 solution points P and Q of the LS, then the *difference* $\overrightarrow{QP}$ represents a *direction* vector R, *pointing from P to Q*; therefore we write

$$\boxed{R = \overrightarrow{PQ} \stackrel{def}{=} Q - P} \qquad \text{read: } R \text{ is the vector from } P \text{ to } Q$$

For the above example, $P = [7, 2, 0]$ and $Q = [8, 3, 2]$ are two solutions (check it!) of the LS, therefore $R = \overrightarrow{PQ} = Q - P = [8, 3, 2] - [7, 2, 0] = [1, 1, 2]$ is a direction vector. By the way, this triple is 2 times as long as $R = [\frac{1}{2}; \frac{1}{2}; 1]$ from above, but this is irrelevant for specifying a direction. We will explain this fact in more detail later.

○ *MiniExercise*: look for two points (= solutions of the above LS) P and Q with $\overrightarrow{PQ} = Q - P = R = [\frac{1}{2}; \frac{1}{2}; 1]$.

- *Recommendation.* The defining equation $\ell: X = P + t \cdot R$ of a straight line is best interpreted dynamically. For $X, P, Q \in \ell$ we have

$$X = \overrightarrow{OX} = P + t \cdot R = \overrightarrow{OP} + t \cdot \overrightarrow{PQ}$$

which we think as follows: "To go from origin O to point X in ℓ, go first from O to P on ℓ, then turn in direction $\overrightarrow{PQ} = R$ and walk t steps of steplength and stepdirection of R".

P62. linear system with ∞-many Solutions.

$$
\begin{aligned}
x_1 + 3x_2 - 4x_3 &= 1 \\
3x_1 - 2x_2 + x_3 &= 0 \\
-1.5x_1 + x_2 - 0.5x_3 &= 1
\end{aligned}
$$

a. Determine the solution set $\mathbb{L}$ of the linear system by hand calculation.

b. Solve the linear system with EIGENMATH. $\triangleright$ EIGENMATH

c. Plot the set of solutions. $\triangleright$CALCPLOT3D

P63. Solution sets of underdetermined linear systems II.

○ A 3×3 – linear system sometimes has a unique solution.
Geometrically, this solution is a point in space.

○ For the underdetermined 2×3 –LS $\begin{smallmatrix}2x-y+z=2\\2y-6z=0\end{smallmatrix}$ all infinite many points of the set $\mathbb{L} = \{(1+t; 3t; t)|t \in \mathbb{R}\}$ are solutions, filling a straight line in space.

● What, if we increase the 'vagueness' once again and drop another equation?

Look at the 1×3 –LS

$$
2x - y + z = 2 \tag{8.3}
$$

Determine the full solution set of eq. (8.3). Attack the problem in different ways:
– with hand trace
– with Row Reduced Echelon Form and $\triangleright$ EIGENMATH.
– ...

Be self-discipline . . .

. . . and study the following suggested solution of *P63* only after extensive own effort.
Don't turn the page right away: you are missing out on a learning opportunity!
Instead, study the sample solution 8.2.1 again.

Solution: **get the solution set by hand with Row Reduced Echelon Form**

1st step: write the LS as one matrix $\begin{bmatrix} 2 & -1 & 1 & 2 \end{bmatrix}$

2nd step: transfer the LGS to rref $\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$
                                                       $\uparrow$

                                                       pivot column

3rd step: mark the pivot positions ($\uparrow$).
The variable x is marked and is therefore
the **base** variable. The remaining 2 vari- $2x - s + t = 2$ (8.4)
ables y and z are **free** and designated as $y = s$ (8.5)
parameters $t \stackrel{def}{=} z$ and $s \stackrel{def}{=} y$. $z = t$ (8.6)

4th step: solve for the base variable x

$$x = 1 + \frac{1}{2}s - \frac{1}{2}t$$
$$y = s$$
$$z = t$$

5st step: write the solution set $[x, y, z] = [1 + \frac{1}{2}s - \frac{1}{2}t, s, t]$
using the parameters s and t: with $t, s \in \mathbb{R}$ arbitrary
                                                     and $z = t$, $y = s$ according to step 3

Exercise: **Solution of the linear system with** EIGENMATH**'s RREF**

a. ———————— EIGENMATH ————————

```
AB=(2,-1,1, 2)    -- the 1x3-LS  2x-y+z=2
AB

run("gjBox.txt")

AB1=((2,-1,1, 2),  -- the lifted 3x3 - LS
     (0, 1,0, s),  -- with parameter s:=y
     (0, 0,1, t))  -- and parameter t:=z

RREF(AB1)
```
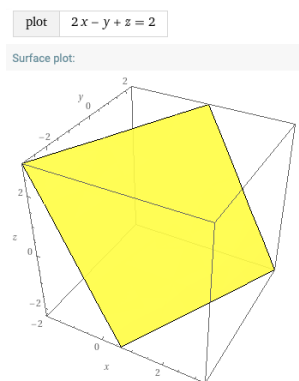
$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}s - \frac{1}{2}t + 1 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \end{bmatrix}$$

EIGENMATH output:

▷   *Click here to run the script.*

b. The solution set $\mathbb{L}$ consists of all *points* in the three-dimensional visual space that have
the form $[x, y, z] = [1 + \frac{1}{2}s - \frac{1}{2}t, s, t]$ with $t, s \in \mathbb{R}$ are arbitrary.

Plot the set $\mathbb{L}$. Use e.g.

- ○   a query at Wolfram'Alpha with input plot 2x-y+z=2
- ○   or use ▷CalcPlot3D.

Wolfram'Alpha output:

Anyway: the solutions of the underdetermined linear system of 1 equation and 3 unknowns

$$2x - y + \ z = 2$$

are not randomly distributed in space $\mathbb{R}^3$, *but obviously fill a* **plane**.

8.4   Planes

The full solution set of the *underdetermined* 1×3 – linear system of the last problem with *2* parameters s and t can be written as an *2–parameter row* vector equation

$$[x, y, z] = [1 + \frac{1}{2}s - \frac{1}{2}t; s; t] = [1, 0, 0] + s \cdot [\frac{1}{2}, 1, 0] + t \cdot [-\frac{1}{2}, 0, 1]$$

or reshaped as *parametrized column* vector equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix}$$

$$\uparrow \qquad\qquad \uparrow \qquad\quad \uparrow$$

a point P   direction vectors U and V

or written down in Eigenmath notation

```
E(s,t) = (1,0,0) + s*(0.5,1,0) + t*(-0.5,0,1)
```

and represents geometrically a *plane* in space $\mathbb{R}^3$. We denote a plane often with the letter E, *Lexicon*: German: Ebene[5] English: plane . You may also think at an Equation for the planE ..

_____

[5]Sometimes we denote a plane also with H, thinking at an **h**yperplane. We avoid the letter 'P' as symbol for a plane, because we prefer to use it to designate a 'P'oint.

8.4.1   Definition: plane

> *Every plane E in space can be described by an vectorial equation of the form*
>
> $$X = P + s \cdot U + t \cdot V \qquad (t \in \mathbb{R})$$
>
> *P is called a **position vector** and U and V the **direction vectors** of E.*

○ One writes briefly:
$$E \colon X = P + s \cdot U + t \cdot V$$
and calls this an *parametric equation* of E with the parameter s and t or a *vector form* of E.

○ P is sometimes called *support vector*, because the arrow $\overrightarrow{OP}$ from $O = (0, 0, 0)$ to $P \in E$ is touching the plane in the point P.

○ P is sometimes also called *reference point*, because P *lies on* E, i.e. its coordinates *fulfill* the associated linear system.

○ $U \neq O$ and $V \neq O$ are called *direction vectors*, because both U and V point from the standpoint $P \in E$ in "two different directions of the plane".

○ *Straights and planes occur as solutions sets of singular linear systems of equations.*

● In contrast to this, an equation like (8.3), e.g.

$$1x + 2y + 3z = 4 \quad \text{or in general} \quad ax + by + cz = d$$

is called a *coordinate equation of plane E*. One writes briefly:

> $$E \colon ax + by + cz = d$$

Here the coefficients a, b, c are not all at the same time zero.

○ We usually use the *coordinate form* for arithmetic calculations (e.g. with EIGENMATH) and the *parameter form* for drawing or for theoretical considerations.

Planes are *infinitely wide on all sides* despite of a plot only showing a triangular part:

3D contour plot:
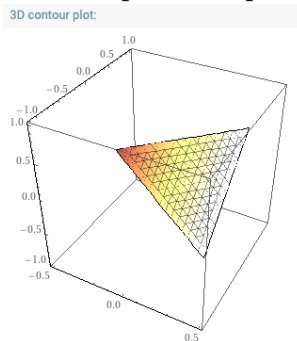
WOLFRAM'ALPHA output:

8.5   Switch between parametric and implicit equations via RREF

Normally the reshaping of a plane from its parametric form into a coordinate ('implicit') equation is laborious. Therefore, study the following script and extract the solution idea.

Given is a plane E in its parametric vector form

$$E : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + r \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + s \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad (E)$$

Find a coordinate equation for $ax + by + cz = d$ for E.
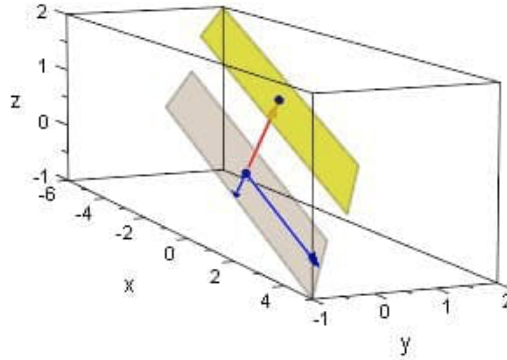
Figure 3:
Yellow: the plane E.
Gray: the plane fixed at the origin O.
Blue: the linearly independent direction vectors of E.
Red: the support vector OP pointing to E.

8.5.1   Solution by hand calculation

The data are chosen 'easy', so that a hand calculation can show the principal heuristic. *First*, split the defining vector equation (E) in 3 separate single equations:

$$-1 + 3 \cdot r + 2 \cdot s \;=\; x \qquad (8.7)$$
$$1 - s \;=\; y \qquad (8.8)$$
$$1 - 1 \cdot r + 0 \cdot s \;=\; z \qquad (8.9)$$

Second, determine the parameter r and s in terms of the other variables using 2 of the 3 equations. Here we select the last two (8.8) and (8.9) and get:

$$-1 + 3 \cdot r + 2 \cdot s \;=\; x \qquad (8.10)$$
$$1 - y \;=\; s \qquad (8.11)$$
$$1 - z \;=\; r \qquad (8.12)$$

Third, eliminate the parameters in the other equation, here (8.7), to get one equation in the variables x, y, z:

$$-1 + 3(1 - z) + 2(1 - y) = x \qquad \rightsquigarrow \qquad \mathbf{E : 1x + 2y + 3z = 4}$$

8.5.2   Compact solution with the help of EIGENMATH's RREF

The geometric idea is to determine the *intersection points* X, Y, Z, where the plane E intersects the coordinate axes, i.e. the so-called *intercepts*. To determine the x–intercept with the x–axis and associated point $X = (x, 0, 0)$ of E , we have the ansatz:

$$-1 + 3 \cdot r + 2 \cdot s = x \tag{8.13}$$
$$1 - s = 0 \tag{8.14}$$
$$1 - 1 \cdot r + 0 \cdot s = 0 \tag{8.15}$$

which we write in matrix shape as augmented linear system:

─────────── EIGENMATH ───────────

```
run("gjBox.txt")
        -- r   s    x RHS          LS for x
X = (( 3, 2, -1,   1),   -- (1')   3*r+2*s-1x =  1
     ( 0,-1,  0, -1),   -- (2')   0*r-1*s+0y = -1
     (-1, 0,  0, -1))   -- (3')  -1*r+0*s+0z = -1

RREF(X)
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

EIGENMATH output: , i.e. intercept $x = 4$, therefore $X = (4, 0, 0)$.

▷   *Click to run the script.*

We could reproduce this same recipe another 2 times, but we prefer to collect all three equations in one linear system, i.e.:

```
run("gjBox.txt")
--       r  s   x  y  z  RHS    LS for x,y,z simultan
A =   (( 3, 2, -1, 0, 0,   1),
       ( 0,-1,  0,-1, 0,  -1),
       (-1, 0,  0, 0,-1,  -1))
RREF(A)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

EIGENMATH output: .

▷   *Click here to run the script.*

We read off $x = 4$, $2y = 4$ and $3z = 4$, and because the coordinate equation must include all three parts we have $E : x + 2y + 3z = 4$.

8.5.3   Plot the plane using the intersection points with the axes

We have calculated the intercepts $x = 4$, $y = 2$ and $z = \frac{4}{3}$, i.e the intersection points $X = (4, 0, 0), Y = (0, 2, 0)$ and $Z = (0, 0, 4/3)$. We use these points to plot the plane easily by hand, look here:
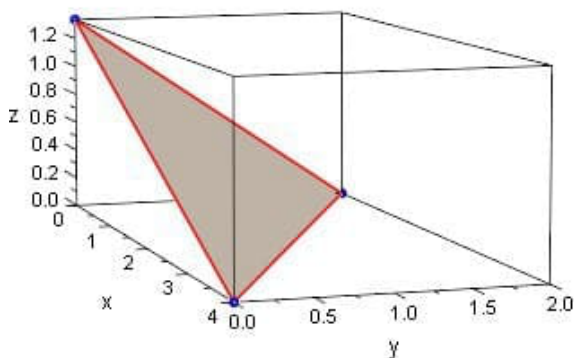

Figure 4:   Red: the polygon XYZ connecting the intercepts.
'Brown': the plane E_{XYZ} stretched between the intercepts.

8.5.4   Exercise

a. Reproduce figure.4 in 8.5.3 using WOLFRAM'ALPHA Widget 3D plot with the data f(x,y): (4-x-2y)/3, x-min: 0, x-max: 5, y-min: 0, y-max: 5 and then press Submit.

b. Reproduce figure.4 in 8.5.3 using CALCPLOT3D, see [10]. Remark: When a plane is written as coordinate equation, it s also called an '*implicit* (defined) surface', so do:

Add to graph: ▷ Select ▷ Implicit surface ▷□ Eq: 1+2x+3y+3z=4 ▷ disable the predefined example. Then Add to graph: the three points X, Y, Z and press Graph.

○   Go to ▷CALCPLOT3D.

c. Use the tree intercept points X, Y, Z to get back a vector equation for plane E.

P64. One again: a more complicated example.   Given is a plane E in its parametric vector form

$$E : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -5 \\ 4 \end{bmatrix} + r \cdot \begin{bmatrix} -25 \\ 20 \\ -6 \end{bmatrix} + s \cdot \begin{bmatrix} 3 \\ 15 \\ 10 \end{bmatrix} \qquad (E)$$

Find a coordinate aka implicit aka 'general' equation $ax + by + cz = d$ for E.

We give some hints.

o The hand calculation in step 1 is laborious. Therefore use EIGENMATH instead to eliminate r and s from the last 2 equations:

```
run("gjBox.txt")
       --    r   s   RHS
R23 = ((20,15, y+5),   --(2) -5+20*r+15*s = y
         (-6,10, z-4))  --(3)   4-6*r+10*s = z
R = RREF(R23)
R                       --(4)
r = R[1,3]
r                       --(5)
s = R[2,3]
s
16+(-25)*r+3*s          --(6) = x    |*10
last*10                 --(7) =10x  therefore: -8y+15z+60=10x
```

▷ *Click here to run the script.*

$$R = \begin{bmatrix} 1 & 0 & \frac{1}{29}y - \frac{3}{58}z + \frac{11}{29} \\[2mm] 0 & 1 & \frac{3}{145}y + \frac{2}{29}z - \frac{5}{29} \end{bmatrix}$$

$$r = \frac{1}{29}y - \frac{3}{58}z + \frac{11}{29}$$

$$s = \frac{3}{145}y + \frac{2}{29}z - \frac{5}{29}$$

EIGENMATH output zu (4) and (5):

Line (6) returns $-\frac{4}{5}y + \frac{3}{2}z + 6$ and (multiplied with 10) we get $-8y + 15z + 60$, which must be $10x$. We get the solution $10x + 8y - 15z = 60$.

a. Use EIGENMATH to get a compact solution at one stroke à la 8.5.2. The result should

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{29} & \frac{3}{58} & \frac{11}{29} \\[2mm] 0 & 1 & 0 & -\frac{3}{145} & -\frac{2}{29} & -\frac{5}{29} \\[2mm] 0 & 0 & 1 & \frac{4}{5} & -\frac{3}{2} & 6 \end{bmatrix}$$

be the EIGENMATH output

o How do you get the coordinate equation $10x + 8y - 15z = 60$ from that output?

b. Read off the 3 intercept points X, Y, Z and make a sketch of the plane by hand.

c. Make a quality plot of plane E using e.g. ▷CALCPLOT3D.

8.6   Relationships between lines and planes

We will now use these representations of lines and planes to show the mutual relation of straight lines to one another or that of a straight line to another line or the relationship of planes. We ask whether straight lines or planes intersect, run parallel or are in a subset relationship.

8.6.1   Representation of lines and planes in 2 resp. 3–points–form

The points $A = [1, -2, 5], B = [4, 6, -2]$ and $C = [1, 2, 3]$ are given. Determine a parameter representation of the straight line ℓ_{AB} through A and B as well as for the plane E_{ABC}. Give a coordinate equation for the plane E_{ABC}.
Draw line and plane in a 3D coordinate system.

Solution. We show a possible solution with EIGENMATH:

```
run("gjBox.txt")

-- parametric equation of line L: X=P+t*R=A+t*vec(AB)
A=(1,-2,5)
B=(4,6,-2)
C=(1,2,3)

L(t)= A+t*(B-A)            -- L fixed in P=A, R=vec(AB)
L(t)
E(r,s)= A+ r*(B-A) +s*(C-A) -- parametric eq for E fixed in A
E(r,s)

-- coordinate implicit equation Ec for E
--       r  s   x  y  z  RHS   LS for x,y,z simultan
Ec = (( 3, 0, -1, 0, 0,  -1),
      ( 8, 4,  0,-1, 0,   2),
      (-7,-2,  0, 0,-1,  -5))
RREF(Ec)
-- read off: 1x+0.5y+1z=5 i.e. 2x+y+2z=-10
```

EIGENMATH output:

$$
\begin{bmatrix}
1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{4}{3} \\[2mm]
0 & 1 & 0 & -\frac{7}{12} & -\frac{2}{3} & -\frac{13}{6} \\[2mm]
0 & 0 & 1 & \frac{1}{2} & 1 & 5
\end{bmatrix}
$$

i.e. $E\colon 1x + \frac{1}{2}y + 1z = 5$.

▷   *Click here to run the script.*

We plot the situation in 3D:
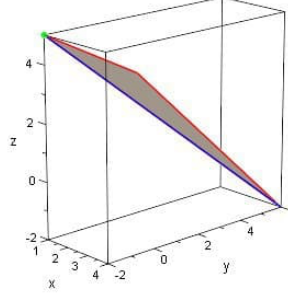
Green: the point A on both ℓ and E.
Figure 5: Blue: the line ℓ_{AB}.
Red: plane E_{ABC} plotted as polygon ABC.

P65. Relative position of two planes - an overview.   Two planes E and E' in space $\mathbb{R}^3$ can be related true *parallel* ('$\|$') or *cutting* ('$\sharp$') or *identical* ('$\equiv$').

a. Here is an systematic overview of the possible relationships of a plane $E : X = P + rU + sV$ to a another plane $E' : X' = P' + s' * U' + t' * V'$ by specifying criterions when the respective case occurs.

$E \sharp E'$	$E \parallel E'$	$E \equiv E'$
The linear system $P + rU + sV = P' + s' * U' + t' * V'$ has ...		
∞ many solutions.	0 solutions.	∞ many solutions.
$\{U, V, U'\}$ or $\{U, V, U'\}$ are linearly independent.	$\{U, V, U'\}$ and $\{U, V, U'\}$ are linearly dependent.	
	$\{U, V, P - P'\}$ are linearly independent	$\{U, V, P - P'\}$ are linearly dependent

◦   *Remember*: Given 3 vectors, you can determine if they are linearly independent by writing the vectors as the columns of a system matrix A, and solve the linear system $A * X = O$.
  – If there are any non-zero solutions, then the vectors are linearly dependent.
  – If the only solution is $X = O$, then the vectors are linearly independent.

◦   Looking back at 6.2 we can say: the system $A * X = O$ has $X = O$ for its only solution, if $det(A) \neq 0$. Using this fact we can rewrite the above criterion list in a way, that is suited for a computer aided decision making:

$E \sharp E'$	$E \parallel E'$	$E \equiv E'$
The linear system $P + rU + sV = P' + s' * U' + t' * V'$ has ...		
$det(UVU') \neq 0$ or $det(UVV') \neq 0$	$det(UVU') = 0$ and $det(UVV') = 0$ and $det(UV\overrightarrow{PP'}) \neq 0$	$det(UVU') = 0$ and $det(UVV') = 0$ and $det(UV\overrightarrow{PP'}) = 0$

b. Define a EIGENMATH function `relEE(P,U,V,Q,W,Z)`, which automatically determines the geometrical **rel**ationship between two planes.
Here is a start. Replace the dots ... with the appropriate code.

─────────── EIGENMATH ───────────

```
-- relation plane E vs plane E'
relEE(P,U,V, Q,W,Z) =
   do( UVW=zero(3,3),
        UVW[1]=U, UVW[2]=V, UVW[3]=W,
        UVZ= ... ,
        UVZ[1]= ... ,
        UVPQ=zero(3,3),
        UVPQ[1]=U, UVPQ[2]=V, UVPQ[3]=P-Q,
        test(
          or( not(det(UVW)==0), not(det(UVZ)==0)),
            "E cut E' = g",
          and( det(UVW)==0, ... , not(det(UVPQ)==0) ),
            "E // E'",
          and(  ... ),
            "E == E'"
          ))
```

Here is a test case. Write down tests for the other two cases.

```
-- test the relation E vs E'
do( P=(2,1,2), U=(-1,2,0), V=(3,1,4) )   -- plane E:  X=P+rU+sV
do( Q=(0,6,4), W=(2,2,4),  Z=(-2,1,3))   -- plane E': X=Q+aW+bZ
relEE(P,U,V, Q,W,Z)
```

▷   *Click here to see the solution.*

c. The test in b. asserts, that the planes E and E' intersects in a line ℓ. Determine an equation of this the line.

Solution: In EIGENMATH we name the second plane E_2 instead of E'.

$$E \overset{!}{=} E2 \quad \rightsquigarrow \quad P + rU + sV = X = Q + aW + bZ$$
$$\rightsquigarrow \quad rU + sV - aW = Q - P + bZ$$
$$\rightsquigarrow \quad r[-1, 2, 0] + s[3, 1, 4] - a[-2, -2, -4] = [-2, 5, 2] + b[-2, 1, 3]$$

```
run("gjBox.txt")
--    r  s   a   RHS b
EE =((-1,3,-2, -2-2b),
     ( 2,1,-2,  5+1b),
     ( 0,4,-4,  2+3b))
```

```
RREF(EE)                        --(1)

E2(a,b)=(0,6,4)+a*(2,2,4)+b*(-2,1,3)    --(2)  2nd plane
E2(-33/4*b-5/2, b)                      --(3)  substitute a=-33/4*b-5/2
```

Output (1):
$$\begin{bmatrix} 1 & 0 & 0 & -4\,b+1 \\ 0 & 1 & 0 & -\frac{15}{2}\,b-2 \\ 0 & 0 & 1 & -\frac{33}{4}\,b-\frac{5}{2} \end{bmatrix}$$
i.e. $a = -\frac{33}{4}b - \frac{5}{2}$. Ergo:
$$\begin{bmatrix} -\frac{37}{2}\,b-5 \\ -\frac{31}{2}\,b+1 \\ -30\,b-6 \end{bmatrix}$$
using (1).

In summa we have in row vector notation $\ell\colon X = [-5, 1, -6] + b \cdot [-37/2, -31/2, -30]$.

▷   *Click here to run the script.*

d. Plot a figure of the situation. ▷CALCPLOT3D

P66. Plane–plane intersection.

a. Investigate whether the planes

$$E : X = [1, 3, 2] + r[1, -2, 0] + s[3, 1, 4]$$

and

$$E' : X = [-1, 5, 2] + a[1, 1, 2] + b[-2, 1, 3]$$

intersect[6]. If possible, determine the intersection line ℓ.
Do the math and without ▷ EIGENMATH.

b. Find the line of intersection of $E_1 : 3x - 4y + z = 1$ and $E_2 : 5x + 2y - 3z = 6$.

c. Find the line of intersection of $E_3 : x - y + 3z = 12$ and[7]

$$E_4 : X = [8.0.2] + r[-4.1.1] + s[5.0, -1]$$

---

[6]see e.g.: Weisstein, Eric W. "Plane-Plane Intersection." From MathWorld–A Wolfram Web Resource. `https://mathworld.wolfram.com/Plane-PlaneIntersection.html`. – In Part 5 of our series we will compute this and other intersections in a unified form using determinants, the CRAMER rule or the so-called 'geometric' product of CLIFFORD geometry.

[7]Solutions: a. $E \cap E' = g : X = [-5, 1, -6] + t[37, 31, 60]$ b. $E \cap E' = g : X = [1, 0.5, 0] + t[5, 7, 13]$;
c. $E \cap E' = g : X = [3, 0, 3] + t[1, 1, 0]$

P67. Relative position of a straight line vs a plane. A line ℓ and a plane E in space $\mathbb{R}^3$ can be related true–*parallel* ('$\|$')[8] or *cutting* ('$\sharp$') or *identical* ('$\equiv$').
Here is an systematic overview of the possible relationships of a line $\ell : X = P + t \cdot U$ to a plane $E : X = Q + r \cdot V + r \cdot W$ by specifying criterions when the respective case occurs.

$\ell \sharp E$	$\ell \parallel E'$	$\ell \subset E$
The linear system $P + tU = Q + rV + sW$ has ...		
1 solution.	0 solutions.	∞ many solutions.
$\{U, V, W\}$ are linearly independent.	$\{U, V, W\}$ are linearly dependent.	
	$\{V, W, P - Q\}$ are linearly independent	$\{V, W, P - Q\}$ are linearly dependent

○ *Remark*: Given 3 vectors, you can determine if they are linearly independent by writing the vectors as the columns of a matrix A, and solve the linear system $A * X = O$.
– If there are non-zero solutions, then the vectors are linearly dependent. In this case $det(A) = 0$.
– If the only solution is $X = O$, then the vectors are linearly independent. In this case $det(A) \neq 0$.

a. Define a EIGENMATH function `relLE(P,U,Q,V,W)`, which automatically determines the geometrical **rel**ationship between a line g and a plane E. Think of the remark.

b. Investigate the relative position of the straight line $g : X = [4, 0, 8] + r[1, 3, 0]$ to the plane $E_1 : X = [1, 2, -1] + r[2, 3, 1] + s[1, 4, -3]$ and to plane $E_2 : 4x + 3y - 5z = 7$. If possible, determine the point of intersection[9] S, i.e. $\{S\} = g \cap E$.

Do the math with and without EIGENMATH.

P68. Relative position of two straight lines in space $\mathbb{R}^3$.
If two straight lines ℓ and k lie in the coordinate plane $\mathbb{R}^2$, they fall either together (are *identical* '$\equiv$") or intersect ('$\sharp$") or are parallel *parallel* ('$\|$").
If two straight lines lie in the space $\mathbb{R}^3$, then can also lie *skewed* ('$\bowtie$")- i.e. they are neither parallel nor intersecting. Here is an systematic overview of the possible relationships of a line $\ell : X = P + rU$ to a another line $k : X = Q + sV$ in $\mathbb{R}^3$ by specifying a criterion when the respective case occurs.

identical: $\ell \equiv k$	true parallel: $\ell \parallel k$	cutting: $\ell \dagger k$	crooked: $\ell \bowtie k$
The linear system $P + r \cdot U = Q + s \cdot V$ has ...			
∞ many solutions.	0 solutions.	1 solution.	0 solutions.
$\{U, V\}$ are linearly dependent and ...		$\{U, V\}$ are linearly independent and ...	
$\{U, P - Q\}$	$\{U, P - Q\}$	$\{U, V, P - Q\}$	$\{U, V, P - Q\}$
linearly dependent	linearly independent	linearly dependent	linearly independent

[8]i.e. $\ell // E$ but $\ell \not\subset E$.
[9]see: Weisstein, Eric W. "Line-Plane Intersection." From MathWorld–A Wolfram Web Resource. `https://mathworld.wolfram.com/Line-PlaneIntersection.html`. – We treat this in 10.4 and more general in Part 5 of our introduction.

a. Construct a EIGENMATH function `relLL(P,U,Q,V,W)`, which automatically determines the geometrical **rel**ationship between 2 lines.

b. Investigate the relative positions of the straight line $\ell : X = [2, 1, 1] + r[0, 1, 1]$, the line $k : X = [0, -4, 0] + r[2, 4, 0]$ and the line $h : X = [1, 1, 1] + r[0, 2, 2]$ to each other

– using your EIGENMATH function `relLL(P,U,Q,V,W)`
– using paper & pencil to check a.

If possible, determine the point of intersection S, i.e. $\{S\} = \ell \cap k$.[10]

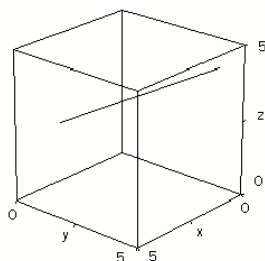**P69. Track of a straight line in space.** Consider the following sketch of a straight line ℓ in space.

Figure 6: A line ℓ in $\mathbb{R}^3$, which cuts the yz–plane and the xz–plane, giving the xz– and the yz–intercept points P_{xz} resp. P_{yz}.

○ The points $A = P_{xz} = [3, 0, 2]$, $B = P_{yz} = [0, 4, 4] \in \ell$ in Fig.6 are called the *track points* aka *plane intercepts* of the straight line ℓ, i.e. the *intersection points* of the coordinate planes with the straight line. In these track points there is cut off a part of the straight line by the coordinate planes, which is similar to a clothes line between two piles.

○ How to easily calculate the track points?

○ Why are the track points ideal suited if a straight line is to be sketched by hand?

○ Do a quality plot of the tracking part, i.e. plot the section $\overline{AB}$ between A and B of ℓ_{AB} and the points A and B with your favorite plotting program.
Your figure should be much better than Fig.6 ... We recommend ▷CALCPLOT3D.

○ Reconstruct an equation of the line drawn.

♡ Construct a EIGENMATH function `track(P,U)`, which automatically determines the track points of a straight line $\ell : X = P + tU$ in space. ▷EIGENMATH

With these experiences about lines and planes in $\mathbb{R}^3$ we are now prepared to do a little more theoretical exploration of great importance.

---

[10]Solution: a. S=(2,0,0).

8.7   Homogeneous and inhomogeneous linear systems

A linear system of equations $A * X = B$ such as

$$
\begin{aligned}
x + y + 2z &= 1 \\
2x + 2y - 4z &= 2
\end{aligned}
$$

is called *inhomogeneous*, because the right-hand side $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ does not consist only of zeros.
The corresponding *homogeneous* linear system $A * X = O$ is

$$
\begin{aligned}
x + y + 2z &= 0 \\
2x + 2y - 4z &= 0
\end{aligned}
$$

their the right-hand side $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ consist only of zeros.
One can assume that there are some connections between $A * X = B$ and $A * X = O$, since
the matrix of coefficients A is the same for both LS. Therefore we examine the question:

 – *Which connections exists between the solution sets of an* inhomogeneous *linear system
   and its associated* homogeneous *LS?*

P70. Inhomogeneous and associated homogeneous linear systems I.

 a. Determine the solution sets of the following linear systems LS1..5 in vector parameter
   form. Remark: Each system is a bundle of 2 or 3 planes in implicit equation form,
   which we input in augmented matrix shape A1 .. A5.
   What are the 'basic' solutions of the solution set?

   ─────────────── EIGENMATH ───────────────

```
run("gjBox.txt")
-- LS1:  {x + y + 2z = 1, 2x + 2y - 4z = 2}
A1 = ((1, 1, 2, 1), (2, 2, -4, 2))

-- LS2: {x - y - z = 1, x + y + 4z = -1, 7x + 6y - z = 7}
A2 = ((1, -1, -1, 1), (1, 1, 4, -1), (7, 6, -1, 7))

-- LS3:  {x + y + 2z = 1, 0.5x + 0.5y + z = 0.5, - 2x - 2y - 4z = 2}
A3 = ((1, 1, 2, 1), (1/2, 1/2, 1, 1/2), (-2, -2, -4, 2))

-- LS4:  {x - y - z = 1, x + y + 4z = -1, 3x + y + 7z = 3}
A4 =  ((1, -1, -1, 1), (1, 1, 4, -1), (3, 1, 7, 3))

-- LS5:  {2x + 2y + 4z = 2, x + y + 2z = 1}
A5 =  ((2, 2, 4, 2), (1, 1, 2, 1))
```
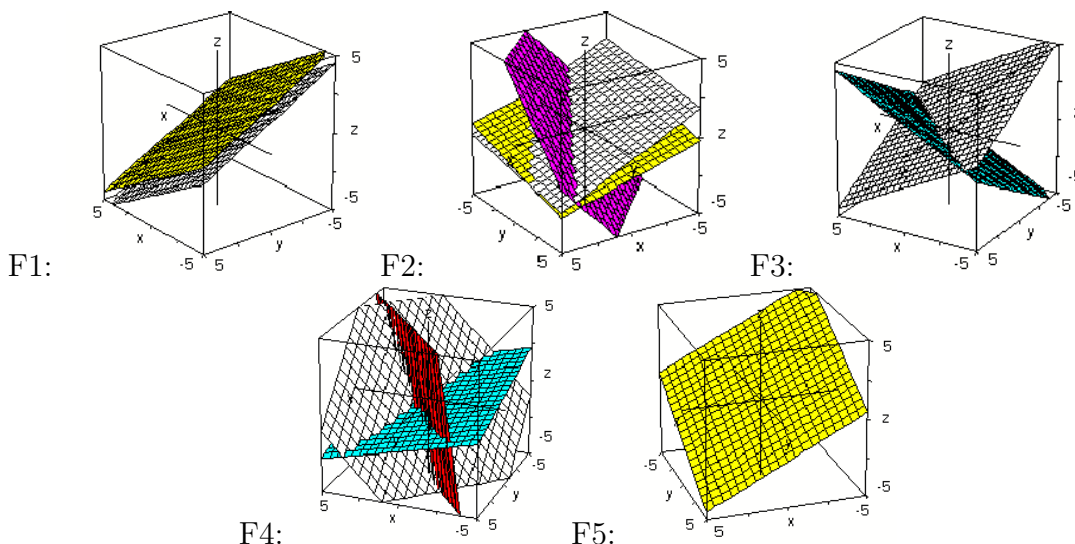
 ▷   *Click here to get the data.*

b. Which of the following graphic representations F1 ... F5 of the solution sets belongs to
which of the linear systems from **a.** Why?

F1:

F2:

F3:

F4:

F5:

○ Can you see the basic solutions in the 3D scene? Describe your observations.

c. Replace the right-hand sides of the above LS's with the zero vector $[0, 0]$ resp. $[0, 0, 0]$.
Determine the respective solution set of these homogeneous LS and draw the corre-
sponding graphic representation.

♡ Look for systematic connections and relationships. Formulate a hypothesis as a result.

8.7.1   Inhomogeneous and associated homogeneous linear systems.

Here we look again at the *inhomogeneous*[11] systems of equations LS1..5 from the previ-
ous exercise and their solution sets in parametric form. At the same time we solve the
corresponding *homogeneous*[12] systems of equations.
Here we solve in detail problem *P70* to show the handling of EIGENMATH in this context.

Step **1** : We discuss the 1st system $A1$.
We name the solution phases A-B-C-D, each corresponds to a matrix transformation.
─────────── EIGENMATH ───────────

```
run("gjBox.txt")
-- LS1: {x+y+2z=1, 2x+2y-4z=2}      case: cut of two planes
A1 = ((1,1,2, 1), (2, 2,-4, 2))
B1 = Ge(A1)    --(1) liften!  RREF is ko!
B1
C1 = ((1,1,2, 1), (0,1,0, d), (0,0,-8, 0))  --(1') lift row 2
```

─────────────────────────────
[11]i.e. the right hand side of the LS does not consist entirely of zeros.
[12]i.e. the right side of the LS is the zero vector $[0, 0, ..., 0]$.

```
D1 = RREF(C1)          --(1") read off solution
D1
A1h = ((1,1,2, 0),(2,2,-4, 0))  -- A1h homogen, analog
B1h = Ge(A1h)   -- lift!
B1h
C1h = ((1,1,2, 0), (0,1,0, d), (0,0,-8, 0))
D1h = RREF(C1h)        -- read off!
D1h
```

First we observe that the call `RREF(A1)` fails (try it!). Why? Now, going a step backwards and using a GAUSS elimination step by calling $\text{Ge(A1)} \rightsquigarrow B_1$, we see that `RREF` is missing the third row for a free variable, which we name d. After inserting this 'dummy' row in `Ge(A1)=B1` we have a 'lifted' 3×3–LS with augmented matrix $C1$. Now `RREF(C1)` succeeds and allows to read off the solution (set) $D1$ of the original system $A1$. – The same is done in the (h)omogeneous case.

$$B_1 = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & -8 & 0 \end{bmatrix} \qquad\qquad B_{1h} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -8 & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & -d+1 \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D_{1h} = \begin{bmatrix} 1 & 0 & 0 & -d \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We get: `RREF(C1)=` and `RREF(C1h)=`.

From these final statui we read off the solution (sets):[13]

$$soli : X = (1,0,0) + d(-1,1,0)$$
$$solh : X = (0,0,0) + d(-1,1,0) = d(-1,1,0)$$

Figure $F3$ belongs to system $A1$. The solution sets are straight lines.
▷   *Click here to run this script.*

Step **2** : We discuss the 2nd system $A2$.
——————— EIGENMATH ———————

```
run("gjBox.txt")
-- LS2: {x-y-z=1,    x+y+4z=-1,    7x+6y-z=7}
A2 = ((1,-1,-1, 1), (1,1,4, -1), (7,6,-1, 7)) -- inhomog. system
B2 = RREF(A2)  --ok
B2

A2h = ((1,-1,-1, 0), (1,1,4, 0), (7,6,-1, 0)) -- homog. system
B2h = RREF(A2h)
B2h
```

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{39}{53} \\ 0 & 1 & 0 & \frac{12}{53} \\ 0 & 0 & 1 & -\frac{26}{53} \end{bmatrix} \qquad\qquad B_{2h} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Output: $RREF(A2) \rightsquigarrow$ and $RREF(A2h) \rightsquigarrow$

---

[13]Lexicon: $soli = \underline{sol}$ution set of $\underline{i}$nhomogenous linear system; $solh = ..$ of $\underline{h}$omogeneous LS.

We read off the single solution point from the 4th column of the final status:

$$\begin{aligned} soli : X &= (39/53, 12/53, -26/53) \\ solh : X &= (0,0,0) \end{aligned}$$

Figure $F4$ belongs to system $A2$. The solution sets are single points as cut of 3 planes in 'general' position.   ▷   *Click here to run the script.*

Step **3** : We discuss the 3rd system $A3$.
——————— EIGENMATH ———————

```
run("gjBox.txt")
-- LS3: {x+y+2z=1, 0.5x+0.5y+z=0.5,  -2x-2y-4z=2}
A3 = ((1,1,2, 1), (1/2,1/2,1, 1/2), (-2,-2,-4, 2))
B3 = doGi(2,3,doGm(1,A3))   -- thereafter Gi ..
B3

A3h = ((1,1,2, 0), (1/2,1/2,1, 0), (-2,-2,-4, 0))
B3h = doGi(2,3,doGm(1,A3h))
B3h
C3h = ((1,1,2,0), (0,1,0,a), (0,0,1,b)) -- 2 dummy rows
D3h = RREF(C3h)
D3h
```

Output B_3 is reached after an selective elimination in column 1 and a follow-up switch of row 2 and 3. The GAUSS-JORDAN algorithm detects a hidden zero row and a contradiction in row 2: $0 = 4$. Therefore the inhomogeneous system has an empty solution set, but the homogeneous one a full plane:[14] :

$$B_{3h} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Output: $Gi..Gm..(A3) \rightsquigarrow$ $B_3 = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $RREF(C3h) \rightsquigarrow$ $D_{3h} = \begin{bmatrix} 1 & 0 & 0 & -a-2b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \end{bmatrix}$

Therefore we have:

$$\begin{aligned} soli : X &= (\,) \\ solh : X &= (0,0,0) + r(-1,1,0) + s(-2,0,1) = r(-1,1,0) + s(-2,0,1) \end{aligned}$$

Figure $F2$ belongs to this cut of 3 planes. Use ▷CALCPLOT3D to check it!
▷   *Click here to run the script.*

———————————————————

[14]... there is no contradiction anymore in the homog. system: $0=0$ is true!

Step **4** : We discuss the 4th system $A4$.
────────────── Eigenmath ──────────────

```
run("gjBox.txt")
-- LS4: {x-y-z=1,     x+y+4z=-1,   3x+y+7z=3}
A4 =  ((1,-1,-1, 1), (1,1,4, -1), (3,1,7, 3))
B4 = Ge(A4)
B4

A4h = ((1,-1,-1, 0), (1,1,4, 0), (3,1,7, 0))
B4h = doGu(2,Ge(A4h))
B4h
C4h = ((1,0,3/2,0), (0,-2,-5,0), (0,0,1,a)) -- dummy = a
D4h = RREF(C4h)
D4h
```

`RREF(..)` succeeds only for the lifted homogeneous case $C4h$, returning a parametrized
solution set – a straight line as intersection of 3 planes. In the inhomogeneous case a
Gauss-Jordan step detects a contradiction '0=4' leading to an empty solution: Output:

$$
Ge(A4) \rightsquigarrow \quad B_4 = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 5 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad ..Ge(A4h) \rightsquigarrow \quad B_{4h} = \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ... \; RREF(C4h) \rightsquigarrow \quad D_{4h} = \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2}a \\ 0 & 1 & 0 & -\frac{5}{2}a \\ 0 & 0 & 1 & a \end{bmatrix}
$$

Therefore

$$
\begin{aligned}
soli : X &= \;() \\
solh : X &= \;(0,0,0) + r(-3/2, -5/2, 1) = r(-3/2, -5/2, 1)
\end{aligned}
$$

Figure $F1$ belongs to this cut of 3 planes. Use ▷CalcPlot3D. to check it!
▷ *Click here to run the script.*

Step **5** : We discuss the 5st system $A5$.
────────────── Eigenmath ──────────────

```
run("gjBox.txt")
-- LS5:{2x+2y+4z=2, x+y+2z=1}
A5 = ((2,2,4, 2), (1,1,2, 1))
B5 = Ge(A5)
B5
C5 = ((2,2,4, 2), (0,1,0,a), (0,0,1,b)) -- 2 dummy rows
D5 = RREF(C5)
D5

A5h = ((2, 2, 4, 0), (1,1,2, 0))
B5h = Ge(A5h)
B5h
```

```
C5h = ((2,2,4, 0), (0,1,0, a), (0,0,1, b)) -- 2 dummy rows
D5h = RREF(C5h)
D5h
```

In the inhomogeneous case RREF(..) succeeds not until we lift the intermediate status B_5 to D_5, then it returns a 2–parametrized solution set: a plane as intersection of 2 planes, i.e. both planes must be identical.

Output: $Ge(A5) \rightsquigarrow$ $B_5 = \begin{bmatrix} 2 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, ergo $RREF(C5) \rightsquigarrow$ $D_5 = \begin{bmatrix} 1 & 0 & 0 & -a-2b+1 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \end{bmatrix}$.

In the homogeneous case the same system $B_5 = B_{5h}$ returns the same 2-parameter part, but the 'special' point is now the origin O.

Output: $Ge(A5h) \rightsquigarrow$ $B_{5h} = \begin{bmatrix} 2 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, ergo $RREF(C5h) \rightsquigarrow$ $D_{5h} = \begin{bmatrix} 1 & 0 & 0 & -a-2b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \end{bmatrix}$.

Therefore

$$\begin{aligned} soli : X &= (1,0,0) + r(-1,1,0) + s(-2,0,1) \\ solh : X &= (0,0,0) + r(-1,1,0) + s(-2,0,1) = r(-1,1,0) + s(-2,0,1) \end{aligned}$$

Figure $F5$ belongs to this 'intersection' of 2 identical planes. Use *CalcPlot3D* to check it!
▷ *Click here to run the script.*

Behind our observations during this investigation there is the following mathematical

Fact:

> The solution set of a *inhomogeneous* LS $A * X = B \neq 0$ consist of all solutions of the associated *homogeneous* LS $A * X = O$, added to a *special* solution of the inhomogeneous.

In short:

$$\begin{aligned} \mathsf{Sol}(A * X = B) &= \mathsf{P} + \mathsf{Sol}(A * X = 0) \\ \mathsf{Sol}(A * X = B) &= \mathsf{P} + \mathsf{kern}(A) \\ "\mathsf{Soli} &= \mathsf{solp} + \mathsf{Solh}" \end{aligned}$$

○ Obviously the solution set of an homogeneous system $A * X = O$ is called the *kernel* of the system matrix A, i.e.

$$\mathsf{kern}(A) \stackrel{def}{=} \{X \in \mathbb{R}^3 : A * X = O\}$$

9   The scalar product

We start with the exploration of a geometric phenomenon, that leads naturally to a working definition of the so-called *scalar product*. We use our knowledge about in/homogeneous linear systems.

P71. Homogeneous linear systems II.
We make a first observation, which has a geometric consequence. We know that the solutions of the underdetermined inhomogeneous 1×3–LS $1x - 2y = 3$ form a *straight line* ℓ in the xy–plane $\mathbb{R}^2$. In this geometric context the (single) row of the system matrix $A = [1, -2]$ is called the *normal vector* of the line ℓ, noted $N = [1, -2]$.
The solution set ℓ can be described with *a free parameter* t as follows.

$$
\begin{aligned}
[x, y] \in \ell \quad &\Leftrightarrow \quad [x = t, y = 0.5(t - 3)] \\
&\Leftrightarrow \quad [x, y] = [t, 0.5 \cdot (t - 3)] \\
&\Leftrightarrow \quad [x, y] = [t, 0.5t - 1.5] \\
&\Leftrightarrow \quad [x, y] = [0, -1.5] + t[1, 0.5] \\
&\Leftrightarrow \quad \mathbf{X = P + t \cdot U} \\
&\qquad \text{with} \quad P := [0, -3] \text{ and } \quad U := [1, 0.5]
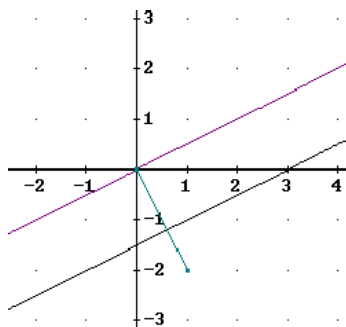\end{aligned}
$$

Figure 7:
Green: the tip N of the vector $\vec{ON}$ has coordinates $[1, -2]$.
Magenta: the direction vector ("kernel") $U = [1, \frac{1}{2}]$ of line ℓ.
Black: the solution set ℓ of the inhomogeneous linear system.

a. Explore Figure.7 and describe your observation in feedback with the term $\ell \colon [x, y] = [0, -1.5] + t \cdot [1, 0.5] = P + tU$. Compare $U = [1, 0.5]$ with the normal $N = [1, -2]$.

b. What is the relationship between the **basic** solution[15] U and the equation's system matrix $A = [1, -2]$ of $A * X = B$? How to describe this relationship *algebraically* resp. *geometrically*?

---

[15]Remember, that the **basic solution** U is per definionem the solution of the *homogeneous* LS $A*X = O$, where you exchanged the 'old' right side B of the "inhomogeneous" LS $A * X = B$ through the zero column $O = [0, 0, ..., 0]$.

P72. Homogeneous linear systems III.
The solutions of the 1×3–linear system

$$1x + 1y + 2z \;=\; 1$$

fill a *plane E* in space, because

$$
\begin{aligned}
[x, y, z] \in E \;\;&\Leftrightarrow\;\; 1x + 1y + 2z = 1 \\
&\Leftrightarrow\;\; [x, y, z] = [1 - 2s, -r, s] \\
&\Leftrightarrow\;\; [x, y, z] = [1, 0, 0] + r[-1, 1, 0] + s[-2, 0, 1] \\
&\Leftrightarrow\;\; \mathbf{X = P + t \cdot U + s \cdot V} \\
&\quad \text{with} \quad P = [1, 0, 0] \quad \text{and} \quad U = [-1, 1, 0], V = [-2, 0, 1]
\end{aligned}
$$

The set of solutions E is given in parametric form $X = P + tU + sV$ with *two free parameters* r and s. These *two* significantly different *basis solutions* $U = [-1, 1, 0]$ and $V = [-2, 0, 1]$ point in two different geometrically **basic** *directions* in the solution plane, i.e. $\{U, V\}$ are linearly independent.
Consider:
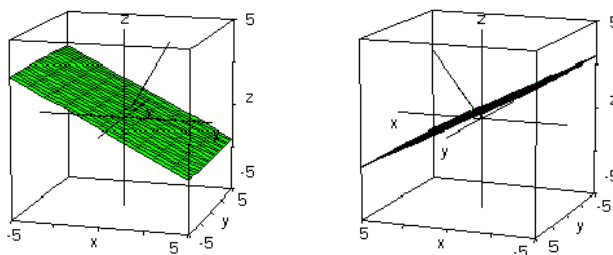
Figure 8: The figure shows the same graphical scene from two viewpoints.
Green: the solution set E is spanned by the two linearly indepen-
dent vectors $U = [-1, 1, 0]$ and $V = [-2, 0, 1]$.
Black: the vector $N = [1, 1, 2]$ (alias system matrix A) pointing
from origin O to "1 o'clock p.m." in the right figure.

a. The figure is not optimal plotted. Reconstruct it with CalcPlot3D and watch the geometric object by turning the plot with the mouse in all directions.

b. Describe your observations.

c. What is the relationship between $U,\ V$ and the normal $N{=}A$ of $A * X = B$?

   How to describe this relationship *algebraically* or *geometrically*?

P73. Homogeneous linear systems IV.

The solutions of the LS

$$x + y + 2z = 1$$
$$2x + 2y - 4z = 2$$

fill a *line* in space. The set of solutions can be described by using *a free parameter* $y = t$:

$$[x, y, z] \in \ell \iff [x = 1 - y, y, z = 0]$$
$$\iff [x, y, z] = [1 - t, t, 0]$$
$$\iff [x, y, z] = [1, 0, 0] + t[-1, 1, 0]$$
$$\iff \mathbf{X} = \mathbf{P} + \mathbf{t} \cdot \mathbf{U}$$
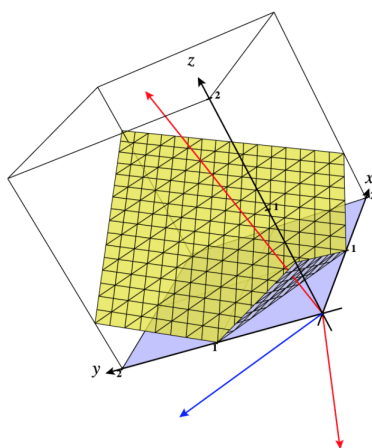$$\text{with} \quad P := [1, 0, 0] \quad and \quad U := [-1, 1, 0]$$

Consider:

Figure 9: Red: the row vectors $A1 = [1, 1, 2]$ and $A2 = [2, 2, -4]$ of the system matrix A, also called the plane's normal vectors $N1$ resp. $N2$.
Blue: the basis vector $U = [-1, 1, 0]$ of the kernel of A.

a. The figure was created with CalcPlot3D. Reconstruct it.

b. In this geometric setting each row $A1$ resp. $A2$ of the system matrix A represents a plane in space. The coordinates of each row constitutes in a geometric phrasing the normal vectors $N1$ resp. $N2$ of the respectively planes.
Do you see a relationship between the direction vector[16] $\mathbf{U}$ of line ℓ and the normal vectors N_i of the planes?

c. How can we describe this relationship *algebraically* or *geometrically*?

---

[16]i.e. the basis vector of the kern(A), i.e. the line ℓ translated to the origin O. Not plotted here.

P74. Homogeneous linear systems V.

Now study the following terms. What do the factors represent? How could one interpret
the result? Search for connections. Hypothesis?

───────────── Eigenmath ─────────────

```
-- P71
A = ((1, -2))
U = (1,0.5)
do(A, U, dot(A, U) )    --(1)

-- P72
A = ((1,1,2), (2,2, -4))
U = (- 1,1,0)
A
U
dot(A,U)    --(2) A*U

-- P73
A = ((2,2,4), (1,1,2))
U = ((1, -2), (-1,0), (0,1))
do(A, U, dot(A, U))    --(3)  A*U
```

Output: (1): 0 (2): $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (3): $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
▷   *Click here to run the script.*

9.0.1   A geometric–algebraic view on homogeneous linear systems.

- *Algebraic observation* in problems *P71* ff.:

$$\begin{bmatrix} 1 & -2 \end{bmatrix} * \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \;=\; \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & -4 \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} * \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \;=\; \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{A} \quad * \quad \mathbf{U} \;=\; \mathbf{O}$$

Our observations show an algebraic relationship:

> *The matrix product of the row vectors of the system matrix A with the* basic solution
> *columns U always results in a zero matrix:*
>
> $$A * U = O$$

• *Geometric observation:* What does the relation $A * U = O$ between the row vectors of the system matrix A and the basic solution columns means geometrically?
Looking back at *P72* we plot the scene again with the same data, but without the plane. In figure.10 the 2 red vectors U, V span the plane E [17] and N.
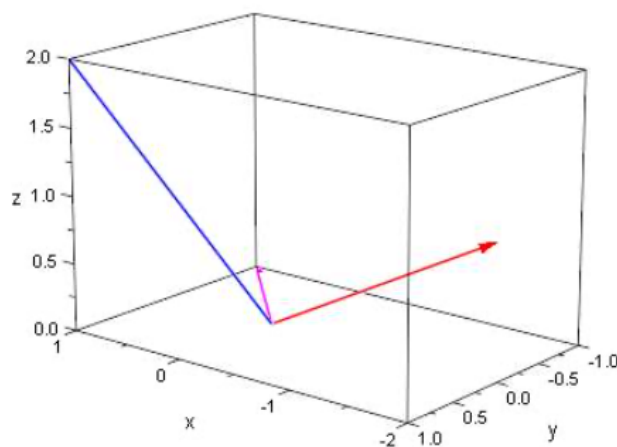Tip: a closer view on *P72*:

Figure 10:
Magenta: the 3D vector $U1$ from O to $[1, -1, 0]$.
Red: the 3D vector $U2$ from O to $[-2, 0, 1]$.
Blue: the 3D vector N from O to $[1, 1, 2]$, the normal vector of E.

Our observations show also an *geometric* relationship:

> *The normal vector N is always* perpendicular *to both span vectors U, V of a plane E.*

N ⊥ U and N ⊥ V

Remark.

◦ From both facts we conclude, that we can check the orthogonality alias perpendicularity with the nullity of a matrix product. In a more theoretic oriented mathematical text a proof would be given. We content ourselves here with this strong evidence.

---

[17]think of U and V as edges of a parallelogram i.e. a small piece of plane E

9.1   The normal form of a plane

The solution set of an underdetermined 1×3 - linear system like

$$1x + 2y + 3z = 4$$

geometrically forms a plane E in the viewing space. Such an equation is called an *implicit representation* of the plane, because the solution set is in its geometric-vector (parametric) form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + s \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

as $E : X = P + t \cdot U + s \cdot V$ has yet to be determined. This vector form allows the well-known geometric interpretation of the solution set and is therefore also called an *explicit representation* of the plane E. Now we have gained a second geometric interpretation:

• *The coefficients $1, 2, 3$ in the implicit representation of the plane $E : 1x + 2y + 3z = 4$ are geometrically nothing but the coordinates of the normal vector $N(\perp E)$.*

9.1.1   Visualization of a plane E with its normal vector N

We denote the coefficients (i.e. the system matrix A) of the plane $E : 1x + 2y + 3z = 4$ for the further investigation with $N := [1, 2, 3]$ and call it the *normal vector* of the plane:
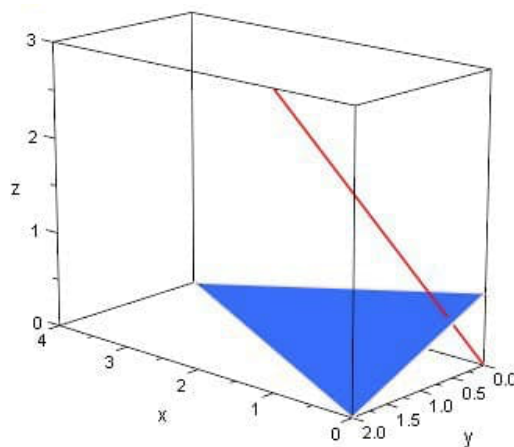

Figure 11: Red: the 3D normal vector N from O to $[1, 2, 3]$ is orthogonal to E.
Blue: the plane $E : 1x + 2y + 3z = 4$

Exercise: Reconstruct Figure.11 with `CalcPlot3D`. Use both the Select ▷ Implicit surface entry as well as the Parametric surface entry to learn to distinguish between both representations of E.

P75. Planes and its corresponding normal vectors.
Examine the geometric position of the normal vector to the plane for the following planes and visualize the plane and the corresponding normals appropriately.

a. $E_1 : x + y + z = 5$

b. $E_2 : x - 2y - 5z = 0$

c. $E_3 : y = 0$

d. Now interpret E_1, E_2 and E_3 by omitting the z component as the solution set of an underdetermined 1×2 – linear system in $\mathbb{R}^2$ and investigate the geometric situation. Is there a normal vector even then? Plot.

9.1.2 Normal form of a plane

Here a plane E is given as a coordinate equation

$$E : 1x + 2y + 3z = 4$$

The following reshaping shows how the so-called *normal form* (NF) of this plane can be calculated:

$$
\begin{aligned}
X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E \quad &\Leftrightarrow \quad 1x + 2y + 3z = 4 \\
&\Leftrightarrow \quad [1,2,3] * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 4 \\
&\Leftrightarrow \quad [1,2,3] * \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 4 = 0 \\
&\Leftrightarrow \quad [1,2,3] * \begin{bmatrix} x \\ y \\ z \end{bmatrix} - [1,2,3] * \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 0 \\
&\Leftrightarrow \quad [1,2,3] * \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right) = 0 \\
&\Leftrightarrow \quad \mathbf{N} * (\mathbf{X} - \mathbf{P}) = \mathbf{0} \qquad \text{(NF)}
\end{aligned}
$$

The description of a plane E by a vector equation of the form

$$\color{red}\mathbf{N} * (\mathbf{X} - \mathbf{P}) = \mathbf{0}$$

is called a *normal form of the plane E with position vector $P \in E$ and normal vector N.*

Notes:

1. In the equation (NF) the $*$ denotes the matrix multiplication of two *matrices*; in this context, this special case of matrix multiplication is often called the *scalar product* $\bullet$ of two *vectors* and then we write instead of

$$[1,2,3] * \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right) = 0$$

now

$$[1, 2, 3] \bullet ([1, 2, 3] - [0, 2, 0]) = 0$$

or

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right) = 0$$

More about the arithmetic operation *scalar product* $\bullet$ between two *vectors* below.

2. The normal form of the plane E also allows a geometric and algebraic characterization of all points X in space, that lie on the plane E:

A point X lies in the plane E ($X \in E$) if and only if

○   the vector $X - P = \overrightarrow{PX}$ from P to point X is perpendicular to N, i.e. $(X - P) \perp N$

○   the scalar product of $\overrightarrow{PX}$ with the normal vector N is zero, i.e. $(X - P) \bullet N = 0$

P76. Forms of representation of a plane in space.

a. *Normal form ▷ coordinate equation:* A plane through $P = [4, 1, 3]$ has the normal vector $N = [2, -1, 5]$. Give an equation of the plane in normal form and determine the implicit coordinate equation for E.

b. *parameter form ▷ normal form:* The plane $E : X = [5, 2, 3] + r[1, 0, 2] + s[0, -5, 8]$ is to be specified in normal and implicit form.

c. *normal form ▷ parameter form:* The plane $E : (X - [1, -1, 2]) \bullet [2, 3, 4] = 0$ is to be specified in parameter form.

d. Draw the plane by hand with the help of the track points.

9.1.3   Summary: geometric view on homogeneous linear system

Our investigation of the relationships between the basic solutions U of a homogeneous linear system $A * X = O$ and its system matrix A resulted in:

1. The **basic** solution columns U are perpendicular to the **row** vectors of A.

2. The **base** directions U ”span” the kernel of the solution set.

3. *Algebraically* we have $A * U = O$.
   Geometrically we say 'A *is perpendicular to* U' and define symbolically:

$$A \perp U \quad \overset{def}{\Longleftrightarrow} \quad A * U = O$$

4. In particular: *the normal vector N of a plane E is perpendicular to this plane*: $N \perp E$.

9.2   The scalar product

In the special case of the matrix product of a *single-row* matrix X with a *one-column* matrix Y such as $[2\ 2\ 4] * \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = [0]$ one also writes $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$ and speaks of the $\bullet$ *dot product* of the two *vectors* $X = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$:

$$X \bullet Y \quad \overset{def}{\Longleftrightarrow} \quad X^t * Y$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

dot product   matrix product

Notes:

○ We have the following explicit expression for the scalar product of two vectors in $\mathbb{R}^3$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \overset{def}{=} [a_1, a_2, a_3] * \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

○ To distinguish it from the matrix product, this *scalar product of two vectors* is denoted with a thick point $\bullet$, therefore the name '*dot*' product. In the end, however, one calculates with the corresponding single–row/single–column matrices: X^t denotes the transpose of X. In this algebraically respect, this is nothing new.

○ The EIGENMATH dot(.) function is programmed for use in both cases: it works as realization of the matrix product '$*$' as well as realization of the scalar product $\bullet$.

Lexicon:

Math	EIGENMATH
$A * B$	dot(A,B)
$X \bullet Y$	dot(X,Y)
$\overrightarrow{PX} \perp N$	dot(X-P,N)=0

○ Fact: *two vectors U and V are perpendicular to each other, if its scalar product is 0.*

○ With the scalar product we can express *geometrical situations* like the *vertical position* of geometrical objects by an *algebraic relation* $\cdots \bullet \cdots = 0$. We will explore this in more detail in the following tasks.

○ How to extend the vertical position as 90° to a general angle definition is developed below.

● Investigate as exercise: which algebraic properties has the scalar product $\bullet$ of two vectors, i.e. think about commutativity, associativity, distributivity etc.

Here are two prominent applications of the scalar product.

P77. Distance of a point to a straight line.   In the following express the calculations using the ● resp. EIGENMATH's dot() function.

a. Calculate the distance from the point $P(3;4)$ to the origin $O(0;0)$.

b. Calculate the distance from the point $P(3;1)$ to the point $Q(2;4)$.

c. Show: the *distance* of a point $P(x,y)$ to the origin $O(0,0)$ is

$$dist(O,P) = \sqrt{x^2 + y^2}$$

Express this definition in EIGENMATH.

d. Show: the distance between two points $P(x,y)$ and $Q(v,w)$ is

$$dist(P,Q) = \sqrt{(x-v)^2 + (y-w)^2}$$

Express this definition in EIGENMATH.[18]

e. What is the distance from P to Q in

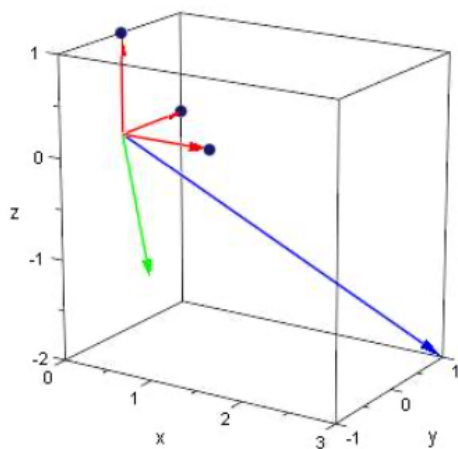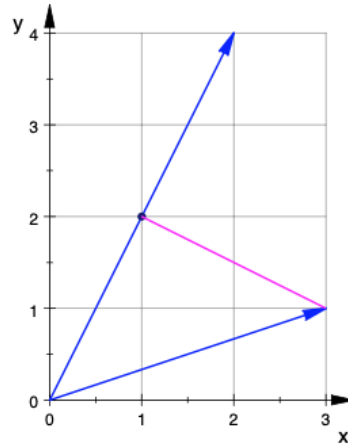

Red: the coordinate systems unit vectors.
Figure 12: Blue: the vector $OP = [3,1,-2]$.
Green: the vector $OQ = [1,-1,-1]$.

---

[18]We should not use the name 'd' for distance in EIGENMATH, because 'd' is reserved for the *derivative* operator.

f. What is the distance between the point $P(3; 1)$ and the line with equation $\ell : y = 2x$?

○ Use `CalcPlot3D` to check the results!

Solution:

─────────── EIGENMATH ───────────

```
-- c. and d.
dist(P,Q)=sqrt(dot(P-Q,P-Q))
dist(P-Q)
-- a.
P=(0,0)
Q=(3,4)
dist(P,Q)
-- b.
P=(3,1)
Q=(2,4)
dist(P,Q)
float(last)   -- its decimal value
-- e. later in Part 4 ;)
```

▷   *Click here to run the script.*

This motivates the following definition.

9.3   Length and distance

- For two vectors $A = [A_1, A_2, ..]$ and $B = [B_1, B_2, ..]$ the real number

$$|\mathbf{A}| \stackrel{def}{=} \sqrt{A_1^2 + A_2^2 + \ldots} = \sqrt{A \bullet A}$$

  is called the *length of the vector* A or the *2–norm* [19] of the vector A.

- The real number

$$\mathbf{dist(A, B)} \stackrel{def}{=} |A - B| = \sqrt{(A_1 - B_1)^2 + (A_2 - B_2)^2 + \ldots}$$
$$= \sqrt{(A - B) \bullet (A - B)}$$

  is the *distance* between A and B or the length of the *direction vector* $\overrightarrow{AB} \stackrel{def}{=} B - A$.

<div align="center">

Lexicon:

Math	Eigenmath
$|A|$	`abs(A)`
$dist(A, B)$	`abs(A-B)`

</div>

P78. Distance calculations.
Calculate the distances from the corner points to the zero point and the length of the sides in the triangle ABC with the points with and without Eigenmath.

a. $A = (1; 1), B = (4; 1), C = (0; 3)$

b. $A = (1; 1; 2), B = (4; 1; 0), C = (0; 3; 3)$

c. Calculate $|\overrightarrow{A_1U_1}|$, dist(U1,U2), $|A1|$ in this CAS MuPAD plot:

```
A1 := Line3d ( [0,0,0], [1,1,2], Color = RGB::Blue):
U1 := Arrow3d( [0,0,0],[1,-1,0], Color = RGB::Magenta):
U2 := Arrow3d( [0,0,0],[-2,0,1], Color = RGB::Red):

plot( A1,U1,U2, Scaling=Constrained );
```
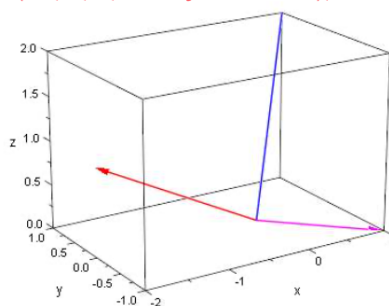

○   Use `CalcPlot3D` to check the results!

---

[19]One also speaks of a *norm,* as this is a scale for the *normed* length measurement. One writes often '2–norm', because of the square root of the sum of squares $A^2 + ...$.

P79. HESSE**'s distance formula.**[20]
Let $P = [x_0, y_0]$ be a point of the plane $\mathbb{R}^2$ and $\ell\colon ax + by = 0$ a straigth line with $[a, b] \neq [0, 0]$.

a. Show that the distance from P to ℓ is

$$dist(P, g) \overset{def}{=} \frac{|ax_o + by_o|}{\sqrt{a^2 + b^2}}$$

b. Program HESSE's distance formula in EIGENMATH.

c. Calculate the distance from $[3, 1]$ to the straight line $y = 2x$ using the HESSE formula w/o EIGENMATH. See *P77.f*.

d. Derive an analogous formula for the general straight line $\ell\colon ax + by + c = 0$ and use it to calculate the distance from $[3, 1]$ to the straight line $y = 2x + 1$.

e. Derive an analogous formula for a plane $E : (X - P) \bullet N = 0$ in space and use this to calculate the distance of $R = [1, 6, 2]$ to plane $E : x - 2y + 4z = 1$.
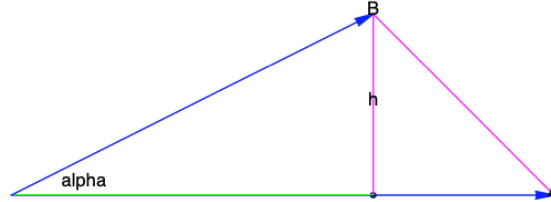
   Program your formula in EIGENMATH.

○ Make a quality plot of the results.

---

[20]Otto HESSE, German mathematician.

9.4   Angle between two vectors

> With the help of the scalar product • and the length derived from it one can define an *angle measure.*

Consider the triangle with the corner points OAB and the edge vectors $\overrightarrow{OA} = A - O$, $\overrightarrow{OB} = B - O$ and $\overrightarrow{AB} = B - A$ and $\alpha \stackrel{def}{=} \angle AOB$:

In the following four steps we derive a formula for calculating the angle between two vectors.

Step 1.

Here is a knowledge count-down:

$$
\begin{align}
h &= |B| \cdot \sin(\alpha) \tag{9.1} \\
h^2 &= |B|^2 \cdot (1 - \cos^2(\alpha)) \tag{9.2} \\
h^2 &= |A - B|^2 \cdot (|A| - |B| \cdot \cos^2(\alpha)) \tag{9.3} \\
|B - A|^2 &= |A|^2 + |B|^2 - 2|A| \cdot |B| \cdot \cos(\alpha) \tag{9.4} \\
|B - A|^2 - |A|^2 - |B|^2 &= -2|A| \cdot |B| \cdot \cos(\alpha) \tag{9.5}
\end{align}
$$

The statement (9.4) is well-known as the *Cosinus–theorem.*

Step 2.

Now consider the following general calculation, which gives the length of side AB in a different, algebraic way. We let EIGENMATH help us:

```
-- derivation of the angle formula

A = (a1, a2)
B = (b1, b2)

dot(A-B, A-B)    -- = |A-B|^2
dot(A, A)        -- = |A|^2
dot(B, B)        -- = |B|^2

--    |A-B|^2 - |A|^2   - |B|^2
dot(A-B,A-B) - dot(A,A) - dot(B,B)    -- (9.5)
```

$$a_1^2 - 2\,a_1\,b_1 + a_2^2 - 2\,a_2\,b_2 + b_1^2 + b_2^2$$

$$a_1^2 + a_2^2$$

$$b_1^2 + b_2^2$$

$$-2\,a_1\,b_1 - 2\,a_2\,b_2$$

Step 3.

Equating both formulae (9.5) we get $a_1 \cdot b_1 + a_2 \cdot b_2 = |A| \cdot |B| \cdot \cos(\alpha)$, i.e. p.def. $A \bullet B = |A| \cdot |B| \cdot \cos(\alpha)$ and therefore arrive at *the angle formula*:

$$\cos(\alpha) = \frac{A \bullet B}{|A| \cdot |B|} \qquad \text{with } \alpha = \measuredangle AOB$$

○ Or explicit:

$$\alpha = \arccos\left(\frac{A \bullet B}{|A| \cdot |B|}\right)$$

○ In EIGENMATH :

```
Alpha(A,B) = arccos( dot(A,B) / sqrt(dot(A,A))
                               / sqrt(dot(B,B)) )
```

P80. Angle of intersection.

a. Redo the derivation using the triangle

$$OAB = [[0,0,0],[1,2,3],[-2,1,2]]$$

and calculate the angle $\varphi = \measuredangle AOB$.

b. Calculate the lengths of the sides and the interior angles in the triangle $ABC = [[4,1,4],[-2,3,5],[-2,4,3]]$.

c. What could we understand under the *intersection angel* of two straight lines g and h?

Which intersection angle φ form the straight lines $\ell\colon X = [2,1,-1] + r[1,3,2]$ and $k\colon X = [5,3,0] + s[-2,1,1]$ with each other?

d. We define the *intersection angle* of the planes E_1 and E_2 as the angle between their normal vectors N_1 and N_2, i.e.

$$\measuredangle(E_1, E_2) \stackrel{def}{=} \measuredangle(N_1, N_2)$$

Which intersection angle φ is between the planes $E_1\colon 3x - 4y + 5z = 10$ and $E_2\colon 3x + 2y - z - 4 = 0$?

e. Determine the angle between U1 and U2 in *P78*.

$$\bowtie$$

To prepare a collection of formulas for explicit calculations and decisions of case distinctions (e.g. questions of linear in/dependence), we need two more product formations: the cross and the triple ("spar") product.

9.5   The cross product

The normal vector of a plane is perpendicular to E and can be read from the co-efficient matrix of its normal form. How can we calculate the normal vector N, if the plane is explicitly given in parameter representation – without converting to the normal form?

9.5.1   A deductive determination

We specify the question a bit more general. Let $A = [a1, a2, a3[$ and $B = [b1, b2, b3]$ be two vectors in space $\mathbb{R}^3$. Determine a vector $C = [c1, c2, c3]$, which is perpendicular to both A and B.   *Solution*: choose

$$C \stackrel{def}{=} [a2 \cdot b3 - a3 \cdot b2, a3 \cdot b1 - a1 \cdot b3, a1 \cdot b2 - a3 \cdot b1]$$

and call this defining term the *cross product* term for A and B, noted $C = A \times B$. We verify the conditions using EIGENMATH:

```
-- to the CROSS product

A=(a1,a2,a3)
B=(b1,b2,b3)

C=(a2*b3-a3*b2, a3*b1-a1*b3, a1*b2-a2*b1)
C

dot(A,C)
dot(B,C)
```

$$C = \begin{bmatrix} a_2\,b_3 - a_3\,b_2 \\ -a_1\,b_3 + a_3\,b_1 \\ a_1\,b_2 - a_2\,b_1 \end{bmatrix}$$

$$0$$

$$0$$

Because of the last 2 lines we have $A \bullet C = B \bullet C = 0$, therefore $C \perp A$ and $C \perp B$, q.e.d. Of course, EIGENMATH knows about the cross product:

```
A=(a1,a2,a3)
B=(b1,b2,b3)

cross(A,B)
```

.. giving back the same term C. But: how did you come up with this term? Look here ...

9.5.2   A constructive determination

The 2 conditions $C \perp A$ and $C \perp B$ lead to an underdetermined 2×3 linear system for the unknown cross vector C, i.e. (1), (2) and (3)::

```
A=(a1,a2,a3)        -- 1st vector
B=(b1,b2,b3)        -- 2nd vector
C=(c1,c2,c3)        -- unkown orthogonal vector 'cross'

-- a1*c1+a2*c2+a3*c3=0       (1) A |_ C orthogonal
-- b1*c1+b2*c2+b3*c3=0       (2) B |_ C
```

```
--      c1 c2 c3
C0 = ((a1,a2,a3, 0),      -- (3) underdetermined LS for c1,c2,c3
      (b1,b2,b3, 0))


C01 = ((a1,a2,a3, 0),
       (b1,b2,b3, 0),
       ( 0, 0, 1, c3))    -- (4) lift up to 3x3


-- how to choose c3? We are free ..

C1 = ((a1,a2,a3),         -- (5) system matrix of lifted system
      (b1,b2,b3),
      ( 0, 0, 1))


Sol = dot(inv(C1), (0,0,c3))          -- (6) general solution to (4)
Sol


Sol1=dot(inv(C1), (0,0,a1*b2-a2*b1))  --(7) special solution for c3=a1*b2-a2*b1
Sol1


-- terms equal?
a1*a2*b2*b3 - a1*a3*b2*b2 - a2*a2*b1*b3 + a2*a3*b1*b2  --(8)
(a2*b3-a3*b2)*(a1*b2-a2*b1)                            --(9)
```

▷   *Click here to run the script.*

Using the 3rd coordinate c_3 as free variable, we lift the system up to an 3×3 system. This is solvable using the inverse matrix (5). The output (6) gives an indication of how one should choose the free variable c_3:

$$S_{ol} = \begin{bmatrix} \dfrac{a_2\,b_3\,c_3}{a_1\,b_2\,-\,a_2\,b_1} - \dfrac{a_3\,b_2\,c_3}{a_1\,b_2\,-\,a_2\,b_1} \\[2ex] -\dfrac{a_1\,b_3\,c_3}{a_1\,b_2\,-\,a_2\,b_1} + \dfrac{a_3\,b_1\,c_3}{a_1\,b_2\,-\,a_2\,b_1} \\[2ex] c_3 \end{bmatrix}$$

In (7) we choose $c_3 \overset{def}{=} a_1 b_2 - a_2 b_1$, i.e. the common denominator in S_{ol}. This gives the output:

$$S_{ol1} = \begin{bmatrix} \dfrac{a_1\,a_2\,b_2\,b_3}{a_1\,b_2\,-\,a_2\,b_1} - \dfrac{a_1\,a_3\,b_2^2}{a_1\,b_2\,-\,a_2\,b_1} - \dfrac{a_2^2\,b_1\,b_3}{a_1\,b_2\,-\,a_2\,b_1} + \dfrac{a_2\,a_3\,b_1\,b_2}{a_1\,b_2\,-\,a_2\,b_1} \\[2ex] -a_1\,b_3 + a_3\,b_1 \\[2ex] a_1\,b_2 - a_2\,b_1 \end{bmatrix}$$

Now, in (8) and (9), we let EIGENMATH verify, that the nominator $a_1 a_2 b_2 b_3 - a_1 a_3 b_2 b_2 - a_2 a_2 b_1 b_3 + a_2 a_3 b_1 b_2$ is equal to $(a_2 b_3 - a_3 b_2) \cdot (a_1 b_2 - a_2 b_1)$, which is shortened to $a_2 b_3 - a_3 b_2$.
- q.e.d.

9.5.3   Example of using cross

in (9.1) we looked at the plane E given in explicit parametric form $E : X = P + rU + sV =$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + s \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

This plane had the implicit coordinate equation $E : 1x + 2y + 3z = 4$. Here we will reconstruct their normal vector using `cross()`.

```
A=(3, 0,-1)
B=(2,-1, 0)
C=(c1,c2,c3)    -- unknown cross vector

-- FIRST solution using RREF
C1 = ((3, 0,-1,    0),
      (2,-1, 0,    0),
      (0, 0, 1,   c3))

run("gjBox.txt")
R=RREF(C1)
R
sol1=subst(3,c3, R)    -- substitute 3 for free c3 in R
sol1

-- SECOND solution using inverse matrix
C2 = ((3, 0,-1),
      (2,-1, 0),
      (0, 0, 1))
Sol2 = dot(inv(C2), (0,0,3))
Sol2

-- THIRD solution using build-in cross product
sol3 = cross(A,B)
sol3
```

▷   *Click here to run the script.*
Result: in all cases the normal vector is $C = [1, 2, 3] = N$.

**Remark.** Because our main interest is to show the usefulness of EIGENMATH in Analytic Geometry, we do not discuss the mathematical properties of '×'. The orientation of $A \times B$ we assume known, too. Instead we will program important facts in the collection of formulae for the benefit of the user. The following facts about the cross product will therefore suffice.

1. *Fact*: (geometric interpretation) the cross product is as long as the area of the parallelogram with sides $A$ and $B$ is big, i.e. $|A \times B| = 2 \cdot \triangle(A, B)$.

2. *Fact*: $(A, B)$ is linearly dependent $\Longleftrightarrow A \times B = 0$, because then there is no area between $A$ and $B$.

## 9.6   The box product

"The *scalar triple product* (also called the mixed product, *box product*, or triple scalar product) is defined as the dot product of one of the vectors with the cross product of the other two. Geometrically, the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is the (signed) volume of the parallelepiped defined by the three vectors given. Here, the parentheses may be omitted without causing ambiguity, since the dot product cannot be evaluated first. If it were, it would leave the cross product of a scalar and a vector, which is not defined. [21]
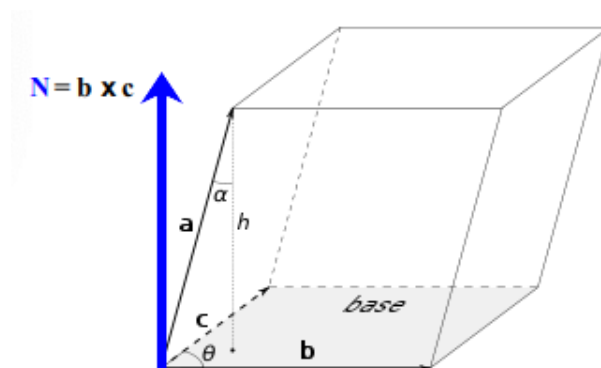


### 9.6.1   *Definition*: Box product

Let $A, B, C$ be tree vectors in space $\mathbb{R}^3$. Then we define the box[22] product

$$Box(A, B, C) \stackrel{def}{=} \mathbf{A} \bullet (\mathbf{B} \times \mathbf{C})$$

| | *Math* | Eigenmath |
|---|---|---|
| Box(A,B,C): | $A \bullet (B \times C)$ | `dot(A, cross(B,C))` |

### 9.6.2   *Example*: Box product

In the Figure above, let $a = [1, 1, 4], b = [1, 0, 0], c = [1, 3, 0]$.
Calculate Box(a,b,c) w/o Eigenmath.
*Solution*:

```
Box(A,B,C) = dot(A, cross(B,C))          -- definition Box product

do( a=(1,1,4), b=(1,0,0), c=(1,3,0) )    -- data
Box(a,b,c)
```

Result: 12, which is the volume of the parallelepiped spanned by a, b and c.

---

[21] Figure adapted from wikipedia, `https://en.wikipedia.org/wiki/Triple_product`
[22] We prefer the name 'box', because it invokes the typical picture of a 'feldspare' in your head. In German speaking counties it is therefore often called 'spat product'.

9.7   Active Summary: the theoretical minimum.

① **An under-determined linear system** $A * X = B$

  – y/n?

  – □□ has an inverse of its matrix of coefficients.

  – □□ has more equations than unknowns.

  – □□ has a solution set, which is a line or plane or ...

② **The solution set of an underdetermined linear system** $A * X = B$

  ○ is *read off* from the RREF final state (GAUSS-JORDAN algorithm, by hand too) ...

  ○ is *displayed* mostly in parameter form

  ○ is *visualized* as a *straight line* in 2D or a *plane* in the 3D

③ **The dot product** $\cdots \bullet \ldots$ of two vectors

  ○ is *defined* by $X \bullet Y \equiv X * Y^t$

  ○ *serves* to calculate lengths and angles

  – y/n?

  – □□ is *commutative*

  – □□ is *associative*

④ **Formula collection:**

(1) *orthogonality* test: $X \perp Y \Leftrightarrow X \bullet Y = 0$

(2) *length* of vector A: $|A| \equiv \sqrt{A \bullet A}$

(3) *distance* between points A and B: $\operatorname{dist}(A, B) \equiv |A - B|$

(4) *angel* formula – angel between vectors $A = \overrightarrow{OA}$ and $B = \overrightarrow{OB}$: $\angle(A, B) \equiv \arccos\left(\frac{A \bullet B}{|A| \cdot |B|}\right)$

(5) *cross product*: $\operatorname{cross}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} bz - cy \\ az - cx \\ ay - bx \end{bmatrix}$

(5) *Box product*: $\operatorname{Box}(A, B, C) \equiv A \bullet (B \times C)$

$$\bowtie$$

10   Formulae Collection for Analytic Geometry

In this chapter we collect some formulae of Analytic Geometry and program them in
EIGENMATH. Each formula is accompanied by a running example. All code is collected
in a toolbox `geoBox.txt` for easy use in problem solving. We make heavy use of the cross
and Box products.

10.1   Linear Dependence checks using products

Imagine the Box product figure of 9.7. We 'see' vividly, that the three vector a, b, c forming
the spar, point in different directions in space, i.e. they are linearly independent. This is
reflected by the fact that the volume of the associated spar (parallelepiped) is not zero.
Also, the vectors b, c in the base plane are linearly independent, because the area of the
parallelogram spanned by a and b is not zero. Therefore we have the

10.1.1   Fact

Let $U, V, W \in \mathbb{R}^3$ be three[23] vectors in space. Then we have

$$(U, V) \text{ linearly dependent} \iff U \times V = \vec{0}$$
$$(U, V, W) \text{ linearly dependent} \iff U \bullet (V \times W) = 0$$

10.1.2   *Example*

```
yes =1
no  =0
Box(A,B,C) = dot(A, cross(B,C))

linDep2(U,V)   = test( cross(U,V)==(0,0,0) )
linDep3(U,V,W) = test( Box(U,V,W)==0 )   -- pay attention to the double '=='

do( u =(1,2,0),   v =(2,4,0),  v2=(2,1,0) )

linDep2(u,v)          -- two vectors on a line: linDep=1=yes
linDep2(u,v1)         -- two vectors in a plane with diff. directions
linDep3(u,v,v1)       -- three vectors lying in a plane: linDep=1=yes

do( a=(1,1,4), b=(1,0,0), c=(1,3,0))
linDep3(a,b,c)        -- 3 vectors forming a spar
```

Result: 1 0 1 0, meaning linDep= yes no yes no.
▷   *Click here to run the script.*

---

[23]Sometimes we write $\{U, V, W\} \subset \mathbb{R}^3$, if we want to emphasize, that the order of the vectors does not
matter. In our context of EIGENMATH it seems irrelevant.

10.2   Orthogonality Tests

Invoke the Box product figure 9.7 in your head. Then you 'see' vividly before your inner eye, that the normal vector n is orthogonal to the base plane spanned by b and c. In $\mathbb{R}^2$ there is no cross product, so we have to construct a 'normal' vector to another vector in $\mathbb{R}^2$ ad hoc.

10.2.1   Fact

Let $U, N \in \mathbb{R}^3$ be two vectors in space. Then we have

$$U \perp N \iff U \bullet N = 0$$

10.2.2   *Example*

```
perp(U,V)   = test( dot(U,V)==0 )   -- perp(endicular

u =(1,2)
v =(2,-1)
w =(2,1)

perp(u,v)            -- perp=1=yes
perp(u,w)            -- perp=0=no
```

Result: 1 0, i.e. $u \perp v$ and $u \not\perp w$.
▷   *Click here to run the script.*

10.2.3   *Exercise*

We solve *P77.f.* What is the distance between the point
$P(3; 1)$ and the line with equation $\ell : y = 2x$?
▷   *Click here to run the script.*


```
L(t) = (0,0)+ t*(1,2)
L(t)
N=(-2,1)             -- see figure: 2 steps west, 1 step north
n(t)=(3,1)+s*N       -- normal line through P in R^2
n(t)
LS=((1, 2, 3),       --   t = -2s+3 i.e. 1t+2s=3
    (2,-1, 1))       -- 2t =   s+1 i.e. 2t -s=1
A=((1,2),(2,-1))     -- system matrix
B=(3,1)              -- RHS
S=dot( inv(A), B)    -- solve for (t,s)
S                    -- =(1,1)
L(1)                 -- =(1,2).    END of EX.
```

10.3   Incidence checks

In *P65, P66, P67* and *P68* we gave a systematic overview of the possible incidences of lines
and planes. We revise these checks making them more smart using the functions `cross`
and `Box`. We show here one example implementation[24] and refer for the other functions to
our toolbox `geoBox.txt`.

10.3.1   *Example*: **Incidence relation of line and plane** – `relLP`

```
run("geoBox.txt")
                 ----- Line: X=Q+tW  vs Plane: X=R+rU+sV
relLP(Q,W, R,U,V) =
        test(linDep3(W,U,V)==1,          -- (1)
            test(linDep3(U,V,R-Q)==1,    -- (2)
                "line in plane",         -- L in plane
                "line // plane"),        -- L // plane
            1)                           -- 1 = intersect  (3)

-- tests:
do( q=(4,4,4), w=(-1,2,-1), p=(1,0,2), u=(-1,2,-1), v=(0,3,-2))
relLP(q,w, p,u,v)    -- Result: line // plane

do( q=(2,2,1), w=(1,-1,1), p=(1,1,5), u=(2,0,1), v=(-1,-1,3))
relLP(q,w, p,u,v)    -- Result: 1 , i.e. both lines cut in a point.
```

Comment. `relLP` implements the table test from *P67* and solves *P67.a*. In (1) we ask,
if (W, U, V) are lin.dependent by using a box and if -yes- we ask the 2nd criterion (2). If
the 1st test (1) is true and (2) false, we have 'L // plane'. In the 2nd example, test (1) is
false, so because of (3) a '1' is returned: 'line cuts plane'. This returned '1' is used later in
a next step for computing the intersection point, see 10.4. ▷ *Click here to run the script.*

Remark. 1. The toolbox `geoBox.txt` contains also the check `relLL(R,U,Q,V)` for two
lines $X = R+tU$ and $X = Q+sV$ and analogously `relPP(R,U,V,R1,U1,V1)` for 2 planes.
It suffices to know these interfaces to use all three functions in concrete problem solving.
The implementation details are visible for the interested user.
2. In concrete problem solving situations you one need the last 2 lines of code! The
implementation code of `relLP` is shown here as surplus for inspection, but it is already and
silently included in the call `run("geoBox.txt")`.

10.3.2   *Exercise*

○ Check the results of *P65.b*, control *P66* and *P67.b*. using the `rel..`– functions.
○ Check the results with a quality plot.

[24]see [5, pp. 153–155] for an implementation in DERIVE.

10.4   Intersection of Lines and Planes

If the incidence test shows that two geometric objects intersect, one can try to determine
the intersection set. For lines and planes as objects of analytical geometry we therefore
implement three functions:

○   `cutLL(R,U, Q,V)` to determine the intersection of Line X=R+rU and Line X=Q+sV

○   `cutLP(Q,W, R,U,V)` to calculate the intersection of Line X=Q+tW and Plane X=R+rU+sV

○   `cutPP(R,U,V, R1,U1,V1)` for intersection of PPlanes X=R+aU+bV and X=R1+rU1+sV1

10.4.1   *Example: intersection of line and plane −* `cutLP`

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + s \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ is the blue plane of figure.11 in 9.1.2 and
the red normal line is $X = [0, 0, 0] + a \cdot [1, 2, 3]$. Both geometrical
objects intersect in a point in the blue triangle. Determine this
'cut' point S using `cutLP()`.
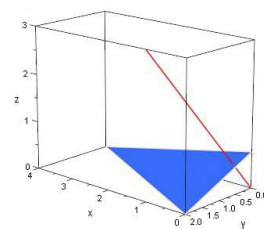
Solution: use EIGENMATH:

```
run("geoBox.txt")

cutLP(Q,W, R,U,V) =                    -- intersection line-plane
    test( relLP(Q,W,R,U,V)==1,   -- if intersection test ok
          Q + Box(U,V,R-Q)/Box(U,V,W) * W )          --(1)

do( q=(0,0,0), w=(1,2,3),  r=(-1,1,1), u=(3,0,-1), v=(2,-1,0) )
float( cutLP(q,w,r,u,v) )
```

*Result*: $S \approx [0.29, 0.57, 0.86]$, which is plausible by looking at the figure.

PS: the calculation of factor a in (1) as a quotient of two volumina is an application of
CRAMER's rule, which we discuss in Part 5 of this series.

▷   *Click here to run the script.*

10.4.2   *Exercise*

a. Redo problems *P65.c, P66* and *.68.b.*

b. Make quality plots to check the solutions using `CalcPlot3D`.

c.

In *P.77* the red vector $OP = [3, 1, -2]$ and the green vector
$OQ = [1, -1, -1]$ span a plane, which intersects the base $xy-$
plan. Determine the intersection line using `cutPP(.)`.
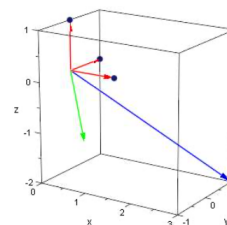
10.5   Distances between Points, Lines, Planes

Most distance formulae invoke more or less the HESSE normal form[25]. Because of the many different representation forms of lines and planes we give here the analogous interfaces. The implementation is straight forward.

Distance formulae in $\mathbb{R}^3$

	Math	EIGENMATH
distance R to plane (X-Q)*N=0 in **n**ormal form	$\left\|(R - Q) \bullet \frac{N}{\|N\|}\right\|$	distRPn(R,Q,N)
distance R to **p**arametric plane X=Q+sU+tV		distRPp(R,Q,U,V)
distance R to plane through 3 points A,B,C		distRP3(R,A,B,C)
distance R to plane in normal form (R-Q)*U=0	$\sqrt{\|R - Q\|^2 - ((R - Q) \bullet \frac{U}{\|U\|})^2}$	distRL(R,Q,U

10.5.1   *Example: distance point to plane*

```
distRPn(R,Q,N)    = abs( dot(R-Q, N/abs(N)) )
distRPp(R,Q,U,V) = distRPn(R,Q,cross(U,V))
distRP3(R,A,B,C) = distRPp(R,A,B-A,C-A)

-- distance of point (9,4,-3) to plane (X-(1,-3,1))*(1,2,2)=0
p=(9,4,-3)
q=(1,-3,1)
n=(1,2,2)
float( distRPn(p,q,n) )
float(14/3)        -- Result: 14/3 ca. 4.66

-- High of pyramid with base A=(4,0,0), B=(4,5,1), C=(0,0,1)
--                 and top D=(-1,4,3)
do( A=(4,0,0), B=(4,5,1), C=(0,0,1), D=(-1,4,3) )
h=distRP3(D,A,B,C)
float(h)       -- Result: 19/21 ca. 0.905
```

▷   *Click here to run the script.*

10.5.2   *Exercise: distance point to line*

Solve *P77.f* again – this time using a distance formula. *Hint*: embed the situation in the xy–plane in $\mathbb{R}^3$, if you plan to use distRPp(..) or distRP3(..).

What is the distance between the point $P(3; 1)$ and the line with equation $\ell : y = 2x$?   ▷ *To invoke* EIGENMATH

---

[25]see e.g.: Weisstein, Eric W. "Hessian Normal Form." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/HessianNormalForm.html

10.5.3   Example: distance of two skew lines

```
run("geoBox.txt")
-- 1: distance of point R to line Q+tU
R=(2,-3,5)
Q=(4,3,3)
U=(2,1,-1)
float(distRL(R,Q,U))
float(2*sqrt(5))

-- 2: distance of two skew lines
      -- L1: X=(6,1,-4)+t*(4,1,-6)
      -- L2: X=(4,0,3)+t*(0,-1,3)
U=(4,1,-6)
V=(0,-1,3)
Q=(6,1,-4)
R=(4,0,3)
distRPp(R,Q,U,V)    -- Result: 10/3
```

▷ *To the script.*
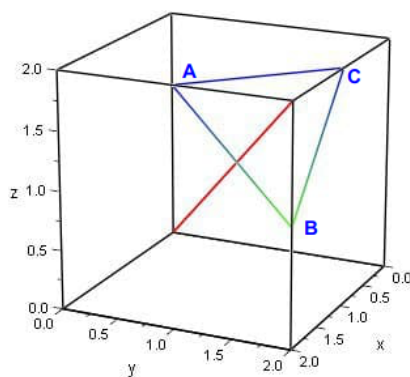
P81. Distance of a point to a plane.

Red: the line L from $(0, 0, 0)$ to $(2, 2, 2)$.
Figure 13: Blue: the plane $\triangle$ hold by the 3 points $A = (2, 1, 2)$,
$B = (2, 2, 1)$, $C = (1, 2, 2)$.

    a. Determine the intersection S of line L and planE E.

    b. What is $\mathrm{dist}(S, \triangle)$?

    c. What is $\mathrm{dist}((1, 1, 1), \triangle)$?

    d. Ask more question and give the solutions.

    e. Make a quality plot using `CalcPlot3D` .

10.6   Switching between representation forms of planes

To calculate distances it is sometimes preferable to use the normal form of a plane in order to set up the HESSE distance formula. Therefore we present some formulae to change from a parametric form of a plane to a normal form. We show the interfaces and in selected cases the implementation, too.

<div align="center">

Switch between representations of planes

(EIGENMATH procedures returns the LHS only!)

</div>

	Math	EIGENMATH
parametric eq. of plane	$X = Q + t \cdot U + s \cdot V$	`paraP(Q,U,V)`
parametric eq. of plane through 3 points A, B, C	$X = A + t \cdot (B - A) + s \cdot (C - A)$	`paraP3(A,B,C)`
normal form of plane through Q perp. to N	$(X - Q) \bullet N = 0$	`normalP(Q,N)`
normal form of plane $X = R + t*U + s*V$	$(X - R) \bullet (U \times V) = 0$	`normalP2(R,U,V)`
normal form of plane through 3 points	$(X - A) \bullet ((B - A) \times (C - A)) = 0$	`normalP3(A,B,C)`
HESSE normal form of plane through Q perp. to N	$(X - Q) \bullet \frac{N}{|N|} = 0$	`Hesse(Q,N)`
HESSE normal form of plane $X = Q + t*U + s*V$	$(X - Q) \bullet \frac{U \times V}{|U \times V|} = 0$	`Hesse2(Q,U,V)`
HESSE normal form of plane through 3 points A, B, C		`Hesse3(A,B,C)`
HESSE's distance of $R \notin$ plane to plane through Q	$\left| (R - Q) \bullet \frac{N}{|N|} \right|$	`HesseDist(R,Q,N)`

10.6.1   *Example script*

```
X=(x,y,z)           -- do NOT give X another value here
normalP(Q,N)    = dot(X-Q,N)
normalP2(R,U,V) = normalP( R, cross(U,V))
normalP3(A,B,C) = normalP2(A,B-A,C-A)

-- plane though q perpendicular to n
q=(4,1,3)
n=(2,-1,5)
normalP(q,n)

-- normal form of plane X=(5,2,3)+r*(1,0,2)+s*(0,-5,8)
R=(5,2,3)
U=(1,0,2)
V=(0,-5,8)
normalP2(R,U,V)

-- normal form of a plane going through 3 points
do( A=(1,1,0),B=(1,0,1), C=(0,1,1))
normalP3(A,B,C) = normalP2(A,B-A,C-A)
normalP3(A,B,C)
```

▷   *Click here to run the script.*

10.6.2   *semi–automatic switching of representation forms*

a. To change an implicit coordinate form of a plane to a parametric one, we first determine three 'easy' points of the plane (e.g. the intersections with the coordinate axes) and then invoke `paraP3(.)`.

```
-- semi-automatic
-- Find parametric form of plane 2x-y+4z=6
E(x,y,z) = 2x-y+4z-6    -- = 0

paraP(Q,U,V)    = Q+t*U+s*V
paraP3(A,B,C)   = paraP(A,B-A,C-A)

-- 1. determine cut points
E(0,0,z)         --i.e. z=6/4 =3/2 =1.5, gives point A
E(0,y,0)         --i.e. y=-6,  gives point B
E(x,0,0)         --i.e. x=3, gives point C=(3,0,0)
-- 2. invoke paraP3 with A,B,C
paraP3((0,0,1.5),(0,-6,0),(3,0,0))
```

▷   *Click here to run the script.*

b. Here is another solution of a. without the `geoBox` functions, using a previous reordering of the coordinate equation of the plane:

```
-- semi-automatic transformation of plane 2x-y+4z=6 into parametric
-- 2x-y+4z = 6
--      2x = 6+y-4z
--       x = 2+0.5y-2z
E(y,z) = (2+0.5y-2z, y, z)
E(1,1)

-- you may rename the parameter y,z to r,s
-- and split the term into three vector components
E(r,s) = (2,0,0)+r*(0.5,1,0)+s*(-2,0,1)
E(1,1)
```

▷   *Click here to run the script.*

10.6.3   *Using* HESSE *normal form (HNF) – the Hessian*

Warning: the HESSE normal form needs as input data a point Q, *which must lie on the given plane.*

Example.

```
X=(x,y,z)
normalP(Q,N)     = dot(X-Q,N)              -- to be = 0
HesseNF(Q,N)     = dot(X-Q, N/abs(N))   -- = 0, with Q _in_ plane
HesseDist(R,Q,N) = mag( dot(R-Q, N/abs(N)) )

-- distance of R(1,6,2) to H: x-2y+4z=1
--            R               N(1,-2, 4)

H(x,y,z)=x-2y+4z-1  -- the given (hyper)plane as function term
R=(1,6,2)           -- R must be outside of plane H
N=(1,-2,4)

H(1,6,2) == 0       --(1) test returns 0=FALSE, i.e. R is _outside_ H
                    --    Now look for a special point Q _inside_ H
H(1,0,0) == 0       --(2) test returns 1=TRUE, therefore Q=(1,0,0) in H
Q=(1,0,0)

-- Now invoke Hesse normal form
HNF = HesseNF(Q,N)  -- needs Q in H, which is ok here
HNF                 -- (3)

-- compute the distance of R to H by plugging in R for X in HNF
HesseDist(R,Q,N)         --(4)
float(HesseDist(R,Q,N))  --(5) distance R to H ca. 0.87
```

EIGENMATH output: display of results of commands in line (1) .. (5):

$$0$$

$$1$$

$$H_{NF} = -\frac{1}{3^{1/2}\,7^{1/2}} + \frac{x}{3^{1/2}\,7^{1/2}} - \frac{2\,y}{3^{1/2}\,7^{1/2}} + \frac{4\,z}{3^{1/2}\,7^{1/2}}$$

$$\frac{4}{3^{1/2}\,7^{1/2}}$$

$$0.872872$$

▷   *Click here to run the script.*

Remark. Functions `Hesse2(Q,U,V)` returns the HNF of a plane $X = Q + t \cdot U + s \cdot V$ given in vector form and `Hesse3(A,B,C)` returns the HNF, if the plane is given via 3 different points.

10.7   Volumina of Box and Pyramid

The formulae for the volumina of a parallelepiped and a pyramid are straight forward and given here for the purpose of completeness. This brings our investigation of some formulae of elementary Analytic Geometry to an end.

<div align="center">

Volumia of box and pyramid in $\mathbb{R}^3$

</div>

	Math	Eigenmath
volume of box spanned by edges U,V,W	$\lvert (U \times V) \bullet W \rvert$	VolBox(U,V,W)
volume of box with 4 points A,B,C,D		VolBox4(A,B,C,D)
volume of pyramid with the 4 points A,B,C,D		VolPyr(A,B,C,D)

10.7.1   Eigenmath *Example*

```
---- implementation of volume formulae
Box(A,B,C)       = dot(A, cross(B,C))

VolBox(U,V,W)    = mag( Box(U,V,W) )
VolBox4(A,B,C,D) = VolBox(B-A,C-A,D-A)
VolPyr(A,B,C,D)  = 1/6 * VolBox4(A,B,C,D)

-- interface examples:
--(1) volume of box spanned by U=(2,3,5),V=(2,-1,7), W=(3,9,2)
do( U=(2,3,5),V=(2,-1,7), W=(3,9,2) )
VolBox(U,V,W)

1/6 * VolBox(U,V,W)     -- volume of associated pyramid

--(2) volume pyramid: base A=(4,0,0), B=(4,5,1), C=(0,0,1) top: D=(-1,4,3)
do( A=(4,0,0), B=(4,5,1), C=(0,0,1), D=(-1,4,3) )
VolPyr(A,B,C,D)
float(last)      -- Result: 19/21 ca. 0.905float(last)
-- Result: 19/21 ca. 0.905
```
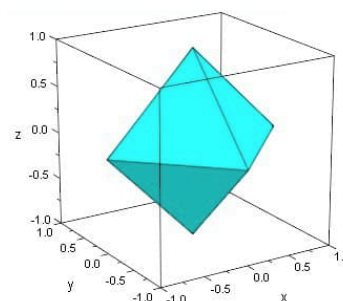
▷   *Click here to run the script.*

10.7.2   *Exercise*

What is the volume of the double regular pyramid?
Calculate the area of a regular face.
Where does the line though $(1, 0, 0)$ and $(-1, -1, 1)$ intersects
the pyramid's base?

⋈

The important concept of *orthogonal projection* incl. a study of *overdetermined linear systems* and best fit problems is a main topic of Part 4 in this series. More about volumina, determinants and other product concepts will be found in Part 5 about *Geometric Algebra*, where we also discuss a modern view on Analytic Geometry in low dimensional Clifford algebras – using EIGENMATH!

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Links checked 1.1.2021, wL

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January 2021